MAY 1980

PPPL-1655

UC-20g

DR. 1219

MASTER

RESISTIVE INSTABILITIES AND
FIELD LINE RECONNECTION

BY

R. B. WHITE

PLASMA PHYSICS
LABORATORY

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U.S. Department of Energy
Contract No. DE-AC02-76-CHO 3073. Reproduction, transla-
tion, publication, use and disposal, in whole or in part,
by or for the United States government is permitted.

Resistive Instabilities and Field Line Reconnection

by

R. B. White

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544

DISCLAIMER

This document contains information of a confidential nature. It is the property of the United States Government and is loaned to you by the United States Government. It is not to be distributed outside the United States Government. It is not to be used for any purpose other than that for which it was loaned. It is not to be copied, reproduced, or otherwise disclosed in any manner. It is not to be used in any way that would reflect unfavorably on the United States Government. It is not to be used in any way that would be prejudicial to the national defense. It is not to be used in any way that would be detrimental to the national health, safety, or interest. It is not to be used in any way that would be contrary to the public interest. It is not to be used in any way that would be in violation of any law, regulation, or executive order. It is not to be used in any way that would be in violation of any policy, procedure, or guideline. It is not to be used in any way that would be in violation of any contract, agreement, or understanding. It is not to be used in any way that would be in violation of any duty, obligation, or responsibility. It is not to be used in any way that would be in violation of any law, regulation, or executive order. It is not to be used in any way that would be in violation of any policy, procedure, or guideline. It is not to be used in any way that would be in violation of any contract, agreement, or understanding. It is not to be used in any way that would be in violation of any duty, obligation, or responsibility.

CB

OUTLINE

1. Introduction
 - 1.1 Motivation and scope of review
 - 1.2 Magnetic surfaces and plasma-field coupling
2. Linear Theory in Slab Geometry
 - 2.1 The rippling mode
 - 2.2 The gravitational interchange mode
 - 2.3 The tearing mode
 - 2.4 Magnetic energy of the tearing mode
3. Cylindrical Geometry
 - 3.1 The inverse aspect ratio expansion
 - 3.2 Single helicity equations
 - 3.3 The resistive kink mode
 - 3.4 The resistive internal kink mode
 - 3.5 High beta and multiple helicities
4. Nonlinear Behavior
 - 4.1 Introduction and survey
 - 4.2 Rutherford behavior
 - 4.3 Nonlinear cylindrical modes $m > 1$
 - 4.4 Nonlinear cylindrical modes $m = 1$
 - 4.5 Multiple tearing modes
 - 4.6 Externally-driven reconnection
5. Ergodicity
 - 5.1 Ergodic transition
 - 5.2 Kolmogorov entropy and field diffusion
 - 5.3 Ergodicity and transport

1. INTRODUCTION

1.1 Motivation and Scope of Review

Laboratory plasma devices are generally designed with ideal magnetohydrodynamic stability in mind, so it is not surprising that the next most rapid class of modes, the resistive instabilities, figure prominently in the description of the behavior of the plasmas achieved. In this chapter we review the linear and nonlinear plasma behavior associated with simple scalar resistivity.

The essential feature of resistive instabilities is that they violate the basic topological restriction of the infinite conductivity analysis; namely, the preservation of the distinct identity of separate magnetic flux surfaces. The magnetic surfaces form topologically distinct regions. Magnetic reconnection consists of the growth of one or more regions, in terms of the total magnetic flux in the region, at the expense of other regions. Heuristically, the process consists of the "breaking" of magnetic lines of force in one region and their recombination in a way to make the resulting lines a part of another region. The tearing mode, for example, consists of the growth of a magnetic island configuration. It occurs when an increase in island size leads to a state of lower magnetic energy. Since the state is not accessible without magnetic reconnection, the time scale for the mode is a hybrid of the magnetohydrodynamic time scale associated with the magnetic driving forces τ_A , and the resistive time scale

associated with the reconnection, τ_R . The time scale for resistive modes is given by $S\tau_A$, with $S = \tau_R/\tau_A$ having values greater than 10^6 in present-day fusion research devices, and p being a fractional power which depends on the nature of the mode and varies from 0 to 1.

The resistive diffusion of the magnetic field with respect to the fluid, which allows changes in magnetic topology, occurs at a significant rate only in a narrow region of space surrounding a particular singular surface in the field, the "resistive layer", and the width of this layer tends to zero in the limit of zero resistivity. Thus, the solution of the general resistive problem consists of solutions of the ideal magnetohydrodynamic equations in the "outer" or infinite conductivity region, joined across the layer by "interior" or resistive solutions.

There are two types of problems of interest for laboratory plasmas, the distinction having to do with the specification of the boundary conditions. The first type consists of problems with fixed external field conditions with no externally imposed plasma flow; the plasma possessing a nearby state of lower magnetic energy which is, however, inaccessible without magnetic reconnection. This situation is of primary interest in the study of equilibria of toroidal confinement devices. A typical example consists of an axisymmetric ideal magnetohydrodynamic equilibrium with magnetic surfaces consisting topologically of nested tori, for which a nearby lower energy state exists which possesses one or more magnetic islands. With boundary conditions held fixed so that the only driving force is due to the energy difference between the two

states, the transition to the lower energy state is limited by the rate of magnetic reconnection.

The second type of problem of interest involves external driving forces, either in the form of adjustable external field coils, such as in the formation of a doublet or "figure eight" configuration from an initial configuration consisting simply of nested tori, or in the form of a fixed externally driven flow, as in the case of models for the description of a solar flare or magnetic reconnection in planetary magnetospheres. In these latter cases, the rate of reconnection is primarily determined by the external driving forces, and the plasma can be imagined to be advancing adiabatically from one state of minimum energy to the next, or even to be essentially in a steady state configuration. This physical situation is similar to that of standard hydrodynamic shocks, and seems suitable to a formulation consisting of the replacement of the resistive layers with suitable jump conditions joining the ideal outer solutions.

Much of the history of a typical tokamak discharge is known, or conjectured, to be due to tearing mode activity. In the initial stages of a discharge, the double tearing mode is thought to play a role in producing rapid current penetration into the plasma. During the discharge, there occur sawtooth oscillations and Mirnov oscillations, which are understood to be due to tearing modes, and the observed anomalous electron thermal transport is possibly due to small magnetic islands and resulting stochastic field behavior produced by microinstabilities. Finally, abrupt termination of a

discharge, through what is referred to as a major disruption, is understood to be due primarily to tearing mode activity.

The organization of this chapter is as follows: In Section 1.2 we review the preliminary notions of the existence of magnetic surfaces, the coupling of field and plasma, and the relaxation of this strict coupling brought about by finite scalar resistivity.

Section 2 consists of a review of the linear theory of reconnection for a plane current layer. The three basic modes are the Rippling Mode (Section 2.1), the Gravitational Interchange Mode (Section 2.2), and the Tearing Mode, discussed in Section 2.3. In Section 2.4 there is a derivation of the magnetic field energy which provides the driving force for the tearing mode. The existence of this energy and hence the stability of the mode is a question involving the global geometry of the field. In Section 3 the necessary concepts for the analysis of tearing modes in cylindrical geometry are introduced. The equations governing tearing mode evolution in a tokamak are expanded to lowest order in the inverse aspect ratio in Section 3.1, resulting in differential equations for the analysis of the linear stability of these modes in cylindrical geometry and in reduced equations for the nonlinear evolution of a single helicity (Section 3.2). The tearing mode in a toroidal device is closely related to the ideal magnetohydrodynamic kink mode, and this relationship is stressed in the derivations of the linear growth rates for modes with poloidal mode number $m > 2$ (Section 3.3) and for the quite different $m = 1$ mode (Section 3.4). In Section 3.5 are derived the equations

governing the nonlinear coupling of different helicities and the effect of plasma pressure.

In Section 4 we review the nonlinear theory of tearing mode development and the implications of this theory for the understanding of toroidal magnetic confinement devices. Section 4.1 consists of a survey of the nonlinear behavior which is reasonably understood and a brief description of the analytical and numerical methods employed. Section 4.2 treats the transition from exponential to algebraic growth, which occurs when the island width exceeds the width of the tearing layer. Section 4.3 discusses mode saturation, which can occur when the island width is comparable to the scale length defined by the shear. This scale length is generally determined by the size of the device itself, and thus the global geometry is relevant in this process. The nonlinear behavior of the $m = 1$ mode, discussed in Section 4.4, is quite different from that of modes with $m > 1$. In Section 4.5, multiple tearing modes are considered. The relevance of the nonlinear behavior of all these modes in the interpretation of the behavior of magnetic confinement fusion devices is briefly discussed. Section 4.6 consists of a discussion of other nonlinear methods used in resistive reconnection analysis. These include steady state treatments which employ standing waves and shock fronts, the use of adiabatic, long-time scale methods, replacement of the resistive layer with jump conditions, and postulation of invariants for resistive evolution.

In Section 5 we consider the implications of the presence of a background of small magnetic islands for plasma properties, in general, and for magnetic confinement, in particular. Section 5.1 is a discussion of the transition to ergodic field behavior, Section 5.2 treats the geometrical characterizations of fields once they have become ergodic, and Section 5.3 discusses the implications of ergodicity for transport.

We use rationalized Gaussian units with $c = 1$.

1.2 Magnetic Surfaces and Plasma-Field Coupling

The existence of magnetic surfaces in magnetic confinement devices, or at least the existence of approximate magnetic surfaces over a large fraction of the plasma volume, is an essential requirement for long-term confinement. The existence of such surfaces has been shown only under fairly restrictive conditions (Morozov and Solov'ev 1966). They are known to exist everywhere or in all but a small part of the plasma volume only when there exists a symmetry or approximate symmetry. This is easily demonstrated in cylindrical geometry for translational, axial, and helical symmetry. Writing the field in terms of the vector potential, $\vec{B} = \vec{\nabla} \times \vec{A}$, the surfaces are defined by $A_z(r, \theta) = \text{constant}$ in case $\vec{A}$ is translationally invariant in z , $rA_\theta(r, z) = \text{constant}$ in case $\vec{A}$ is independent of θ , and $A_z(r, \theta - \alpha z) + \alpha rA_\theta(r, \theta - \alpha z) = \text{constant}$ in case $\vec{A}$ is helically invariant; i.e., depends only on the variables $r, \theta - \alpha z$. It is readily verified that these equations define surfaces to which $\vec{B}$ is tangent.

According to a theorem by Kolmogorov (1957), small perturbation of a symmetric case leaves well defined magnetic surfaces existing everywhere except in a small volume proportional to the perturbation, where the field assumes a stochastic character. The onset of stochasticity will be further discussed in Section 5.1.

Surfaces are also known to exist in the vicinity of a closed field line (a fixed point of the mapping defined by the field). Consider a fixed point at $\vec{r} = (x, y) = 0$; i.e., $\vec{B}_1 = (B_x, B_y) = 0$ for $0 < z < L$, with periodicity imposed on the field, $\vec{B}(x, y, 0) = \vec{B}(x, y, L)$. In the vicinity of $r = 0$ the equation for the field line is

$$\frac{d\vec{r}}{dz} = \frac{\vec{B}_1}{B_z} = (\vec{r} \cdot \vec{\nabla}) \vec{B}_1 \quad (1)$$

Assuming that $\vec{\nabla} \cdot \vec{B}_1 \neq 0$, it can be approximated by its value at $\vec{r} = 0$, giving a linear first order differential equation for $\vec{r}$ with coefficients periodic in z . Suppose we have found two solutions to this equation $\vec{r}_1(z)$, $\vec{r}_2(z)$ with the initial values $\vec{r}_1(0) = (1, 0)$, $\vec{r}_2(0) = (0, 1)$. Then the solution for initial condition $\vec{r}(0) = (a, b)$ is given by $\vec{r} = a\vec{r}_1(z) + b\vec{r}_2(z)$. Tracing a field line in successive passes through the cylinder is thus reduced to a finite difference equation for the coordinates (x_n, y_n) of $\vec{r}$ after n passes

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = (\alpha_{ij}) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad (2)$$

with $\alpha_{ij} = [\vec{r}_i(L), \vec{r}_j(L)]$. From $\vec{\nabla} \cdot \vec{B} = 0$ we have that $\det(\alpha_{ij}) = 1$. Attempt a solution of the form $x_n = C_1 \lambda^n$, $y_n = C_2 \lambda^n$. Substituting, we find the characteristic equation

$$\lambda^2 + (\alpha_{11} + \alpha_{22})\lambda + 1 = 0 \quad (3)$$

with a general solution of Eq. (2) given in terms of the two roots of Eq. (3) by

$$x_n = a_1 \lambda_1^n + a_2 \lambda_2^n \quad (4)$$

$$y_n = b_1 \lambda_1^n + b_2 \lambda_2^n .$$

The form of the trajectory depends on the magnitude of the trace of α_{ij} . If $|\alpha_{11} + \alpha_{22}| < 2$, let $\cos \mu = (\alpha_{11} + \alpha_{22})/2$, and the characteristics become $\lambda_{1,2} = \exp(\pm i\mu)$. A typical trajectory is given by

$$x_n = a \cos(\mu n) \quad (5)$$

$$y_n = b \sin(\mu n)$$

and the fixed point is referred to as elliptical. The surface mapped out by Eq. (5) either forms a discrete set of points or is dense on the ellipse depending on whether μ is a rational multiple of π .

If $|\alpha_{11} + \alpha_{22}| > 2$, let $\cosh \mu = (\alpha_{11} + \alpha_{22})/2$, and the characteristics become $\lambda_{1,2} = \exp(\pm \mu)$. A typical trajectory is then given by

$$x_n = a \cosh(\mu n) \quad (6)$$

$$y_n = b \sinh(\mu n)$$

and the fixed point is referred to as hyperbolic. In either case the surfaces are well defined. The approximation of linearizing $\vec{B}_1$ in the vicinity of $\vec{r} = 0$ assumes that the sum over higher harmonics will not destroy the surfaces found. This has been shown by Arnol'd (1963) and Moser (1962). They proved that nonlinear terms only lead to a λ dependence of the coordinate; i.e., a distortion of the surfaces, but not to their destruction.

The first step in determining the degree to which such surfaces are destroyed is the determination of all instabilities which lead to magnetic reconnection. This was first carried out in a systematic manner by Furth, et al. (1963). Special cases had been considered earlier by various authors cited by them. Numerical confirmation of the analytically obtained growth rates was carried out by Wesson (1966). A number of simplifying assumptions are made in this analysis. The hydromagnetic approximation is assumed and a simplified Ohm's law is used, of the form $\vec{E} + \vec{u} \times \vec{B} = \eta \vec{J}$, where η is a scalar resistivity (Spitzer 1953).

The consideration of resistive modes in noncollisional plasmas, and of kinetic theory effects is beyond the scope of this review. The reader is referred to papers by Laval et al. (1965), Hazeltine and Ross (1978), Drake and Lee (1977), and Ross et al. (1979), Waddell et al. (1977) and Ora et al. (1978).

From Maxwell's equations $\partial \vec{B} / \partial t = - \vec{\nabla} \times \vec{E}$, $\vec{\nabla} \times \vec{B} = \vec{j}$ and Ohm's law, we find

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B} \cdot \vec{\nabla}) \vec{u} - (\vec{u} \cdot \vec{\nabla}) \vec{B} - \vec{B} (\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \times \eta (\vec{\nabla} \times \vec{B}) \quad (7)$$

which for $n = 0$ describes the convection (Terms 1 and 2 on the right), and compression (Term 3) of the magnetic field with the fluid.

To illustrate the basic functioning of Eq. (7) in hydromagnetic stability problems, we consider the interchange mode (Furth, 1963).

Consider perturbations of the form

$$f_1 = f_1(y) \exp[\gamma t + i(k_x x + k_z z)] \quad (8)$$

in the (plane) equilibrium configuration

$$\vec{B}_0 = \hat{x} B_{x0}(y) + \hat{z} B_{z0}(y) \quad (9)$$

If we set $n = 0$, we obtain for the y component of Eq. (7), simply,

$$\gamma B_{y1} = i(\vec{k} \cdot \vec{B}_0) u_{y1} \quad (10)$$

If B_0 is unidirectional over all space, we can orient $\vec{k}$ so that $\vec{k} \cdot \vec{B}_0 = 0$, and the perturbation then does not distort magnetic field lines (i.e., $B_{y1} = 0$). The mere displacement of the undistorted magnetic field lines in the y plane does not alter their energy content. Thus we find the familiar result that a fluid in a plane unidirectional magnetic field has neutral stability against the interchange mode.

Let us now consider the more general magnetic field, where the direction of B_0 is y dependent. For any choice of $\vec{k}$, we can then divide $\vec{B}_0$ into a component $\vec{B}_{0\perp}$ orthogonal to $\vec{k}$ and a component $\vec{B}_{0\parallel}$ parallel to $\vec{k}$. We note that $\vec{\nabla} \cdot \vec{B} = 0$ holds for $\vec{B}_{0\perp}$ and $\vec{B}_{0\parallel}$ individually, so that we may treat them as distinct magnetic fields. The behavior of the component $\vec{B}_{0\perp}$ during an interchange perturbation is identical with the behavior of the unidirectional $\vec{B}_0$. Therefore, this component has no influence on stability. The component $\vec{B}_{0\parallel}$ appears in Eq. (10), and its persistence implies that any interchange perturbation increases the magnetic energy of the system via B_y generation. In this way finite "magnetic shear" stabilizes against weakly driven interchange modes; e.g., those of a fluid with a mass-density gradient $\partial \rho_0 / \partial y$ in a weak, y directed gravitational field. In a perfectly conducting fluid, where field and fluid are coupled by Eq. (10), it takes a finite destabilizing force to overcome a finite magnetic shear.

Let us now see to what extent fluid resistivity can effect a decoupling. There is no need to decouple the fluid from $\vec{B}_{0\perp}$ since the latter does not impede the motion and so we may restrict ourselves to consideration of $\vec{B}_{0\parallel}$. Since $(\partial \vec{B} / \partial t) = -\vec{\nabla} \times \vec{E}$, a sufficient condition for decoupling is $\vec{E}_\perp = 0$. In that case $\vec{j} = \vec{E} + \vec{u} \times \vec{B}$ yields

$$\vec{j}_\perp = \frac{\vec{u}_\perp \times \vec{B}_{0\parallel}}{\eta}$$

and there is a restoring force $\vec{F}_1 = \vec{j}_\perp \times \vec{B}_{0\parallel}$

with a y component

$$F_{y1} = - \frac{u_{y1} B_{0||}^2}{\eta}$$

for $\eta \rightarrow 0$, we find that $-F_{y1}/u_y$ becomes infinite everywhere except at the nulls of $\vec{B}_{0||}$; this fact is, of course, consistent with the earlier result that $B_{y1} \equiv 0$ is not possible. For any finite η , however, there is a finite region ϵL about each null of $\vec{B}_{0||}$, where the restoring force is arbitrarily weak. In this region the magnetic field is decoupled from the fluid and slips through it. The width of this recombination region must be calculated for each mode, and is related to the growth rate. The rate at which work is done on the fluid is

$$P = -u_y \cdot (\vec{j} \times \vec{B}) = \frac{u_{y1}^2 (B')^2 (\epsilon L)^2}{\eta} \quad (11)$$

where we have used $B_{||} \sim B' \epsilon L$. In general, the instability wavelength is much larger than ϵL and thus the fluid kinetic energy in the $\vec{k}$ direction is dominant. Equating the rate of change of this energy to the driving power we have

$$\gamma \rho u_y^2 / k^2 \epsilon^2 L^2 = u_{y1}^2 (B' \epsilon L)^2 / \eta$$

and thus

$$\epsilon L \sim \left(\frac{\gamma \rho \eta}{k^2 (B')^2} \right)^{1/4} \quad (12)$$

There are two time scales associated with resistive reconnection; the ideal magnetohydrodynamic time τ_A proportional to $\rho^{1/2}$ and the resistive time τ_R , proportional to $1/\eta$. The ratio, $S = (\tau_R/\tau_A) = 1/(\eta\rho^{1/2})$ is very large in most plasmas of interest. The growth rate we will find to be proportional to $S^p\eta$ for some fractional power p and thus from Eq. (12) the reconnection region is very narrow.

