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Free Boundary Skin Current MHD Equilibria

By

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Free boundary skin current MHD equilibria

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Abstract

Function theoretic methods in the complex plane are used to develop simple parametric hodograph formulae which generate sharp boundary equilibria of arbitrary shape. The related method of Gorenflo and Merkel is discussed. A numerical technique for the construction of solutions, based on one of the methods is presented. A study is made of the bifurcations of an equilibrium of general form.

MASTER

I. Introduction

The study of a magnetohydrodynamic (MHD) equilibrium or stability problem for a plasma of given geometric form is often begun by finding a similar skin current equilibrium because of the analytic simplification this affords. One solution technique is the hodograph method of Helmholtz¹ and Kirchhoff,² which solves the free boundary, MHD equilibrium problem using conformal mappings. This technique is also widely used in fluid mechanics.³ Solutions obtained by this method are restricted to the Cartesian plane. However, in many cases this is a valid limit for a more general geometry, and the solution can be perturbed to obtain results of wider validity as in the high beta, large aspect ratio limit studies of Goedbloed.⁴

The MHD equilibrium equations include a force balance equation relating the pressure P to the current density $\vec{J}$ and magnetic field $\vec{B}$,

$$\vec{\nabla} P = \vec{J} \times \vec{B}. \quad (1)$$

The electric field does not enter as the plasma is assumed neutrally charged. Only two of Maxwell's equations are needed in steady state to make up the rest of the MHD system.

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}, \\ \vec{\nabla} \cdot \vec{B} &= 0. \end{aligned} \quad (2)$$

The nonlinear elliptic MHD system given by Eqs. (1) and (2) can only be solved under special circumstances. Here it is assumed that the solution and boundary conditions are independent of the cartesian z axis and that the solution has the appearance of a simply connected plasma region Ω_P with boundary $\partial\Omega_P$ surrounded by a doubly connected annular vacuum region Ω_V (Fig. 1). This vacuum region is in turn bounded by a perfectly conducting wall $\partial\Omega_V$. It is also assumed that the pressure is a given constant P in the plasma and that the magnetic field along the cartesian z axis is a given constant B_i in the plasma and, perhaps, a different constant B_e outside the plasma. The poloidal field

(the component lying in the x - y plane) is assumed to vanish within the plasma. Under these conditions current flows only in $\partial\Omega_P$ and $\partial\Omega_V$, and the poloidal field in the annular vacuum region outside the plasma Ω_V obeys

$$\vec{\nabla} \times \vec{B}_P = \vec{\nabla} \cdot \vec{B}_P = 0. \quad (3)$$

$\vec{B}_P$ can be written in terms of either the scalar potential Φ or the flux Ψ as,

$$2\pi\vec{B}_P = \vec{\nabla}\Phi = \hat{z} \times \vec{\nabla}\Psi, \quad (4)$$

where the flux is related to the vector potential by $\Psi = -2\pi A_z$.

The flux and scalar potential satisfy the Cauchy conditions for the conjugate parts of an analytic function as well as Laplace's equation. Define the analytic flux function as

$$F(z) = \Psi(x, y) + i\Phi(x, y), \quad (5)$$

where $z = x + iy$ is the complex independent variable of the physical plane and should not be confused with the third cartesian axis $\hat{z}$, which only enters here in vector form. The poloidal magnetic field is then determined by the analytic field function

$$W = \frac{dF}{dz} = \frac{\partial\Psi}{\partial x} - i\frac{\partial\Psi}{\partial y}. \quad (6)$$

Since $\partial\Omega_P$ is a constant pressure surface, it is a surface of constant Ψ . The integral equivalent of Eq. (1) tells us that $P + B^2/2\mu_0$ is constant across $\partial\Omega_P$ or,

$$\left| \vec{\nabla}\psi \right| = |W| = 2\pi(2\mu_0 P + B_i^2 - B_e^2)^{1/2} = \lambda, \quad (7)$$

for all $z \in \partial\Omega_P$, where λ is a constant. The Neumann boundary condition of Eq. (7) is normally supplemented by the Dirichlet boundary conditions: $\Psi = \Psi_P$ for $z \in \partial\Omega_P$ and $\Psi = \Psi_V$ for $z \in \partial\Omega_V$, where Ψ_P and Ψ_V are given constants. The position of the vacuum boundary, $\partial\Omega_V$, being given, the imposition of an extra boundary condition over the normal Dirichlet problem serves to determine the position of the plasma surface free boundary, $\partial\Omega_P$.

In general, this problem has a non-unique solution, as may be seen by considering the cylindrically symmetric solution $F(z) \propto \text{Log}(z)$,

$$\Psi(r) = \Psi_P + \lambda a \log\left(\frac{r}{a}\right), \quad (9)$$

where a is the unknown radius of the plasma free boundary.

