RADIATION TEMPERATURE OF NON-EQUILIBRIUM PLASMAS

V. Arunasalam

PPPL--2782

DE91 016217

Princeton University,
Princeton Plasma Physics Laboratory,
Princeton, New Jersey, 08543.

ABSTRACT

In fusion devices measurements of the radiation temperature $T_r(\omega, k)$ near the electron cyclotron frequency ω_c and the second harmonic $2\omega_c$ in directions nearly perpendicular to the confining magnetic field B (i.e., $k \approx k_\perp$) serve to map out the electron temperature profiles $T_e(r, t)$. For optically thick plasmas at thermodynamic equilibrium $T_r = T_e$. However, there is increasing experimental evidence for the presence of non-equilibrium electron distributions [such as a drifting Maxwellian with appreciable values of the streaming parameter $\xi = v_d/v_t$, a bi-Maxwellian, an anisotropic Maxwellian with $T_\perp \neq T_\parallel$, etc.,] in tokamak plasmas, especially in the presence of radio-frequency heating. Here, we examine [both non-relativistically and relativistically] the dependence of T_r on ξ , $T_\perp/T_\parallel$, T_h/T_c , n_h/n_c , etc., where n_c , n_h , T_c , T_h are the densities and

MASTER

temperatures, respectively, of the bulk and the hot components of the bi-Maxwellian plasma. Our bi-Maxwellian results predict that the ratio T_r/T_e is a very sensitive function of the ratios n_h/n_D and T_h/T_D . Further, these relativistic and non-relativistic results satisfy the well-known "limit $c \rightarrow \infty$ correspondence principle", showing that the intensity of the emission and absorption line is independent of the line broadening mechanism.

I. INTRODUCTION

In fusion research there is considerable interest in understanding the emission near the electron cyclotron frequency ω_c and its harmonics. Such emission could constitute a significant power loss in a reactor,¹ can prescribe the proper conditions for efficient application of electron cyclotron resonance (ECR) heating through the reverse process (Kirchhoff's law²⁻⁴), and, of more immediate interest, can provide an independent measurement of the properties of the emitting electrons^{2,5} (e.g., electron temperature T_e and/or presence of runaways). In hot dense tokamak plasmas where the runaway⁶ or the slideaway⁷ population is small, the intensity of this emission is observed to be at the blackbody level for the second harmonic extraordinary (X) mode and the fundamental ordinary (O) or extraordinary (X) mode.⁸ Thus the emission intensity is proportional only to the local electron temperature and is independent of the density. Indeed, for optically thick plasmas in thermodynamic equilibrium the radiation temperature T_r (for $\omega \approx \omega_c$, and $\omega \approx 2\omega_c$) is equal to the local electron temperature T_e . For optically thick plasmas, this radiation temperature T_r is defined to be the temperature of an equivalent blackbody that has the same emission intensity as the observed plasma emission.^{2,4}

Recently, however, there is increasing experimental evidence that

the electron distribution in tokamak plasmas, especially in the presence of electron cyclotron resonance heating (ECRH), include a small subpopulation of very hot particles.⁹⁻¹³ In fact, the early⁹ incoherent Thomson scattering measurements from the Alcator device showed the presence of non-Maxwellian distribution function when the ratio of the drift velocity to the thermal velocity $\xi = (v_d/v_t) > 0.1$. More recently,¹¹ X-ray spectra measurements from the Alcator A device reflected the presence of a non-Maxwellian distribution when this streaming parameter $\xi > 0.03$. A non-Maxwellian distribution in the Frascati Tokamak (FT) has also been inferred¹² for $\xi > 0.025$ from the measurements of $T_r(\omega \approx 2\omega_c)/T_e$ as a function of this streaming parameter ξ . When the experimentally measured ratio T_r/T_e of an optically thick plasma is larger than unity, it is an early warning of the presence of non-equilibrium electron distributions in the plasmas under study. Non-thermal electron cyclotron emission (ECE) from the Princeton TFTR tokamak has also been reported in recent literature.¹³

According to the neoclassical transport theory of plasma confinement in toroidal systems¹⁴ such as tokamaks, spherators, etc., if $v_{er} < \omega_b$, the bulk of the plasma electrons are magnetically trapped between the mirrors and execute banana type orbits, and this regime is usually called the neoclassical banana regime or the trapped electron regime. If $v_{er} > \omega_b$, the bulk of the plasma electrons are not trapped and this regime is usually called the collisional regime. Here, v_{er} and

ω_D are the effective collision frequency for scattering a trapped electron out of the trapped region, and the average bounce frequency between mirror trapping, respectively. If we apply an external electric field E so as to draw an Ohmic heating current along the confining magnetic field $B = B_z$, the behavior of the plasma in these two regimes is quite different.^{15,16} In the collisional regime, the current is carried by the bulk of the plasma electrons and consequently, the net electron distribution function is approximately a displaced Maxwellian (or more precisely of the Spitzer-Harm type¹⁷) with a drift velocity v_d . This $v_d \propto E$ if E is less than the Dreicer runaway field⁶ E_D . However, in the banana regime, the bulk of the plasma electrons are magnetically trapped between the mirrors and consequently, the current is carried only by the freely circulating electrons. These freely circulating electrons which carry the entire current form only a small fraction of the total electron population and have longitudinal velocities $v_{||} = v_z > v_t$. Hence, in the banana regime, the trapped fraction can be approximated by a regular Maxwell-Boltzmann distribution, while the circulating electrons form a displaced Maxwellian. According to Coppi *et.al.*,^{7,15} if the streaming parameter $\xi = v_d/v_t$ is large enough these circulating electrons will slide away from the bulk electrons even when $E < E_D$. Furthermore, it has been shown from basic thermodynamic entropy arguments¹⁸ and equivalently from the quasilinear theory¹⁹ that an initially isotropic ($T_{\perp} = T_{||}$) plasma will be forced into an anisotropic ($T_{\perp} \neq T_{||}$) state in any

resonance frequency heating.

It is not clear how such non-equilibrium distributions (i.e., a drifting Maxwellian with a large value of ξ , a bi-Maxwellian, an anisotropic Maxwellian, etc.) can be maintained for long periods of time in the presence of Coulomb collisions, or what other mechanisms may be responsible for the production of such non-equilibrium distributions in tokamak discharges. Here, we put aside such questions to consider the simpler issue of more immediate practical interest: what is the radiation temperature $T_r(\omega, k)$ of such non-equilibrium plasmas?. In particular, we wish to obtain closed-form analytic expressions for the dependence of T_r on $T_\perp/T_\parallel$, $\xi = v_d/v_t$, T_h/T_b , n_h/n_b , etc., where n_b , n_h , T_b , T_h , are the densities and temperatures, respectively, of the bulk and the hot components of a bi-Maxwellian plasma.

Most of the existing theoretical work²⁰⁻²² on the electron cyclotron absorption and emission in non-thermal plasmas is based on the direct evaluation of the dielectric tensor for a model distribution function which possesses the relevant non-thermal features and the numerical solution of the corresponding dispersion relation to obtain the absorption coefficient (and the absorption line shape), followed by the application of a suitably modified Kirchhoff's law²¹ or by some ad hoc definition of the source function²⁰ to obtain the emission coefficient (and the emission line shape). In the case of streaming

electrons some authors²¹ have made use of the Lorentz transformation of both the dielectric tensor and the Kirchhoff's law from the co-moving to the laboratory frame in their study of such non-thermal absorption and emission coefficients and line shapes. In this paper we approach the problem from an altogether different point of view. First, we evaluate in the conventional way the fundamental Einstein A and B coefficients from the golden rule of the quantum theory. These emission coefficients in the weak dispersion limit [i.e., $(1/2\omega)(\partial/\partial\omega)(\omega^2 K) \approx K$, where K is the dielectric coefficient] agree with those of the classical cyclotron emission theory of Trubnikov^{2,23}. Then, we calculate the radiation temperature via the steady-state solution of the photon master equation of non-equilibrium statistical mechanics and kinetic theory. We then derive both the non-relativistic and the relativistic closed form expressions for the radiation temperature, and show that these two results agree in their common limit in accordance with the "limit $c \rightarrow \infty$ correspondence principle". We also find that our bi-Maxwellian results predict that the ratio T_r/T_e is a very sensitive function of the fractional hot electron population n_h/n_b and their temperature ratio T_h/T_b .

11. A GENERAL DEFINITION OF $T_r(\omega, k)$.

Let us first seek a general definition of $T_r(\omega, k)$ in terms of the Einstein A and B coefficients via the principle of detailed balance and the photon master equation.²⁴ For this purpose we will consider the emission and absorption of photons of frequency ω and wave vector k by a particle (say an atom or a Landau electron^{25,26}) of mass m moving with a velocity v and having a bound state energy E . Then the energy and momentum conservation for the emission process yield:

$$\hbar\omega + E_f + (mv_f^2/2) = E_i + (mv_i^2/2) \quad . \quad (1)$$

and

$$\hbar k + mv_f = mv_i \quad . \quad (2)$$

respectively. Here, i and f stand for the initial and the final states of the emission process under study. Since $(v_i^2 - v_f^2) = (v_i - v_f) \cdot (v_i + v_f)$, and the resonance frequency $\omega_{fi} = (E_i - E_f)/\hbar$, on making use of Eq.(2) in Eq.(1) we get

$$\omega - \omega_{fi} - k \cdot (v_i - \hbar k/2m) = 0 \quad . \quad (3)$$

This is the familiar Doppler shift including the Compton recoil shift for the emission process under study. Let $N(E, v)$ be the number of particles with the internal (bound state) energy E , and the translational

velocity v , and $N(\omega, k)$ be the number of photons of frequency ω and wave vector k . Then the rate at which the photons are absorbed is

$$[dN(\omega, k)/dt]_{ab} = \sum B_{fi} N(E_f, v_f) N(\omega, k) \quad , \quad (4)$$

and the rate at which the photons are emitted is

$$[dN(\omega, k)/dt]_{em} = \sum A_{if} N(E_i, v_i) + \sum B_{if} N(E_i, v_i) N(\omega, k) \quad , \quad (5)$$

where B_{fi} and B_{if} are the coefficients (called the Einstein induced transition probabilities) describing the absorption and the induced emission of photons, respectively, A_{if} is the Einstein spontaneous emission probability, and the sum is over all possible transitions which involve photons of frequency ω and wave vector k . The spontaneous emission goes on even in the absence of an external radiation field; if in addition the system is bathed in radiation we have by the *principle of detailed balance* an induced emission of photons at the rate $\sum B_{if} N(E_i, v_i) N(\omega, k)$, where $B_{if} = B_{fi}$ by the familiar hermiticity requirement on the matrix elements for the transition. From Eq.(3), it is clear that these Einstein A and B coefficients are proportional to $\delta[\omega - \omega_{fi} - k \cdot (v_i - \hbar k/2m)]$, where δ is the Dirac δ -function. Thus from Eqs.(4) and (5), the net rate of increase in the number $N(\omega, k)$ of photons of frequency ω and wave vector k is given by

$$[dN(\omega, k)/dt] = [dN(\omega, k)/dt]_{em} - [dN(\omega, k)/dt]_{ab} \quad (6)$$