The approximation of fluid incompressibility is generally valid for tokamak plasmas because of the large toroidal field. Viscosity is neglected, so that the equation of motion takes the form

$$\vec{\nabla} \times \mu \frac{d\vec{u}}{dt} = \vec{\nabla} \times [(\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{g}\rho] \quad (13)$$

where ρ is the mass density and $\vec{g}$ the acceleration due to gravity. Ohmic heating is neglected. The initial state is assumed to have zero flow ($\vec{u}_0 = 0$) which implies neglecting motion on the resistive diffusion time scale. The modification of growth rates in the presence of an equilibrium flow has been investigated by Killeen and Shestakov (1978).

Three basic modes are found; a long wavelength "tearing" mode corresponding to breakup of the current layer along current flow lines; a short wave "rippling" mode due to the flow of current across the resistivity gradients of the layer; and a low g gravitational interchange mode. In the present review we will concentrate on the tearing mode, for which the gravitational term is not important. This mode is most unstable for large

wavelengths, and thus plays an important role in the overall stability of magnetically confined plasmas.

2. LINEAR THEORY IN SLAB GEOMETRY

2.1 The Rippling Mode

This mode is caused by the local channeling of the current through ripples in the resistivity. Ohm's law is modified by the resistivity change through

$$\eta_0 \vec{j}_1 = -\eta_1 \vec{j}_0 + \vec{u} \times \vec{B} \quad (14)$$

where η_1 is given by the convective law $\eta_1 = -(\vec{u}/\gamma) \cdot \vec{\nabla} \eta_0$. Within the reconnection layer $\vec{E} = 0$ and the η_1 term in Eq. (14) gives rise to a force $F_d = \vec{j}_1 \times \vec{B} = (\vec{u} \cdot \vec{\nabla} \eta_0 / \gamma \eta_0) (\vec{j}_0 \times \vec{B})$ that changes sign at the singular surface. Hence, F_d is stabilizing on the side of higher resistivity and destabilizing on the side of lower resistivity. An unstable mode results if the region of decoupled flow lies on the lower resistivity side and has a width ϵL such that the driving power

$$\vec{u} \cdot \vec{F}_d \sim u_y^2 \eta_0' (B')^2 \epsilon L / \eta_0 \gamma \quad (15)$$

dominates the work done on the fluid P . Comparing Eqs. (11) and (15) we find

$$\epsilon L = \eta_0' / \gamma \quad (16)$$

From Eq. (12) we then find

$$\gamma \sim \left(\frac{(\eta')^2 k_B'}{(\eta \rho)^{1/2}} \right)^{2/5} \sim S^{2/5} \eta \quad (17)$$

This mode is thus most rapidly growing for small wavelengths.

2.2 The Gravitational Interchange Mode

This mode occurs in the presence of a mass density gradient, a y directed 'gravitational' field. The driving force is of the form

$$F_d = \rho_1 g = - \frac{u_y \rho_o' g}{\gamma} \quad (18)$$

which is destabilizing if g points toward decreasing density. Comparing $u_y F_d$ with Eq. (11) gives

$$SL = \frac{(\rho_o' g \eta)^{1/2}}{B \gamma^{1/2}} \quad (19)$$

from which we find

$$\gamma \sim \left(\frac{k g \rho_o'}{B} \left(\frac{\eta}{\rho} \right)^{1/2} \right)^{2/3} \sim S^{2/3} \eta \quad (20)$$

This mode is also most unstable for small wavelengths.

2.3 The Tearing Mode

In this section we derive the growth rate for a magnetic island in slab geometry (Furth, et al. 1963). The zero order field is fixed in time and all perturbations are required to vanish far from the singular surface.

Take the initial field of the form

$$\begin{aligned} B_{y0} &= \bar{B}F(x) \\ B_z &\gg \bar{B} \\ \rho_{0y} &= 0 \end{aligned} \quad (21)$$

where $F(x) = \vec{k} \cdot \vec{B}_0 / k B_0 \approx x$ for $x \ll 1$ and $F(x)$ is odd with $F(x) \rightarrow 1$ for $x \gg 1$. The effect of an asymmetric current layer has been considered by Dubois and Samain (1979). The distance from the singular surface x is normalized to the width of the current sheet $L^{-1} = (1/\bar{B})(\partial B_y / \partial x)$, and B_z is constant in space. The plasma is initially stationary, $\vec{u}_0 = 0$. According to Section 1.2, a singular surface exists at $x = 0$ for modes propagating in the y direction.

Use $\vec{\nabla} \cdot \vec{B} = 0$ to introduce a flux function $\psi(x, y)$ with

$$\vec{B} = \vec{\nabla} \psi \times \hat{z} + B_z \hat{z} \quad (22)$$

The scalar function ψ satisfies $\vec{B} \cdot \vec{\nabla} \psi = 0$; i.e., $\vec{B}$ is tangent to surfaces of constant ψ . We also have from Ampere's law $\vec{\nabla} \times \vec{B} = \vec{j}$

$$\nabla^2 \psi = -j_z \quad (23)$$

The simplified form of Ohm's law is

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \quad (24)$$

Taking the z component and using Faraday's law $\vec{\nabla} \times \vec{E} = -(\partial B / \partial t) \hat{z}$ we find for ψ ,

$$\frac{\partial \psi}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \psi = \eta \nabla^2 \psi + E \quad (25)$$

where E is any externally imposed constant field which we will take equal to zero. For the velocity, use the electron equation of motion

$$m \frac{d\vec{u}}{dt} = \vec{j} \times \vec{B} - \vec{\nabla} p \quad (26)$$

where (d/dt) is the convective derivative, $(d/dt) = (\partial/\partial t) + \vec{u} \cdot \vec{\nabla}$, and operate on this equation with $\hat{z} \cdot \vec{\nabla} \times$, also using the incompressibility of the plasma in the x - y plane (large B_z) to introduce the velocity stream function ϕ with $\vec{u} = \vec{\nabla} \phi \times \hat{z}$. We find

$$\eta \frac{d}{dt} \nabla^2 \phi = -\hat{z} \cdot [\vec{\nabla} \psi \times \vec{\nabla} (\nabla^2 \psi)] \quad (27)$$

Thus we are left with two coupled second order partial differential equations for the scalar potentials ϕ and ψ . The zero order state is completely described by $\psi_0(x) = -\tilde{B} F(x)$ and $\phi_0 = 0$. The initial current density profile is given by $j_{z0} = -\psi_0''$.

We now linearize Eqs. (27) and (25), introducing a perturbation which will "interact" with the singular surface at $x \approx 0$. Write

$$\psi(x, y) = \psi_0(x) + \psi_1(x) \cos ky \quad (28)$$

$$\phi(x, y) = (\gamma/k\bar{B})\phi_1(x) \sin ky$$

where $\psi_1(x)$, $\phi_1(x)$ are assumed to vary in time as $e^{\gamma t}$. In Fig. 1 is shown the resulting pattern of magnetic field surfaces, $\psi = \text{constant}$, forming a chain of islands. The value of ψ on the separatrix is found easily at the x point $\psi_s = \psi_0(0) - \psi_1(0)$ and setting $\psi(x, y)$ equal to this value for $y = 0$ and expanding $\psi_0(x)$ in a Taylor series about $x = 0$ gives for the island width

$$W = 4(-\psi_1/\psi_0'')^{1/2} \quad (29)$$

Substituting expressions (28) into Eqs. (25) and (27) we find two coupled second order ordinary differential equations for $\psi_1(x)$, $\phi_1(x)$.

$$\psi_1(x) - F(x)\phi_1(x) = (1/\gamma\tau_R)[\psi_1''(x) - k^2\psi_1(x)] \quad (30)$$

$$-\gamma^2\tau_A^2[\phi_1''(x) - k^2\phi_1(x)] = F(x)[\psi_1''(x) - k^2\psi_1(x)] - F''(x)\psi_1(x) \quad (31)$$

where the primes denote d/dx and $\tau_A = \rho^{1/2}/(k\bar{B})$, $\tau_R = L^2/\eta$ are the characteristic Alfvén and resistive times for the problem. We make at this point a number of assumptions concerning the solution of these equations which must be verified a posteriori. We assume that the growth rate is intermediate between ideal magnetohydrodynamic and resistive; i.e., $(1/\tau_R) \ll \gamma \ll (1/\tau_A)$.

Further, we assume that the resistivity is relevant only in a narrow layer $x_T \ll 1$ where reconnection or tearing takes place.

Examine first the exterior region, where resistivity is assumed negligible. Assuming that scale lengths for the solution are on the order of the shear length, and using $\gamma\tau_A \ll 1$ we have

$$\psi_1(x) = \phi_1(x)F$$

$$F(\psi_1'' - k^2\psi_1) = F''\psi_1 \quad (32)$$

which for the particular profile $F(x) = \tanh x$ has the solution

$$\psi_1(x) = \exp(\mp kx)[1 \pm (\tanh x)/k] \quad (33)$$

the upper (lower) signs corresponding to x positive (negative). The function $\psi_1(x)$ has a discontinuous derivative at $x = 0$, and we readily find

$$\Delta' = \frac{\psi_1'(0+) - \psi_1'(0-)}{\psi_1(0)} = 2[(1/k) - k] \quad (34)$$

Note that Δ' is positive for $k < 1$; i.e., when the shear length is short compared to the wavelength of the mode. The exterior solutions $\psi_1(x)$, $\phi_1(x)$ are shown in Fig. 2. Now we must solve the full resistive equations in the interior region and match the solutions at the boundary. Use the fact that $x_T \ll 1$ so that within the tearing layer $F(x) \approx x$. We also assume, from the nature of the exterior solution, that within the tearing layer $\psi_1(x)$ is

approximately constant (the "constant ψ " approximation) although, of course, ψ_1' must be changing. Since $x_T \ll 1$, we can take $\psi_1'' \gg k^2 \psi_1$, $\phi_1'' \gg k^2 \phi_1$. Thus Eqs. (30), (31) give

$$\psi_1(0) - x\phi_1 = \frac{\psi_1''}{\gamma\tau_R} \quad (35)$$

$$\gamma^2 \tau_A^2 \psi_1'' = -x\phi_1'' \quad (36)$$

and Eq. (36) gives, for the necessary matching to the exterior solution

$$-\frac{\gamma^2 \tau_A^2}{\psi_1(0)} \int_{-\infty}^{\infty} \frac{\phi_1''}{x} dx = \Delta' \quad (37)$$

where we have made an assumption about the asymptotic behavior of ϕ_1 to extend the integral to $\pm\infty$ which will have to be checked a posteriori. The convergence of this integral guarantees that, asymptotically the interior solution ψ_1 has constant slope, and thus can be matched to the exterior solution.

Introducing a new variable z through $x = (\gamma\tau_A^2/\tau_R)^{1/4} z$ and writing $\phi_1 = -(\tau_R/\gamma\tau_A^2)^{1/4} \psi_1(0) \chi(z)$ we find upon combining Eqs. (35) and (36)

$$\chi'' - z^2 \chi = z, \quad (38)$$

which has the solution (Rutherford and Furth, 1971)

$$\chi = -\frac{z}{2} \int_0^{\pi/2} d\phi \sin^{1/2}\phi \exp\left(-\frac{z^2 \cos\phi}{2}\right) = -\frac{z}{2} \int_0^1 d\mu e^{-(z^2/2)\mu} (1-\mu^2)^{-1/4} \quad (39)$$

which can be verified by substitution. The solution is odd and also asymptotically equals $-(1/z)$. The matching condition becomes

$$\gamma^{5/4} \tau_R^{3/4} \tau_A^{1/2} \int_{-\infty}^{\infty} \frac{dz}{z} \chi'' = \Delta' \quad (40)$$

A numerically obtained solution of Eq. (38) is shown in Fig. 3, and the integrand χ''/z of Eq. (40) is shown in Fig. 4. Note that the growth rate is now determined except for a constant,

$$\gamma = \tau_R^{-3/5} \tau_A^{-2/5} (\Delta')^{4/5} \quad (41)$$

and that $\gamma \tau_R = S^{2/5} \gg 1$ and $\gamma \tau_A = S^{-3/5} \ll 1$ as assumed. Further, we note that the integration in Eq. (40) can be truncated at $z = 2$ with very little error. Thus, we will arbitrarily set $x_T = 2(\gamma \tau_A^2 / \tau_R)^{1/4}$ to be the tearing width, which is seen to scale as $S^{-2/5}$ times the shear length. The significance of the point $x = x_T$ will become apparent in Section 2.4.

To evaluate the coefficient in Eq. (41) we need to evaluate $I = \int_{-\infty}^{\infty} (dz/z) \chi''$. Substituting Eq. (39) we find

$$I = \frac{1}{2} \int_{-\infty}^{\infty} dz \int_0^1 d\mu (1-\mu^2)^{-1/4} (3\mu - z^2\mu^2) \exp(-z^2\mu/2) \quad (42)$$

Integrating over z this becomes $I = (\pi/2)^{1/2} \int_0^1 d\mu \mu^{1/2}/(1-\mu^2)^{1/4}$ which is equal to $\pi\Gamma(3/4)/\Gamma(1/4)$. We thus find

$$\gamma = \left(\frac{\Gamma(1/4) \Delta' L}{\pi \Gamma(3/4)} \right)^{4/5} \tau_R^{-3/5} \tau_A^{-2/5} \quad (43)$$

Note from Eq. (34), that the mode is stabilized by shear, or equivalently, long wavelength modes are the most unstable. Note that

$$\gamma \tau_R x_T = 2 \Delta' L^2 \quad (44)$$

and substituting this into the definition of x_T we find

$$x_T = (4 \Delta')^{1/5} S^{-2/5} \quad (45)$$

In Fig. 5 are summarized the first order fields and flow velocities.

2.4 Magnetic Energy of the Tearing Mode

In this section we calculate the magnetic energy available due to the formation of an island in slab geometry (Adler et al. 1980). The energy gained in island formation is proportional to Δ' . Although some transfer of energy takes place in the exterior region, the net contribution from this region is zero.

The change in magnetic energy, $(1/2) \int |B|^2 dx dy$, in terms of the notation of Section 2.3 is given correctly to second order by

$$M = (1/4) \int dx [(\psi_1')^2 + k^2 \psi_1^2 + 4\psi_0' \psi_{20}'] \quad (46)$$

where $\psi(x, y) = \psi_0(x) + \psi_1(x) \cos ky + \psi_{20}(x) + \psi_{22}(x) \cos 2ky$ and terms linear in $\psi_1(x)$ and $\psi_{22}(x)$ average to zero over y . Now we must find $\psi_{20}(x)$. Averaging the second order part of Eq. (25) over y we find

$$2\psi_{20}(x) + \frac{1}{2} \frac{d}{dx} [\psi_1(x) \phi_1(x)] = (\gamma_R)^{-1} \frac{d^2 \psi_{20}(x)}{dx^2} \quad (47)$$

where we have taken $\eta = \text{constant}$ for simplicity. Terms proportional to $\nabla^2 \eta$ can be shown to be negligible. The solution to Eq. (47) is

$$\psi_{20} = -\frac{1}{2\alpha} \int_{-\infty}^{\infty} dx' \exp(-\alpha|x-x'|) \frac{\partial}{\partial x'} K(x') \quad (48)$$

with

$$K(x') = \frac{\gamma_R}{2} \psi_1(x') \phi_1(x'), \quad \alpha = (2\gamma_R)^{1/2}.$$

The asymptotic form of Eq. (48) is readily evaluated, giving

$$\psi_{20}(x) \approx -\frac{1}{4} \frac{d}{dx} \left(\frac{\psi_1^2(x)}{F} \right) \quad (49)$$

which is also directly obtained from Eq. (47) by neglecting the resistivity. However, unlike the exterior solutions for $\psi_1(x)$, $\psi_{20}(x)$, this expression is only valid for $x \gg (x_T/\Lambda)^{1/2}$; i.e., the skin depth of the current sheet, and cannot be used in the vicinity of x_T .

To evaluate the change in the magnetic energy M , first consider the last term

$$M_3 = \int_{-\infty}^{\infty} \psi_{20}^2 dx \quad (50)$$

and split it into two parts

$$M_3 = -2 \int_0^{\epsilon} x \psi_{20}^2 dx + \frac{1}{2} \int_{\epsilon}^{\infty} F \frac{d^2}{dx^2} \left(\frac{\psi_1^2}{F} \right) dx \quad (51)$$

where $x_A^{1/2} \ll \epsilon \ll 1$. Using $F(x) \approx x$ in the interior region and the asymptotic expression for $\psi_{20}(x)$ in the exterior region and integrating by parts we find for the total change in magnetic energy

$$M = \frac{1}{4} \int_{-\infty}^{\infty} dx \left[(\psi_1')^2 + k^2 \psi_1^2 + (F''/F) \psi_1^2 \right] + R \quad (52)$$

Integrating the first term by parts we find that the integral in

Eq. (52) is equal to $-(1/4) \Delta' \psi_1^2(0)$. Further $|R| \ll \Delta' \psi_1^2(0)$ provided $\epsilon > (x_T)^{1/3}$, a reflection of the fact that the asymptotic form of ψ_{20} used for $x > \epsilon$ is not valid for ϵ less than this value. Thus

$$M = - \frac{\Delta' \psi_1^2(0)}{4} \quad (53)$$

The magnetic energy density can be directly evaluated in the interior region. Differentiate Eq. (48), obtaining

$$\frac{d\psi_{20}}{dx} = - \frac{1}{2\alpha} \int_{-\infty}^{\infty} dx' \exp(-\alpha|x - x'|) \frac{\partial^2}{\partial x'^2} K(x') \quad (54)$$

which, upon integration by parts twice gives

$$\frac{d\psi_{20}}{dx} \approx K(x) \quad (55)$$

plus terms of order αx , which are negligible provided $x < x_T$. We then find for the magnetic energy density

$$\frac{\delta B^2}{2} = \frac{\Delta'}{2x_T} \psi_1^2(0) z\chi(z) \quad (56)$$

which has a value of $-(1/2) (\Delta'/x_T) \psi_1^2(0)$ at $x = x_T$. The total change in magnetic energy within $|x| < x_T$ is given by

$$M(x_T) = \frac{\Delta'}{2x_T} \psi_1^2(0) \int_0^{x_T} z\chi(z) dx \quad (57)$$

and using Eq. (40) we find $M(x_T) = -\psi_1^2(0)\Delta'/4$, i.e., the entire

change in magnetic energy. In Fig. 6 is shown schematically the change in magnetic energy density as a function of x , as found numerically using Eqs. (46) and (48). The surface terms in the integrations for $|x| < \epsilon$ and $|x| > \epsilon$ result from the energy transfer which takes place for $|x| > x_T$. The total contribution from this region is, however, zero. Also shown is the divergent energy density obtained by using the exterior solution for ψ_{20} for $x < \epsilon$.

Furth (1963) derived a quadratic form which leads to a variational principle for the full resistive equations. He shows that a term gives a negligible contribution inside the tearing layer. In the exterior region this quadratic form is given by

$$V_0 = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[(\psi_1')^2 + k^2 \psi_1^2 + \frac{F''}{F} \psi_1^2 \right] \quad (58)$$

which is equal to M , from Eq. (52). The resulting "energy" density D_v is shown in Fig. 6. It is seen to agree with the magnetic energy density for large x and also has an integrated value of $-\Delta' \psi_1^2(0)/4$, although it is quite unlike the magnetic energy density for small x . The quadratic form V_0 is related to the infinite conductivity energy W_∞ (Bernstein et al. 1958) by a singular term, $W_\infty = V_0 + F'(\epsilon) \psi_1^2(0)/[2 F(\epsilon)]$, which is the result of integrating the last term in Eq. (58) by parts.