Applying the boundary conditions with a given wall radius $r = b$ one has.

$$-\frac{a}{b} \log\left(\frac{a}{b}\right) = \frac{\Psi_V - \Psi_P}{\lambda b}. \quad (10)$$

Equation (10) has no solutions, one solution, or two solutions for the free boundary radius a depending on the magnitude of the constant on the right-hand side. This one parameter family of solutions has a bifurcation point at a plasma radius $a = b/e$. At this point there is one solution, above it no solutions, and below it two solutions. When $a > b$, a single unique solution is determined for the inverted situation of a plasma enveloping but confined away from a cylindrically symmetric central conductor.

This problem can be posed in a different way so as to determine a unique solution by giving Ψ_P , λ , b , and the radius of the free boundary a . A unique Ψ_V can then be found from Eq.(10) over the whole parameter range. This behavior of the free boundary problem will be examined in more generality later where it will be shown that specification of Ω_P results in a unique exterior solution.

The hodograph technique of Helmholtz and Kirchoff assumes that the range of the complex field function W and the range of F are known beforehand. Provided that the range of W can be conformally mapped onto the range of F , one has, in principle, solved the problem as

$$z(F) = z_0 + \int_{F_0}^F \frac{dF}{W(F)}, \quad (11)$$

which yields $z = x + iy$ as an analytic function of F . The discovery of a suitable $W(F)$ is the goal of this procedure. This technique is limited, in practice, to cases where both ranges are simple. A variation is to describe W parametrically, by some mechanism, and then to find values for the parameters which map the ranges. This is essentially the method of this paper. Even when $W(F)$ is known, it may be impossible to obtain a closed-form expression for the integral of Eq. (11), or to invert the result to obtain $F(z)$, and one must be content with a numerical representation.

Some work in this subject calculates the equilibria which result from field sources of given location and magnitude. In perhaps the earliest application of complex variables to plasma equilibria Berkowitz⁵ and Berkowitz et alii⁶ discuss the simple cusped equilibria produced by four or more conductors. Cusped equilibria were of particular interest at the time because of their absolute MHD stability.⁷ Grad et alii⁸ solve the problem of a finite net current plasma immersed in a uniform vertical field. A closed form solution, in terms of elliptic integrals, of formidable algebraic complexity, was found by Kadish and Stevens⁹ for a rectangular vacuum boundary. Shercliff¹⁰ gives an exhaustive analysis in closed form of smooth and cusp boundary plasmas in the field of two parallel conductors. Becker¹¹ exhibits a solution of the semi-infinite slab which is the same solution as given in the text book of Morse and Feshbach for a variable condenser.¹² Spencer¹³ treats the bifurcation of cusped equilibria in the field produced by n symmetrically disposed conductors.

Other work gives the shape of the plasma boundary and solves the resultant Cauchy problem for the exterior field. An outline of a general method is given by Kerner, Pfirsch and Tasso.¹⁴ They also discuss the elliptical solution of Gajewski.¹⁵ Friedburg and Grossman¹⁶ base a stability analysis on sharp boundary equilibria of finite toroidicity. Their equilibrium solution is trivial, on account of the absence of a wall boundary.

Methods for constructing the mapping of doubly connected regions onto the annulus exist, specifically the nonlinear integral equations of Theodorsen¹⁷ and Garrick.¹⁸ An

exposition of these is given in the textbook by Henrici¹⁹ and in his paper on the use of the fast Fourier transform in their numerical solution.²⁰ Goedbloed⁴ has adapted Henrici's method for the solution of the free boundary equilibrium problem of a high beta tokamak with a circular wall. His problem is more complicated than the problem of this paper, in that the poloidal magnetic field is not constant on $\partial\Omega_P$.

Gorenflo²¹ and Merkel²² have developed a very general method of solution using function theoretic methods, which are related to the ideas of this paper. This connection will be discussed explicitly.

In this paper two very general variations, a Blaschke product form and a parametric exponential form, of the hodograph are given from which the solution to a variety of plasma and boundary shapes can be worked out. The Blaschke product form can yield closed-form results, but is not readily adapted to problems with given shape. The exponential form does not usually give closed-form answers, but it is intuitive, in that the effect of parameter variation on shape is straightforward. It is also easily implemented numerically. The relationship of these mappings to the method of Gorenflo and Merkel yields further insight into the latter in view of the fundamental connection of their work with Blaschke products. Finally, the bifurcation problem is treated in general and in the context of our mapping.

II. The Blaschke Product Form

Assume a non-vanishing total plasma current I_P along the cartesian z axis and a smooth plasma boundary. Disregard cusped solutions. Then Φ increases monotonically in the poloidal direction and Ψ increases monotonically in the "radial" direction. Ψ can be thought of as the radial variable and Φ as the poloidal angle variable. It is then natural to consider the mapping

$$\eta = \exp[2\pi(F - \Psi_P)/\Phi_T] = \rho e^{i\phi}. \quad (12)$$

Here

$$\Phi_T = \oint d\vec{l} \cdot \vec{\nabla} \Phi = 2\pi\mu_0 I_P, \quad (13)$$

so that $0 \leq \phi \leq 2\pi$. If the solution exists, then the mapping of Eq. (12) takes the z plane into a cylindrically symmetric annulus in the η plane. The plasma boundary maps into the unit circle $|\eta| = 1$, while the wall boundary maps into the circle of radius $\rho_V = \exp(2\pi(\Psi_V - \Psi_P)/\Phi_T)$. Note further that $\Phi_T = \lambda L$ where L is the poloidal length of the plasma boundary. This mapping would be self-intersecting in the cases which have been excluded.