In statistical mechanics and kinetic theory,²⁴ this result of Eq.(6) is usually referred to as the *master equation* for the photon number density. In nonlinear and quasilinear plasma kinetic theory^{27,26} Eq.(6) is one of the coupled pair of equations which determines the way in which the radiative steady state is established between the electrons (i.e., the particle oscillators) and their radiation field (i.e., the field oscillators). The other member of this coupled pair describes the time evolution of the electron (i.e., particle) distribution function. Here, of course, the transition probabilities (i.e., the Einstein A and B coefficients) are usually evaluated in the *random phase* approximation. On making use of Eqs.(4) and (5), the solution of Eq.(6) for a uniform homogeneous system may be written

$$N(\omega, k) = N_0(\omega, k) [1 - \exp(-\gamma t)] \quad (7)$$

where the steady-state (i.e., $[dN(\omega, k)/dt] = 0$) solution $N_0(\omega, k)$ is

$$N_0(\omega, k) = \sum A_{if} N(E_i, v_i) / \sum B_{if} [N(E_f, v_f) - N(E_i, v_i)] \quad (8)$$

and the energy damping rate $\gamma(\omega, k)$ is

$$\gamma(\omega, k) = \sum B_{if} [N(E_i, v_i) - N(E_f, v_f)] \quad (9)$$

The optical depth $\tau(\omega, k)$ is usually defined as

$$\tau(\omega, k) = \int dl \cdot (\chi/v_g) \quad , \quad (10)$$

where $v_g = (\partial\omega/\partial k)$ is the group velocity, and dl is the incremental path length of the radiation field in the medium under study. In thermodynamic equilibrium at a temperature T ,

$$[N(E_f, v_f)/N(E_i, v_i)] = \exp(\hbar\omega/kT) \quad , \quad (11)$$

where we have made use of Eq.(1). Since, in the Fermi's *golden rule* approximation $A_{if} = B_{if}$, from Eqs.(8) and(11), we get the familiar Bose-Einstein statistics at thermodynamic equilibrium:

$$N_O(\omega, k) = [\exp(\hbar\omega/kT) - 1]^{-1} \quad . \quad (12)$$

Hence at thermodynamic equilibrium, both the particles (i.e., electrons) and their radiation field assume the common temperature T . That is, $T = T_r(\omega, k) = T_e$ at thermodynamic equilibrium. Hence, from Eqs.(8) and (12), a *general definition* of the radiation temperature $T_r(\omega, k)$ may be written

$$[\exp(\hbar\omega/kT_r) - 1] = \sum B_{if}[N(E_f, v_f) - N(E_i, v_i)]/\sum A_{if}N(E_i, v_i). \quad (13)$$

Since, in the linear and quasilinear approximations $A_{if} = B_{if}$, Eq.(13) yields

$$\chi T_r(\omega, k) = \hbar \omega \left(\ln \left[\sum A_{if} N(E_f, \nu_f) / \sum A_{if} N(E_i, \nu_i) \right] \right)^{-1}. \quad (14)$$

For plasmas, this result of Eq.(14) is equivalent to the result found in Eq.(2.39) of Bekerfi.² However, Eqs. (13) and (14) are applicable to any physical system of particles (i.e., electrons, ions, atoms, molecules, etc.) and their radiation field. It may be noted from Eq.(13) or (14) that if $N(E_i, \nu_i) = N(E_f, \nu_f)$, then $T_r = \infty$; and when $N(E_i, \nu_i) > N(E_f, \nu_f)$, T_r is negative.^{3,4,24} That is, at the quasilinear steady-state where the populations of the initial and the final states are equal, the radiation temperature is infinite; and when the population is inverted as in a laser or a maser, the radiation temperature is negative. At thermodynamic equilibrium the radiation temperature T_r must, of course, be equal to the system temperature T , and this can easily be seen from Eqs.(11), (13), and (14).

It may be noted that Eq.(8) is the steady-state solution of Eq.(6) if and only if the Einstein A and B coefficients of Eqs.(4) and (5) are independent of the energy density $\varepsilon(\omega, k) = \hbar \omega N(\omega, k)$ of the radiation field. This is true only in the linear and the quasilinear theories, since in these theories one makes use of the *golden rule* of the quantum time-dependent perturbation theory.²⁸ In this golden rule

approximation one obtains the linear particle-wave resonance condition of Eq.(3) which, in turn, yields the conventional causal dampings such as the Landau damping, cyclotron damping, etc. of the linear plasma wave theory. However, this golden rule approximation is only the first term of the series solution of the Heitler-Ma integral equation of the time-dependent perturbation theory. The higher order Heitler-Ma terms yield all the nonlinear processes²⁹ such as the nonlinear mode-coupling, nonlinear Landau damping, nonlinear decay interactions, nonlinear particle-wave resonance broadening, Karplus-Schwinger nonlinear saturation mechanism, particle trapping, etc. Thus in the nonlinear theory, the transition probabilities A and B depend on the intensity of the radiation field $\epsilon(\omega, k) = \hbar\omega N(\omega, k)$. In this case it is not clear whether one can seek a steady-state solution of the form given by Eqs.(8) and (13) to the set of Eqs.(4),(5), and (6). That is, it is not clear how one can define the nonlinear radiation temperature $T_r[\omega, k, N(\omega, k)]$. Here, we will restrict ourselves only to the golden rule evaluation of the Einstein A and B coefficients.

Moreover, in this paper we will be concerned only with the classical range of frequencies, $\hbar\omega \ll$ particle energy κT , and it is to this limit we shall take the above mentioned Eqs.(13) and (14). Since $T_r \geq T$, where the equality is for the case of thermodynamic equilibrium, $\hbar\omega \ll \kappa T_r$. Then, in Eq.(13), $[\exp(\hbar\omega/\kappa T_r) - 1] \approx (\hbar\omega/\kappa T_r)$. Also, by Taylor's series expansion, we get

$$N(E_f, v_f) - N(E_i, v_i) \approx - [\hbar \omega_{fi} (\partial / \partial E_i) + (\hbar k / m) \cdot (\partial / \partial v_i)] N(E_i, v_i), \quad (15)$$

where $\hbar \omega_{fi} = E_i - E_f$, and we have made use of Eq.(2). Thus, for $\hbar \omega \ll \kappa T_r$, Eq.(13) becomes

$$\kappa T_r \approx - \omega \sum A_{if} N(E_i, v_i) / \sum A_{if} [\omega_{fi} (\partial / \partial E_i) + (k / m) \cdot (\partial / \partial v_i)] N(E_i, v_i). \quad (16)$$

In the classical limit of $\hbar \rightarrow 0$, the approximate equality of Eq.(16) becomes an exact equality.

III. RADIATION TEMPERATURE FOR CYCLOTRON EMISSION

In all the existing magnetically confined fusion devices such as the tokamaks, measurements of the fundamental ordinary (O) mode $T_r(\omega \approx \omega_c, k \approx k_\perp)$ and the second harmonic extraordinary (X) mode $T_r(\omega \approx 2\omega_c, k \approx k_\perp)$ have been constantly used as an important plasma diagnostic tool to ascertain the time evolution of the electron temperature profiles. Hence, it is now physically instructive to examine the radiation temperature $T_r(\omega, k)$ for frequencies ω near the electron cyclotron frequency ω_c and its harmonics, and for near

perpendicular propagation, i.e., $k_{\perp} \gg k_{\parallel} = k_z$. Here, $\perp$ and $\parallel$ refer to directions perpendicular and parallel, respectively, to the confining magnetic field $\mathbf{B} = B\hat{z}$. Then the bound state energy E is the energy of the quantized electron cyclotron motion perpendicular to $\mathbf{B}$ [i.e., the Landau energy levels^{25,26} $E_j = (j + 1/2)\hbar\omega_c$, where $j = 0, 1, 2, \dots, \infty$], and the translational velocity $\mathbf{v}$ is the free particle motion along $\mathbf{B}$, i.e., $\mathbf{v} = v_{\parallel} = v_z\hat{z}$. Thus, for emission near the n -th harmonic, the initial bound state energy $E_i = (i + 1/2)\hbar\omega_c$, the final bound state energy $E_f = E_{i-n} = (i - n + 1/2)\hbar\omega_c$, the resonance frequency $\omega_{fi} = n\omega_c$, and the particle-wave resonance condition of Eq.(3) becomes $\omega - n\omega_c - k_{\parallel}(v_{\parallel} - \hbar k_{\parallel}/2m)$.

In Sec.II, we defined the Einstein A and B coefficients as the quantum mechanical transition probabilities for emission and absorption of photons of energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$. However, in this paper we are concerned only with the classical range of frequencies, $\hbar\omega \ll$ particle energy $\times T$. For this purpose we define the appropriate classical quantities $\tilde{A} = \hbar A$ and $\tilde{B} = \hbar B$. Then, it is shown elsewhere²⁶ that in the classical limit^{2,23,30}

$$\tilde{A}(n) = \tilde{B}(n) = (4\pi^2 e^2 / L^3 \omega_c) [v_{\perp} J_n'(\lambda)]^2 \delta(\omega - n\omega_c - k_{\parallel} v_{\parallel}) \quad (17)$$

for emission and absorption of the X-mode near the n -th harmonic, and

$$\tilde{A}(n) = \tilde{B}(n) = (4\pi^2 e^2 / L^3 \omega_C) \{[(c \cos\theta - v_{||}) / \sin\theta] J_n(\lambda)\}^2$$

$$\delta(\omega - n\omega_C - k_{||} v_{||}) \quad (18)$$

for the emission and absorption of the O-mode near the n -th harmonic, where $\lambda = (k_{\perp} v_{\perp} / \omega_C)$, $J_n(\lambda)$ is the Bessel function of order n , $J_n'(\lambda) = dJ_n(\lambda)/d\lambda$, θ is the angle between $\mathbf{k}$ and $\mathbf{B}$ (i.e., $\cos\theta = k_{||}/k$), e is the electronic charge, L^3 is the plasma volume under study, and the "dressing" factor³¹ $C = [E^* \cdot (1/2\omega)(\partial/\partial\omega)(\omega^2 K_H) \cdot E / (E^* \cdot E)]$. Here, K_H is the Hermitian portion of the hot plasma dielectric tensor $\mathbf{K}$, $\mathbf{E}$ is the wave electric field, and $*$ means the complex conjugate. This factor C takes account of the screening in the usual "dressed test particle" approach.³² For $k_{||} \ll k_{\perp}$, $[(E^* \cdot K_H \cdot E) / (E^* \cdot E)] \approx \text{Re} [K_{yy} + K_{xy}^2 / K_{xx}]$ for the X-mode and $[(E^* \cdot K_H \cdot E) / (E^* \cdot E)] \approx \text{Re} K_{zz}$ for the O-mode, where Re stands for the real part and we have chosen $\mathbf{k}$ to be in the x - z plane. In Eq.(18), the direct contribution from the z -component of the $\mathbf{E}$ -field $E_z \propto [(c \cos\theta - v_{||}) J_n(\lambda)]^2$. Since $(\mathbf{k} \cdot \mathbf{E}) = 0$ with $\mathbf{k}$ in the x - z plane, we also have^{2,23,26,30} an $E_x = -(\cos\theta/\sin\theta)E_z$. Thus, including this part of the X-mode component (E_x which is proportional to E_z) in Eq.(18), we get the well-known Trubnikov O-mode factor $\{[(c \cos\theta - v_{||}) / \sin\theta] J_n(\lambda)\}^2$. An alternative method of obtaining this same Trubnikov