The total rate of change of energy inside a volume bounded by $a < x < b$ is due to the changes in magnetic and kinetic energy. It is instructive to evaluate these quantities (Adler et al., 1980) and show them equal to the work done by the pressure at the two

boundary surfaces plus the energy radiated through the surfaces. Without loss of generality we will take $0 < a < x_T < \epsilon < b$.

For the kinetic energy $(1/2) \rho u^2$ we readily find

$$\frac{d}{dt} K = \frac{1}{2} \gamma (\gamma_{\tau_A}^2) \int_a^b [(\phi_1')^2 + k^2 \phi_1^2] dx \quad (59)$$

Integrating the first term by parts and using Eq. (52) we find

$$\frac{d}{dt} K = \frac{\gamma}{2} \int_a^b F \phi [\psi_1'' - k^2 \psi_1 - (F''/F) \psi_1] dx \quad (60)$$

where we have neglected the surface term, which is of order $\gamma^2 \tau_A^2$ smaller than the final result.

The change in magnetic energy is given by Eq. (46).

$$\frac{d}{dt} M = \frac{\gamma}{2} \int_a^b [(\psi_1')^2 + k^2 \psi_1^2 - 4F\psi_{20}'] dx \quad (61)$$

Integrating the first term by parts and then adding the kinetic and magnetic energies we find

$$\begin{aligned} \frac{d}{dt} (K + M) &= \frac{\gamma}{2} \int_a^b (\phi_1 F - \psi_1) (\psi_1'' - k^2 \psi_1) dx \\ &\quad - \frac{\gamma}{2} \int_a^b (4F\psi_{20}' + \phi_1 F'' \psi_1) dx + \frac{\gamma}{2} \psi_1 \psi_1' \Big|_a^b \end{aligned} \quad (62)$$

The first term on the right hand side of this equation is easily seen to be negligible. To evaluate the second term on the right hand side, split the range of integration into two parts according to whether $x > \epsilon$, with $\epsilon \gg x^{1/2}/\Delta$, and use the

exterior solution for ψ_{20} for $x > \epsilon$. Integrating by parts then gives

$$\int_{\epsilon}^b (4F\psi'_{20} + \phi_1 F'' \psi_1) dx = \left[-F \frac{d}{dx} \left(\frac{\psi_1^2}{F} \right) + F' \frac{\psi_1^2}{F} \right]_{\epsilon}^b \quad (63)$$

In the interior region use $F \approx x$ and integrate both terms by parts to find

$$\int_a^{\epsilon} (4F\psi'_{20} + \phi_1 F'' \psi_1) dx = 4x\psi_{20} \Big|_a^{\epsilon} - \int_a^{\epsilon} 4\psi_{20} dx \quad (64)$$

Now integrate Eq. (47) and discard the term $(\gamma_T R)^{-1} \psi'_{20}$, which is negligible, giving

$$\begin{aligned} \frac{d}{dt} (K + M) &= \frac{\gamma}{2} \left[F \frac{d}{dx} \left(\frac{\psi_1^2}{F} \right) - \frac{F' \psi_1^2}{F} \right]_{\epsilon}^b \\ &+ \frac{\gamma}{2} \left[-4x\psi_{20} - \psi_1 \phi_1 \right]_a^{\epsilon} + \frac{\gamma}{2} \psi_1 \psi_1' \Big|_a^b \end{aligned} \quad (65)$$

Using $x^{1/2}/\Delta' \ll \epsilon \ll 1$ we note that the terms evaluated at ϵ vanish as they must, giving

$$\frac{d}{dt} (K + M) = \frac{\gamma}{2} \left[-4F\psi_{20} - F' \psi_1 \phi_1 + \psi_1' \psi_1 \right]_a^b \quad (66)$$

There are two contributions to the energy balance at the surface, that due to the Poynting flux and the work done by the pressure. To evaluate the Poynting flux use

$$P_x = (\vec{E}_1 \times \vec{B}_1)_x + (\vec{E}_2 \times \vec{B}_0)_x \quad (67)$$

Using Eq. (24) to evaluate E_z , and Eq. (23) to evaluate j_z we find that $E_{12}B_{1y} = \gamma\psi_1'\psi_1 \cos^2 ky$ and $E_{2z}B_{0y} = -2\gamma\psi_{20} F$, giving

$$P_x = \gamma(-\psi_1'\psi_1 \cos^2 ky + 2F\psi_{20}) \quad (68)$$

The work done by the pressure is given at each surface by pu_x since the fluid is incompressible. To evaluate the pressure use $\nabla p = (j \times B)$ neglecting the density term in Eq. (26) which is of order $\gamma^2 \tau_A^2$ to find

$$\frac{dp}{dx} = \frac{d}{dx} (F'\psi_1) \cos ky \quad (69)$$

where we have used Eq. (31) to simplify $j \times B$ and have discarded a term of order $\gamma^2 \tau_A^2$. The work done is then $pu_x = (\gamma/2)\phi_1 F'\psi_1$ and we thus find

$$\frac{d}{dt} \int_V (K + M) dx dy = - \int_S \vec{P} \cdot d\vec{S} - \int_S p \vec{u} \cdot d\vec{S} \quad (70)$$

as was to be shown.

Shown in Fig. 7 is the second order y independent current density $j_z = -\psi_{20}''$. It arises physically through $\vec{u}_1 \times \vec{B}_1$, and opposes j_{z0} inside the tearing layer and reinforces it outside. It is responsible for the change in the magnetic energy density by the induced change in B_y . From expression (55) the value of δj_z at $x = 0$ is seen to be $\delta j_z(0) = -\bar{B}K'(0) \approx -\Delta'\psi_1^2(0)/(x_T^3)$ where we have used the numerically obtained value of $(d\chi/dz) = -0.55$ (see Fig. 3).

3. CYLINDRICAL GEOMETRY

3.1 Inverse Aspect Ratio Expansion

In this section we develop the equations used for the analysis of linear and nonlinear tearing mode behavior in toroidal devices. An examination of nonlinear behavior demands the consideration of the interaction of the island with the tearing layer itself (Section 4.2), and with the shear (Section 4.3).

As originally pointed out by Shafranov (1970), the ideal magnetohydrodynamic kink instabilities are so strongly growing that toroidal curvature has very minor effect on the modes, and they can be analyzed to good approximation in cylindrical geometry. The same is true for the tearing mode, since its behavior is primarily governed by the magnetic energy driving it. It was shown by Furth et al. (1973) that the toroidal curvature, for very small resistivity, does result in a small reduction of the tearing mode growth rate. Glasser et al. (1975) further showed that toroidicity improves the stability of the mode through the introduction of a threshold.

As in Section 2.3, we begin with Faraday's law

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad , \quad (71)$$

Ampere's law

$$\vec{\nabla} \times \vec{B} = \vec{j} \quad , \quad (72)$$

the simplified form of Ohm's law

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \quad , \quad (73)$$

and the equation of motion, Eq. (26). Tearing mode behavior is governed primarily by the current profile, but can also be influenced by pressure, density, and resistivity profiles. When one or more of these is taken into account, we also need equations which describe their evolution. Realistic treatment of the evolution of the resistivity profile requires the consideration of transport, ohmic heating, radiation and impurities, and is beyond the scope of this review.

The geometry and notation for variables is shown in Fig. 8. Tokamaks are characterized by values of the safety factor $q(r) = |rB_\phi / (RB_\theta)|$ of order unity and values of the inverse aspect ratio $\epsilon = a/R_0$ much less than unity. The equations are thus expanded in the inverse aspect ratio, allowing the ratio of plasma pressure to magnetic field pressure $\beta = 2p/B^2$ to be of order ϵ or smaller (Rosenbluth et al., 1976; Strauss, 1976). Choosing B_ϕ and q of order unity, we find $B_\perp \sim O(\epsilon)$. The divergence of the flow, $\vec{\nabla} \cdot \vec{u}$ is found to be of order ϵ^3 which is important numerically since it eliminates the magnetosonic wave from the system, the fastest wave present being the incompressible Alfvén wave. The following consistent ordering scheme for resistive magnetohydrodynamic motion is found.

$$O(1): B_\phi, \rho, \nabla_\perp$$

$$O(\epsilon): B_\perp, j_\phi, u_\perp, \frac{\partial}{\partial t}, \frac{1}{R} \frac{\partial}{\partial \phi}$$

$$O(\epsilon^2): \tilde{B}_\phi, j_\perp, u_\phi \quad (74)$$

$$O(\epsilon^3): \nabla_\perp \cdot u_\perp$$

and the pressure is taken to be either of order ϵ (high beta 3-dimensional equations) or order ϵ^2 (low beta 3- or 2-dimensional equations). Here $\nabla_\perp = \hat{r}(\partial/\partial r) + \hat{\theta}(1/r)(\partial/\partial \theta)$, and more generally the subscript $\perp$ denotes perpendicularity to $\hat{\phi}$.

3.2 Single Helicity Equations

Considerable success has been achieved by restricting consideration to following the nonlinear development of perturbations of a fixed helicity, thus reducing the number of dimensions of the problem from three to two (Rosenbluth et al. 1976). The nonlinear coupling of modes of different helicities appears not to play an essential role except in the final stages of development of large islands, which occur for particular current profiles. These phenomena, related to tokamak disruption, will be discussed in Section 4.3.

Fixed helical symmetry, together with cylindrical boundary conditions and equilibrium, implies that all quantities are functions of τ , r , and t only, where $\tau = m\theta + kz$ and $k = n/R_0$. Here m , n are the poloidal and toroidal mode numbers of the initial perturbation which has the form $f(r) \exp[i(m\theta + kz)]$, and the torus has been replaced by the equivalent cylinder with length $L = 2\pi R_0$, and the ϕ coordinate with $z = R_0 \phi$. The helical symmetry allows us to eliminate the z coordinate through $\partial/\partial z = (k/m)(\partial/\partial \theta)$. In addition this symmetry, together with $\vec{\nabla} \cdot \vec{B} = 0$, implies that $\vec{B}$ can be written in terms of a scalar $\psi(r, \theta)$,

$$\vec{B} = \vec{\nabla}\psi \times \hat{z} - \frac{kr}{m} B_z \hat{\theta} + B_z \hat{z} \quad (75)$$

and ψ is a flux function, i.e., $\vec{B} \cdot \vec{\nabla}\psi = 0$. In a cylindrically symmetric state $\psi(r)$ is proportional to the flux through a helical

ribbon defined by the magnetic axis and a helix of constant τ at minor radius r . To see this, calculate $\vec{B} \cdot (\hat{r} \times \vec{h}) / |\hat{r} \times \vec{h}|$ where $\vec{h} = \hat{z} - (kr/m) \hat{\theta}$ is a vector defining the helix (see Fig. 9). We find $\vec{B} \cdot (\hat{r} \times \vec{h}) / |\hat{r} \times \vec{h}| = (\partial\psi/\partial r) / [1 + (k^2 r^2/m^2)]^{1/2}$ and thus the total flux passing through the ribbon inside r is given by

$$\phi = \int_0^r \frac{\vec{B} \cdot (\hat{r} \times \vec{h})}{|\hat{r} \times \vec{h}|} dS = \int_0^r \frac{\partial\psi}{\partial r'} dr' = \psi(r) \quad (76)$$

Vanishing of $(\partial\psi/\partial r)$ at $r = r_s$ means that the field through the helical ribbon vanishes at this point, and this condition determines a singular surface. In addition, ψ is related to the magnetic vector potential by $\psi = A_z - (kr/m)A_0$. The dynamical behavior of ψ follows from Faraday's law [Eq. (71)]

$$\frac{\partial\psi}{\partial t} + \vec{u} \cdot \nabla \psi = -\eta j_z + E \quad (77)$$

where E is an imposed electric field at the tokamak wall. The z component of the current is related to ψ through Ampere's law [Eq. (72)]

$$j_z = -\nabla^2 \psi - \frac{2kB_z}{m} \quad (78)$$

Incompressibility is again used to reduce the number of variables through the introduction of the potential ϕ with $\vec{u} = \vec{\nabla} \phi \times \hat{z}$.

Substituting Eq. (75) and Eq. (72) into the equation of motion and keeping only lowest order in ϵ we find

$$\rho \frac{d}{dt} \vec{u}_\perp + \vec{\nabla}_\perp \left[p + \frac{B^2}{2} - \frac{(\nabla_\perp \psi)^2}{2} + \frac{2k_B z \psi}{m} + \frac{1}{2} \left(\frac{k_B z r}{m} \right)^2 \right] + \nabla_\perp \psi \nabla^2 \psi = 0 \quad (79)$$

which gives, upon operation with $\hat{z} \cdot \nabla \times$,

$$\frac{d}{dt} (\nabla_\perp \cdot \rho \nabla_\perp \phi) = -\hat{z} \cdot [\nabla_\perp \psi \times \nabla_\perp (\nabla_\perp^2 \psi)] \quad (80)$$

These equations have been used in numerical codes to investigate the nonlinear development of modes with $m \geq 2$ (White et al., 1977c; Monticello et al., 1979; Biskamp and Welter, 1978; Waddell et al., 1978; Danilov et al., 1976) and $m = 1$ (Sykes and Wesson, 1976; Waddell et al., 1976; Asumi et al., 1976; Dnestrovskii et al., 1977) and to construct analytic models for the nonlinear behavior of modes with $m \geq 2$ (White et al. 1977a).

Evolution of a mode is generally followed by perturbing a θ independent equilibrium, specified by $j(r)$, or equivalently $q(r)$. From the definition of the safety factor we find

$$q(r) = \frac{r}{R[\psi'(r) + (rc/m)]} \quad (81)$$

which at the singular surface $[\psi'(r_s) = 0]$ equals m/n . Using Eq. (43) and expressing the result in terms of the characteristic Alfvén and resistive times for the torus

$\tau_A = [(\rho)^{1/2} a m / B_z] = [(\rho)^{1/2} a^2 m / \bar{B} R]$, $\tau_R = a^2 / \eta$ and using $k \approx n/R$, $F' = L^{-1} = \psi_0''$ we find

$$\gamma = [\psi_0''(r_s)]^{2/5} (\Delta' a)^{4/5} \tau_R^{-3/5} \tau_A^{-2/5} \quad (82)$$

To determine Δ' and the radial dependence of other quantities in the linear theory, linearize Eqs. (77), (80) through

$$\begin{aligned} \psi &= \psi_0 + \psi_1 \cos m\theta \\ \rho &= \rho_0 + \rho_1 \cos m\theta \\ \phi &= \phi_1 \sin m\theta \\ j &= j_0 + j_1 \cos m\theta \end{aligned} \quad (83)$$

where all first order quantities are assumed to vary as e^{it} , and eliminate all but ψ_1 , ϕ_1 obtaining

$$\gamma \nabla \cdot \rho_0 \nabla \phi_1 = (m/r) (\psi_0' \nabla_m^2 \psi_1 - m \psi_1 \psi_0''') \quad (84)$$

$$\gamma \psi_1 + (m/r) \psi_0' \phi_1 = \eta \nabla_m^2 \psi_1 \quad (85)$$

with $\nabla_m^2 = (1/r) (\partial/\partial r) r (\partial/\partial r) - (m^2/r^2)$. Assuming, as in Section 2.3 that $\gamma \tau_A \ll 1$ we find, away from the singular surface

$$\nabla_{\perp}^2 \psi = - \frac{d j_0}{dr} \frac{\psi_1}{\psi_0} . \quad (86)$$

The stability of a cylindrical plasma to tearing modes is completely determined through Eq. (82) by the quantity Δ' , which as shown in Section 2.4 describes the available magnetic energy for the formation of an island. Shafranov (1970) calculated Δ' for a constant current profile of finite extent, and Furth et al. (1973) carried out a detailed investigation of the dependence of Δ' on the profile shape. The determination of Δ' is made by integration of Eq. (86) in the exterior region. They introduced profiles which they called peaked, rounded, and flat, with associated safety factors $q(r) = q(0)[1 + (r/r_0)^{2p}]^{1/p}$, and $p = 1, 2, 4$ respectively. From Eq. (81) and Eq. (78) we find that the current associated with a given q profile is

$$j = - \left(\frac{2}{Rq} \right) + \frac{rq'}{Rq^2} . \quad (87)$$

which reduces in this case to $j = j(0)[1 + (r/r_0)^{2p}]^{1+1/p}$. As is readily verified, $q(r) \sim r^2$ for large r is associated with asymptotically vanishing $j(r)$. The parameter r_0 describes the width of the current channel. Equivalently, the two free parameters can be chosen to be q on axis and the location of the singular surface r_s . The peaked model is found to be unstable to $m = 2$ and $m = 3$ tearing modes. The rounded model is generally more unstable, the $m = 2, 3$ modes existing over a wider range of r_s and the $m = 4$ mode also becoming unstable. The flat profile is even more unstable. The behavior of Δ' as $r_s \rightarrow 0$ is $\Delta' \rightarrow f(m)/r_s$.

Glasser et al. (1977) investigated the optimization of stability for different current profile shapes, and showed that it is possible to achieve simultaneous stability against all modes. For a review of stability boundaries for different m values and various profiles, see Wesson (1978).

3.3 The Resistive Kink Mode

The free boundary kink, and the tearing mode can be viewed as two different limits of the same instability. In the case of the ideal kink the magnetic island must be located outside the plasma in the surrounding vacuum region, and is felt only as a distortion of the plasma-vacuum boundary. If a resistive plasma is introduced in the vacuum region, the magnetic reconnection is resistively limited and the growth rate of the mode accordingly modified. We present here a simple derivation of the linear growth rate for the resistive kink mode, which makes obvious this close connection (Pogutse, 1977).