One then has in a manner analogous to the hodograph method

$$z(\eta) = z_0 + \frac{\lambda L}{2\pi} \int_{\eta_0}^{\eta} \frac{d\eta}{\eta W}. \quad (14)$$

Observe that if W is given as a function of η , that the problem is, in principle, solved. The main conditions are: 1) $1/W$ must be analytic in the annulus $1 \leq \rho \leq \rho_V$, and 2) the modulus must be a constant $|W| = \lambda$ on the unit circle $\rho = 1$.

One can consider those functions which have unit modulus on the unit circle without loss of generality. A general rational form for such functions has been given by Blaschke.²³ Define

$$B(m, n, \phi_0, \alpha_1, \alpha_2, \dots, \alpha_n) = e^{i\phi_0} \eta^m \prod_{k=1}^n \frac{\alpha_k^* \eta - 1}{\eta - \alpha_k}, \quad (15)$$

and call these Blaschke products as suggested by Rudin.²⁴ Here ϕ_0 is a real constant and α_k^* is the complex conjugate of α_k . This is the most general form of a rational function of finite order which is unimodular on the unit circle. It is a product of factors, each of which is unimodular on the unit circle. Provided that the product converges, one can also take

$n \rightarrow \infty$. One can further generalize this product form by taking η to some k dependent real power or the quantity $\eta - \alpha_k$ to some k dependent real power in each term of the product. If these powers are all integers, one can always factor such expressions to obtain the form of Eq. (15). Non-integer powers are not considered here.

If one takes $1/W = B(1, n)/\lambda$, conditions 1 and 2 are satisfied if the poles of $B(1, n)$ at $\eta = \alpha_k$ line either within the unit circle or outside $\eta = \rho_1$. Other restrictions must be applied to obtain: 3) closure of the plasma boundary, and 4) a one-to-one mapping of the η plane onto the z plane within the annulus.

Condition 3 requires that the sum of the residues, at poles of the integrand within the unit circle, must vanish. Assuming that all the poles are simple and distinct, the Blaschke product integrand can be expanded in a sum of partial fractions as

$$\frac{B(1, n)}{\eta} = \left(1 + \sum_{k=1}^n \frac{\Delta_k^n}{\eta - \alpha_k}\right) \prod_{k=1}^n \alpha_k^\dagger. \quad (16)$$

If one defines $\delta_k = \alpha_k - 1/\alpha_k^\dagger$, then

$$\Delta_k^n = \prod_{l \neq k}^n \left(1 + \frac{\delta_l}{\alpha_k - \alpha_l}\right) \delta_k. \quad (17)$$

Equation (16) is easily integrated, yielding a term linear in η plus a sum over logarithms. If the sum of Δ_k^n over those poles within the unit circle vanishes, the total residue is zero and the curve is closed.

Condition 4, equivalent to unit winding number with respect to the origin of the η plane or the more stringent condition of unit winding number with respect to points within the annulus, is less easily satisfied. Intuitively, one expects that if the winding number of the integrand is zero, then the winding number of the integral is unity. This requires that the number of zeros and poles of the integrand within the unit circle be equal when weighted by order. Numerical evidence tends to support intuition when the integrand is a

Blaschke product, but the author has been unable to prove this conjecture. However, even with the zero residue condition and zero integrand winding number condition in force, self-intersecting boundaries are sometimes generated by the Blaschke product mapping and one has to resort to trial and error placement of the poles and zeros. It is also difficult to adjust their number and location so as to reproduce a given plasma shape.

The simplest form of Blaschke product field function which satisfies all of these constraints is

$$\frac{1}{W'} = \frac{\eta}{\lambda} \frac{[(\alpha^\dagger \eta)^2 - 1](\eta^2 - \beta^2)}{(\eta^2 - \alpha^2)[(\beta^\dagger \eta)^2 - 1]}. \quad (18)$$

Here it is assumed that $|\alpha| > 1$ and $|\beta| > 1$. The field function has zeros at $\eta = \pm\alpha$ and singularities at $\eta = \pm\beta$. This solution is similar to that of Shercliff¹⁰ for a plasma confined in the field of two parallel line currents. Figure 1 is, in fact, a pinched solution of the form of Eq. (18), where the field zeros lie outside the singularities. The generalization of Eq. (18) to $n > 2$ nonsymmetric line currents and field nulls is straightforward for even n .