O-mode factor via the absorption calculation and Kirchhoff's law is given in the appendix. These emission coefficients of Eqs.(17) and (18) in the weak dispersion limit (i.e., when the dressing factor $C \approx N_r^2$, where N_r is the real part of the index of refraction which for vacuum is unity) agree with those of the classical cyclotron emission theory of Trubnikov.^{23,2} However, the result of Eq.(18) for the O-mode emission do not seem to agree with those given by some authors,^{20,21,33,34} even though the O-mode dressing factor $C \approx N_r^2 \approx (1 - \omega_p^2/\omega^2) \approx 1$ for $\omega^2 \gg \omega_p^2$. In Eq.(16), $N(E_i, v_i) \rightarrow N(E_\perp, v_\parallel) = (L^3 n_e) F(E_\perp, v_\parallel) = (L^3 n_e)(2\pi v_\perp) f(v_\perp, v_\parallel)$, where n_e is the electron number density, $E_\perp = (mv_\perp^2/2)$, and $F(E_\perp, v_\parallel)$ and $f(v_\perp, v_\parallel)$ are the normalized electron distribution functions. Thus, in the classical limit of $\hbar \rightarrow 0$, Eq.(13) becomes

$$\begin{aligned}
 \kappa T_r(\omega, k) &= - \int dE_\perp \int dv_\parallel [\omega \tilde{A}(n) F(E_\perp, v_\parallel) / \int dE_\perp \int dv_\parallel [\tilde{A}(n) (n\omega_c (\partial/\partial E_\perp) \\
 &\quad + (k_\parallel/m)(\partial/\partial v_\parallel)) F(E_\perp, v_\parallel)] \\
 &= - \int v_\perp dv_\perp \int dv_\parallel [\omega \tilde{A}(n) f(v_\perp, v_\parallel)] / \int v_\perp dv_\perp \int dv_\parallel [\tilde{A}(n) \{ (n\omega_c / mv_\perp) \\
 &\quad (\partial/\partial v_\perp) + (k_\parallel/m)(\partial/\partial v_\parallel) \} f(v_\perp, v_\parallel)] \quad (19)
 \end{aligned}$$

for the emission near the n -th harmonic. In Eq.(19), sometimes it is

much easier to carry out the perpendicular integrals over $dE_{\perp}$ using $F(E_{\perp}, v_{\parallel})$ rather than over $dv_{\perp}$ using $f(v_{\perp}, v_{\parallel})$. But for the trapped-electron regime in toroidal plasmas^{15,16} one will have to use the distribution function in terms of $v_{\perp}$ and $v_{\parallel}$ [i.e., $f(v_{\perp}, v_{\parallel})$], since electrons with $v_{\perp} \geq v_{\parallel} \epsilon^{1/2}$ are magnetically trapped between the mirrors. In this banana regime the fraction of electrons that are magnetically trapped $\approx \epsilon^{1/2}$, where $\epsilon \approx (\Delta B/B)$ [$\approx (r/R)$ for tokamaks] is a measure of the mirror ratio, and these trapped electrons do not contribute to the plasma current. In a subsequent paper we will use the form of Eq.(19) in terms of $f(v_{\perp}, v_{\parallel})$ to examine the combined effects of the trapped particles and the fluctuating part of the distribution function $\tilde{f}(v_{\perp}, v_{\parallel})$ on the radiation temperature T_r . Furthermore, in the spatially varying tokamak field $B \propto R^{-1}$, the narrow spectral line shape $\phi(\omega, R_0)$ of the cyclotron resonance at a certain point R_0 will yield an equivalent narrow spatial shape $\phi(\omega_0, R)$ of this resonance at a certain frequency ω_0 . Some effects of this spatial shape of the resonance are briefly discussed in Sec. IV. For non-relativistic energies $kT \ll mc^2$ (i.e., $v \ll c$), $\lambda = (k_{\perp} v_{\perp} / \omega_c) \ll 1$, and $J_n(\lambda) \approx (1/n!)(\lambda/2)^n$. Then, in Eqs.(17) and (18)

$$[v_{\perp} J_n'(\lambda)]^2 \approx (k_{\perp} v_{\perp}^2 / 4\omega_c)^2 = (k_{\perp} E_{\perp} / 2m\omega_c)^2 \quad (20)$$

for the second harmonic (i.e., $n = 2$) X-mode, and

$$\begin{aligned}
[[(c \cos\theta - v_{\parallel})/\sin\theta]J_n(\lambda)]^2 &\approx [[(c \cos\theta - v_{\parallel})/\sin\theta]k_{\perp}v_{\perp}/2\omega_c]^2 \\
&= [[(c \cos\theta - v_{\parallel})/\sin\theta]^2 k_{\perp}^2 E_{\perp}^2 / 2m\omega_c^2] \quad (21)
\end{aligned}$$

for the fundamental (i.e., $n = 1$) O-mode, respectively. It may be noted from Eqs.(17), (18), (20), and (21) that the second harmonic X-mode and the fundamental O-mode emission and absorption are both due to the finite size of the electron Larmor orbits ($\rho = v_{\perp}/\omega_c = \lambda/k_{\perp}$). Furthermore, for $\theta = \pi/2$, the existence of the ordinary wave is solely due to the fact that $v_{\parallel} \neq 0$, a result that has been pointed out by Bekefi.^{2,30}

A. $T_{\perp}$ For a Drifting Anisotropic Maxwellian Distribution

We now consider an anisotropic ($T_{\perp} \neq T_{\parallel}$) Maxwellian distribution drifting with a drift velocity v_d along the confining magnetic field $B = B\hat{z}$. This velocity distribution function may be written

$$f(E_{\perp}, v_{\parallel}) = f_{\perp}(E_{\perp}) f_{\parallel}(v_{\parallel}), \quad (22)$$

where

$$f_{\perp}(E_{\perp}) = (\kappa T_{\perp})^{-1} \exp(-E_{\perp}/\kappa T_{\perp}); \quad 0 \leq E_{\perp} \leq \infty, \quad (23)$$

and

$$f_{\parallel}(v_{\parallel}) = (m/2\pi\kappa T_{\parallel})^{1/2} \exp[-m(v_{\parallel} - v_d)^2/2\kappa T_{\parallel}]; \quad -\infty \leq v_{\parallel} \leq \infty. \quad (24)$$

It may be noted from Eq.(24) that when $T_{\parallel} \rightarrow 0$, $f_{\parallel}(v_{\parallel}) \rightarrow \delta(v_{\parallel} - v_d)$. That is, a beam distribution is a special limiting case of a regular Maxwellian distribution. It is relatively easy to show from Eq.(23) that for positive integral values of p

$$\langle E_{\perp}^p \rangle = \int dE_{\perp} E_{\perp}^p f_{\perp}(E_{\perp}) = (\kappa T_{\perp})^p p! \quad (25)$$

Thus, on making use of Eqs.(17), (18), (20) - (25) in Eq. (19) and carrying out an integration by parts over $dE_{\perp}$, we get for $k_{\parallel} \neq 0$

$$\begin{aligned} \kappa T_F(\omega, k \approx k_{\perp}) &= \omega \langle E_{\perp}^2 \rangle / [2\omega_C \langle 2E_{\perp} \rangle + (\langle E_{\perp}^2 \rangle / 2\kappa T_{\parallel})(\omega - 2\omega_C - k_{\parallel} v_d)] \\ &= \kappa T_{\perp} / [(2\omega_C/\omega) + T_{\perp}(\omega - 2\omega_C - k_{\parallel} v_d)/\omega T_{\parallel}] \end{aligned} \quad (26)$$

for the second harmonic (i.e., $n = 2$) X-mode, and in addition when $[(c \cos \theta - (\omega - \omega_C)/k \cos \theta)/\sin \theta] \approx 0$

$$\begin{aligned} \chi_{T_r}(\omega, k \approx k_{\perp}) &= \omega \langle E_{\perp} \rangle / [\omega_c + (\langle E_{\perp} \rangle / \kappa T_{\parallel}) (\omega - \omega_c - k_{\parallel} v_d)] \\ &= \kappa T_{\perp} / [(\omega_c / \omega) + T_{\perp} (\omega - \omega_c - k_{\parallel} v_d) / \omega T_{\parallel}] \end{aligned} \quad (27)$$

for the fundamental (i.e., $n = 1$) O-mode, respectively; and for $k_{\parallel} = 0$

$$\chi_{T_r}(\omega, k = k_{\perp}) = [(\kappa T_{\perp})^2 \delta(\omega - 2\omega_c)] / [(\kappa T_{\perp}) \delta(\omega - 2\omega_c)] = \kappa T_{\perp} \quad (28)$$

for the second harmonic X-mode, and

$$\chi_{T_r}(\omega, k = k_{\perp}) = [(\kappa T_{\perp})(\kappa T_{\parallel}) \delta(\omega - \omega_c)] / [(\kappa T_{\parallel}) \delta(\omega - \omega_c)] = \kappa T_{\perp} \quad (29)$$

for the fundamental O-mode, respectively. In the right-hand side of Eq.(26), we have divided both the numerator and the denominator by the common factor $\{(k_{\perp}/2m\omega_c)^2 |k_{\parallel}|^{-1} f_{\parallel}[v_{\parallel} = (\omega - 2\omega_c)/k_{\parallel}]\}$, and when $k_{\parallel} \rightarrow 0$, this factor tends to $\{(k/2m\omega)^2 \delta(\omega - 2\omega_c)\}$. Similarly, in the right-hand side of Eq.(27), we have divided both the numerator and the denominator by the common factor $\{(k_{\perp}^2/2m\omega_c^2) [\{(ck_{\parallel}/k) - (\omega - \omega_c)/k_{\parallel} \} / \sin\theta]^2 |k_{\parallel}|^{-1} f_{\parallel}[v_{\parallel} = (\omega - \omega_c)/k_{\parallel}]\}$, and when $k_{\parallel} \rightarrow 0$, this factor tends to $\{-(k/2m\omega)^2 (2\kappa T_{\parallel}) (\omega - \omega_c) \delta'(\omega - \omega_c)\} = \{(k/2m\omega)^2 (2\kappa T_{\parallel}) \delta(\omega - \omega_c)\}$. Thus, we see that Eqs.(28) and (29) are the $k_{\parallel} \rightarrow 0$ limit of Eqs.(26) and (27), respectively. It should be noted that in

arriving at the last equalities of Eqs.(28) and (29) we have divided both the numerator and the denominator of the right-hand side of this equation by a common factor that is proportional to $\delta(\omega - n\omega_c)$. This Dirac δ -function is an "improper singular function", and is an operational quantity which is meaningful only when it occurs under an integral. However, since in this case it is the limit $k_{\parallel} \rightarrow 0$ of the same ordinary Gaussian function, such a division in Eqs.(28) and (29) is mathematically justified. We will later show that this limit $k_{\parallel} \rightarrow 0$ of the non-relativistic result is exactly the same as the limit $c \rightarrow \infty$ of the relativistic result for $k_{\parallel} = 0$.