Consider an equilibrium with a dense plasma and a constant current density extending to $r = r_1$, and a much less dense, resistive plasma, and zero current, for $r > r_1$. The safety factor is then constant for $r < r_1$ and parabolic for $r > r_1$. The singular surface is assumed to lie in the low density resistive plasma. Begin with Eqs. (84), (85). Away from the singular region ($r \approx r_s$) resistivity is unimportant so $\vec{\nabla}\psi \times \vec{\nabla}j_z = 0$, or $j_z = j(\psi)$. Linearizing this we find that

$$j(\psi) = j_0(\psi_0) + \frac{dj_0}{d\psi_0} \frac{1}{r_1} \psi_1 \cos m\theta \quad (88)$$

and thus for a constant current equilibrium we have $j_1 = 0$, which along with Eq. (78) gives

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right) \psi_1 = 0 \quad (89)$$

Thus, solving for $\psi_1(r)$ reduces to matching solutions of Laplace's equation in the three regions $r < r_1$, $r_1 < r < r_s$, and $r_s < r < a$. To find the discontinuity in ψ_1' at $r = r_1$ we integrate Eq. (84) across the surface $r = r_1$, relating the discontinuity in ψ_1' to that in ψ_1 .

$$\rho_0 \phi_1' \Big|_{-}^{+} = \frac{m}{\gamma r_1} \left(\psi_0' \psi_1' \Big|_{-}^{+} + \psi_1 \psi_0'' \Big|_{-}^{+} \right). \quad (90)$$

From Eq. (85) we have, away from the singular surface $\gamma \psi_1 + (m \phi_1 Q/R) = 0$ with $Q = [(1/q) - (n/m)]$. Thus also

$$\gamma \psi_1' + m \phi_1' Q/R = 0 \quad r < r_1, \quad (91)$$

$$\gamma \psi_1' + m \phi_1' Q/R + m \phi_1 Q'/R = 0 \quad r > r_1, \quad (92)$$

and $Q' = -2/(rq)$ just outside the current column. The discontinuity at the singular surface is given by Δ' . Writing the solution to Eq. (84) as

$$\psi_1 = \begin{cases} (r/r_1)^m & r < r_1 \\ C_1 (r/r_1)^m + C_2 (r/r_1)^{-m} & r_1 < r < r_s \\ C_3 (r/r_s)^m + C_4 (r/r_s)^{-m} & r_s < r < a \end{cases} \quad (93)$$

and choosing for simplicity $m/n = q(0) \ll (m/n)$ we find from the matching conditions at $r = r_1$

$$C_2 = (m - nq)^{-1} \left(1 - \frac{\gamma^2 (\tau_+^2 - \tau_-^2)}{2Q} + \frac{\gamma^2 \tau_-^2 n^2}{m^2 Q^2} \right) \left(1 + \frac{\gamma^2 \tau_-^2 n^2}{m^2 Q^2} \right)^{-1} \quad (94)$$

where $\tau_{\pm} = \rho_{\pm}^{1/2} / (nb_z)$ are the characteristic Alfvén times of the inner and outer regions. Similarly from matching at $r = r_s$

$$\Delta' = \frac{(m/r_s) \{-h + hC_2[1 - (r_1/r_s)^{2m}] - 1 + C_2[1 + (r_1/r_s)^{2m}]\}}{1 - C_2[1 - (r_1/r_s)^{2m}]} \quad (95)$$

where $h = [1 + (r_1/a)^{2m}] / [1 - (r_1/a)^{2m}]$. Equation (82), which also relates the quantity Δ' to the growth rate, together with Eqs. (94) - (95), now determine γ . The two limits are readily obtained. Letting $\tau_R \rightarrow 0$ for fixed γ we find that $\Delta' = 0$, which upon combining Eqs. (94) and (95) gives

$$\gamma^2 = \left(\frac{m}{nq} - 1 \right)^2 \frac{2m^2}{[(r_1/a)^{2m} - 1]} + \frac{2m^2}{nq} \left(\frac{m}{nq} - 1 \right) \quad (96)$$

which is the growth rate of the kink mode. In this limit the island growth is impeded by the inertia of the central core rather than by resistive tearing. The tearing mode limit is found by setting the density discontinuity equal to zero in Eq. (94), giving $C_2 = (m - nq)^{-1}$, independent of γ , and resulting in the ordinary tearing mode growth rate. In Fig. 10 is shown the growth rate γ as a function of $S = \tau_R / \tau_A$. The close relation between

these modes is intuitively useful as it indicates that for rather sharply limited current profiles and with rather large resistivity in the region outside the current channel the nonlinear behavior of the tearing mode can be expected to be similar to that of the kink mode (Rosenbluth et al., 1976).

3.4 The Resistive Internal Kink Mode

The internal kink mode, just as the surface kink mode described in Section 3.3, has a resistive version. Recall that the ideal surface kink modes are unstable only when the singular surface $\vec{k} \cdot \vec{B} = 0$ falls within the vacuum region.

Instability of the $m = 1$ ideal internal kink mode does not depend on the singular surface lying within the vacuum. For finite resistivity the resulting mode can be expected to have quite different behavior depending on whether the mode is ideally stable, neutral stable, or unstable. The linear growth rate can be calculated for all three cases in a single analysis (Coppi et al., 1976).

Begin with the reduced equations of Section 3.2, valid to lowest order in kr/m , and perform the usual linearization with $\psi = \psi_0(r) + \psi_1(r) \cos m\theta$, $\phi = \phi_1(r) \sin m\theta$. It is also convenient to express the equations in terms of the radial displacement $\xi(r) = [u_{1r}(r)/\gamma] = (m/\gamma r) \phi_1(r)$, and the radial magnetic field $b(r) = -(m/r) \psi_1(r)$, giving two coupled second order differential equations,

$$\gamma b = m\gamma \xi \frac{\psi_0'}{r} + \eta \left(\frac{1}{r^3} \frac{d}{dr} r^3 \frac{d}{dr} b - \frac{(m^2 - 1)b}{r^2} \right) \quad (97)$$

and

$$\begin{aligned} \rho\gamma^2 \left(\frac{1}{r} \frac{d}{dr} r^3 \frac{d\xi}{dr} - (m^2 - 1)\xi \right) = & -m\psi_0' \left(\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} (rb) - \frac{m^2}{r^2} rb \right) \\ & + br \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rm\psi_0') \right) \end{aligned} \quad (98)$$

where we have used $\nabla^2(rf \cos m\theta) = [(1/r^2)(d/dr)r^3(d/dr) - (m^2-1)/r]f \times \cos m\theta$. Recall that ψ'_0 is simply related to the equilibrium safety factor through

$$\psi'_0 = \left(\frac{m}{nq} - 1 \right) r \quad (99)$$

We follow the standard method introduced in Section 2.3; i.e., we assume the resistivity negligible except in a layer surrounding $r = r_s$, the width of the layer to be determined in the course of the calculation.

First consider the exterior region. For zero resistivity Eq. (97) gives

$$b = \frac{m\psi'_0}{r} \xi \quad (100)$$

which, upon substitution into (98) results in

$$\frac{d}{dr} \left\{ r^3 \left[\rho_Y^2 + \left(\frac{m\psi'_0}{r} \right)^2 \right] \frac{d\xi}{dr} \right\} = g\xi \quad (101)$$

with $g = [(m\psi'_0)^2/r + \rho_Y^2 r](m^2 - 1)$. Thus for $m = 1$ the right hand side of Eq. (101) vanishes, so a solution is $\xi(r) = \text{constant}$. Neglecting ρ_Y^2 with respect to $(m\psi'_0/r)^2$ the singular nature of the differential equation $[\psi'_0(r_s) = 0]$ allows a discontinuity at the point $r = r_s$, and thus the solution to lowest order in (kr/m) consistent with the boundary condition $\xi(a) = 0$ is

$$\epsilon_0(r) = \begin{cases} \epsilon_\infty & r < r_s \\ 0 & r > r_s \end{cases} \quad (102)$$

valid in the exterior region ($\rho\gamma^2$ negligible).

Notice that the change in ξ (and ψ_1) of 100% across the inner region renders invalid the analysis of Section 2.3, based on the constant ψ approximation.

Now we must solve Eqs. (97), (98) in the vicinity of the singular surface $r = r_s$ and match the inner solution to the exterior solution. The matching turns out to be impossible to do with the lowest order solution Eq. (102) because of the identical vanishing of g . It is necessary to find the exterior solution to the next order in the basic expansion parameter of Section 3.1, kr/m . Write the solution to Eq. (101) as $\xi(r) = \xi_0(r) + \xi_1(r) + \dots$ where the expansion parameter is kr/m . To obtain $\xi_1(r)$ we need to find $g(r)$ to order $(kr/m)^2$. To do this return to the equation of motion and linearize, giving

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} p_1 + \vec{j}_1 \times \vec{B}_0 + \vec{j}_0 \times \vec{B}_1 \quad (103)$$

Now apply $\vec{B} \cdot \vec{\nabla}$ x, giving on the right hand side

$$\vec{B} \cdot \vec{\nabla} (\vec{j}_1 \times \vec{B}_0) + \vec{B} \cdot \vec{\nabla} (\vec{j}_0 \times \vec{B}_1) \quad (104)$$

To all order in kr/m we have

$$\vec{B}_0 = \hat{z} B_z - \psi'_0 \hat{\theta} - (kr/m) \hat{\theta} \quad (105)$$

$$\vec{\nabla} = \hat{r}(\partial/\partial r) + \hat{\theta}(1/r)(\partial/\partial \theta) + \hat{z}(\partial/\partial z)$$

and substituting $\vec{j}_0 = \vec{\nabla} \times \vec{B}_0$ and $\vec{j}_1 = \vec{\nabla} \times \vec{B}_1$ with $\vec{B}_1 = \vec{\nabla} \psi_1 \times \hat{z}$ we find to order $(kr/m)^2$ that

$$g = \left(\frac{(\psi'_0)^2}{r} + \rho_Y^2 r \right) (m^2 - 1) + \left(\frac{kr}{m} \right)^2 \frac{(\psi'_0)^2}{r} \quad (106)$$

We can now find the solution ξ_1 in the external region by integrating Eq. (101)

$$\frac{d\xi_1}{dr} = \begin{cases} \frac{\xi_\infty}{r(m\psi'_0)^2} \int_0^r g \, dr & r < r_s \\ \frac{\xi_\infty}{r(m\psi'_0)^2} \int_0^{r_s} g \, dr & r > r_s \end{cases} \quad (107)$$

from which it is apparent that

$$\frac{1}{\xi_\infty} \frac{d\xi_1}{dr} \rightarrow - \frac{\lambda_H}{\pi x^2} \quad (108)$$

near $r = r_s$, where $x = (r - r_s)/r_s$ and

$$\lambda_H = \frac{-\pi}{r_s [mr_s q' (x_s)/q]^2} \int_0^{r_s} g \, dr \quad (109)$$

Here we have used $\psi''_0 = (-q'/q)(kr/m)$ at $r = r_s$. The

quantity λ_H , having the dimensions of an inverse length, gives the effective width of the second order exterior solution.

Now consider the interior region. In the vicinity of $x = 0$, as long as $S = \tau_R/\tau_A \gg 1$, we can simplify Eqs. (97) and (98) as

$$d^2 \xi / dx^2 = (x/\lambda^2) (d^2 \beta / dx^2) \quad (110)$$

$$\beta = -\xi x + (s\lambda)^{-1} (d^2 \beta / dx^2)$$

where $\beta = b(r) (m/\psi_0'')$, $\lambda = \gamma \tau_H$ and τ_H^{-1} is the ideal free boundary kink growth rate, $\tau_H = \rho^{1/2} / \psi_0''$. Now we look for a solution of Eq. (110) which can be smoothly joined to the exterior solution, Eq. (107). Combining the equations we find a fourth order equation for ξ

$$\frac{\lambda}{Sx} \xi'''' - \frac{2\lambda}{Sx^2} \xi''' + \left(\frac{2\lambda}{Sx^3} - x - \frac{\lambda^2}{x} \right) \xi'' - 2\xi' = 0. \quad (111)$$

One solution is $\xi' = 0$. The asymptotic behavior of the other three independent solutions for large $|x|$ are $\xi' \approx (1/x^2)$ and $\xi' \approx \exp[\pm x^2/(4\lambda/S)^{1/2}]$. From Eq. (108) the matching solution is a combination of the first ($\xi = \text{const.}$) and second ($\xi' \approx 1/x^2$). Note that the asymptotic behavior of the other two solutions depends on η , and they therefore cannot be matched to the exterior solution. Write

$$\xi = (1/2)\xi_\infty + \xi_{\text{odd}} \quad (112)$$

and impose the condition $\lim_{x/\lambda \rightarrow -\infty} (x^2/2) (\xi'_{\text{odd}}/\xi_{\text{odd}}) = -(\lambda_H/\pi)$ for $\xi_{\text{odd}} \rightarrow \xi_{\infty}/2$.

Numerical solution of Eq. (110) was carried out using $\hat{x} = xS^{1/3}$, $\hat{\beta} = \beta S^{1/3}$, $\hat{\lambda} = \lambda S^{1/3}$. The results for ξ_{odd} are shown in Fig. 11a for various values of $\hat{\lambda}$. The boundary conditions imposed were that $\hat{\beta}(0) = 1$, $\xi_{\text{odd}}(0) = 0$, and $d\xi(0)/dx$ was adjusted until Eq. (109) was satisfied. In Fig. 11b is shown the growth rate as a function of the ideal magnetohydrodynamic growth rate measured by $\hat{\lambda}_H$. Ideal marginal stability ($\hat{\lambda}_H = 0$) gives $\hat{\lambda} = 1$ or $\lambda = S^{-1/3}$. If $\lambda_H \gg S^{-1/3}$, we find $\lambda = \lambda_H$; i.e., the mode has the ideal growth rate.

These results can be reproduced analytically by introducing the even function which is the first integral of Eq. (110)

$$\gamma(x) = x \frac{d\beta}{dx} - \beta = \lambda^2 \frac{d\xi}{dx} + \chi_{\infty} \quad (112)$$

Then

$$\xi = \lambda^{-2} \int_x^{\infty} dx (\chi - \chi_{\infty}) \quad (114)$$

$$\beta = -\chi - x \int_x^{\infty} \frac{dx}{x} \frac{d\chi}{dx} \quad (115)$$

also $\xi_{\infty} = 2 \int_0^{\infty} (dx/x) (d\chi/dx)$ and $(d\xi/dx) \rightarrow -(\chi_{\infty}/x^2)$. Thus Eq. (108) reduces to

$$\chi_{\infty} = \lambda_H \frac{2}{\pi} \int_0^{\infty} \frac{dx}{x} \frac{d\chi}{dx} \quad (116)$$

and Eq. (110) leads to a second order differential equation for χ ,

$$\frac{\lambda}{S} \left(\frac{d^2 \chi}{dx^2} - \frac{2}{x} \frac{d\chi}{dx} \right) - (x^2 + \lambda^2) \chi = -x^2 \chi_{\infty} \quad (117)$$

Now solve this equation in the three limits, ideally unstable ($\lambda_H \gg S^{-1/3}$), marginally stable ($\lambda_H = 0$), and stable ($\lambda_H < 0$). In the ideally unstable case we find $\chi = \chi_{\infty} x^2 / (x^2 + \lambda^2)$ and $\lambda = \lambda_H$. The marginally stable case has $\chi_{\infty} = 0$ and the solution is

$$\chi = -\chi_{\infty} (S^{-1/3}/\sqrt{2}) \exp(-x^2 S^{2/3}/2) \quad (118)$$

and $\lambda = S^{-1/3}$. For a general solution write Eq. (117) as

$$4 \left(\frac{d^2 \chi}{d\zeta^2} + \frac{2}{\zeta} \frac{d\chi}{d\zeta} \right) - \left(\frac{\hat{\lambda}^{3/2}}{\zeta} + 1 \right) \chi = -\chi_{\infty} \quad (119)$$

where $\zeta = x^2 / \lambda^{1/2}$. The solution can be obtained by expansion in Laguerre polynomials, and is given by

$$\chi = \frac{-\chi_{\infty}}{(\hat{\lambda}^{3/2} - 1)} \left\{ 1 - \frac{\hat{\lambda}^{3/2}}{\sqrt{2}} \int_0^1 dy y^p \frac{d}{dy} \left[(1+y)^{1/2} \exp\left(-\frac{\zeta(1-y)}{2(1+y)}\right) \right] \right\} \quad (120)$$

where $p = (\hat{\lambda}^{3/2} - 1)/4$,

which can be verified by direct substitution. Evaluating

$\int_0^{\infty} (dx/x) (d\chi/dx)$ we find

$$\lambda = \lambda_H \left(\frac{\hat{\lambda}^{9/4}}{8} - \frac{\Gamma[(\hat{\lambda}^{3/2} - 1)/4]}{\Gamma[(\hat{\lambda}^{3/2} + 5)/4]} \right) \quad (121)$$

We thus obtain the growth rate in the three cases.

Ideally unstable: $\gamma \sim (a/R)^2 (1/\tau_H)$

Ideally marginally stable: $\gamma \sim (a/R)^2 S^{-1/3} (1/\tau_H)$

Ideally stable: $\gamma \sim (a/R)^2 S^{-3/5} (1/\tau_H)$

Note that in the case of ideal stability the growth rate has the usual tearing mode scaling found in Section 2.3. This scaling can be obtained, however, only if additional physical effects are present which can add stabilizing terms to $\int g(r) dr$.

3.5 High Beta and Multiple Helicities

If the plasma pressure is taken to be of order ϵ the equilibrium is θ dependent, modes of different helicities are coupled, and a full three dimensional analysis is necessary. To find the equilibrium state, write the field in terms of the poloidal flux and the gradient operator in terms of the variables of Fig. 8, $\vec{B} = \vec{\nabla} A \times (\phi R_0/R) + B_\phi \hat{\phi}$, with $\vec{\nabla} = \hat{R}(d/dR) + \hat{\phi}(1/R)(d/d\phi) + \hat{y}(d/dy)$. We then find

$$\vec{\nabla} \times \vec{B} = - \frac{\hat{\phi} R_0}{R} \Delta^* A + \vec{\nabla}(R B_\phi) \times \frac{\hat{\phi}}{R} \quad (122)$$

where $\Delta^* = R(\partial/\partial R)(1/R)(\partial/\partial R) + (\partial^2/\partial y^2)$. Substituting this expression into $\vec{\nabla} p = \vec{j} \times \vec{B}$ and using $p = p(A)$ we find

$$p' \vec{\nabla} A = -(R_0^2/R^2) \Delta^* A \vec{\nabla} A - R B_\phi (1/R^2) \vec{\nabla} (R B_\phi) \quad (123)$$

from which we conclude that $R B_\phi$ is also a function of A , in fact, equal to the total poloidal current. We thus find the equilibrium equation (Grad and Rubin, 1958, Shafranov, 1966).

$$-\Delta^* A = \frac{R^2}{R_0^2} p'(A) + \frac{I I'(A)}{R_0^2} \quad (124)$$

To lowest order in the inverse aspect ratio ϵ the operator Δ^* reduces to $(\partial^2/\partial x^2) + (\partial^2/\partial y^2) = \nabla_\perp^2$, and the Grad-Shafranov equation becomes

$$\nabla_\perp^2 A + \frac{2x}{R_0} \frac{\partial p}{\partial A} = \frac{I I'(A)}{R_0^2} + p'(A) \quad (125)$$

To lowest order in ϵ the expression for the magnetic field becomes

$$\vec{B} = \vec{\nabla} A \times \hat{\phi} + B_\phi \hat{\phi} \quad (126)$$

and $\vec{j} = \vec{\nabla} \times \vec{B}$ gives

$$j_\phi = -\nabla_\perp^2 A \quad (127)$$

The incompressibility allows the velocity to be expressed as usual in terms of a scalar

$$\vec{u} = \vec{\nabla} W \times \hat{\phi} \quad (128)$$

To complete the description of the plasma evolution, we find a closed set of coupled differential equations for the three scalars W , A , and p . The toroidal field $B_\phi = I/R$, to lowest order, can be written as $B_\phi = B_0 + B_0 R_0/R + I_1/R_0$, where the two corrections to the zero order constant toroidal field are due respectively to toroidal curvature and plasma current. To lowest order in ϵ we find $\vec{\nabla}_\perp = \vec{j} \times \vec{B}$ and thus $\vec{\nabla}(p + B^2/2) = (\vec{B} \cdot \vec{\nabla})\vec{B}$, which implies

$$\vec{\nabla}(p + I^2/2R_0^2) = 0 \quad (129)$$

Applying $\vec{\nabla} \cdot \vec{\nabla} \times$ to the equation of motion then gives

$$\frac{d}{dt} (\nabla_\perp^2 W) = \hat{e}_\phi \cdot (\vec{\nabla} A \times \vec{\nabla} j_\phi) + \frac{\partial}{\partial \phi} \nabla_\perp^2 A + 2 \left(\frac{\partial p}{\partial r} \sin \theta + \frac{1}{r} \frac{\partial p}{\partial \theta} \cos \theta \right) \quad (130)$$

For the potential A , begin with $\vec{\nabla}_\perp \cdot \vec{B}_\perp = 0$, or $(\partial B_\perp / \partial t) = \vec{\nabla} \times \vec{K}$ which gives $(\partial A / \partial t) = \vec{K} \cdot \hat{e}_\phi$. But $(\partial \vec{B}_\perp / \partial t) = (\vec{B} \cdot \vec{\nabla}) \vec{u}_\perp - (\vec{u} \cdot \vec{\nabla}) \vec{B}_\perp = \vec{\nabla} \times [\vec{u}_\perp \times \vec{B}_\perp + (B_0/r) (\partial W / \partial \phi) \hat{e}_\phi]$ and thus $(\partial A / \partial t) = (\vec{B} \cdot \vec{\nabla}) W$ or

$$\frac{\partial A}{\partial t} = \frac{\partial W}{\partial \phi} \quad (131)$$

Finally, for the pressure we have

$$\frac{dp}{dt} = 0 \quad (132)$$

Equations (78), (79), and (80) have been used in three dimensional numerical codes to investigate nonlinear tearing and ballooning modes (White et al., 1979) and in a low β version (Callen et al., 1979 ; Biskamp and Welter, 1978 ; Dnestrovskii, 1979) to examine the nonlinear interaction of tearing modes of different helicities. Various numerical methods have been employed, the most successful uses Fourier decompositions in θ and ϕ , and a radial grid. One method of advancing the equations in time is a second order accurate explicit difference scheme similar to the two-step Lax-Wendroff method. Results obtained using these codes are discussed in following sections.