III. An Exponential Parametric Form

A particularly convenient parametric form for the hodograph is obtained by considering the exponential form

$$\frac{1}{W'} = \frac{\eta e^{G(\eta)}}{\lambda}. \quad (19)$$

The real part of $G(\eta)$ must vanish for $|\eta| = 1$ and G must have the Laurent series form

$$G(\eta) = i\phi_0 + \sum_{n=1}^{\infty} A_n \eta^n - \frac{A_n^\dagger}{\eta^n}. \quad (20)$$

This hodograph is quite general, since one can show that any smooth, closed, non-self-intersecting curve has the representation given by substituting Eq. (19) into Eq. (14), and

then taking $\rho = 1$. A proof is given in the appendix. Equation (19) can therefore generate the exterior solution for any plasma boundary of this type. Further, this form is locally equivalent to an infinite Blaschke product and every Blaschke product equivalent to an exponential form up to simple powers of η , as may be shown by expanding both forms into Laurent series.

In general, the residue of this mapping does not vanish. When it is finite, the resultant plasma boundaries are of the picket fence type.²⁵ These solutions are not considered here. One easily proves that for pure modes in which only one of the A_n , $n \neq 1$ is nonzero or for mixtures of even n modes the residue of Eq. (19) vanishes. For arbitrary mixtures of modes the A_1 coefficient can be iteratively adjusted to ensure closure.

One nice feature of this mapping is that for $|A_n|$ and ρ small enough the winding number with respect to the origin of the η plane and points in the annular solution region always seems to be unity, although no analytic proof of this result has been made. Another important feature of this hodograph is the simple relationship between the A_n and the resultant plasma and wall shapes.

If all the A_n are zero, one recovers the cylindrically symmetric solution. For a pure $n = 2$ mode with small $|A_2|$ one obtains the quasi-elliptical shape of Fig. 2. In this and the following figures the innermost curve is the plasma boundary and successive outer curves correspond to choices of Ψ_V at successively larger values. As $|A_2|$ increases one obtains a doublet shape similar to that of Fig. 1. Figures 3, 4, and 5 correspond respectively to pure modes with A_n nonzero only for $n = 3, 4$, and 5. Figure 6 illustrates an even mixed mode example.

If $|A_n|$ or Ψ_V become large enough, then non-topological, self-intersecting curves are generated as shown in the outermost wall boundary of Fig. 6. This happens more quickly at high mode numbers since the field strength falls off as r^{n-1} . Conductors which cause a shape to vary rapidly with poloidal angle must, therefore, lie close to the plasma surface.

When the mapping cannot be continued indefinitely, as in Fig. (6), a critical maximum radius exists for the placement of conductors which are to induce a finite deformation in plasma shape. This is usually the case, and contradicts the notion that arbitrary deformations of the plasma can be obtained with coil systems placed arbitrarily far away from the plasma. The latter fallacious view arises from a moments treatment of the external field.

This is a point of some subtlety and much misunderstanding. A given field moment can be thought of as arising from a coil system placed arbitrarily far away from the plasma. These moments are a "complete" set in that an arbitrary field can be expanded as a sum in terms of the moments. What is often forgotten is that the radius of convergence of such expansions is generally finite.

The simplest example of a solution, other than a circle, which can be continued indefinitely is the elliptical shape obtained with a quadrupole field.¹⁰ The simplest example of a solution which cannot be indefinitely continued is obtained by placing a coil very near the plasma so that a portion of the boundary is concave. However, convexity of the boundary is not sufficient to allow infinite continuation. For example, Greene²⁶ has shown that an almost circular configuration can require a field singularity at a finite position.

More interesting than the exact singularities required by an exactly given boundary are the necessary singularities required by a boundary which can deviate by a small amount from a specified position. Clearly, some singularities required by the exact boundary could be moved to infinity by allowing the boundary to relax somewhat and replacing these non-necessary singularities with field moments. Other singularities, the essential ones, have to be retained, but their required strength might be reducible by this process.

In the design of doughnut-shaped, tokamak fusion reactors, it is important to know the location and minimum possible strength of such necessary singularities. The poloidal field coils, which determine the shape of the plasma or doughnut cross section, must often

be placed outside the toroidal field coils, and far away from the plasma. This requires that huge currents flow in the poloidal coils if the plasma cross section deviates considerably from a circle, and these large currents are a problem in such designs. A method for the determination of optimal necessary singularities for a given plasma shape would be quite useful, but does not exist to the author's knowledge.

IV. Fitting Specified Boundaries

A numerical code based on the parametric exponential form for the hodograph has been written to fit given plasma or wall boundaries. The chief feature of this code is its simplicity. Given plasma boundaries and given wall boundaries are treated separately. In both cases the boundary $z = x - iy$ is given in exact parametric form as distinct non-conjugate Fourier series for $x(\theta)$ and $y(\theta)$ as a function of θ where $0 \leq \theta \leq 2\pi$. The code then attempts to find the mapping $\theta(\phi)$ and the coefficients A_n of the G function. In the case of a given wall boundary the ratio of radii or annular modulus ρ_V is also determined, which requires an additional input, the length of the plasma boundary L .