In the conventional non-relativistic collisionless hot plasma theory which is valid for $(k_{\parallel}/k) > (v_t/2c)$ both the emission and absorption lines undergo the familiar Gaussian Doppler broadening. But when $(k_{\parallel}/k) < (v_t/2c)$, one has to correct this theory so as to take account of the line broadening due to the relativistic variation of mass with velocity. Also if the electron collision frequency is large enough, then one has to correct the emission and absorption line shapes accordingly. It may be pointed out that, in general, the intensity of the emission and absorption lines is independent of the line broadening mechanism. However, it is apparent from Eqs.(26), (27), (28), and (29) that for a Maxwellian plasma the radiation temperature T_r is also independent of the line broadening mechanism. The physical reason for

this is that T_r is the ratio of spontaneous emission to net absorption (i.e., absorption minus the induced emission) and all these three processes of spontaneous emission, induced emission, and absorption all undergo the same line broadening. Thus, we believe that the results of Eqs.(26), (27), (28) and (29) are valid regardless of the line broadening mechanism. Further, it is interesting to note from these two equations that T_r depends on the familiar cyclotron-overstability term $[(n\omega_c/\omega) + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel]$ of the conventional collisionless hot plasma theory.^{2,30,35}

If $\epsilon(\omega, k)$ represents the energy in the electromagnetic wave of frequency ω and wave vector k , then $\epsilon(\omega, k) = \hbar\omega N(\omega, k)$. Thus, from Eqs. (7), (8), (13), and (19), it is relatively easy to see that the radiation temperature $T_r(\omega, k)$ is also a measure of the electromagnetic energy density $\epsilon(\omega, k)$ inside the plasma under study when its optical depth $\tau(\omega, k) \gg 1$. That is, for a classical plasma at thermodynamic equilibrium $\epsilon_0(\omega, k) = \hbar\omega N_0(\omega, k) = \kappa T_r$. Blackbody emission occurs only at radiative equilibrium, where the emitted and the reabsorbed radiation field $\epsilon_0(\omega, k)$ is in thermodynamic equilibrium with the radiating electrons. When a system is in radiative equilibrium, equipartition theorem must be valid. According to the classical equipartition theorem the energy per degree of freedom must be $\kappa T/2$. Since both the fundamental O-mode and the second harmonic X-mode emission and absorption are due to the finite size $\rho = v_\perp/\omega_c$ of the

Larmor orbits, it is the transverse motion of the electrons which is in equilibrium with its own radiation field. This transverse motion has two degrees of freedom, one associated with v_x and the other with v_y . Hence, for blackbody emission at thermodynamic equilibrium $\varepsilon_0(\omega, \mathbf{k}) = \kappa T_r = 2(\kappa T_\perp/2) = \kappa T_\perp$. Since the cyclotron-overstability term $[(n\omega_c/\omega) + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel] = 1$ for $\omega = n\omega_c$ and $v_d = 0$, it is clear that the results of Eqs. (26), (27), (28), and (29) are in complete agreement with the classical equipartition theorem.

We now wish to point out an apparent paradox associated with the limit $k_\parallel \rightarrow 0$ of the cyclotron-overstability term $[(n\omega_c/\omega) + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel]$ in Eqs.(26) and (27). Obviously, the straightforward $k_\parallel \rightarrow 0$ limit of this term is $[(n\omega_c/\omega) + T_\perp(\omega - n\omega_c)/\omega T_\parallel]$. However, this is not quite right for $k_\parallel = 0$. We will now show that the correct limit of this term in Eqs.(26) and (27) for $k_\parallel = 0$ is simply unity. This can be seen from the $k_\parallel \rightarrow 0$ limit of Eqs.(17), (18), and (19). In Eqs.(17) and (18) $\delta(\omega - n\omega_c - k_\parallel v_d) \rightarrow \delta(\omega - n\omega_c)$, and in Eq.(19) the term $(k_\parallel/m)(\partial/\partial v_\parallel) \rightarrow 0$ as $k_\parallel \rightarrow 0$. It is this term $(k_\parallel/m)(\partial/\partial v_\parallel)$ in Eq.(19) that yields the factor $T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel$ in the cyclotron-overstability term $[(n\omega_c/\omega) + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel]$ of Eqs.(26) and (27). Hence, the correct $k_\parallel = 0$ limit of this term in

Eq.(26) and (27) is $(n\omega_c/\omega) = 1$. This is also the dual limit $k_{\parallel} \rightarrow 0$ and $\omega \rightarrow n\omega_c$ of this cyclotron-overstability term.³⁶ The physical reason for this is the term $(k_{\parallel}/m)(\partial/\partial v_{\parallel})$ in Eq.(19) is due to the conservation of parallel momentum in the emission and absorption process. But for $k_{\parallel} = 0$ there is no parallel wave momentum.

B. T_F For a Drifting Anisotropic Bi-Maxwellian Distribution

We now consider the general case of a two-component plasma. We will assume that each of these components can be described by a drifting anisotropic Maxwellian of the type discussed in the previous subsection IIIA. The bulk of the plasma electrons have the number density n_b , the parallel and the perpendicular temperatures $T_{\parallel b}$ and $T_{\perp b}$, and the parallel drift velocity v_{db} . Similarly, the hot component takes the corresponding values n_h , $T_{\parallel h}$, $T_{\perp h}$, and v_{dh} , respectively. Then the total electron number density $n_e = n_b + n_h$. Let $\eta = (n_h/n_e) < 1$. Then this drifting anisotropic bi-Maxwellian distribution may be written

$$\begin{aligned}
 f(E_{\perp}, v_{\parallel}) &= (1 - \eta) f_b(E_{\perp}, v_{\parallel}) + \eta f_h(E_{\perp}, v_{\parallel}) \\
 &= (1 - \eta) [f_{\perp b}(E_{\perp}) f_{\parallel b}(v_{\parallel})] + \eta [f_{\perp h}(E_{\perp}) f_{\parallel h}(v_{\parallel})], \quad (30)
 \end{aligned}$$

where $f_{\perp b}(E_{\perp})$ and $f_{\perp h}(E_{\perp})$ are given by Eq.(23) with $T_{\perp}$ replaced by $T_{\perp b}$ and $T_{\perp h}$, respectively; and $f_{\parallel b}(v_{\parallel})$ and $f_{\parallel h}(v_{\parallel})$ are given by Eq.(24) with $(T_{\parallel}, v_d)$ replaced by $(T_{\parallel b}, v_{db})$ and $(T_{\parallel h}, v_{dh})$, respectively. Here again one may note that when $T_{\parallel h} \rightarrow 0$, $f_{\parallel h}(v_{\parallel}) \rightarrow \delta(v_{\parallel} - v_{dh})$, a distribution appropriate to that of a beam. Similarly, from Eq.(25), for a positive integer p

$$\langle E_{\perp}^p \rangle_b = (\kappa T_{\perp b})^p p! \quad \text{and} \quad \langle E_{\perp}^p \rangle_h = (\kappa T_{\perp h})^p p! \quad (31)$$

Thus, on making use of Eqs.(17), (18), (20) - (25), (30) and (31) in Eq. (19), after a certain amount of algebra we get for $k_{\parallel} \neq 0$

$$\begin{aligned}
 T_r(\omega, k \approx k_{\perp}) &= [(1 - \eta)(T_{\perp b}^2/T_{\parallel b}^{1/2}) \exp(-\alpha_{2b}^2) + \eta (T_{\perp h}^2/T_{\parallel h}^{1/2}) \\
 &\quad \exp(-\alpha_{2h}^2)] / [(1 - \eta)(T_{\perp b}/T_{\parallel b}^{1/2}) A_{2b} \exp(-\alpha_{2b}^2) + \\
 &\quad \eta (T_{\perp h}/T_{\parallel h}^{1/2}) A_{2h} \exp(-\alpha_{2h}^2)] \quad (32)
 \end{aligned}$$

for the second harmonic (i.e., $n = 2$) X-mode, and in addition when $[(c$

$$\cos\theta - (\omega - \omega_c)/k \cos\theta / \sin\theta] = 0$$

$$T_r(\omega, k \approx k_\perp) = [(1-\eta)(T_{\perp b}/T_{\parallel b})^{1/2} \exp(-\alpha_{1b}^2) + \eta (T_{\perp h}/T_{\parallel h})^{1/2} \exp(-\alpha_{1h}^2)] / [(1-\eta)(1/T_{\parallel b})^{1/2} A_{1b} \exp(-\alpha_{1b}^2) + \eta (1/T_{\parallel h})^{1/2} A_{1h} \exp(-\alpha_{1h}^2)] \quad (33)$$

for the fundamental (i.e., $n = 1$) O-mode, respectively, where

$$\alpha_{nj} = (m/2kT_{\parallel j})^{1/2} [(\omega - n\omega_c - k_{\parallel} v_{dj})/k_{\parallel}], \quad (34)$$

and

$$A_{nj} = [(n\omega_c/\omega) + T_{\perp j}(\omega - n\omega_c - k_{\parallel} v_{dj})/\omega T_{\parallel j}] \quad (35)$$

with $j = b$ and h for the bulk and the hot component, respectively.

Similarly, for $k_{\parallel} = 0$ we get

$$T_r(\omega, k=k_\perp) = [(1-\eta)T_{\perp b}^2 \delta(\omega - 2\omega_c) + \eta T_{\perp h}^2 \delta(\omega - 2\omega_c)] / [(1-\eta)T_{\perp b} \delta(\omega - 2\omega_c) + \eta T_{\perp h} \delta(\omega - 2\omega_c)] \quad (36)$$

for the second harmonic X-mode, and

$$T_r(\omega, k=k_\perp) = [(1-\eta) T_{\perp b} T_{\parallel b} \delta(\omega - \omega_c) + \eta T_{\perp h} T_{\parallel h} \delta(\omega - \omega_c)] /$$

$$[(1-\eta) T_{\parallel b} \delta(\omega - \omega_c) + \eta T_{\parallel h} \delta(\omega - \omega_c)] \quad (37)$$

for the fundamental O-mode, respectively. Here again we have divided both the numerator and the denominator of Eqs.(32) and (33) by their common factors $\{(k_\perp/2m\omega_c)^2 (m\kappa/2\pi)^{1/2} |k_\parallel|^{-1}\}$ and $\{(k_\perp^2/2m\omega_c^2) [((ck_\parallel/k) - (\omega - \omega_c)/k_\parallel)/\sin\theta]^2 (m\kappa/2\pi)^{1/2} |k_\parallel|^{-1}\}$, respectively.