4. Nonlinear Theory

4.1 Introduction and survey

Linear stability theory of the tearing mode gives only an incomplete description of the behavior of the mode, since in most laboratory plasmas nonlinear effects become important in the very early stages of development of the mode.

There are two separate nonlinear effects. The first, investigated by Rutherford (1973), and described in Section 4.2, occurs when the magnetic island width exceeds the width of the tearing layer. At this point, the growth slows from exponential to algebraic, the island width increasing linearly in time. (The $m = 1$ mode, in cylindrical geometry, is an exception to this because of being ideally neutral stable.) The second effect occurs when the island size becomes comparable to the shear length associated with the initial field. At this point, the behavior of the mode is dominated by the gross geometry, or equivalently by external driving forces. In cylindrical geometry with fixed boundary conditions complete saturation of the mode can occur (White et al., 1977a). The saturation is due to the existence of a geometry dependent minimum magnetic energy state for a particular island width.

When the magnetic island is large compared to the tearing width, the plasma inertia becomes negligible and the further evolution of the mode is governed by a nonlinear elliptic equation of state and an equation for the flux-line averaged time derivative

of the magnetic flux. The properties of these nonstandard equations have been investigated extensively by Grad (1975). In this phase, development of the mode, and response to variation of external boundary conditions, results in adiabatic deformation of the state. Postulating that energy dissipation takes place only in a boundary layer near the separatrix, with the interior domains remaining classically adiabatic allows the formulation of the problem in terms of jump conditions across the separatrices, similar to the Hugoniot conditions for an ideal shock (Grad, 1975). Evolution of the state to a minimum in magnetic energy subject to postulated constraints was proposed by Taylor (1974). These formulations will be briefly discussed in Section 4.6.

4.2 Rutherford Behavior

In this section we review the transition from exponential growth on a hybrid hydrodynamic-resistive time scale to algebraic growth on the resistive time scale (Rutherford 1973). As shown in Section 2.4 there is a γ independent eddy current $\delta \vec{j}_z$ which opposes the zero order current near the singular surface (see Fig. 7). This eddy current produces a third order nonlinear force through $\delta \vec{j}_z \times \vec{B}_1$ which impedes the growth of the mode, and since it is nonlinear, at some mode amplitude it replaces the inertia as the force opposing growth. To estimate the island width at which this will happen, we consider Eq. (27)

$$\mu \frac{d}{dt} \nabla^2 \phi = -\hat{z} \cdot [\vec{\nabla} \psi \times \vec{\nabla} (\nabla^2 \psi)] \quad (133)$$

Consider the interior region for simplicity. Using the notation of Section 2.3 to write Eq. (133) in this region we find

$$\gamma \tau_A^2 \phi_1'' = -x \psi_1'' - \frac{\psi_1 \psi_{20}''}{\bar{B}} \quad (134)$$

where the first term on the right hand side is the linear driving force and the second is due to δj_z . We also have from Section 2.4 that $\psi_{20} \sim K(x) = (\gamma/\eta \bar{B}) \psi_1 \phi_1$. Substituting this into Eq. (134) and making use of the constant ψ approximation we find that the nonlinear term will equal the left hand side when $\gamma \tau_A^2 = \psi_1^2 / \eta \bar{B}^2$, or, using Eq. (29)

$$w \approx x_{,T} \quad . \quad (135)$$

Furthermore, the sign of the nonlinear term is such as to replace the inertia term. In terms of Eq. (133), neglect of the inertia means that

$$\vec{\nabla}\psi \times \vec{\nabla}j = 0 \quad (136)$$

or $j = j(\psi)$, this constraint on the form of j replacing Eq. (133). The fluid velocity can also be eliminated from Eq. (25) for the magnetic flux by introducing the average along a flux line through $\langle F \rangle = \oint (d\ell/\nabla\psi) F / \oint (d\ell/\nabla\psi)$. Noting that $\langle u \cdot \nabla\psi \rangle = 0$ we then find the further evolution of the mode is governed by the two equations

$$\nabla^2\psi = -j(\psi) \quad (137)$$

$$\left\langle \frac{\partial\psi}{\partial t} \right\rangle = -\langle \eta \rangle j(\psi) + E \quad . \quad (138)$$

Differential equations of this type were first investigated by Grad (1970). The approximate evolution of the mode in this regime is easily obtained. We can consider the island arbitrarily small (compared to the shear length) and use the linear approximations for ψ , j . Substitute the expression for the mode,

$\psi = \psi_0 + \psi_1$ cosky and evaluate Eq. (138) at the o point, where the average over flux lines consists of the contribution from a single point, giving

$$\frac{\partial \psi_1(0)}{\partial t} = -\eta(0) j_1(0) \quad . \quad (139)$$

Approximating j_1 as constant across the island and integrating Eq. (137) across the island by approximating $\nabla^2 \psi_1$ cosky with ψ_1'' cosky we find

$$\frac{\partial \psi_1(0)}{\partial t} = \frac{\psi_1 \Delta'}{w \tau_R} \quad (140)$$

and using Eq. (29),

$$(dw/dt) = \Delta' / \tau_R \quad . \quad (141)$$

A more accurate integration of the flux surface averaged equations yields a factor of $\pi/2$ on the right-hand side of Eq. (141).

This analysis was extended to the collisionless ($\gamma \gg \nu$) and semicollisional ($\gamma \ll \nu$) regimes by Drake and Lee (1977). In these regimes the Doppler shift in the electron response prevents the complete shorting out of the field E_z by electron flow. They found, however, that the collisionless and collisional modes evolve nonlinearly into the semicollisional regime, and that in this regime the mode again assumes the algebraic growth found by Rutherford.

4.3 Nonlinear Cylindrical Modes, $m > 1$

Even algebraic growth as discussed in Section 4.2 will typically produce a magnetic island which, barring some saturation mechanism, will fill the device in a time which is short compared to hoped for confinement times. In Fig. 12 are shown flux surfaces for a large $m = 2$ island. The destructive properties of magnetic islands vis-a-vis confinement are apparent, magnetic surfaces providing rapid thermal conduction across the island and thus removing a large fraction of the insulating volume between the hot plasma center and the walls. There is, however, a natural saturation of the mode.

The saturation of the tearing mode occurs simply through the vanishing, as the island grows, of the source of magnetic energy driving the instability in the first place. The saturation depends also on the form of the resistivity, but is rather insensitive to the exact form, provided the resistivity profile increases radially with approximately the scale length of the minor radius, as is normal. The fact that nonlinear coupling to other modes is not important in the saturation process makes a single helicity analysis possible. An analytical saturation model (White et al. 1977a) is obtained as a quasilinear extension of the work of Rutherford described in Section 4.2. Beginning with the reduced equations for a mode of a single helicity derived in Section 3.2, and taking flux surface averages as in Section 4.2, we find two equations which are the equivalent of Eqs. (137), (138) for cylindrical geometry.

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle = -\langle \eta \rangle j + E \quad (142)$$

$$\nabla^2 \psi = -j(\psi) - (2kB_z/m) \quad (143)$$

For the flux topology of interest (a single island) the boundary conditions require the value of ψ at the o point as well as at the wall or equivalently, the island width. The flux contours for an island in cylindrical geometry are shown in Fig. 13. It is readily verified that the perturbation expansion employed in Section 2.3,

$$\psi = \psi_0(0) + \epsilon \psi_1(r) \cos m\theta + \epsilon^2 (\psi_2 \cos 2m\theta + \delta \psi_0) + \dots \quad (144)$$

$$j = j_a(\psi_0 + \Delta\psi) + \epsilon j^1(\psi_0 + \Delta\psi) + \dots \quad (145)$$

is not applicable in the island interior. Here, $j_a(\psi_0)$ is the initial current profile. Note that $j_a(\psi)$ possesses many harmonics. It is evident from Fig. 14 that the separatrix is a highly singular point for the function $j(\psi)$. The harmonic expansion of ψ is, however, rapidly convergent for all r even in the case of fairly large islands. Write

$$j(\psi) = \begin{cases} j_a(\psi) & \psi < \psi_s \\ j_b(\psi) & \psi > \psi_s \end{cases} \quad (146)$$

where ψ_s is the value of ψ on the separatrix, and model the current in the island interior through

$$j_b(\psi) = a + b\psi \quad (147)$$

which can be regarded as a truncation of the Taylor expansion of $j(\psi)$. The form of the model was arrived at by examination of results of numerical advancement of the equations of Section 3.2. It was observed that saturation occurred with significant changes taking place only in the first and zeroth harmonic in the island vicinity. The coefficients in Eq. (147) can be determined by matching the interior and exterior regions, much as in Section 2.3, only now interior refers to the island, not to the tearing layer. In terms of these harmonics Eq. (143) takes the form

$$\begin{aligned} \delta\psi_0'' &= -(\delta\psi_0'/r) - \delta j_0 \\ \psi_1'' &= (m^2/r^2)\psi_1 - (1/r)\psi_1' - j_1 \end{aligned} \quad (148)$$

where the harmonics of the current are given by

$$\begin{aligned} j_0(r) &= (m/\pi) \int_0^{\pi/m} d\theta j(\psi) \\ j_1(r) &= (2m/\pi) \int_0^{\pi/m} d\theta \cos m\theta j(\psi) \end{aligned} \quad (149)$$

and $\delta j_0 = j_0(r) - j_a[j_0(r)]$.

We now integrate Eq. (148) across the island and match to the exterior solution. This uniquely determines the state by determining the constants a, b in terms of the island width, which is a free parameter. Once this state is determined, Eq. (142) can be used to determine its growth rate. To lowest order in ϵ this procedure reproduces the algebraic growth results of Section 4.2, and carried out to the next order it also predicts saturation widths, through

$$(dw/dt) = 1.66 \gamma(r_s) \Delta'(w) - \epsilon w \quad (150)$$

where γ depends on the resistivity profile, but is practically negligible if this profile is increasing radially with a scale length given by the minor radius. The major role in this equation is played by the quantity $\Delta'(w)$, obtained from the exterior solution, which is the equivalent of Δ' in the linear theory; i.e., $\Delta'(w) = [\psi_1(r_2) - \psi_1(r_1)] / \psi_1(r_x)$, where r_1, r_2 are the island edges and r_x the x point position. For typical resistivity profiles the mode thus saturates approximately when $\Delta'(w)$ vanishes and the saturated island width can be determined by a numerical evaluation of $\Delta'(w)$. As we saw in Section 3.2, the most unstable modes were those with low values of m . The largest saturated island states also correspond to modes with $m = 2$. The dependence of the saturation width on the island location, r_s , is shown in Fig. 15. This is to be contrasted with the dependence of Δ' on r_s , $\Delta' \rightarrow [f(m)]/r_s$ for small r_s . An example of the dependence of

the saturation width on the current profile is shown in Fig. 16 (Monticello et al., 1979). Here, it is apparent that the flatness of the current profile, i.e., the absence of shear, is important for large island development. In this figure the island position ($q = 2$ surface) has been held fixed at $r_s = 0.7a$ so that increasing $q(0)$, as well as increasing the parameter p , corresponds to flattening the profile. As long as the current profile is not too flat, the state of minimum energy consists of an island state with island width roughly ten percent of the minor radius. Inclusion of finite Larmor radius effects does not change the saturation width of this state, but causes a mode rotation at the diamagnetic frequency (Monticello et al., 1979; Monticello and White, 1980; Biskamp, 1979). The resulting $m = 2$ oscillations give a theoretical interpretation of the experimentally observed Mirnov oscillations (Mirnov and Semenov, 1971).

The major disruption occurring in tokamak discharges appears to be primarily due to a large $m = 2$ magnetic island. There are a variety of guises in which the disruption can appear. A canonical sequence, however, is the following: some time is spent executing sawtooth oscillations (see Section 4.4), after which these oscillations cease and the so-called precursor oscillation, an $m = 2$ mode, is observed to grow at a rate consistent with the Rutherford rate for a magnetic island [Eq. (141)]. As shown by White et al. (1977c), a small change of a current profile associated with sawtooth oscillations [$q(0) \approx 1$] can lead from a case in which the $m = 2$ mode is stable, to one in which the saturation width is a large fraction of the minor radius.

Furthermore, the growing island, by increasing radial transport, decreasing the current, and thus increasing the value of q , produces a profile unstable to even larger island growth, as seen in Fig. 16. This process, associated with the precursor oscillation, would, of course, result in the termination of the sawtooth oscillations with the elimination of the $q = 1$ surface. The coupling of the precursor oscillation to other modes, particularly to $m = 1$ through toroidal coupling, and the sensitivity of this coupling to the current profile (Bussac et al., 1977) provides for a wide variety of satellite modes in the predisruptive phase. The precursor oscillation is followed by the disruption itself; a rapid drop in central temperature and, often, a total loss of plasma. The disruption itself is thought to be caused by an onset of ergodicity induced by this $m = 2$ island together with some combination of nonlinear coupling to other modes (Welter and Biskamp, 1978; Waddell et al., 1979; Callen et al., 1979; Biskamp, 1978; Dnestrovskii 1979; Biskamp and Welter, 1979) toroidal coupling (Finn, 1975; 1977), and contact with the limiter or walls (Pollard and Wesson, 1979). None of the proposed models give a complete description of the final stages of the disruption. The mode coupling calculations, although giving rise to an explosive growth on a rapid time scale when the $m = 2, n = 1$ island overlaps an $m = 3, n = 2$ island, do not produce a rapid drop of central plasma temperature. The resulting ergodic region flattens the temperature between the $q = (3/2)$ and $q = 2$ surfaces, but some further mechanism is required to destroy confinement in the central region. Significant mode coupling occurs only for profiles which

are unstable to fairly large $m = 2$ island development, so the mode coupling model complements this description of the disruption rather than replacing it. The role of limiter contact in the disruption is not yet certain. Model calculations including the interaction of an $m = 2$ mode with the limiter do show plasma loss, but the time scale for this process with realistic plasma parameters is not known. Present experimental evidence makes it very plausible that there are many different forms of disruption, involving some combination of these, and perhaps also other, processes (Lackner and Zehrfeld, 1979).

Control of the disruption by feedback has been studied both experimentally (Arsenin and Chuyanov, 1977; Arsenin et al., 1978; Karger et al., 1977) and theoretically (Monticello et al., 1979; Callen et al., 1979).

4.4 Nonlinear Cylindrical Modes, $m = 1$

If the toroidal current in a tokamak is made sufficiently large that the safety factor is less than one on axis, the x-ray emission typically shows a characteristic sawtooth behavior. At the plasma center, the x-ray intensity rises in a time consistent with ohmic heating, and then suddenly drops. Outside the $q = 1$ surface the sawtooth is reversed; i.e., the x-ray intensity rises abruptly and decays slowly (Sauthoff et al., 1978).

As originally conjectured by Kadomtsev (1975b) and Monticello (1975), the $m = 1$ mode is capable of evolving nonlinearly through a sequence of states which ends with helical flux surfaces forming concentric cylinders, just as in the original state, but which results in a flattening of the current profile inside the $q = 1$ surface, and releases a corresponding amount of magnetic energy. Since the final state of this process can again acquire a central current peak through ohmic heating and the ensuing decrease in resistivity, the process can in principle be cyclic. The reconnection sequence proposed by Kadomtsev and Monticello is shown in Fig. 17. The first sketch (17a) shows the initial helical flux contours with the $q = 1$ surface shown as a dotted line. An initial $m = 1$ perturbation causes the displacement of the central region, reconnection taking place at the x point. The two surfaces labeled 1 reconnect and withdraw from the x point (17c). The subsequent evolution of the area inside surface 1 does not involve the resistivity, so this area is conserved, as is the value of ψ since $(d\psi/dt) = 0$ neglecting resistivity. The same process occurs with

the surfaces labeled 2, 3 (17d, e) until finally the initial o point (labeled 4) has been expelled through the x point and the flux contours have again returned to an axisymmetric equilibrium state. The resulting changes in the helical flux profile, the q profile, and the current profile are shown in Fig. 18. The flux surface labeling in Fig. 18a corresponds to that in Fig. 17. The new ψ profile is easily obtained by recalling that ψ is conserved following the field in this process and that surfaces joining in the reconnection process have initially the same value of ψ . In the final state there is a discontinuity in $(d\psi/dr)$ at r_4 . The safety factor profile (Fig. 18b) is readily obtained by noting that $q(r) = [1 + (R/r)(d\psi/dr)]^{-1}$. In the final state $q(0) = 1$ and $q(r)$ is appreciably flattened out to r_4 , where it is discontinuous. The current profile (Fig. 18c) is readily obtained from Eq. (87) and is quite flat out to r_4 where it possesses a reverse current sheet. All three profiles are unchanged for $r > r_4$. The final state has much lower energy than the initial state, as is readily verified, but it is not clear from this analysis whether this final state is accessible nor at what rate it would proceed during its highly nonlinear evolution.

The nonlinear two-dimensional codes described in Section 3.2 were used to investigate the process in detail, and showed (Sykes and Wesson, 1976; Waddell et al., 1976; Asumi et al., 1976; Danilov et al., 1976; Dnestrovskii et al., 1977) that this process was in fact nonlinearly possible as well as energetically favorable, and that the experimental time scale for the rapid phase of the sawtooth was consistent with the nonlinear behavior of the mode.

The sequence of helical flux surfaces for the process obtained from a numerical code is shown in Fig. 19. In order for the experimental observations for the rapid phase of the sawtooth to be described by the mode it is necessary that it continue growing at essentially its linear growth rate for the entire nonlinear process. This is what is observed with the numerical codes. The mode has a linear growth rate substantially larger than that of the $m = 2$ mode, because of being ideally neutral stable (Sec. 3.4). The invalidity of the constant ψ approximation makes the Rutherford analysis (Section 4.2) invalid.

A simple estimate can be made (Dubois and Samain, 1979) for the time necessary to complete the full nonlinear evolution shown in Fig. 19. Consider the helical surface (see Fig. 9) bounded by the x point and the o point. The total flux passing through this surface, as shown in Section 3.2, is given by $\psi(x) - \psi(o)$. During the evolution of the mode the x and o points pinch together, and the flux through this surface changes from its initial value to zero. We also have from Eq. (77) that $(d\psi/dt) = j_z/r_R$, and assuming that the o and x points move with the fluid and taking the resistivity η constant for simplicity we find

$$\Delta\psi = - \frac{1}{r_R} \int_0^t [j_x(t) - j_o(t)] dt \quad (151)$$

where $\Delta\psi = \psi(o) - \psi(x)$ evaluated at $t = 0$. Now, if it is assumed that the difference in the current between the x and o points remains approximately constant, or is at least bounded by its value at $t = 0$ one obtains

$$\Delta\psi < \frac{\Delta t}{\tau_R} |j_x(0) - j_0(0)| \quad (152)$$

and thus the total time for the evolution of the mode is given by the resistive time scale, $\Delta t > c\tau_R$, quite different from a scaling produced by the linear growth rate, $\Delta t \sim S^{1/3}\tau_A$. The resolution of this apparent paradox is readily found by an examination of the results of the numerical codes. In Fig. 20 is shown the current profile corresponding to the sequence of flux surfaces shown in Fig. 18. What is observed is that a current sheet develops at the x point, the magnitude of which scales as $S^{2/3}$, so as to allow the entire transition to proceed on the time scale of the linear growth rate. The narrow width of the current sheet prohibits numerical examination of the full evolution of the mode for very large values of S , small scale effects not included in these codes (finite Larmor radius) would make such investigations academic in any case. Most of the evolution has been examined for values of S as large as 10^7 , and no deviation from the linear growth rate has been observed. These results are summarized in Fig. 21. Attempts to analyze the $m = 1$ mode using flux surface averaging and a variational principle, (Dubois and Samain, 1979) produce results which indicate that this mode should possess algebraic behavior, with the width increasing as $t^{1/2}$, until the acute angle at the x point vanishes. At this point, the inertia again is found to be nonnegligible. The island width for this transition is not estimated, and it is not clear whether or not these results are in agreement with the numerical code results.