The average plasma radius is defined as $a = L/2\pi$. Differentiation of Eq. (14) with respect to ϕ for fixed ρ , taking the log of the result, splitting it into real and imaginary parts, and exponentiation of the real part yields two equations,

$$\frac{d\theta}{d\phi} = \frac{a \rho e^{G_R(\phi)}}{r(\phi)}, \quad (21)$$

$$G_I(\phi) = \gamma(\phi) - \phi. \quad (22)$$

Here G_R and G_I are the real and imaginary parts of G while

$$-i \frac{dz}{d\theta}(\phi) = r(\phi) e^{i\gamma(\phi)}. \quad (23)$$

Writing $A_n = \alpha_n + i\beta_n$, where α_n and β_n are real quantities, one has,

$$G_R(\phi) = \sum_{n=1}^N (\rho^n - \rho^{-n})(\alpha_n \cos n\phi - \beta_n \sin n\phi). \quad (24)$$

$$G_I(\phi) = \phi_0 + \sum_{n=1}^N (\rho^n + \rho^{-n})(\beta_n \cos n\phi - \alpha_n \sin n\phi). \quad (25)$$

Observe that, in the vicinity of the plasma boundary, G_R is a weaker function of the α_n and β_n than is G_I . This suggests using Eq. (21) to determine θ as a function of ϕ and Eq. (22) to determine the α_n and β_n . The numerical code "Hoodoo" which implements these ideas is described below.

For the given plasma boundary case $\rho = 1$, $G_R = 0$, and a direct method suffices. a is then the average radius of the given boundary,

$$a = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, r(\theta). \quad (26)$$

The inverse of Eq. (21) is integrated with $r = r(\theta)$ to obtain $\phi(\theta)$. This function is inverted to get $\theta(\phi)$. The right-hand side of Eq. (21) is positive definite so that a one-to-one relationship between θ and ϕ is guaranteed. Now γ is expressed as a function of ϕ and Eq. (22) can be solved for the α_n and β_n by Fourier transformation. Note that, like the circular case, no freedom in the position of the wall boundary exists. Once the plasma boundary and ρ_V are given, the wall boundary is uniquely determined.

The given plasma boundary case is only slightly more complicated. An iterative method is used. a is the given average radius of the plasma boundary and is held invariant while iterating. $\rho_V > 1$ is initially unknown. Starting values for θ , α_n , and β_n are assumed, i.e., $\theta = \phi$, and $\alpha_n = \beta_n = 0$.

I - New $r(\phi)$ and $\gamma(\phi)$ are defined using the input boundary function and the current $\theta(\phi)$. This cannot be done with fast Fourier transform techniques because of the unequal spacing of θ points. G_R is defined with the current α_n and β_n .

II - Equation (21) is numerically integrated with $\rho = 1$ to get a monotonic function proportional to $\theta(\phi)$. A ρ is calculated such that $\theta(2\pi) = 2\pi$ and the function is rescaled to yield new values for θ , while ρ_V is set to this value of ρ .

III - The right-hand side of Eq. (22) is Fourier transformed. The result is truncated to the desired number of modes. Using the current values for ρ_V , new values for α_n and β_n are calculated. ϕ_0 , the phase factor, which equals the average value of $\gamma - \phi$ should be zero for up/down symmetric plasmas.

IV - The average change in θ and the α_n and β_n from the last to the current iteration is examined. If it is small enough or if some maximum number of iterations has been exceeded, then the iteration is terminated. Otherwise steps I through IV are repeated.

Finally, the average errors of Eqs. (21) and (22) are calculated. The hodograph is integrated in ϕ for the output value of ρ_V , and the mapped and input boundaries are compared at successive ϕ locations to generate an overall fit error. The plasma boundary is then found by integrating the hodograph formula for $\rho = 1$.

The current version of this code is primitive. The most sophisticated technique being used is a fast Fourier transform. Even so, experience shows that it functions very well. No convergence difficulties are had for modest ρ_V . The iteration converges quickly to a precision limited by the mode truncation number, which approaches the precision limit of the machine if enough modes are kept in one hundred iterations.

The major restriction of code in its present form is to boundaries without small curvature regions. Such regions are almost stagnation points for the poloidal field and as a result θ is changing rapidly with respect to ϕ . To resolve this rapid change one must use

a large number of ϕ points. In the present equally spaced ϕ point code this is inefficient. An adaptive integration technique could be used for Eq. (21). The present code uses the trapezoidal method. Otherwise, adaptive, irregular, ϕ point spacing should help but would increase the computation time by not allowing the use of the FFT.

The small curvature regions cause equivalent rates of change in γ with respect to ϕ . To resolve these one needs to keep a large number of modes. At modest ρ , this is no problem but as ρ increases the ρ dependent factors in G_R and G_I grow quickly. The code, therefore, does not retain α_n and β_n in their basic form. Instead it keeps them multiplied by the ρ dependent factor of Eq. (25). To evaluate G_R , one then multiplies them by a ratio of the ρ dependent terms of Eqs. (24) and (25),

$$R_n(\rho) = \frac{1 + \rho^{-2n}}{1 - \rho^{-2n}}. \quad (27)$$

At large ρ and mode number these ratios are unity plus a small correction term that underflows harmlessly. This technique also improves the general accuracy of the code.