Thus, in the limit $k_\parallel \rightarrow 0$, Eqs.(32) and (33) become

$$T_r(\omega, k \rightarrow k_\perp) = [(1-\eta) T_{\perp b}^2 \delta_b(\omega - 2\omega_c) + \eta T_{\perp h}^2 \delta_h(\omega - 2\omega_c)] /$$

$$[(1-\eta) T_{\perp b} \delta_b(\omega - 2\omega_c) + \eta T_{\perp h} \delta_h(\omega - 2\omega_c)] \quad (36a)$$

and

$$T_r(\omega, k \rightarrow k_\perp) = [(1-\eta) T_{\perp b} T_{\parallel b} (\omega - \omega_c) \delta_b'(\omega - \omega_c) + \eta T_{\perp h} T_{\parallel h} (\omega$$

$$- \omega_c) \delta_h'(\omega - \omega_c)] / [(1-\eta) T_{\parallel b} (\omega - \omega_c) \delta_b'(\omega - \omega_c) + \eta T_{\parallel h}$$

$$(\omega - \omega_c) \delta_h'(\omega - \omega_c)], \quad (37a)$$

respectively, where

$$\delta_j(\omega - n\omega_c) = \lim_{k_{\parallel} \rightarrow 0} [|k_{\parallel}|^{-1} (m/2\pi k T_{\parallel j})^{1/2} \exp(-\alpha_{nj}^2)], \quad (38)$$

and $j = b$ or h , since $[(\pi\varepsilon)^{-1/2} \exp(-x^2/\varepsilon)] = \delta(x)$ when $\varepsilon \rightarrow 0$. Further, since $x\delta'(x) = -\delta(x)$, we see that Eqs.(36) and (37) are the limit $k_{\parallel} \rightarrow 0$ of Eqs.(32) and (33), respectively. However, Eqs.(36a), (37a) and (38) show that in Eqs.(36) and (37) one cannot divide both the numerator and the denominator by a common factor of $\delta(\omega - n\omega_c)$ as was done in Eqs.(28) and (29). Further, if Coulomb collisions are responsible for the line broadening at $k_{\parallel} = 0$, then in Eqs.(36a) and (37a) one has to replace $\delta_j(\omega - n\omega_c)$ by the factor $\{(v_j/2\pi)/[(\omega - n\omega_c)^2 + v_j^2/4]\}$, where the Coulomb collision frequency $v_j \propto T_j^{-3/2}$. In Sec.IIIC we have examined the situation where the relativistic Doppler broadening (or equivalently, the broadening due to the relativistic variation of mass with speed) is dominant at $k_{\parallel} = 0$. It is relatively easy to see that Eqs.(32) and (33) reduces to Eqs.(26) and (27), respectively, either when $\eta = 0$ or when $\eta = 1/2$ and $b = h$, as they should.

C. $T_{\perp}$ For Relativistic Isotropic Maxwellian and Bi-Maxwellian Distributions

In this subsection, we wish to briefly examine the relativistic case of isotropic Maxwellian and bi-Maxwellian distributions. In the fully relativistic case, the distribution function cannot be written as a product of perpendicular and parallel distributions as was done in Eqs.(22) and (30). Further, it is not at all clear whether one can even find an appropriate relativistic anisotropic (with $T_{\perp} \neq T_{\parallel}$) Maxwellian distribution function. In recent literature some authors^{20-22,37} have used approximate anisotropic model distribution functions. However, for an isotropic plasma with $T_{\perp} = T_{\parallel} = T$, one can use the known² relativistic Maxwellian distribution function

$$f(p_{\perp}, p_{\parallel}) = [4\pi(mc)^3 K_2(\mu)]^{-1} \mu \exp(-\mu\gamma), \quad (39)$$

where $K_2(\mu)$ is the MacDonald function (i.e., the modified Bessel function of the second kind), $\gamma = (1 - v^2/c^2)^{-1/2} = (1 + p^2/m^2c^2)^{1/2}$, and $\mu = mc^2/\kappa T$. For the weakly relativistic case, $\mu \gg 1$ and $p \ll mc$. Then $K_2(\mu) \approx (\pi/2\mu)^{1/2} \exp(-\mu)$, and $\mu\gamma \approx \mu(1 + p^2/2m^2c^2) = (\mu + p^2/2m\kappa T)$. Thus for the weakly relativistic case, Eq.(39) becomes³⁷⁻³⁹

$$f(p_{\perp}, p_{\parallel}) \approx (2\pi m\kappa T)^{-3/2} \exp(-p^2/2m\kappa T), \quad (40)$$

where $p^2 = p_{\perp}^2 + p_{\parallel}^2$. For the relativistic case, Eqs.(17) and (18) remain valid with ω_c replaced by ω_c/γ to account for the relativistic mass variation with speed². That is, the relativistic particle-wave resonance condition is $\delta(\omega - n\omega_c/\gamma - k_{\parallel}v_{\parallel})$. For any isotropic distribution f , $[p_{\perp}^{-1}\partial f/\partial p_{\perp}] = [p_{\parallel}^{-1}\partial f/\partial p_{\parallel}] = [p^{-1}\partial f/\partial p] = 2\partial f/\partial p^2$, and $[(m\gamma n\omega_c/\gamma p_{\perp})(\partial f/\partial p_{\perp}) + k_{\parallel}(\partial f/\partial p_{\parallel})] = [mn\omega_c + k_{\parallel}p_{\parallel}] (2\partial f/\partial p^2) = (2m\gamma\omega)(\partial f/\partial p^2)$, where we have used the resonance condition $\gamma\omega = n\omega_c + \gamma k_{\parallel}v_{\parallel} = n\omega_c + k_{\parallel}p_{\parallel}/m$. Remembering that in Eq.(19) one has to replace m and ω_c by their relativistic values γm and ω_c/γ , respectively, we find that the relativistic form of Eq.(19) for an isotropic f may be written

$$\kappa T_r(\omega, k) = -\int p_{\perp} dp_{\perp} \int dp_{\parallel} [\omega \tilde{A}(n)f] / \int p_{\perp} dp_{\perp} \int dp_{\parallel} [2m\gamma\omega \tilde{A}(n)\partial f/\partial p^2]. \quad (41)$$

For the f of Eq.(39), $(\partial f/\partial p^2) = (\partial f/\partial \gamma)(\partial \gamma/\partial p^2) = -(\mu/2\gamma m^2 c^2)f = -(1/2m\gamma\kappa T)f$. Thus, the use of Eq.(39) in Eq.(41) yields that $T_r(\omega \approx n\omega_c, k) = T$. Hence, for a Maxwellian f and when the optical depth $\tau(n\omega_c, k)$ of Eq.(10) is very much larger than one, $T_r(\omega \approx n\omega_c, k) = T$ both in the non-relativistic and in the relativistic formalisms, in agreement with the Kirchhoff's law. One also sees from the classical limit of Eq.(12) that at thermodynamic equilibrium the electromagnetic energy density $\epsilon_0(n\omega_c, k) = \hbar\omega N_0(n\omega_c, k) = \kappa T$, in agreement with the classical

equipartition theorem. Thus, our method of analysis based on the photon master equation shows that the Kirchhoff's law is simply a manifestation of (a more general principle) the equipartition theorem. Further, we see that the radiation temperature for a Maxwellian f is independent of the emission and absorption line shapes, since in the non-relativistic theory the Gaussian line shape is a consequence of the non-relativistic Doppler shift $\omega' = \omega - \mathbf{k} \cdot \mathbf{v}$, while in the relativistic theory the line shape should be determined by the relativistic Doppler shift $\omega' = \gamma(\omega - \mathbf{k} \cdot \mathbf{v})$ or equivalently by the relativistic mass variation with speed.

We now consider a relativistic isotropic bi-Maxwellian of the form $f = (1 - \eta)f_D + \eta f_H$, where the bulk component f_D and the hot component f_H are given by Eqs.(39) or (40) with T replaced by T_D and T_H , respectively. For convenience in treating the particle-wave resonance condition $\delta(\gamma\omega - n\omega_c - k_{\parallel}p_{\parallel}/m) = \gamma^{-1}\delta(\omega - n\omega_c/\gamma - k_{\parallel}v_{\parallel}) = |\omega^{-1}| \delta(\gamma - n\omega_c/\omega - k_{\parallel}p_{\parallel}/m\omega)$, here we will follow very closely the analytic method and the notation used by Chu and Hui.³⁹ Thus, on making use of Eqs.(17), (18), (20), (21), and (39) or (40) in Eq.(41) we get

$$\kappa T_r(\omega, \mathbf{k}) = \{ (1 - \eta)(\kappa T_D) \operatorname{Im} G_D(0.2, k_{\parallel}) + \eta (\kappa T_H) \operatorname{Im} G_H(0.2, k_{\parallel}) \} /$$

$$[(1 - \eta) \operatorname{Im} G_D(0.2, k_{\parallel}) + \eta \operatorname{Im} G_H(0.2, k_{\parallel})] \quad (42)$$

for the second harmonic (i.e., $n = 2$) X-mode, and

$$\begin{aligned}
 \chi T_r(\omega, k) = & \{(\cos\theta)^2 \{ (1 - \eta) (\chi T_D) \operatorname{Im} G_D(0, 1, k_{\parallel}) + \eta (\chi T_H) \operatorname{Im} G_H(0, 1, k_{\parallel}) \} \\
 & - (2\cos\theta) \{ (1 - \eta) (\chi T_D) \operatorname{Im} G_D(1, 1, k_{\parallel}) + \eta (\chi T_H) \operatorname{Im} G_H(1, 1, k_{\parallel}) \} \\
 & + \{ (1 - \eta) (\chi T_D) \operatorname{Im} G_D(2, 1, k_{\parallel}) + \eta (\chi T_H) \operatorname{Im} G_H(2, 1, k_{\parallel}) \} \} / \\
 & \{ (\cos\theta)^2 \{ (1 - \eta) \operatorname{Im} G_D(0, 1, k_{\parallel}) + \eta \operatorname{Im} G(0, 1, k_{\parallel}) \} - (2\cos\theta) \\
 & \{ (1 - \eta) \operatorname{Im} G_D(1, 1, k_{\parallel}) + \eta \operatorname{Im} G_H(1, 1, k_{\parallel}) \} + \{ (1 - \eta) \\
 & \operatorname{Im} G_D(2, 1, k_{\parallel}) + \eta \operatorname{Im} G_H(2, 1, k_{\parallel}) \} \} \quad (43)
 \end{aligned}$$

for the fundamental (i.e., $n = 1$) O-mode, respectively, where the imaginary part of the G-function is given by

$$\begin{aligned}
 \operatorname{Im} G(s, q, k_{\parallel}) = & (2\pi^2 m c^2 / \chi T) (m c)^{-s-2q} \int p_{\perp} dp_{\perp} \int dp_{\parallel} [p_{\parallel}^s p_{\perp}^{2q} \\
 & \delta(\gamma - n\omega_c/\omega - k_{\parallel} p_{\parallel}/m)] f(p_{\perp}, p_{\parallel}). \quad (44)
 \end{aligned}$$