Further analysis of the complete sawtooth period, including the ohmic heating phase, and in particular the scaling of this period on various plasma parameters, has been attempted (Waddell et al., 1978; Callen and Jahns, 1977; Jahns et al., 1978; McGuire and Robinson, 1979) with reasonable agreement obtained for the overall sawtooth period as well as for the rapid $m = 1$ phase.

4.5 Multiple Tearing Modes

In the initial or skin phase of a tokamak discharge the current profile assumes a hollow form, and the safety factor can assume the same rational value at two values of the minor radius. This leads to the possibility of a pair of magnetic islands with the same helicity developing in a sequence of flux contours such as that shown in Fig. 22 for $m = 2$. This is of interest because the mixing of the two islands produces a complicated flux topology, which should strongly affect radial heat and particle transport (Stix 1973, 1976). Redistribution of the plasma current along the convoluted lines would appear as fast radial current penetration. Numerical simulation of this process was carried out by several authors (White et al., 1977b; Carreras et al., 1979; Schnack and Killeen, 1979).

The constant ψ approximation is not applicable for these modes for typical plasma parameters since during the skin phase of the discharge the tearing layer is quite wide. For values of S of 10^4 the islands are observed to continue exponential growth over most of their history. The merging of the two modes as shown in Fig. 22 is not a necessary conclusion of their development, and Carreras et al. (1979) found a simple condition which the equilibrium must satisfy for the islands to merge before they separately saturate through the same mechanism leading to saturation of the $m = 2$ mode in a monotonely decreasing current profile (Section 4.3). If S is sufficiently large, the islands are observed to enter a Rutherford regime as described in Section 4.2.

Detailed comparison of the process with the observed anomalous current penetration in tokamak discharges has been done by Hawryluk et al. (1980).

4.6 Externally Driven Reconnection

In this section we consider models for nonlinear reconnection in which additional assumptions are made, allowing the elimination of the rapid time scale transients, and the determination of a quasi steady state or adiabatically evolving state. One class of models of this sort has been proposed for the description of solar flares and the release of energy in the earth's magnetosphere (Cross and Van Hoven, 1971; Sweet 1958; Parker 1963; Petschek 1964; and Leboeuf et al., 1978). These models consist of essentially steady-state descriptions of the reconnection process with the rate determined by boundary conditions placed on the plasma flow far from the reconnection region. We sketch here one of the first such models, proposed by Petschek (1964). For a review of recent work in this field see Sonnerup (1979) and Vasylunas (1975). Shown in Fig. 23 are the flux surfaces for the magnetic field. All quantities in this model are translation invariant in the z direction. Note that the field makes an abrupt change in direction at a wave or shock front, which is stationary in space, its propagation in the fluid being balanced by the fluid flow. The conservation of mass requires that

$$u_{x0} y = v \delta \quad (153)$$

where both V , the flow away from the x point, and δ , the width of the region between the two shock fronts, are functions of y . Conservation of momentum in y gives

$$(d/dy) (\rho v^2 \delta) = -B_{y0} B_x \quad (154)$$

where B_x is the x component of the magnetic field within the boundary layer and is also a function of y . Combining these equations gives

$$M_o^2 \frac{d}{dy} \left(\frac{y^2}{\delta(y)} \right) = -b_x \quad (155)$$

where $M_o = (u_{x0}/V_A)$, $b_x = (B_x/B_{y0})$, and $V_A = (B_{y0}/\sqrt{\rho})$ is the local Alfvén speed. Equating the flow velocity into the boundary layer with the wave propagation gives

$$M_o = |b_x| \quad (156)$$

since the propagation speed depends only on the normal component of the field. Substituting this into Eq. (155) gives

$$\delta = M_o |y|, \quad (157)$$

thus the shock front positions are given approximately by straight lines as is shown in Fig. 23. In the vicinity of the x point the rate is governed by diffusion, not by wave propagation. To calculate the diffusion rate, first note that $E_z = \eta j_z$. Since we

are considering a state of steady flow $\vec{v} \times \vec{E} = 0$ so E_z is constant in space. For large x $\vec{E} = -(\vec{v} \times \vec{B})$ so $E_z = -u_{x0} B_{y0}$. Near $y = 0$ the change in B_y across the boundary is $2B_{y0}$, so the total current in the boundary layer is given by $2\delta j_z = 2B_{y0}$. Thus, we find that the diffusion rate is determined by the resistivity through $u_{x0} = (\eta / \delta)$. Setting the flow and diffusion velocities equal we find

$$M_0 = \eta / v_A \delta \quad (158)$$

which replaces (156) near the x point and determines the thickness of the boundary layer at the origin. Approximating this thickness by its value at the origin and substituting into Eq. (155) we find that b_x varies linearly with y in the vicinity of the x point:

$$b_x = -2M_0^3 v_A y / \eta \quad (159)$$

and this becomes equal to the value in the wave dominated region at the point $y = y^*$ where

$$y^* = \eta / 2v_A M_0^2 \quad (160)$$

is the size of the diffusion region, field annihilation being wave dominated for values larger than this.

In the exterior region we must construct a flow pattern and magnetic field which can be joined to the solution in the diffusion region. Attempt a solution which departs slightly from uniform flow,

$$\vec{u} = u_{x0}(\hat{x} + \vec{u}') \quad (161)$$

$$\vec{B} = B_{y0}(\hat{y} + \vec{B}') \quad (162)$$

with u' , B' small compared to unity. In this region we neglect resistivity, so $\vec{E} + \vec{u} \times \vec{B} = 0$. Taking the curl of this equation twice and using $\vec{\nabla} \times \vec{E} = 0$ we find

$$\frac{\partial}{\partial x} (\vec{\nabla} \times \vec{B}') = - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{u}') \quad (163)$$

which is clearly satisfied if both $\vec{B}'$, $\vec{u}'$ are curl free. This choice is also consistent with the magnetohydrodynamic equations in this region since $\vec{\nabla} \times \vec{B}' = 0$ implies no magnetic forces, and thus the decoupling of the fluid equations and the magnetic field equations. The vorticity is conserved in an incompressible, inviscid two-dimensional fluid so the condition $\vec{\nabla} \times \vec{u}' = 0$ outside the boundary layer is consistent with the boundary conditions.

The condition $\vec{E} + \vec{u} \times \vec{B} = 0$ then specifies a relation between the potential $\vec{B}'$ and the one for $\vec{u}'$. We consider the calculation of $\vec{B}'$. In the boundary layer analysis Eqs. (160), (163) specify B_x along the y axis in terms of conditions far from the boundary. Since B_x satisfies Laplace's equation, this is sufficient to

determine it everywhere. To determine B_x immediately outside the boundary layer, note that across the edge of the boundary layer the normal component of B is conserved, thus

$$B'_x + \frac{d\delta}{dy} = b_x \quad (164)$$

In the wave dominated region $y > y^*$ Eqs. (156), (157) give $b_x = (d\delta/dy) = M_0$ and we find $B'_x = -2M_0 (y/|y|)$ for $y > y^*$. In the diffusion dominated region b_x varies linearly with y , and $(d\delta/dy)$ can be approximated by b_x within this region. Thus $B'_x = -2M_0 (y/y^*)$ for $y < y^*$. We then have from Laplace's equation

$$\vec{B}'(r) = \frac{1}{\pi} \int_{-L}^L \frac{B'_x(s) (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} ds \quad (165)$$

determining the field in the exterior region. The linear approximation is readily verified to be justified ($B' \ll 1$) provided that the flow speed is small $M_0 \ll 1$. An examination of nonlinear terms in this process lead one to the conclusion that also nonlinearly the flow speed would not exceed the Alfvén speed ($M_0 < 1$).

The wave propagation mechanism reduces the region over which the diffusion mechanism must operate. Note that the fluid, and hence reconnection, can be driven at any velocity up to the limiting velocity of the standing waves, i.e., the shock speed.

Numerical simulation of externally driven reconnection models has been done by Sato (1979a, 1979b). He found flow velocities limited only by the local Alfvén speed, identified the shocks which formed as slow shocks, and verified that the Rankine-Hugoniot conditions were well satisfied. In steady state the reconnection rate is found to be determined by the boundary conditions, independent of the form of the resistivity.

A much more ambitious program has been initiated by Grad (1975, 1978). If one considers the limit of zero resistivity, so that all resistive singular layers become arbitrarily thin, it is appealing to attempt to replace the singular layer with a set of jump conditions, similar to the Hugoniot conditions in ordinary hydrodynamics. It is then reasonable to suppose that externally driven reconnection could always be forced to proceed at an arbitrary rate, and that the size of the diffusion region and the diffusion mechanism would adjust to accommodate this rate. In the example given, we have seen that any flow velocity can be imagined, restricted only by the local (Alfvén) speed of the shock fronts. Even if the limiting speed has been reached, an increase in the flow velocity would, of necessity, cause magnetic field to pile up at the neutral point, thus increasing the local Alfvén speed until the system was again in balance at the new flow rate (Adler and Kulsrud 1979).

The difficulty of the proposed scheme is in formulating jump conditions sufficiently general to describe correctly all possible topologies and changes in topologies.

The jump conditions match regions in which the resistivity is assumed negligible, thus the equilibrium equations have the form

$$V^2 = F(\psi) \quad (166)$$

where $F(\psi) = -(dp/d\psi) - f(df/d\psi)$ and $p(\psi)$, $f(\psi)$ determine the pressure and toroidal field. The set of jump conditions was formulated by Grad (1975), by assuming conservation of volume, mass, momentum, and toroidal flux. Although the model is termed "adiabatic," the jump conditions depend on the direction of change. Growth of regions 1, 2 in the poloidal flux diagram shown in Fig. 24 is called splitting whereas shrinking of this area is called mixing. Imagine the flux tube bordering regions 1, 2 to become a part of these regions. Mass conservation gives, $\delta m_0 = \delta m_1 + \delta m_2$ and using $\delta m = \rho \delta V$ and also the fact that the flux is conserved, $\delta \psi_0 = \delta \psi_1 = \delta \psi_2$ we find

$$\frac{\rho_0}{\psi_0} = \frac{\rho_1}{\psi_1} + \frac{\rho_2}{\psi_2} \quad (167)$$

Here $\psi' = (\partial \psi / \partial V)$ and $V(\psi)$ represents the volume inside the flux surface ψ (area in two dimensions), and $\psi(V)$ is the inverse function. Toroidal flux conservation associated with the same flux tubes gives, from $\delta \phi = B_T \delta V$, where B_T is the toroidal field,

$$\frac{B_{T0}}{\psi_0} = \frac{B_{T1}}{\psi_1} + \frac{B_{T2}}{\psi_2} \quad (168)$$

The adiabatic law in each region gives $B_T = v(\psi) \psi'$ so $v_0 = v_1 + v_2$.

From pressure balance we have $[p + (B^2/2)] = 0$ where the square brackets indicate the jump across any surface. Near the x point the poloidal field vanishes so $B^2 = B_T^2$, and since p, B_T are functions of ψ only, we have for the entire separatrix

$$p_0 + (B_{T0}^2/2) = p_1 + (B_{T1}^2/2) = p_2 + (B_{T2}^2/2) \quad (169)$$

which is equivalent to demanding that the poloidal field be continuous across the separatrix. But since $(1/\psi)' = \oint (dl/B_p)$ we find

$$(1/\psi_0)' = (1/\psi_1)' + (1/\psi_2)' \quad (170)$$

Another geometrical quantity determined by continuity of B_p is K , the inductance, defined through $K(V) = \langle |\nabla V|^2 \rangle$ where the angular brackets denote flux surface averaging through $\langle \phi \rangle = \oint \phi (ds/|\nabla V|)$ with $\oint (ds/|\nabla V|) = 1$. It follows that $\langle \nabla^2 \psi \rangle = \langle K \psi' \rangle'$ and thus the Grad-Shafranov equation gives $\langle K \psi' \rangle' = -(\partial/\partial \psi)[p + (B_T^2/2)]$. Integration across the separatrix then gives

$$K_0 \psi_0' = K_1 \psi_1' + K_2 \psi_2' \quad (171)$$

Now examine a mixing process. Given $\rho_1, \rho_2, p_1, p_2, B_{T1}, B_{T2}, \psi_1', \psi_2', K_1, K_2$, we must solve for these quantities in region ϕ , which are uniquely given by Eqs. (167) to (171). The

resulting solution is in general inconsistent with entropy conservation, and the process is irreversible.

For a splitting process these equations are insufficient to determine the states in regions 1, 2. Following individual fluid elements across the separatrix we find from toroidal flux conservation $(d/dt)(B_T/\rho) = 0$ that

$$(B_{T0}/\rho_0) = (B_{T1}/\rho_1) = (B_{T2}/\rho_2) \quad (172)$$

Similarly, if the adiabatic law holds, $(d/dt)(p/\rho^\gamma) = 0$ we have

$$(p_0/\rho_0^\gamma) = (p_1/\rho_1^\gamma) = (p_2/\rho_2^\gamma) \quad (173)$$

Now all quantities are determined, and combining Eqs. (173), (172), and (169) we find that $[v] = [p] = [B_T] = 0$.

In cases of asymmetric islands and more complicated topology the justification of the jump conditions is quite complicated. They must be constructed in an ad hoc manner, and it is not apparent whether they can be derived from basic principles. Confirmation of the validity of the jump conditions using numerical codes capable of reproducing the processes occurring within the resistive layer has not been done, with the exception of the work by Sato described in this section.

There have been developed a number of codes which make use of flux surface averaging and the neglect of inertia to investigate the development of magnetic islands in the nonlinear regime (Grad and Hogan, 1970; Grad 1978; Waddell et al., 1975; and Hirshman and

Jardin, 1979). These codes efficiently follow the evolution of plasma configurations on a long time scale, following changes in plasma shape and flux topology due to mode growth, transport, and changes in external field constraints. They are termed adiabatic codes since they do not include wave motion. They have been used to investigate various topologies including single island, doublet, and the reversed field mirror.

Another approach (Taylor, 1974) is to postulate a global invariant for the adiabatic relaxation. The state of the plasma is then determined by minimizing the magnetic energy subject to the constraint imposed by the invariant. It was shown that the choice of the invariant $K = \int_V \vec{A} \cdot \vec{B} \, d\tau$ where A is the vector potential and B the magnetic field leads to reasonable agreement with observed toroidal pinch data. This "invariant" has been numerically followed in the evolution of the double tearing mode with one of the two-dimensional codes described in Section 3.2 (Carreras et al. 1979). It was found that K decreased monotonically with time when the islands saturated without overlapping, but when they overlapped it suddenly increased and then decreased.

Research on the relaxation rate of invariants in turbulent states has also been carried out by Montgomery et al. (1979).

5. ERGODICITY

5.1 Ergodic Transition

Consider the equations for the magnetic field trajectories in a cylinder

$$\frac{dr}{d\phi} = \frac{RB_r}{B_z}, \quad \frac{d\phi}{d\phi} = \frac{RB_\phi}{rB_z} \quad (174)$$

where the components of B (but not the trajectories) are periodic in $\phi \rightarrow \phi + 2\pi$. We use the geometry of Fig. 8, with the cylindrical approximation, $r \ll R$. Most of the concepts in these sections are readily generalized to toroidal geometry. By integrating these equations and plotting the intersection of the field line with the r, θ plane for $\phi = 0, 2\pi$, etc., we obtain a set of points lying in the magnetic surface to which that line pertains. However, if B contains more than one helicity, the points can fill an area and magnetic surfaces are no longer well defined. When magnetic surfaces exist the transformation defined by integrating Eq. (174) through $\Delta\phi = 2\pi$ preserves the area inside a magnetic surface by virtue of $\vec{\nabla} \cdot \vec{B} = 0$. In the simplest case of a cylindrically symmetric equilibrium we find the points given by $r = \text{constant}$ and $\theta = [2\pi n/q(r)]$, $n = \text{integer}$. If $q(r)$ is not rational, these points are dense on the curve $r = \text{constant}$, for q rational they form a discrete set.

A general perturbation from cylindrical symmetry can be written

$$\delta \vec{B} = \sum_{m,n} \vec{b}_{mn}(r) \exp[i(m\theta - n\phi)] + \text{c.c.} \quad (175)$$

which produces magnetic islands at surfaces r_{mn} defined by $q(r) = (m/n)$. From Eq. (29), using $\psi'_0 = (r/R)(q'/q^2)$ and $B_r = (m c_1/r)$ we find the width of the island at surface r_{mn} is given by

$$w = 4 \left(2 |b_{mnr}| \frac{q^2 R}{q m} \right)^{1/2} \quad (176)$$

where all expressions are to be evaluated at $r = r_{mn}$. The transition to ergodicity occurs very suddenly, magnetic surfaces in the region between two islands of different helicity being destroyed when the islands associated with these surfaces overlap. The central problem in ergodic theory is the determination, for a given field, of the nature and extent of the ergodic regions and the existence of well-defined magnetic surfaces.

Introduce a Hamiltonian given by $H = \int^r (B_\theta/r) dr$ with canonical action-angle variables $I = \int^r (B_\phi)/R dr$, θ , and ϕ the time variable. Hamilton's equations

$\dot{\theta} = (\partial H / \partial I)$, $\dot{I} = -(\partial H / \partial \theta) + (\partial I / \partial \phi)$ give the field trajectories. In particular in the cylindrical approximation (lowest order in r/R) they reduce to

$$\frac{dr}{dt} = \frac{RB_\theta}{rB_z}, \quad \frac{dr}{d\phi} = \frac{RB_r}{B_\phi} \quad (177)$$

We wish to exploit this correspondence between the field trajectories and orbits of a Hamiltonian system. In the case of cylindrical symmetry the Hamiltonian reduces to $H = H_0(I)$ and the trajectories lie on the one-dimensional tori $r = \text{constant}$, integrating to $\psi = \omega\phi + \psi(0)$ with $\omega = (\partial H / \partial I) = 1/q(r)$. We assume that ω is not simply a constant but depends on r , so the Hamiltonian is that of a nonlinear oscillator. Introduction of a perturbing field which depends on r, θ, ϕ is equivalent to perturbing the initial Hamiltonian through

$$H = H_0(I) + V(I, \theta, \phi) \quad (178)$$

In 1954 Kolmogorov stated a theorem (Kolmogorov 1957) later proved by Arnol'd (1963) and Moser (1962) regarding such a perturbation. The theorem is actually more general, applying to a Hamiltonian with N variables. It states that if V is sufficiently small then most system trajectories continue to be on smooth tori. Small perturbations of a cylindrical symmetric field continue to give a field structure which has well-defined magnetic surfaces except for a finite volume proportional to V .