Figure 7 is an illustration of a solved asymmetric bean-like given wall case which was purposefully selected to illustrate these difficulties. The plasma boundary is the internal curve. The given and mapped outer boundaries have been overlayed for comparison and no visible difference is evident. Several intermediate flux surfaces and constant ϕ lines are plotted between the two. To obtain this degree of relative accuracy of $3 \cdot 10^{-5}$ between the given and mapped wall boundaries, 1024 ϕ points and 128 Fourier modes were used. The sharp dependence of γ on ϕ near the small curvature region is shown in Fig. 8, and one should note that γ need not be a monotonic function of ϕ . Figure 9 gives the mode content of this mapping and demonstrates a surprisingly rapid convergence of the Fourier series with mode number.

The modification of this numerical method to allow varying poloidal magnetic field on the plasma boundary, as in the work of Goedbloed,⁴ seems possible. A comparison of

the method with that of Garrick, Theodorsen, and Henrici,¹⁷⁻¹⁹ as applied by Goedbloed, has not been made.

V. The Method of Gorenflo and Merkel

The hodograph forms given above are intimately related to the method of Gorenflo and Merkel.^{21,22} Their procedure assumes the existence of a conformal mapping $z = g(\eta)$ which takes the plasma boundary into the unit circle in the η plane. In the η plane one assumes a form for a flux function $f(\eta)$ such that $df/d\eta$ is a rational function. The z plane flux function is then $F(z) = f[g^{-1}(z)]$. Typically,

$$f(\eta) = \sum_{m=1}^M I_m \log(\eta - b_m). \quad (28)$$

Then, $df/d\eta$ can be brought to the form,

$$\frac{df}{d\eta} = f_0 \frac{\prod_{m=1}^{M-1} (\eta - a_m)}{\prod_{m=1}^M (\eta - b_m)}. \quad (29)$$

Since

$$W = \frac{dF}{dz} = \frac{df/d\eta}{dg/d\eta}, \quad (30)$$

one need only select $dg/d\eta$ so that this ratio is proportional to a Blaschke product, which can be done by inspection. The resultant $g(\eta)$ solves the problem.

In Merkel's paper $dg/d\eta$ was required to be holomorphic outside the unit circle to allow the field to be defined throughout the region exterior to the plasma boundary and to preserve the number of singular points of the field. However, if one is content with a solution valid within some limited annular region and are not concerned with the number of field singularities introduced, one can let $dg/d\eta$ be meromorphic apart from some neighborhood

of the unit circle. Then from a given flux function $f(\eta)$ an infinite set of solutions can be generated by appending appropriate Blaschke terms to the product over and above those necessary to bring it to an initial Blaschke form.

The family of mappings given in this paper correspond to the simple choice of $f(\eta) \propto \log(\eta)$ but allow $g(\eta)$ to have this extended form.

VI. Bifurcation

Bifurcation of the free boundary problem will now be examined, first in general, and then for the case of the exponential hodograph form. The procedure is via a variational principle similar to those of Grad²⁷ and Garabedian.²⁸ Consider the positive functional

$$W = \frac{1}{2} \int_{\Omega_V} d^2x [(\nabla^2 \Psi)^2 - \lambda^2]. \quad (31)$$

One subjects Eq. (31) to variations in Ψ and $\partial\Omega_P$,

$$\begin{aligned} \Psi(\vec{x}) &= \Psi_0(\vec{x}) + \epsilon \Psi_1(\vec{x}) + \epsilon^2 \Psi_2(\vec{x}), \\ \vec{r}(l) &= \vec{r}_0(l) + \epsilon \vec{r}_1(l) + \epsilon^2 \vec{r}_2(l), \end{aligned} \quad (32)$$

and carries the algebra out to second order. The technique for performing the variation is that of Morse and Feshbach¹² applied to a two-dimensional problem. Fixed constant Ψ boundary conditions are assumed on $\partial\Omega_P$ and $\partial\Omega_V$ and the position of the latter wall boundary is held fixed.

The result to first order is

$$W_1 = \frac{1}{2} \oint_{\partial\Omega_P} dl \, \nu \left(\lambda^2 - \left(\frac{\partial \Psi_0}{\partial n} \right)^2 \right) - \int_{\Omega_V} d^2x \, \Psi_1 \nabla^2 \Psi_0, \quad (33)$$

where $\nu = \hat{n} \cdot \vec{r}_1$ is the normal component of the boundary displacement.

Vanishing of W_1 for arbitrary vacuum flux perturbations Ψ_1 requires $\nabla^2 \Psi_0 = 0$ for all $\vec{x} \in \Omega_V$ while for arbitrary plasma boundary displacements, ν , Eq. (7) is recovered up to a sign. Vanishing of the first variation, therefore, yields the equilibrium problem.