It may be pointed out that our Eq.(44) is identical to the imaginary part of the G-function defined in Eq.(11) of Chu and Hui.³⁹ In the

non-relativistic limit of Eq.(44) for an isotropic (i.e., $T_{\perp} = T_{\parallel} = T$) Maxwellian distribution, from Eq.(12) of Chu and Hui:³⁹

$$\text{Im}G^{\text{NR}}(0,2,k_{\parallel}) = 8(2\mu)^{-1/2} (\omega/ck_{\parallel}) \text{Im}Z(\alpha_2), \quad (45a)$$

$$\text{Im}G^{\text{NR}}(0,1,k_{\parallel}) = (2\mu)^{1/2} (\omega/ck_{\parallel}) \text{Im}Z(\alpha_1), \quad (45b)$$

$$\text{Im}G^{\text{NR}}(1,1,k_{\parallel}) = (\omega/ck_{\parallel}) \text{Im}Z'(\alpha_1), \quad (45c)$$

$$\text{Im}G^{\text{NR}}(2,1,k_{\parallel}) = 2(2\mu)^{-1/2} (\omega/ck_{\parallel}) \alpha_1 \text{Im}Z'(\alpha_1), \quad (45d)$$

where the imaginary part of the hot plasma dispersion function^{40,35} $\text{Im}Z(\alpha_n) = \pi^{1/2} \exp(-\alpha_n^2)$. From Eqs.(45b-d), it is relatively easy to show that

$$\begin{aligned} & [(\cos\theta)^2 \text{Im}G^{\text{NR}}(0,1,k_{\parallel}) - (2\cos\theta) \text{Im}G^{\text{NR}}(1,1,k_{\parallel}) + \text{Im}G^{\text{NR}}(2,1,k_{\parallel})] \\ & = \{(2\pi\mu)^{1/2} (\omega/ck_{\parallel}) [\cos\theta - (\omega - \omega_c)/ck_{\parallel}]^2 \exp(-\alpha_1^2)\} \end{aligned} \quad (46)$$

Thus from Eqs.(45a) and (46), one can easily show that Eqs.(32) and (33) are the corresponding non-relativistic limit of Eqs.(42) and (43), respectively.

For $\mu = mc^2/kT \gg 1$, the relativistic effects on the emission and

absorption of both the fundamental O-mode and the second harmonic X-mode enter only as a line broadening mechanism. Hence, these effects are only important when $\cos\theta = k_{\parallel}/k < v_t/2c$. It is therefore physically instructive to examine the mathematically simplest case of exactly perpendicular propagation (i.e., $k = k_{\perp}$ or $\cos\theta = 0$) for which the relativistic mass variation with speed is the dominant line broadening mechanism. In this case, following Chu and Hui³⁹, we shall convert the integral variables from $(p_{\perp}, p_{\parallel})$ to $(\gamma, p_{\parallel})$ by substituting $p_{\perp}^2 = m^2 c^2 (\gamma^2 - 1) - p_{\parallel}^2$, and $p_{\perp} dp_{\perp} = m^2 c^2 \gamma d\gamma$ in Eq.(44). Also we set $p^2 = m^2 c^2 (\gamma^2 - 1)$ in Eq.(40). Further, since $p_{\perp}^2 \geq 0$, $-mc(\gamma^2 - 1)^{1/2} \leq p_{\parallel} \leq mc(\gamma^2 - 1)$, and $1 \leq \gamma \leq \infty$. Then Eq.(44) becomes

$$\text{Im}G(s, q, 0) = (2\pi^2 m c^2 / \kappa T) (m c)^{2-s-2q} \int d\gamma \gamma f(\gamma) \delta(\gamma - n\omega_c/\omega) \int dp_{\parallel}$$

$$p_{\parallel}^s [m^2 c^2 (\gamma^2 - 1) - p_{\parallel}^2]^q. \quad (47)$$

On making use of Eqs.(39) or(40) in Eq.(47), we get

$$\text{Im}G(0, 2, 0) = (16\pi\omega_c/\mu) L(0, 2, 0), \quad (48a)$$

$$\text{Im}G(2, 1, 0) = (2\pi\omega_c/\mu) L(2, 1, 0), \quad (48b)$$

$$\text{Im}G(0, 1, 0) = (2\pi\omega_c) L(0, 1, 0), \quad (48c)$$

and

$$\text{Im}G(1,1,0) = 0, \quad (48d)$$

where

$$L(0,2,0) = [\mu^3 \Delta_2^5 \exp(-\mu 2\omega_c/\omega) / 15\omega \kappa_2(\mu)], \quad (49a)$$

$$L(2,1,0) = [\mu^3 \Delta_1^5 \exp(-\mu \omega_c/\omega) / 15\omega \kappa_2(\mu)], \quad (49b)$$

and

$$L(0,1,0) = [\mu^2 \Delta_1^3 \exp(-\mu \omega_c/\omega) / 3\omega \kappa_2(\mu)] \quad (49c)$$

for the exact relativistic distribution function f of Eq.(39), and

$$L(0,2,0) = [(2/\pi)^{1/2} \mu^{7/2} \Delta_2^5 \exp(-\mu \Delta_2^2/2)/15\omega], \quad (50a)$$

$$L(2,1,0) = [(2/\pi)^{1/2} \mu^{7/2} \Delta_1^5 \exp(-\mu \Delta_1^2/2)/15\omega], \quad (50b)$$

and

$$L(0,1,0) = [(2/\pi)^{1/2} \mu^{5/2} \Delta_1^3 \exp(-\mu \Delta_1^2/2)/3\omega] \quad (50c)$$

for the Synge's^{38,39} weakly relativistic distribution function of Eq.(40). Here, $\Delta_n = [(\hbar\omega_c/\omega)^2 - 1]^{1/2}$, and $\Delta_n^2 \geq 0$. For $\mu = mc^2/\kappa T \gg 1$, the exact relativistic line shape functions of Eq.(49) reduce to the Chu and Hui's³⁹ weakly relativistic line shape functions of Eq.(50). Further, since $[(\mu/\pi)^{1/2} \exp(-\mu x^2)] = \delta(x)$ when $\mu \rightarrow \infty$, $x\delta'(x) = -\delta(x)$, $\delta(ax) = a^{-1} \delta(x)$, and $\delta(x^2 - a^2) = (2a)^{-1} [\delta(x - a) + \delta(x + a)]$, one can show from Eq.(50) that when $\mu \rightarrow \infty$

$$L(s, n, k_{\parallel}=0) \rightarrow (2/\omega) \delta[(n\omega_c/\omega)^2 - 1] = [\delta(\omega - n\omega_c) + \delta(\omega + n\omega_c)]. \quad (51)$$

Thus, on making use of Eq.(51) one sees readily that the non-relativistic limit of Eq.(48) is identically equal to the limit $k_{\parallel} \rightarrow 0$ of Eq.(45) [i.e., $\text{Im}G(s, n, 0)$ of Eq.(48) with $\mu \rightarrow \infty$ in $L(s, n, 0)$ is equal to $\text{Im}G^{\text{NR}}(s, n, k_{\parallel} \rightarrow 0)$ of Eq.(45)]. Physically this means that for $\mu \gg 1$, the relativistic effects on the emission and absorption of both the fundamental O-mode and the second harmonic X-mode enter only as a line broadening mechanism, while the total emission and absorption [i.e., the strength of the resonance which is proportional to the frequency integral of $\text{Im}G(s, n, k_{\parallel})$] is determined only by the non-relativistic finite Larmor radius effects.

For exactly perpendicular propagation (i.e., $k_{\parallel} = 0$), on making use of Eq.(48) in Eqs.(42) and (43) we get

$$T_r(\omega, k=k_{\perp}) = [(1-\eta) T_D^2 L_D(0, 2, 0) + \eta T_H^2 L_H(0, 2, 0)] / [(1-\eta) T_D L_D(0, 2, 0) + \eta T_H L_H(0, 2, 0)] \quad (52)$$

for the second harmonic ($n = 2$) X-mode, and

$$T_r(\omega, k=k_\perp) = [(1-\eta) T_b^2 L_b(2,1,0) + \eta T_h^2 L_h(2,1,0)] / [(1-\eta) T_b L_b(2,1,0) + \eta T_h L_h(2,1,0)] \quad (53)$$

for the fundamental ($n = 1$) O-mode, respectively. It is clear from Eq.(51) that in the non-relativistic limit of $\mu \rightarrow \infty$ (i.e., $c \rightarrow \infty$), Eqs.(52) and(53) reduce to Eqs.(36) and (37) [or equivalently, Eqs.(36a) and (37a)] with $T_\perp = T_\parallel = T$, respectively.. These results of Eqs.(32), (33), (36), (37), (52), and (53) again show that the emission and absorption of the fundamental O-mode and the second harmonic X-mode are both due to the non-relativistic finite Larmor radius effects, and relativistic effects enter only as a line broadening mechanism.^{2,41} Relativistic Doppler broadening, or equivalently, the broadening due to the relativistic variation of mass with speed is the dominant line broadening mechanism when $\cos\theta = k_\parallel/k \ll v_t/2c = (2\mu)^{-1/2}$. In some situations collisional broadening may dominate over the relativistic Doppler broadening. When $\cos\theta \gg (2\mu)^{-1/2}$, the line shape is determined by the non-relativistic Doppler broadening. However, the strength of the resonance (i.e., the frequency integral under the resonance curve) is independent of these line broadening mechanisms. In general, all line broadening mechanisms preserve the resonance strength which is only determined by the appropriate emission and absorption mechanism.^{2,41}

IV. DISCUSSION

It is clear from Eqs.(26), (27), (32), (33), (41), (42) and (43) that for complete thermodynamic equilibrium [i.e., $v_d = 0$, $\eta = 0$, and $T_{\perp} = T_{\parallel} = T$] the radiation temperature $T_r(\omega \approx n\omega_c, k \approx k_{\perp}) = T$ is totally independent of the Doppler line broadening, and hence independent of the frequency ω as long as ω is near $n\omega_c$ so that $\tau(\omega, k) > 1$. The physical reason for this is that T_r is a measure of the ratio of spontaneous emission to absorption minus the induced emission. By Kirchhoff's law^{2,4} this ratio should be fairly insensitive to any line broadening mechanism, since all these three processes of spontaneous emission, induced emission, and absorption all undergo the same line broadening. However, for non-equilibrium plasmas there is a weak dependence of T_r on ω or ω/ω_c via both the cyclotron-overstability term A_{nj} and the Doppler broadening term $\exp(-\alpha_{nj}^2)$. It should be noted that the T_r results of Eqs.(26), (27), (32), and (33) are meaningful only when the optical depth τ of Eq.(10) is larger than unity. Usually, $\tau > 1$ only in the region of frequency space where $\alpha_{nj} < 1$, and in this region $\exp(-\alpha_{nj}^2) \approx 1 - \alpha_{nj}^2 \approx 1$ and $A_{nj} = [(n\omega_c/\omega) + (T_{\perp} k_{\parallel} \alpha_{nj} / \omega T_{\parallel j})(2 \times T_{\parallel j} / m)^{1/2}] \approx 1$. Thus, even for non-equilibrium plasmas $T_r(\omega \approx n\omega_c, k \approx k_{\perp})$ is only a weak function of ω and ω/ω_c .