To see the effect of such a perturbation, consider $V(I, \theta, \phi)$ consisting of a single harmonic.

$$H = H_0(I) + \epsilon V(I) \cos(m\psi - n\phi). \quad (179)$$

Change variables from I, ψ to I, p, ϕ using a canonical transformation with generating function $F(I, \psi, \phi) = -\epsilon(I - I_r)[(\psi + n\phi)/m]$, giving $\psi = -(\partial F/\partial I) = (\psi + n\phi)/m$, $p = -(\partial F/\partial \psi) = (I - I_r)/m$, and the new Hamiltonian $K = H + (\partial F/\partial \phi) = H - (n/m)(I - I_r)$. Examine the Hamiltonian in the vicinity of $I = I_r$, expanding H_0 to second order in p , and choose I_r so that the terms linear in p vanish giving

$$K = (1/2)H''_0(I_r)m^2p^2 + \epsilon V(I_r) \cos \phi. \quad (180)$$

This is time (ϕ) independent, and has the canonical phase space of an island. The separatrix is defined by $K = 0$, and in the original variables defines the trajectories

$$I - I_r = 2 \left(\frac{\epsilon V}{H''_0} \right)^{1/2} \sin \left(\frac{m\psi - n\phi}{2} \right), \quad H'_0(I_r) = \frac{n}{m}. \quad (181)$$

The island width is thus given by $w = 4(\epsilon V/H''_0)^{1/2}$. As shown by Kolmogorov, far from the resonance point the tori are only slightly perturbed; their topological form is unchanged.

If $V(I, \psi, \phi)$ contains more than a single harmonic, the same analysis can be used to find the form of the trajectories in the vicinity of each resonant surface, thus giving rise to separate island structures. This analysis, of course, fails if the islands overlap with one another, the fields in the overlap regions becoming highly stochastic in nature.

Now consider a general perturbation $V(I, \theta, \phi)$, and expand it in harmonics

$$H = H_0(I) + \sum_{m,n} V_{mn}(I) \exp[i(m\theta - n\phi)] + \text{c.c.} \quad (182)$$

In the vicinity of each resonant surface I_{mn} trajectories have the form of islands with width $w = 4(\epsilon V_{mn}/H'_0)^{1/2}$. It might be thought that since the rational surfaces I_{mn} are dense in the range of I , any finite island width associated with each surface would lead to overall stochasticity, but this is not the case. The condition $H'_0(I_{mn}) = (n/m)$ and the condition that the resonant surface I_{mn} lie within a finite specified region $I < I_{\max}$ restricts n in the sum to $n < n_{\max}$ with $n_{\max} = mH'_0(I_{\max})$. Adding up the widths of all islands with $n < n_{\max}$ we find

$$\Delta W_m = \sum_{n=1}^{n_{\max}} 4 \left(\frac{\epsilon V_{mn}}{H'_0(I_{mn})} \right)^{1/2}. \quad (183)$$

Now if $V(I, \theta, \phi)$ is analytic and periodic in θ , V_{nm} is bounded by a constant times $e^{-\sigma m}$, giving $\Delta W_m < \epsilon A m e^{-\sigma m/2}$. The total region occupied by the islands is thus bounded by $\epsilon A \sum_{m=1}^{\infty} m e^{-\sigma m/2}$, which is finite, and proportional to ϵ . Thus there is a critical value of ϵ for which the trajectories become stochastic in the entire region. For a review of ergodic behavior and its relevance to statistical mechanics see Ford (1975), Balescu (1975), Zaslavsky and Chirikov (1971; 1979) Whiteman (1977), and Penrose (1979).

The transition to ergodic field behavior was studied by numerical integration of the field trajectory equations by Gelfand et al. (1963), Morozov and Solov'ev (1964), and Graev et al. (1965). Rosenbluth et al. (1966) and Filonenko et al. (1967) used a perturbation theory analogous to that of quasi-linear theory to solve the Liouville equation for the field lines. They investigated the width of the stochastic region, and confirmed that as long as the magnetic islands associated with different values of q are non overlapping the stochastic regions are narrow bands located in the vicinity of the separatrices, but when V is sufficiently large so that neighboring islands overlap the stochastic regions can be quite wide.

The numerical investigation of stochastic properties of area preserving transformations has been enormously simplified by the discovery that the complicated structures and ergodic behavior given by integration of field line equations are exhibited by discrete area preserving transformations. Some of the remarkable complexity exhibited by simple two-dimensional transformations is discussed by Ford (1975). To see how such a transformation can also approximate a real system, consider Eq. (174) and approximate the shear as constant, giving

$$\frac{dx}{d\phi} = \frac{b(\theta)}{2\pi}, \quad \frac{d\theta}{d\phi} = x \quad (184)$$

where x represents the (radial) distance from the singular surface, and we have taken $(db/d\theta) \gg (db/dx)$ and neglected the x dependence. The equivalent Hamiltonian for this system is

$H(x, \theta, \phi) = (x^2/2) - \int_0^\theta (d\theta/2\pi) b(\theta, \phi)$, and the action variable is x itself. Now convert this to a finite difference equation [$\Delta\phi = (2\pi/N)$] through

$$\begin{aligned} y_{i+1} &= y_i + x_i/N \\ x_{i+1} &= x_i + b(y_{i+1}, \phi_{i+1})/N \end{aligned} \quad (185)$$

where $y = \theta/2\pi$. Since the Jacobian of this transformation is unity, it is area preserving and furthermore in the limit of large N it can be expected to give a reasonable approximation to the differential equations. The simplest case of interest is that of a single mode $b(\theta, \phi) \approx \epsilon \sin(\theta - \phi)$ with $N = 1$ in which case $b(\phi_i, \phi_i)$ is independent of ϕ and $b(\theta) = \epsilon \sin(\theta)$. [Taylor (1969), Chirikov (1969)]. The equivalent Hamiltonian for the discrete map Eq. (185) is readily seen to be

$$H(x, \theta, \phi) = \frac{x^2}{2} + \frac{\epsilon}{2\pi} \sum_{n=-\infty}^{\infty} \cos(\theta - n\phi) \quad (186)$$

since $\dot{x} = -(\partial H/\partial \theta) = (\epsilon/2\pi) \sum_{n=-\infty}^{\infty} \sin(\theta - n\phi) = \epsilon \sin \theta \delta_p(\phi)$ where $\delta_p(\phi) = (1/2\pi) \sum_{n=-\infty}^{\infty} \cos n\phi$ is a periodic δ function of period 2π .

From our previous analysis, islands are produced at all integer x with width $W = 4(\epsilon/2\pi)^{1/2}$. The overlap criterion for stochasticity then gives $W = 1$ or $\epsilon = (\pi/8) = 0.39$. Chirikov expresses this in terms of the parameter $K = 2\pi\epsilon$, giving $K = 2.5$ for overlap.

In Fig. 25 is shown the result of successive applications of this map for $\epsilon = 0.01$. The structure is periodic in x and y , with an elliptic point at $x = 0, y = (1/2)$ and a hyperbolic point at $x = 0, y = 0$. In Fig. 26 is shown the resulting island structure for $\epsilon = 0.15$.

Since a good magnetic surface prohibits a trajectory from crossing it, the existence of any such surfaces can be examined numerically by finding the largest value of ϵ for which passage from $x = 0$ to $x = 1$ (for example) is not observed. A numerical investigation of this sort yields the value of $K = 0.989$, well below the value given by the overlap criterion. There are, however, three effects which are apparent in Fig. 26 which we have neglected. First, besides the islands at $x = \text{integer}$, there are higher harmonics produced by the nonlinearity of H at $x = (1/2), (1/3)$, etc.; second, the islands twist and distort due to mutual interaction as they approach one another; and third, the separatrices themselves have finite width. Taking account only of part of the first effect by including second and third harmonics gives an improved overlap criterion of $K = 1.35$. How do the discrete mapping results differ from an integration of the differential equations? Return to the discrete map with arbitrary step size, Eq. (185), with $b(\theta, \phi) = \epsilon \sin(\theta - \phi)$. Now the symmetry in x is for $x \rightarrow x + N$, and the stochastic threshold occurs when neighboring island chains overlap. In the limit $N \rightarrow \infty$ there is no stochastic behavior in this model. This corresponds to integration of the differential equations for the

case of a single helicity, for which magnetic surfaces are preserved for any island width.

The transition from the existence of magnetic surfaces to ergodicity, and the demarcation of the ergodic region (the so-called KAM boundary) has been further studied by Greene (1968). He studies the transformation by linearizing it in the vicinity of a fixed point and classifying fixed points as elliptic or hyperbolic. A hierarchy of such points is constructed. It is found that when closed invariant curves (magnetic surfaces) exist, an infinite number of the fixed points are elliptic. In ergodic regions all but a finite number of the fixed points of the given order of the hierarchy are hyperbolic.

5.2 Kolmogorov Entropy and Field Diffusion

Within a stochastic field region the field behavior can be characterized by appropriately defined statistical moments. First a given field line diffuses radially as it is traced in z , the diffusion constant being defined through

$$D = \lim_{\substack{z \rightarrow \infty \\ L \rightarrow \infty}} \frac{1}{Lz} \sum_{n=1}^L x_n^2 \quad (187)$$

where $x_n = r_n - r_0$ is the distance the n^{th} line diffuses, L lines being used in the average, all initially at $r = r_0$.

Second, on the average two neighboring lines diverge from one another, the mean distance between them increasing as $d = \exp(z/L_c)$, where L_c is defined to be the correlation length for the field. This divergence is important in calculating transport since it prevents the reversible wandering of a particle back and forth along a single field line as velocity is reversed by collisions (Stix, 1978; Rechester and Rosenbluth, 1978). The value $1/L_c$ is called the Kolmogorov entropy (Kolmogorov 1958) and is defined through

$$h = \lim_{\substack{d_0 \rightarrow 0 \\ z \rightarrow \infty}} \frac{1}{z} \ln \left(\frac{d(z)}{d_0} \right) \quad (188)$$

where d_0 is the initial separation of the lines. In regions where

magnetic surfaces are well defined this quantity is easily seen to equal zero.

The evaluation of the quantities h , D by means of numerical integration of the differential equations for the field lines is prohibitive since in general both h and D approach their limits very slowly in z , and great accuracy is necessary in order to avoid numerical error for such an integration. Thus, these quantities have been studied using discrete mappings. Successive application of a discrete transformation will normally lead to values of d/d_0 exceeding the bounds of any numerical system, and to avoid this Chirikov and Israelev (1973) introduced a renormalization of the distance between the two transformed points after each step, preserving their relative orientation. They confirmed numerically that if d_0 is sufficiently small the limit Eq. (188) exists and is independent of the initial position within the ergodic region. This method was also used by Casartelli et al. (1976).

For the simple discrete map given by Eq. (185) with $b(\theta_i) = \epsilon \sum_{p=0}^{N-1} \sin[2\pi y_i - (2\pi p/N)]$ the Kolmogorov entropy can be calculated directly. For two nearby points with separation x, y one finds

$$\begin{pmatrix} \delta y_{i+1} \\ \delta x_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{N} \\ k & 1 + \frac{k}{N} \end{pmatrix} \begin{pmatrix} \delta y_i \\ \delta x_i \end{pmatrix} \quad (189)$$

with $k = (2\pi/N)\epsilon \sum_{p=0}^{N-1} \cos[2\pi y_{i+1} - 2\pi(i+1)p/N]$. This mapping is called the tangential transformation. It has eigenvalues

$$\lambda = 1 + (k/2N) \pm [(k^2/4N^2) + (k/N)]^{1/2} \quad (190)$$

and for large k/N , $\lambda_{\pm} = (k/N)$, (N/k) , with stretching taking place primarily in the x direction. Thus, for large ϵ the exponentiation becomes one dimensional and the Kolmogorov entropy is readily seen to be $(N/2) \langle \ln \lambda^2 \rangle$, where the bracket indicates averaging along the trajectory. This gives

$$h = \frac{N}{\pi} \int dW \ln \left(\frac{2\pi\epsilon \cos W}{N} \right)^2 = N \ln \left(\frac{\pi\epsilon}{N} \right) \quad (191)$$

This logarithmic behavior is, however, an artifact of the correlations existing between the overlapping islands. The stochasticity in this case is due to the magnetic island overlapping with images of itself due to the symmetry under $x \rightarrow x + N$. For fixed island width and N sufficiently large the ergodic regions vanish, and the Kolmogorov entropy is zero.

A model corresponding to the presence of many different helicities was studied by Rechester et al. (1979). The transformation is given by

$$\begin{aligned} Y_{i+1} &= Y_i + x_i/N \\ x_{i+1} &= x_i + f(Y_{i+1})/N \end{aligned} \quad (192)$$

$$\text{where } f(Y_{i+1}) = \epsilon \sum_{m=1}^M \sum_{p=0}^{N-1} \sin 2\pi[m(Y_{i+1} + \phi_{mp}) - (i+1)P/N]$$

and the ϕ_{mp} are fixed random phases. This model exhibits ergodic behavior when overlap occurs between islands of different m values, which occurs for $\epsilon > 2/M^3$, and the random relative phases of these modes precludes the occurrence of special correlations such as occurred in the case of a single mode, until ϵ is much larger. The results of numerically iterating the transformation Eq. (192) and calculating h through Eq. (188) are shown in Fig. 27. Approximately 10^4 steps were required to obtain reasonable accuracy. Provided $h < N$ the results were fit very well by

$$h = 0.54 M \epsilon^{2/3} \quad (193)$$

which was also obtained analytically by Rechester et al. (1979) by solving a Fokker-Planck equation for the moments of a probability distribution for the field lines. This calculation demonstrated the possibility of the use of a probability description for ergodic fields. This power dependence of h on ϵ was also obtained by Krommes et al. (1978).

When $h > N$ the islands begin to overlap with their own images and the Kolmogorov entropy again assumes a logarithmic dependence. Once again h is given by $(1/2)N \langle \ln \lambda^2 \rangle$ where λ is the large eigenvalue of the mapping of two nearby points. If $M \gg 1$, the randomness of the ϕ_{mp} can be used to invoke a central limit theorem and perform the averaging with a Gaussian distribution $P(\lambda) = \exp(-\lambda^2/2\lambda_0^2)/(2\pi\lambda_0^2)^{1/2}$ where $\lambda_0^2 = (2/3)\pi^2\epsilon^2(M/N)^3$. A simple integration then gives for $h > N$

$$h = (1/2)N \ln[(\pi^2/3) \exp(-c) (M/N)^3 \epsilon^2] \quad (194)$$

where $c = 0.577$ is the Euler constant. This equation is in agreement with numerically obtained values of h for $h > N$.

The approximate, or quasilinear diffusion of a field line under the action of transformation (189) is easily calculated. Assuming each field line moves in x a random direction a distance $\delta x = \epsilon \sin\psi$, i.e., complete random phases, we find

$$D = \lim_{i \rightarrow \infty} \frac{(x_i - x_0)^2}{2i} = \frac{\epsilon^2}{4} \quad (195)$$

which is approximately what is found numerically. Rechester and White (1980) applied a probability method to calculate the diffusion more exactly. The diffusion constant shows oscillatory dependence on the parameter ϵ , approaching the quasilinear value for large ϵ . The ratio of D to the quasilinear value is shown in Fig. 28.

5.3 Ergodicity and Transport

A detailed discussion of plasma transport is beyond the scope of this chapter. We confine our consideration to a brief examination of the effects of ergodic field behavior on plasma transport. There are two major phenomena of interest. First, it has been suggested that the anomalous electron heat transport, which is one or two orders of magnitude larger than the neoclassical value (Galeev and Sagdeev, 1973), could be due to small scale stochasticity arising from small magnetic islands (Kadomtsev 1975b). The second phenomenon is the major tokamak disruption, discussed in Section 4.3. As noted there, the development of a large magnetic island can produce field lines which, for the purposes of transport, effectively short circuit large portions of the minor radius (Hazelton and Strauss, 1976). In the presence of multiple islands of differing helicity or islands coupled through toroidal effects the effect is even stronger. It is possible for the field to become ergodic in much of the confinement region (Finn 1975, Rechester and Stix, 1976).

Present research has been confined to the examination of the changes in confinement and in the scaling of confinement produced by given ergodic fields. The problem of a self consistent treatment of the fields including saturation of microinstabilities due to the ergodicity and turbulence is largely unaddressed.

Stix (1978) and Rechester and Rosenbluth (1977) made estimates of the electron diffusion constant in the presence of an ergodic field. Thermal electrons reverse their parallel momentum many times during a typical tokamak confinement, due to pitch angle scattering. Due to this fact, the Kolmogorov entropy becomes relevant for the diffusion.

First consider the collisionless case. Without collisions the particles simply diffuse with the field, and

$$D_{\perp} = |V_{\parallel}| D_m \quad (196)$$

where D_m is the stochastic diffusion of the field given by Eq. (187). This equation should be directly applicable to runaway electrons.

In the collisional case, consider an electron which suffers a collision, reversing its parallel velocity and displacing its guiding center by a small amount Δ_0 . The electron then retraces its path for a distance of the order of a mean free path for a 90° scattering, $\lambda_{\parallel}$. In the absence of ergodicity the perpendicular diffusion would be $\Delta_0 N^{1/2}$ after N iterations. If stochasticity is present, the perpendicular diffusion will instead be given by

$$\Delta_N = \Delta_0 N^{1/2} \exp \frac{N^{1/2} \lambda_{\parallel}}{L_c} \quad (197)$$

with $L_c^{-1} = h$, the Kolmogorov entropy. For some value of $N = N_c$, Δ_N will be equal to the correlation distance, across $\vec{B}$, for the stochastic portion of $\vec{B}$. The trajectory memory is thus erased

after a time $t \approx N_C (\lambda_{||} / v_{th}^e)$, by which time $|z - z_0| = N_C^{1/2} \lambda_{||}$. We then find

$$D_1^2 = \frac{\langle (\Delta x)^2 \rangle}{\Delta t} = \frac{1}{(N_C)^{1/2}} |v_{th}^e| D_m \quad (198)$$

where N_C is the solution of Eq. (197) with Δ_N = characteristic plasma scale length or mode wavelength.

ACKNOWLEDGMENTS

The author is greatly indebted to D. A. Monticello, M. N. Rosenbluth, A. B. Rechester, T. H. Stix, J. M. Greene, R. M. Kulsrud, E. A. Adler, J. Hammer, F. W. Perkins, and C. Oberman for many useful discussions. The author also wishes to express his appreciation to Ms. E. G. Ferreras for capably typing this manuscript.

This work was partly supported by United States Department of Energy Contract No. DE-AC02-76-CHO3075.

REFERENCES

- Adler, E. A. and R. M. Kulsrud
Princeton Plasma Physics Lab. PPPL-1425 (1979).
- Adler, A. E., R.M. Kulsrud, and R. B. White
Phys. Fluids, (1980).
- Ara, G., B. Basu, B. Coppi, G. Laval, M. N. Rosenbluth,
and B. V. Waddell, Annals of Phys. 112 (1978), 443.
- Arnol'd, V. I., Russ. Math. Sur. 18 (1963), 9.
- Arsenin, V. V., L. I. Arkmenkov, N. V. Ivanov,
and A. M. Kakurin, PPPL-Trans 126 (1978).
- Arsenin, V. V. and V. A. Chuyanov, Usp. Fiz. Nauk. 123
(1977), 83.
- Asumi, M., Japan Atomic Energy Research Institute,
Ibaraki, Japan, private communication (1976).
- Balescu, R., Equilibrium and Nonequilibrium Statistical
Mechanics (Wiley-Interscience, New York, 1975).
- Bernstein, I. B., E. A. Frieman, M. D. Kruskal,

and R. M. Kulsrud, Proc. Roy. Sci. (London) A244, 17 (1958).

Biskamp, D., Nucl. Fusion 19, 777 (1979).

Biskamp, D. and H. Welter

Bull. Am. Phys. Soc. 23, 872 (1978).

Biskamp, D., Nucl. Fusion 18, 1059 (1978).

Biskamp, D. and H. Welter, Bull. Am. Phys. Soc.

22, No. 9, p. 1171, paper 862 (1977).

Bussac, M. N., D. Ederly, R. Pellat, and J. L. Soule

Plasma Physics and Controlled Nuclear Fusion

Research, 1976, IAEA Vienna, 1977, p. 607.