Assuming that a solution is had and the first variation vanishes, after some algebra, one finds a simple result for the second variation.

$$W_2 = \frac{1}{2} \int_{\Omega_1} d^2x (\vec{\nabla} \Psi_1)^2 + \frac{1}{2} \oint_{\partial\Omega_P} dl \lambda^2 \nu^2 \kappa. \quad (34)$$

All dependence on Ψ_2 and $\vec{r}_2$, the second order perturbations, has vanished identically. Here κ is the curvature of the plasma boundary defined so that $\kappa > 0$ when the boundary is convex and $\kappa < 0$ when the boundary is concave, i.e., $\kappa = 1/a > 0$ for a plasma inside a circle of radius a .

Equation (34) is not manifestly positive because of the second curvature-dependent term unless κ is everywhere negative, as in the inverted configuration of a plasma which surrounds and is confined away from a central conductor. In this latter case Eq. (34) is positive and nearby bifurcated solutions are absent as the variation is at a local minimum. This agrees with the circular case.

When κ is not negative definite, bifurcations are had whenever the second variation W_2 vanishes. One has $\Psi_1 = -\lambda\nu$ on the plasma boundary because of the assumption of constant total flux, so the second variation vanishes whenever

$$\nabla^2 \Psi_1 = 0, \quad \vec{x} \in \Omega_V, \quad (35)$$

and

$$\begin{aligned} \frac{\partial \Psi_1}{\partial n} + \kappa \Psi_1 &= 0, \quad \vec{x} \in \partial\Omega_P, \\ \Psi_1 &= 0, \quad \vec{x} \in \partial\Omega_V. \end{aligned} \quad (36)$$

This homogeneous mixed boundary condition problem does not always possess non-trivial solutions. Note that a non-trivial solution for the circular case exists when $a = b/e$, as stated in the introduction, but that the other singular $\kappa \rightarrow \infty$, bifurcation point, where

a solution appears out of the origin, is not predicted. Yet another type of singularity is found in the boundaries of cusped solutions which otherwise have the negative curvature property desired for locally unique solutions. Global uniqueness of such negative curvature solutions would seem to be implied and has been claimed by Garabedian,²⁸ although via a different variational principle. Global uniqueness has not been established here. In order to do so one would have to avoid the assumption that the boundary and flux are differentiable and show that all solutions in a suitably restricted parameter space are connected, that is, they can be reached from a given solution by a continuous variation of the parameters of the space. An absence of bifurcations would then ensure uniqueness.

The above problem, although linear, is almost as hard to solve as the original equilibrium problem. One method of solution is to treat λ as an eigenvalue and recast it as an eigenvalue problem. Bifurcated solutions can otherwise be sought by the numerical method given above. Recall that the vacuum wall $\partial\Omega_V$ and average plasma radius a are the input conditions for the method. Varying a while holding the vacuum wall fixed generates a one parameter family of solutions. If two solutions of this family end up having the same value of $a \log(\mu_V)$, they are a bifurcated pair. If they are geometrically close, a bifurcation point lies nearby.

In general, the numerical method of Sec. IV does not yield unique solutions in the given wall case. Only the shallow and deep solutions analogous to the circular case are usually found. However, other solutions may be present, particularly in waisted wall configurations, corresponding to second or higher-order bifurcation. Methods for finding these higher-order solutions might make use of the above linear formalism.

Conclusions

Two novel methods, based on the hodograph technique of Helmholtz and Kirchoff, have been given for generating solutions to the sharp boundary MHD equilibrium problem.

One of these, an exponential hodograph form, proved particularly suitable for numerical adaption and an outline of a code which successfully implements it was presented. The relation of the hodograph forms given in this paper to the work of Gorenflo and Merkel was pointed out. The fundamental connection of the latter work and that of this paper with Blaschke products was also discussed. A brief outline of an analytical derivation of the bifurcation conditions of this problem was presented. A conjecture with regard to the winding number of certain functions was made. A proof of this conjecture would be of interest. Several avenues for additional research on connected problems were mentioned.

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Appendix: Generality of the Exponential Parametric Hodograph

Theorem: Assume a smooth, simple closed curve in the complex plane described by $z(\theta) = x(\theta) + i y(\theta)$, where θ is a suitable parameter such that $0 \leq \theta \leq 2\pi$ and $x(\theta), y(\theta) \in C^2$, so that $z(2\pi) = z(0)$ and $\frac{dz}{d\theta}(2\pi) = \frac{dz}{d\theta}(0)$.

Then, there exist a continuous, monotonic, real function $\phi(\theta)$, such that $\phi(0) = 0$ and $\phi(2\pi) = 2\pi$, and $0 \leq \phi \leq 2\pi$ whenever $0 \leq \theta \leq 2\pi$, and a real, continuous, single-valued, periodic function $g(\phi)$, such that,

$$z(\phi(\theta)) = z_0 + i a \int_0^\phi d\varphi e^{i(\phi - g(\phi))}, \quad (A1)$$

where

$$a = \int_0^{2\pi} \frac{d\theta}{2\pi} \left| \frac{dz}{d\theta} \right|. \quad (A2)$$

Proof: Define a radius function $r(\theta)$ and a phase function $\gamma(\theta)$ by

$$-i \frac{dz}{d\theta} = r(\theta) e^{i\gamma(\theta)}. \quad (.43)$$

It follows that

$$a = \int_0^{2\pi} \frac{d\theta}{2\pi} r(\theta). \quad (.44)$$

Note that, $r(\theta) > 0$ for any reasonable parameter θ , and that, without loss of generality, one may arrange $\gamma(0) = 0$ by a suitable rotation of coordinates. Then, γ is the angle between the x axis and the intrinsic normal vector of the curve.