We remarked earlier that there is an apparent paradox associated

with the limit $k_{\parallel} \rightarrow 0$ of the cyclotron-overstability term $A_n = [(n\omega_c/\omega) + T_{\perp}(\omega - n\omega_c - k_{\parallel}v_d)/\omega T_{\parallel}] = [(n\omega_c/\omega) + (T_{\perp}k_{\parallel}\alpha_n/\omega T_{\parallel})(2kT_{\parallel}/m)^{1/2}]$. This is due to the fact that according to the non-relativistic particle-wave resonance condition $\delta(\omega - n\omega_c - k_{\parallel}v_{\parallel})$, when $k_{\parallel} \rightarrow 0$, $(\omega - n\omega_c)$ also tends to zero.³⁶ That is, in the non-relativistic hot plasma theory $(\omega - n\omega_c)$ and $k_{\parallel}$ are functions of each other and are not two independent variables (especially in the anomalous dispersion region). However, in the relativistic hot plasma theory this particle-wave resonance condition becomes $\delta(\omega - n\omega_c/\gamma - k_{\parallel}v_{\parallel})$, where $\gamma = (1 - v^2/c^2)^{-1/2} \approx 1 + (v^2/2c^2)$ for $v \ll c$. Thus the relativistic line broadening will dominate over the non-relativistic Doppler broadening when $(k_{\parallel}/k) < (v_t/2c)$. In this case for an isotropic plasma with $T_{\perp} = T_{\parallel} = T$ one can use the known^{2,42} relativistic Maxwellian distribution function $f(\gamma) \propto \exp(-\mu\gamma)$, where $\mu = (mc^2/kT)$. However, for an anisotropic plasma with $T_{\perp} \neq T_{\parallel}$, it is not at all clear whether one can even find an appropriate relativistic anisotropic Maxwellian distribution function. Nevertheless, for $(k_{\parallel}/k) > (v_t/2c)$, the non-relativistic cyclotron-overstability terms A_{nj} of Eqs.(26), (27), (32), and (33) are indeed correct.

It is apparent from Eqs.(26) and (27) that the effects of the drift velocity v_d (or the streaming parameter $\xi = v_d/v_t$) on the radiation

temperature T_r appears only via this cyclotron-overstability term $A_n = [(n\omega_c/\omega) + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel]$. For exact resonance, $\omega = n\omega_c$ and $A_n = [1 + (T_\perp k_\parallel v_d/\omega T_\parallel)] \approx [1 + \xi(T_\perp k_\parallel v_t/T_\parallel kc)]$. Thus, for $k \cdot v_d = k_\parallel v_d > 0$, $T_r < T_\perp$; and for $k \cdot v_d < 0$, $T_r > T_\perp$. However, the effect of this v_d or ξ on $(T_r/T_\perp)$ is rather small. The physical reason for this is that for these radiations the phase velocities are approximately c , and $c \gg v_d$. For example, for $T = T_\perp = T_\parallel = 5$ keV and $\theta = 80^\circ$, i.e., $\cos\theta = (k_\parallel/k) \approx 0.174$, this correction $\xi(T_\perp k_\parallel v_t/T_\parallel kc) \approx 0.05\xi$. Thus, for $\xi \approx 0.1$, this correction is only 0.5%. We should perhaps point out that when this streaming parameter $\xi = (v_d/v_t)$ is comparable to or larger than the Dreicer runaway value^{6,7} ξ_D , then one will not have a simple Maxwellian but instead one is likely to have a bi-Maxwellian, since the low velocity electrons being highly collisional will form the bulk Maxwellian component while the high velocity electrons tend to runaway and form the hot component. Thus, as we will see later that even though the direct effect of ξ on T_r is rather small, its indirect effect via the hot electron population is significantly large.

Let us now examine the frequency dependence of T_r of Eqs.(26) and (27) near the resonance $\omega \approx n\omega_c$. Since $A_n = \{[(n\omega_c/\omega) - 1]\{1 - (T_\perp/T_\parallel)\} + \{1 - (T_\perp k_\parallel v_d/\omega T_\parallel)\}\}$, it is clear that: (1) If $T_\perp > T_\parallel$, then $T_r > T_\perp$ for $\omega > n\omega_c$ and $T_r < T_\perp$ for $\omega < n\omega_c$, and (2) If $T_\perp < T_\parallel$, then

$T_r < T_\perp$ for $\omega > n\omega_c$ and $T_r > T_\perp$ for $\omega < n\omega_c$. In the actual radiation temperature measurements in fusion devices one usually measures the power P received by an electron cyclotron emission (ECE) antenna in a bandwidth $\Delta\omega$ centered around the frequency ω_0 , i.e., $\omega_0 - \Delta\omega/2 \leq \omega \leq \omega_0 + \Delta\omega/2$. This is given by^{2,41}

$$P = \int d\omega \kappa T_r (1 - \zeta) [1 - \exp(-\tau)] / [1 - \zeta \exp(-\tau)] , \quad (54)$$

where the integral is from $\omega = (\omega_0 - \Delta\omega/2)$ to $\omega = (\omega_0 + \Delta\omega/2)$, ζ is the wall reflection coefficient. In general, it is extremely difficult to do the frequency integral in Eq.(54) analytically since T_r , τ , and possibly ζ are all functions of ω , and one must do this integral numerically with the aid of computers. Here, we will always assume that $\zeta = 0$. For $\tau \gg 1$ and for a plasma in a uniform magnetic field and in complete thermodynamic equilibrium we have $P = \Delta\omega \kappa T_r = \Delta\omega \kappa T$ since T_r is then independent of ω . However, for non-equilibrium tokamak plasmas we may take

$$n\omega_c(x) = \omega [1 - (x - x_0)/R] , \quad (55)$$

where R is the major radius of the torus, and x_0 is the location of the n th-harmonic cyclotron layer [i.e., $\omega = n\omega_c(x_0)$]. Let $y = (x - x_0)$, then

$[(\omega - n\omega_c)/\omega] = (y/R)$. Hence,

$$A_n = [1 - (y/R)(1 - T_{\perp}/T_{\parallel}) - (T_{\perp}k_{\parallel}v_d/\omega T_{\parallel})] . \quad (56)$$

Also, by Taylor's series expansion

$$T_{\perp}(x) \approx T_{\perp 0} + y(\partial T_{\perp}/\partial x)_0 + (y^2/2)(\partial^2 T_{\perp}/\partial x^2)_0 + \dots \quad (57)$$

where $T_{\perp 0} = T_{\perp}(x_0)$. Then the T_r of Eqs.(26) and (27) may be written

$$T_r = [T_{\perp}(x)/A_n] \approx T_{\perp 0} [1 + (y/R)\{(R/T_{\perp 0})(\partial T_{\perp}/\partial x)_0 + (1 - T_{\perp}/T_{\parallel})\} + (y/R)^2 \{(R^2/2T_{\perp 0})(\partial^2 T_{\perp}/\partial x^2)_0 + (1 - T_{\perp}/T_{\parallel})^2\} + (T_{\perp}k_{\parallel}v_d/\omega T_{\parallel})] . \quad (58)$$

Thus, on making use of Eqs.(26), (27), (56), (57), and (58) in Eq.(54), the measured electron cyclotron emission (ECE) temperature $T(\text{ECE})$ for $\tau \gg 1$ may be written

$$T(\text{ECE}) = (P/k\Delta\omega) \approx T_{\perp 0} [1 + (1/3)(\Delta\omega/2\omega)^2 \{(R^2/2T_{\perp 0})(\partial^2 T_{\perp}/\partial x^2)_0 + (1 - T_{\perp}/T_{\parallel})^2\} + (T_{\perp}k_{\parallel}v_d/\omega T_{\parallel})] , \quad (59)$$

where we have made use of the relations $\int d\omega' (1/\omega') = [\log(\omega + \Delta\omega/2) -$

$\log(\omega - \Delta\omega/2)] \approx (\Delta\omega/\omega)$ for $\omega - \Delta\omega/2 \leq \omega' \leq \omega + \Delta\omega/2$. $\int dy y = 0$, and $\int dy y^2 = (2/3)(R\Delta\omega/2\omega)^3$ for $(-R\Delta\omega/2\omega) \leq y \leq (R\Delta\omega/2\omega)$. Thus, the measured ECE temperature $T(\text{ECE})$ will differ slightly from the resonant layer perpendicular electron temperature $T_{\perp 0} = T_{\perp}(x_0)$, and this is given by Eq.(59). The first correction $(1/3)(\Delta\omega/2\omega)^2(R^2/2T_{\perp 0})(\partial^2 T_{\perp}/\partial x^2)_0$ comes from a combination of the effects of the finite receiver bandwidth, toroidal field gradient, and the local gradient of the temperature gradient. This correction should exist even when the plasma is at thermodynamic equilibrium. The second correction $(1/3)(\Delta\omega/2\omega)^2(1 - T_{\perp}/T_{\parallel})^2$ is due to the temperature anisotropy and the finite receiver bandwidth. The third correction $(T_{\perp}k_{\parallel}v_d/\omega T_{\parallel})$ is due to Doppler shift arising from the plasma streaming velocity v_d and the finite value of $k_{\parallel}$. It is interesting to note from Eq.(58) that for $v_d = 0$ the leading correction to T_r is linear in y , but this term gives no contribution to $T(\text{ECE})$ of Eq.(59). Only the second order correction to T_r which is quadratic in y in Eq.(58) gives the leading contribution to $T(\text{ECE})$ of Eq.(59). Thus, even though the electron temperature profile and the cyclotron overstability corrections to T_r may be large, their corresponding corrections to $T(\text{ECE})$ can be small, since by the symmetry of the integral limits the $T(\text{ECE})$ -corrections can only depend on even powers of $(\Delta\omega/\omega)$. If we now assume a Gaussian form for the temperature profile, i.e., $T(r) \approx T_0 \exp(-\alpha_t r^2/a^2)$, then $(1/T)(\partial^2 T/\partial r^2) = -(2\alpha_t/a^2)[1 - (2\alpha_t r^2/a^2)]$, and by the principle of profile

consistency⁴³ $\alpha_t \approx (2/3)(q_a + 0.5)$, where q_a is the limiter safety factor. Thus, for example, in the TFTR tokamak $R \approx 250\text{cm}$, $a \approx 80\text{cm}$, $(R^2/2T_{\perp 0})(\partial^2 T_{\perp}/\partial x^2)_0 \approx -(R^2\alpha_t/a^2) \approx -39.1$ for $r \approx 0$ and $q_a \approx 5.5$, $(\omega/2\pi) \approx 140\text{GHz}$, and $(\Delta\omega/2\pi) \approx 4\text{GHz}$, then the first correction $(1/3)(\Delta\omega/2\omega)^2(R^2/2T_{\perp 0})(\partial^2 T_{\perp}/\partial x^2)_0 \approx -0.3\%$. Even if we assume that $T_{\perp} \approx 2T_{\parallel}$, the second correction $(1/3)(\Delta\omega/\omega)^2(1 - T_{\perp}/T_{\parallel})^2 \approx 0.01\%$. As we pointed out earlier the third correction $(T_{\perp}k_{\parallel}v_D/\omega T_{\parallel})$ is also usually very small. Here, we have assumed that the center frequency ω_0 of the receiver is known exactly. If the fractional uncertainty in ω_0 is $\pm(\Delta\omega/2)$, then it is apparent from Eqs.(54) and (58) that the fractional uncertainty in the measured $T(\text{ECE})$ is $\pm(\Delta\omega/2\omega)\{(R/T_{\perp 0})(\partial T_{\perp}/\partial x)_0 \pm (1 - T_{\perp}/T_{\parallel})\}$ since $(y/R) \approx \pm(\Delta\omega/\omega)$. That is, this uncertainty in $T(\text{ECE})$ measurements is determined by the leading correction to T_r in Eq.(58) which is linear in y , and this depends strongly on both the temperature gradient scale length and the temperature anisotropy. For the Gaussian $T(r)$ profile $(1/T)(\partial T/\partial r) = -(2\alpha_t r/a^2)$. Thus, for $r = a/2$ and the TFTR parameters quoted earlier we find that the first uncertainty term $(\Delta\omega/2\omega)(R/T_{\perp 0})(\partial T_{\perp}/\partial x)_0 \approx -17.86\%$, and the second term $(\Delta\omega/\omega)(1 - T_{\perp}/T_{\parallel}) \approx -1.43\%$. However, in practice the fractional uncertainty in the receiver center frequency ω_0 is usually considerably less than the receiver bandwidth $\pm(\Delta\omega/2)$. Hence the above estimates represent an upper bound on the

uncertainties.