Callen, I. D., B. V. Waddell, B. Carreras, M. Azuni,

P. J. Catto, A. R. Hicks, J. A. Holmes, D. K. Lee,

S. J. Lynch, J. Smith, M. Soler, K. T. Tsang, and

J. C. Whitson, IAEA-CN-F-1-1--Innsbruck,

"Magnetic Islandography" 1979.

Callen, J. D. and G. L. Jahns

Phys. Rev. Lett. 38, 491 (1977).

Casartelli, M., E. Diana, L. Galgani, and A. Scotti

Phys. Rev. A13, 1921 (1976).

Carreras, B., H. R. Hicks,, and B. V. Waddell
Nucl. Fusion 19, 583 (1979).

Chirikov, B. V., Research Concerning the Theory of
Nonlinear Resonance and Stochasticity, Novosibirsk 1969
Section of the USSR Academy of Sciences Report 267
CERN Translation 71-40, Geneva 1971.

Chirikov, B. V. and F. M. Israelev, Some Numerical
Experiments with a Nonlinear Mapping: Stochastic Component,
Colloques Internationaux du C. N. R. S. Transformations
Ponctuelles et leurs Applications (Toulouse, 1973) p. 409.

Chirikov, B. V., Phys. Reports 52, 263 (1979).

Coppi, B., R. Galvao, R. Pellat, M. N. Rosenbluth,
and P. H. Rutherford
Princeton Plasma Physics Laboratory Report 1271 (1976).

Coppi, B., J. M. Greene, J. L. Johnson,
Nucl. Fusion 6, 101 (1966).

Cross, M. A. and G. Van Hoven
Phys. Rev. A4, 2347 (1971).

Danilov, A. F., Yu. N. Dnestrovskii, D. P. Kostomarov,
and A. M. Popin, Fiz. Plazmy (Sov. J. Plasma Phys.)

2, 167 (1976).

Dibase, J. and J. Killeen

J. Comput.. Phys. 24, 158 (1977).

Dnestrovskii, Yu. N., Proceedings of the 7th Int.
Conf. on Plasma Physics and Controlled Nuclear Fusion
Research, Innsbruck 1978, Varenna IAEA 1979.

Dnestrovskii, Yu. N., D. P. Kostomarov,

V. G. Pereverzev, and K. N. Tarasyan

Fiz. Plazmy 4, 1001 (1978)

Sov. J. Plasma Phys. 4(5), 557 (1978).

Dnestrovskii, Yu. N., S. E. Lysenko, and R. Smith

Sov. J. Plasma Phys. 3, 9 (1977).

Drake, J. F. and Y. C. Lee

Phys. Rev. Lett. 39, 453 (1977).

Dubois, M. and A. Samain, EUR-CEA-FC-965

June, 1979, submitted to Nucl. Fusion.

Dubois, M. and A. Samain

Plasma Phys. 21, 101 (1979).

Filonenko, N. N., R. Z. Sagdeev, and G. M. Zaslavsky

Nucl. Fusion 7, 253 (1967).

Finn, J. M., Nucl. Fusion 15, (1975).

Finn, J. M., Phys. Fluids 20, 1749 (1977).

Ford, J. in Fundamental Problems in Statistical Mechanics
edited by E. D. G. Cohen, (North Holland,
Amsterdam, 1975) Vol. 3, p. 215.

Furth, H. P., J. Killeen, and M. N. Rosenbluth
Phys. Fluids 6, 459 (1963).

Furth, H. P., P. H. Rutherford, and H. Selberg
Phys. Fluids 16, 1054 (1973).

Furth, H. P., in Propagation and Instabilities in
Plasmas, edited by W. T. Fetterman, Stanford University
Press, Stanford, California, 1963, p. 87.

Galeev, A. A. and R. Z. Sagdeev, Reviews of Plasma
Physics (in Russian), Vol. 7, Atomizdat, Moscow (1973) p. 205.

Gelfand, I. M., M. I. Graev, N. M. Zueva, M. S. Mikhailova,
and A. I. Morozov, Sov. Phys. Doklady 8, 188 (1963).

Glasser, A. H., H. P. Furth, and P. H. Rutherford

Phys. Rev. Lett. 38, 234 (1977).

Glasser, A. H., J. M. Greene, and J. L. Johnson
Phys. Fluids 19, 567 (1976).

Grad, H., "Reconnection of Magnetic Field Lines in an ideal
Fluid" NYU, Courant Institute, Report MF-92 (1978).

Grad, H. and J. T. Hogan
Phys. Rev. Lett. 24, 1337 (1970).

Grad, H. and H. Rubin, IAEA Geneva Conf. 31, 190 (1958).

Grad, H., P. N. Hu, and D. C. Stevens
Proc. Nat. Acad. Sci. 72, 3789 (1975).

Grad, H., Proceedings of the International Congress
of Mathematicians, Nice, (Gauthier Villars,
Paris, Vol. 3, p. 105, 1970).

Grad, H., Survey of $1\frac{1}{2}D$ Transport Codes,
Courant Institute of Mathematical Sciences,
Magneto-Fluid Dynamics Division C00-3077-154 MF-93 1978.

Graev, M. I., M. S. Mikhailova, and A. T. Morozov
Sov. Phys. Tech. Phys. 10, 920 (1966).

Greene, J. M., J. Math. Phys. 9, 960 (1968).

Hawryluk, R. J., N. Bretz, D. Dimock, E. Hinnov, D. Johnson,
D. A. Monticello, D. McCune, S. Suckewer, to be published, (1980).

Hazeltine, R. D. and D. W. Ross
Phys. Fluids 21, 1140 (1978).

Hazeltine, R. D. and H. R. Strauss
Phys. Rev. Lett. 37, 102 (1976).

Hazeltine, R. D., D. Dobrott, and T. G. Wang
Phys. Fluids 18, 1778 (1975).

Hirschman, S. P. and S. C. Jardin
Phys. Fluids 22(4), 731 (1979).

Jahns, G. L., M. Soler, B. V. Waddell, J. D. Callen,
and H. R. Hicks, Nucl. Fusion 18, 609 (1978).

Kadomtsev, B. B. , Sov. J. Plasma Phys. 6, 512 (1975a)
Fiz. Plazmy 1, 938 (1975a).

Kadomtsev, B. B. and O. P. Pogutse
Zh. Eksp. Teor. Fiz. 65, 575 (1973)
[Sov. Phys. JETP 38, 283 (1974)].

Kadomtsev, B. B. , Fiz. Plazmy 1, 710 (1975b)

[Sov. J. Plasma Phys. 1, 389 (1975b)].

Karger, F., K. Lockner, G. Fussman, B. Cannici,
W. Engelhardt, J. Gernhardt, E. Glock, D. E. Groening,
O. Kluber, G. Lisitano, H. M. Mayer, D. Meisel,
P. Morandi, S. Sesnic, F. Wagner, and H. P. Zehrfeld
in Plasma Phys. and Controlled Nucl. Fusion
Vienna, 1977, Vol. 1, p. 267.

Killeen, J. and A. I. Shestakov
Phys. Fluids 21, 1746 (1978).

Kolmogorov, A. N., in Proceedings of the International
Congress of Mathematicians Amsterdam (North Holland,
Amsterdam, 1957), Vol. 1, p. 315.

Kolmogorov, A. N., Dokl. Akad. Nauk. USSR 119, 861 (1958).

Krommes, J. A., R. G. Kleva, and C. Oberman
Plasma Physics Laboratory, Princeton University
preprint No. 1384, (1978), to be published.

Lackner, K. and H. P. Zehrfeld, Proceedings of the IAEA
Symposium on Current Disruption in Toroidal Devices,
Max-Planck Institut fur Plasmaphysik, Garching, Germany, 1979.

Laval, G., R. Pellat, and M. Vuillemin in Plasma Physics and Controlled Nuclear Fusion Research (IAEA, Vienna, 1965) II, 259.

Leboeuf, J. N., T. Tajima, C. F. Kennel, and J. M. Dawson
Geophys. Res. Lett. 5, 609 (1978).

McGuire, K. and D. C. Robinson, Proceedings of the
Symposium on Disruptive Instabilities in Toroidal Devices,
Garching, 1979.

Mirnov, S. V. and I. B. Semenov,
Soviet Phys. JETP, 1134 (1971).

Mirnov, S. V. and I. B. Semenov
Plasma Physics and Controlled Nuclear Fusion
1976, IAEA Vienna, 1977, p. 291.

Montgomery, D. L. Turner, and G. Vahala,
J. Plasma Phys. 21, 239 (1979).

Monticello, D. A., 1975, private communication.

Monticello, D. A. and R. B. White
Phys. Fluids, to be published, (1980).

Monticello, D. A., R. B. White, and M. N. Rosenbluth

Plasma Physics and Controlled Nuclear Fusion Research
1978, Vol. 1, IAEA Vienna, 1979.

Morozov, A. I. and L. S. Solov'ev
Sov. Phys. JETP 18, 660 (1964).

Morozov, A. I. and L. S. Solov'ev in Reviews of Plasma
Physics, Vol. 2, Consultants Bureau, New York 1966,
Translated from Russian: Voprosy Teorii Plazmy
Atomizdat, Moscow, 1963.

Moser, J., Nachr. Akad. Wiss Gottingen II
Math. Phys. Kl. 1 (1962).

Parker, E. N., Astrophys. J. Suppl., Sr. 77, 8:177
(1963).

Penrose, O., Rep. Prog. Phys. 42, 1937 (1979).

Petschek, H. E., Magnetic Field Annihilation in
W. N. Hess (ed.) AAS-NASA Symposium on
Phys. Solar Flares SP-50, 1964.

Pogutse, O. P., PPPL Trans 121, March 1977.

Rechester, A. B. and T. H. Stix
Phys. Rev. Lett. 36, 587 (1976).

Rechester, A. B., M. N. Rosenbluth, and R. B. White
Phys. Rev. Lett. 42, 1247 (1979).

Rechester, A. B. and M. N. Rosenbluth
Phys. Rev. Lett. 40, 38 (1978).

Rechester, A. B. and R. B. White,
Phys. Rev. Lett., (1980).

Rosenbluth, M. N., R. Z. Sagdeev, and J. B. Taylor
Nucl. Fusion 6, 297 (1966).

Rosenbluth, M. N., D. A. Monticello, H. R. Strauss, and
R. B. White, Phys. Fluids 19, 1987 (1976).

Rosenbluth, M. N. and M. N. Bussac
Nucl. Fusion 19, 489 (1979).

Ross, D. W., H. R. Strauss, S. M. Mahajan, R. D. Hazeltine,
and F. L. Hinton in Plasma Physics and Controlled
Nuclear Fusion Research (IAEA, Vienna, 1979).

Rutherford, P. H., Phys. Fluids 16, 1903 (1973).

Rutherford, P. H. and H. P. Furth,
Princeton Plasma Physics Laboratory Report MATT-872 (1971).

Sato, T., Phys. Fluids 22, 1189 (1979a).

Sato, T., Princeton Plasma Physics Lab. Report
PPPL-1554, 1979b.

Sauthoff, N. R., S. V. Goeler, and W. Stodiek
Nucl. Fusion 18, 1445 (1978).

Schnack, D. and J. Killeen, Nucl. Fusion 19, 877 (1979).

Shafranov, V. D.,
Reviews of Plasma Physics 2, 103 (1966).

Schnack, V. D., Zh. Tekh. Fiz. Vol. 40, No. 2, 240 1970.
[Sov. Phys. Tech. Phys. 15, 175 (1970)].

Sonnerup, B. U. O., in Solar System Plasma Physics, A
Twentieth Anniversary Review, C. F. Kennel et al., eds.
North Holland Publishing Co., 1979.

Spitzer, L. and R. Harm, Phys. Rev. 89, 977 (1953).

Stix, T. H., Nucl. Fusion 18, 353 (1978).

Stix, T. H., Phys. Rev. Lett. 36, 521 (1976).

Stix, T. H., Phys. Rev. Lett. 30, 833 (1973).

Strauss, H. R., Phys. Fluids 19, 134 (1976).

Sweet, P. A., Nuovo Cimento Suppl. 8, Ser. X, 188
(1958).

Sykes, A. and J. A. Wesson
Phys. Rev. Lett. 37, 140 (1976).

Taylor, J. B., Phys. Rev. Lett. 33, 1139 (1974).

Taylor, J. B. CLM-PR-12, Culham Laboratory (1959).

Vasyliunas, V. M., Rev. Geophys. Space Phys.
13, 303 (1975).

Waddell, B. V., G. L. Jahns, J. D. Callen, and
H. R. Hicks, Nucl. Fusion 18, 135 (1978).

Waddell, B. V., B. Carreras, H. R. Hicks, and
J. A. Holmes, Phys. fluids 22, 896 (1979).

Waddell, B. V., D. A. Monticello, M. N. Rosenbluth, and
R. B. White, Proceedings of the Seventh Conference on
Numerical Simulation of Plasmas, June 1975,
Courant Institute, New York.

Waddell, B. V., G. Laval, and M. N. Rosenbluth

Oak Ridge Nat. Lab. Report ORNL/TM-5968, Oak Ridge,
TN (1977).

Waddell, B. V., M. N. Rosenbluth, D. A. Monticello,
R. B. White, and J. Carreras, in Theoretical and
Computational Plasma Physics, IAEA, Vienna, 1978, p. 79.

Waddell, B. V., M. N. Rosenbluth, D. A. Monticello,
and R. B. White, Nucl. Fusion 16, 528 (1976).

Welter, H. and D. Biskamp
Bull. Am. Phys. Soc. 23, 872 (1978).

Wesson, J. A., Nucl. Fusion 6, 130 (1966).

Wesson, J. A., Nucl. Fusion 18, 87 (1978).

White, R. B., D. A. Monticello, M. N. Rosenbluth,
and B. V. Waddell, Phys. Fluids 20, 800 (1977a).

White, R. B., D. A. Monticello, M. N. Rosenbluth, and
B. V. Waddell, Plasma Physics and Controlled Nuclear
Fusion Research, IAEA, Vienna, 1977b.

White, R. B., W. Park, D. A. Monticello, and H. R. Strauss.
Proceedings of the Symposium on Disruptive Instabilities
in Toroidal Devices, Garching, Feb., 1979.

White, R. B., D. A. Monticello, and M. N. Rosenbluth
Phys. Rev. Lett. 39, 1618 (1977c).

Whiteman, K. J., Rep. Prog. Phys. 40, 1033 (1977).

Zaslavsky, G. M. and B. V. Chirikov
Usp. Fiz. Nauk. 105, 3 (1971),
[Sov. Phys. Usp. 14, 549 (1972)].

Fig. 1. A magnetic island. Shown are surfaces of constant ψ . The width of the island is the distance between separatrices perpendicular to the zero order flux surfaces.

792424

792426
 Fig. 2. The exterior solutions for the flux ψ_1 , and the velocity potential ϕ_1 , for positive and negative values of Δ' , for the model given by $F(x) = \tanh x$.

Fig. 3. The interior solution for the velocity potential x .⁷⁹²⁴²¹

Fig. 4. The integrand appearing in the matching condition integral, Eq. (22). Most of the contribution to the integral comes from the region $z < 2$.

792425
Fig. 5. The first order fields and flow pattern associated with a linear tearing mode.

802083
 Fig. 6. The change in magnetic energy density produced by a magnetic island. The integral of this density from x_T to ∞ is zero. Also shown is the divergent density obtained by using the asymptotic form of ψ_{20} for small x . The dotted line is the "energy" density obtained from a variational principle for the full resistive equations.

802060
 Fig. 7. The second order eddy current, which produces the change in the magnetic energy for the mode, and is responsible for the nonlinear algebraic growth.

Fig. 8. Geometry for the description of a toroidal field. 792526

792531
 Fig. 9. The helical ribbon defined by the magnetic axis and a helix of constant $m\theta + kz, r$. At a singular surface the magnetic field vector is tangent to the ribbon.

792525
 Fig. 10. Growth rate of the resistive kink mode as a function of S . Also shown are the tearing and kink mode growth rates.

Fig. 11a. Odd part, $\xi - (1/2)\xi_0$, of the resistive layer solution ξ as a function of $x = (r-r_0)/(r_0 S^{1/3})$ for different values of the normalized growth rate $\lambda = \lambda S^{1/3}$. The eigenfunctions that are represented here have been derived by direct numerical integration of Eq. (110). These agree with the analytical representation (120). Notice the change in the character of the eigensolutions as λ becomes smaller than unity, corresponding to $\lambda_H < 0$.

Fig. 11b. The solid curve represents the growth rate as a function of the ideal MHD growth rate that is measured by $\lambda_H = \lambda S^{1/3}$. This curve has been obtained by the direct numerical integration of Eqs. (110) as indicated earlier and the analytical dispersion relation (121) agrees with it.

Fig. 12. A large $m = 2$ island. Along a flux surface the electrons
792528
rapidly bring the plasma to a state of constant
temperature, so the island acts practically as a thermal
short circuit in the radial heat transport.

762372
Fig. 13. Surfaces of constant $\psi(r, \theta)$ for a magnetic island. Region a, the exterior, and region b, the interior are bounded by the separatrix $\psi = \psi_s$. The x point and o point are shifted outward and inward, respectively, from the resonant surface.

Fig. 14. The current profile $J(\psi)$ in the presence of an island. ⁷⁶²³⁷³
The subscripts refer to the regions of Fig. 13.

Fig. 15. Saturation width predicted using the quasilinear model for $m = 2$. Here $J_a(r)$ is given by the peaked current model and the resistivity $\eta(r) = J_a(r)^{-1}$. The points are results of time stepping the full nonlinear code for the case $r_0 = 0.8$.

782124
Fig. 16. Maximum island width as a function of the safety factor on axis $q(0)$ for a variety of current profiles. Only fairly flat profiles are capable of producing large islands. Here the $q = 2$ surface is fixed at $r = 0.7a$.

Fig. 17. The sequence of magnetic flux contours during the ⁷⁹²⁶¹⁹ nonlinear evolution of the $m = 1$ mode.

Fig. 18. The helical flux (a), safety factor profile (b), and current profile (c) before and after the $m = 1$ evolution. ⁷⁹²⁶²⁰

Fig. 19. Helical flux contours in the poloidal plane at selected times (normalized to the resistive time). The radius of the outermost contour is $r = 0.4a$ and the singular surface s at $r = 0.2a$. The peaked profile was used. Although the final stage looks very complicated, the current profile is quite flat at this point. Inclusion of thermal conductivity allows this state to relax quickly to a configuration of concentric circles.

762030

Fig. 20. The current profile corresponding to Fig. 17, along a line running through the x and o points. The current sheet forming a negative spike is located at the x point. 762034

Fig. 21. The island width versus time for $m = 1$ modes at various values of S , for the profile of Fig. 17. No significant deviation from the linear growth rate is observed.

792527

Fig. 22. The time history of a pair of coupled magnetic islands of the same helicity. For this value of $S(10^4)$ no Rutherford regime is observed. The time is given in units of τ_R .

762226

Fig. 23. Flux surfaces for a steady state reconnection process driven by external boundary conditions. The field satisfies jump conditions at the shock fronts.

792530

Fig. 24. The poloidal flux for a doublet configuration.

792529

792171
 Fig. 25. Magnetic surfaces obtained from the Taylor-Chirikov mapping for $\epsilon = 0.01$, $N = 1$. The initial points are marked with an x. Only small stochastic broadening is observed near the x points.

Fig. 26. The same as Fig. 23 for $\epsilon = 0.15$, $N = 1$. Significant stochastic broadening of the separatrix is observed.

792172

Fig. 27. The numerically obtained Kolmogorov entropy. ⁷⁹²⁰²³ The solid curve is the theoretical result. Also shown are the stochastic transition points ϵ_M .

Fig. 28. The numerically obtained diffusion for ⁸⁰²⁰³³ the Taylor-Chirikov model. Shown is the rate of D to D_{QL} , the quasilinear result. Also shown is the theoretical result obtained for large ϵ .