One must have $\gamma(2\pi) = 2\pi$ since the curve is closed and simple, that is to say non-self-intersecting. Because of the assumed differentiability of the boundary, $\gamma(\theta)$ is a continuous function of θ and has no local jumps in value. On the other hand, $\gamma(\theta)$ takes on multiple values which locally differ by 2π as one traces the boundary around completely and arrives back at the same point, i.e., for $0 \leq \theta \leq 2\pi$, $\gamma(\theta + 2\pi n) = \gamma(\theta) + 2\pi n$, where n is an integer. In the restricted domain $0 \leq \theta \leq 2\pi$, $\gamma(\theta)$ can otherwise have arbitrary values and may not be monotonic.

Define $\phi(\theta)$ by

$$\frac{d\phi}{d\theta} = \frac{r(\theta)}{a} > 0. \quad (.45)$$

The right hand side of the above equation is positive. As a result, $\phi(\theta)$ is a monotonic function of θ and is an invertible one-to-one function in the restricted domain $0 \leq \theta \leq 2\pi$. One can always choose $\phi(0) = 0$, then, $\phi(2\pi) = 2\pi$ given the definition of a .

Note that $\phi(\theta)$ has precisely the same multivalued dependence on θ as does $\gamma(\theta)$, that is, for $0 \leq \theta \leq 2\pi$, $\phi(\theta + 2\pi n) = \phi(\theta) + 2\pi n$. Define $g(\phi)$ as the difference of these two functions.

$$g(\phi) = \gamma(\theta(\phi)) - \phi. \quad (.46)$$

$g(\phi)$ is a continuous, periodic, and single-valued function of ϕ and has a convergent representation in terms of Fourier series.

Now, by construction,

$$-i \frac{dz}{d\theta} = r e^{i\gamma} = a \frac{d\phi}{d\theta} e^{i(\phi+g)}, \quad (A6)$$

or

$$\frac{dz}{d\theta} \frac{d\theta}{d\phi} = i a e^{i(\phi+g)}, \quad (A7)$$

and Eq. (A1) follows.

This theorem implies that the parametric exponential form of the hodograph, given in Sec.III, is a general one, when applied to the problem of constructing the exterior solution for a given plasma boundary. The parameters of that form, the A_n , are obtained by identifying $g(\phi)$ in the above theorem with the imaginary part of the argument of the form, $G_I(\rho, \phi)$, for $\rho = 1$. The construction of the conjugate function $G_R(\rho, \phi)$ and extension of the solution into the exterior region are then trivial tasks. That this exterior solution is unique, follows from the fact that two functions are identical if they are analytic throughout some region and have the same values along an arc in the region.

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Figure Captions

Fig. 1. A simply connected plasma region, Ω_P , with boundary, $\partial\Omega_P$, is surrounded by an annular vacuum region, Ω_V , with outer boundary $\partial\Omega_V$. A simple, pinched plasma, Blaschke product solution is illustrated.

Fig. 2. A quasi-elliptical shape is generated by this pure moment example of the exponential hodograph. Here $\alpha_2 = 0.1$, $\beta_2 = 0$, and all the other α_n and β_n are zero. The innermost curve is the plasma boundary, $\rho = 1$, while the other curves are possible wall boundaries at successively larger ρ .

Fig. 3. An $n = 3$ pure mode example of the exponential hodograph, with $\alpha_3 = 0.5$ and $\beta_3 = 0$. The innermost curve is the plasma boundary in this tri-lobed figure.

Fig. 4. An $n = 4$ pure mode example of the exponential hodograph.

Fig. 5. An $n = 5$ pure mode exponential hodograph.

Fig. 6. A mixed, even mode, exponential hodograph. $\alpha_2 = -0.5$, $\beta_2 = 0$, $\alpha_4 = 0$, and $\beta_4 = 0.3$. Boundaries are plotted for $\rho = 1.00, 1.25$, and 1.50 . The outermost boundary is self-intersecting, indicating that it is too far from the plasma to be physical.

Fig. 7. The given outer boundary was fitted to a relative mean error of $3 \cdot 10^{-5}$ using 1024 ϕ points and 128 Fourier modes with the numerical method of Sec. IV. Constant Ψ and Φ lines are plotted between this and the innermost plasma boundary. The small curvature region in the upper left of the plot is almost a stagnation point for the poloidal magnetic field.

Fig. 8. The sharp dependence of γ on Φ is shown in this plot for the solution of Fig. 7. Note that γ is not a monotonic function of Φ .

Fig. 9. The mode content, of the solution of Fig. 7, shows a rapid decrease with mode number n . The mode content is defined as $(\rho_V^n - \rho_V^{-n})/A_n$.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

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