In deriving the approximate Eq.(59) for a tokamak spatially varying field $B \propto R^{-1}$ from the exact Eq.(54), we have made the following approximations: (1) We have assumed that the wall reflection coefficient $\xi = 0$. In practice, the resonant layer optical depth $\tau(\omega_0, k) \gg 1$, and hence the absorption in the resonant layer is always a single path absorption. Thus $\xi = 0$ is not a bad assumption. (2) We have retained only the leading terms of the Taylor series expansion of the relevant variables around the resonant layer. (3) We have assumed that the optical depth $\tau(\omega, k) \gg 1$ over the entire frequency range $(\omega_0 - \Delta\omega/2) < \omega < (\omega_0 + \Delta\omega/2)$ of the integral in Eq.(54). That is, in Eq.(59) we have neglected the contribution $\Delta P = -\int d\omega \times T_r(\omega, k) \exp[-\tau(\omega, k)]$ in this frequency range. Usually $\tau(\omega, k)$ is a very highly peaked function of ω around its resonant layer value of $\tau(\omega_0, k)$. If the frequency width $\delta\omega$ of the function $\tau(\omega, k)$ is very much less than the receiver bandwidth $\Delta\omega$ and $\tau(\omega_0, k) \gg 1$, then $\Delta p \ll P$ and is approximately zero. These conditions are generally true for the present day tokamak discharges and the ECE receivers.

In a spatially varying field the cyclotron resonance at a certain point R_0 or x_0 has a certain narrow spectral line shape $\phi(\omega, R_0 \text{ or } x_0)$ determined by the appropriate line broadening mechanism (such as the non-relativistic Doppler, relativistic mass variation with speed,

collisional, etc.). Equivalently, at a certain frequency ω_0 the resonance has a narrow spatial shape $\phi(\omega_0, R \text{ or } x)$. For a tokamak $B \propto R^{-1}$, and thus $d\omega = d(\omega - \omega_c) = -d\omega_c = (\omega_c/R) dR$. Hence the frequency integral in P of Eq.(54) is equivalent to a spatial integral along the major radius R or the minor radius x of the torus. In the relativistic regime at perpendicular propagation [since each electron resonates with the wave at its down shifted frequency $n\omega_c(R \text{ or } x)/\gamma$] the spectral line shape $L(s, q, 0)$ of Eqs. (49) and (50) is asymmetric. That is, at perpendicular propagation the high velocity components are resonant at a smaller major radius R (higher B) than the low velocity ones. Hence, if one wishes to look specifically for the effects of nonthermal tail one must view the radiation from the high field side of the torus, since when viewing from the low field side this nonthermal radiation is reabsorbed in the resonant layer of the thermal bulk electrons before reaching the receiving antenna. An elegant discussion of the physics of this problem can be found in Secs. 5.2.6 to 5.2.8 of the book by Hutchinson.⁴¹

It is now of interest to examine in some detail the physical consequences of the results of Eqs.(32) and (33) for the bi-Maxwellian plasma. For simplicity we will assume that the streaming parameters $\xi_j = (v_{dj}/v_{tj}) \ll 1$, $\omega \approx n\omega_c$ so that $\alpha_{nj} \ll 1$, and it is an isotropic bi-Maxwellian so that $T_{\perp j} = T_{\parallel j} = T_j$ for both $j = b$ and $j = h$. Indeed, the optical depth $\tau \gg 1$ only in the regions of frequency space where

$\alpha_{nj} < 1$, at least for $j = b$ when $\eta \ll 1$. Thus, in this approximation we will set $\exp(-\alpha_{nj}^2) \approx 1$, and $A_{nj} \approx 1$. Then, Eqs.(32) and (33) become

$$T_r(\omega \approx 2\omega_c, k \approx k_\perp) \approx T_b [1 + \{\eta/(1-\eta)\}(T_h/T_b)^{3/2}] / [1 + \{\eta/(1-\eta)\}(T_h/T_b)^{1/2}] \quad (60)$$

for the second harmonic ($n = 2$) X-mode, and

$$T_r(\omega \approx \omega_c, k \approx k_\perp) \approx T_b [1 + \{\eta/(1-\eta)\}(T_h/T_b)^{1/2}] / [1 + \{\eta/(1-\eta)\}(T_h/T_b)^{-1/2}] \quad (61)$$

for the fundamental ($n = 1$) O-mode, respectively. These results of Eqs.(60) and (61) are valid only in the region where the non-relativistic Doppler broadening dominates, i.e., when $\cos\theta \gg v_t/2c = (2\mu)^{-1/2}$. Further, by making use of Eq.(45), one can easily show that Eqs.(60) and (61) also follow from Eqs.(42) and (43), respectively, when $\cos\theta \gg (2\mu)^{-1/2}$. These results are illustrated graphically in Fig.1. Here, the top part shows the variation of $[(T_r/T_b) - 1]$ with the ratio (T_h/T_b) for a fixed η value of 10^{-2} ; while the bottom part shows the corresponding variation of $[(T_r/T_b) - 1]$ with η for a fixed (T_h/T_b)

value of 16. For example, if we assume that the hot component of the bi-Maxwellian plasma is about 1% (i.e., $\eta \approx 10^{-2}$), and the energy of this hot component is about sixteen times that of the bulk electrons (i.e., $T_h \approx 16T_b$), then from Eqs.(60) and (61) we get $T_r(\omega \approx 2\omega_c, k \approx k_\perp) \approx 1.58T_b$ and $T_r(\omega \approx \omega_c, k \approx k_\perp) \approx 1.04T_b$. Thus we see that when $\cos\theta \gg (2\mu)^{-1/2}$ the radiation temperature of the second harmonic X-mode is much more sensitive to the fractional hot electron population η than that of the fundamental O-mode. Indeed, in this non-relativistic Doppler regime one can show that, in general, this η -sensitivity of the radiation temperature to the harmonic number n increases monotonically with increasing values of n .

Let us now examine the case of exactly perpendicular propagation (i.e., $k = k_\perp$, or $\cos\theta = 0$). In this case the non-relativistic Doppler shift vanishes, and the dominant line broadening mechanism is either the relativistic mass variation with speed or electron collisions. The exact results for the broadening due to relativistic mass variation with speed is given by Eq.(52) for the second harmonic X-mode and Eq.(53) for the fundamental O-mode. By setting $[\partial L(s,q,0)/\partial \Delta_n] = 0$, it is relatively easy to show from Eq.(50) that $L(0,2,0)$ and $L(2,1,0)$ take their maximum values when $\Delta_2^2 = \Delta_1^2 = (5/\mu)$, and this common maximum value is $\{[(2/\pi)^{1/2} 5^{5/2} \exp(-5/2)] \mu\} \propto T^{-1}$. Hence, using these peak values of $L(s,q,0)$, the approximate form of Eqs.(52) and (53) may be written

$$T_r(\omega, k=k_\perp) \approx T_b [1 + \{\eta/(1-\eta)\}(T_h/T_b)]/[1 + \{\eta/(1-\eta)\}] \quad (62)$$

both for the second harmonic X-mode and the fundamental O-mode. The behavior of Eq.(62) for a given η of 10^{-2} and a given T_h/T_b of 16, respectively, are shown in the top and the bottom halves of Fig.1. Thus, for $\eta \approx 10^{-2}$ and $T_h \approx 16 T_b$, Eq.(62) yields that $T_r(\omega, k=k_\perp) \approx 1.15 T_b$ both for the second harmonic X-mode and the fundamental O-mode. Furthermore, for $\eta \ll 1$, Eq.(62) yields that $[(T_r/T_b) - 1] \approx \eta[(T_h/T_b) - 1]$. Hence, the higher the value of the ratio T_h/T_b the smaller is the value of η that is needed to give the same value for $[(T_r/T_b) - 1]$.

For the sake of completeness we now wish to examine the case where the collisional broadening is dominant when $k_\parallel = 0$. In this case it is apparent that one should replace the line shape function $L_j(s, q, 0)$ of Eqs.(52) and (53) by the factor $\{(\nu_j/2\pi)/[(\omega - n\omega_c)^2 + \nu_j^2/4]\}$, where the Coulomb collision frequency $\nu_j \propto T_j^{-3/2}$. It is apparent this collisional broadening function peaks at $\omega = n\omega_c$, and this peak value is $(2/\pi\nu_j) \propto T_j^{3/2}$ in a fully ionized gas. In a partially ionized gas, electron-neutral collisions dominate over the Coulomb electron-ion collisions. Hence, using these peak values, the

approximate form of Eqs.(52) and (53) for the collisional broadening dominated regime may be written

$$\begin{aligned}
 T_r(\omega, k=k_\perp) &\approx T_D [1 + \{\eta/(1-\eta)\}(T_h^2 v_D / T_D^2 v_h)] / [1 + \\
 &\quad \{\eta/(1-\eta)\}(T_h v_D / T_D v_h)] \\
 &= T_D [1 + \{\eta/(1-\eta)\}(T_h/T_D)^{7/2}] / [1 + \{\eta/(1-\eta)\}(T_h/T_D)^{5/2}] \quad (63)
 \end{aligned}$$

both for the second harmonic X-mode and the fundamental O-mode. The last equality of Eq.(63) is true only for a fully ionized plasma where $v_j \propto T_j^{-3/2}$. Here again the behavior of Eq.(63) for a given value of $\eta \approx 10^{-2}$ and a given value of $(T_h/T_D) \approx 16$, respectively, are shown in the top and the bottom halves of Fig.2. Thus for Coulomb collisions with $\eta \approx 10^{-2}$ and $T_h \approx 16 T_D$, Eq.(63) yields that $T_r \approx 14.7 T_D$. Indeed, according to Eq.(63), for $\{\{\eta/(1-\eta)\}(T_h/T_D)^{5/2}\} \gg 1$, $T_r \rightarrow T_h$ and is independent of T_D for a Coulomb collisional broadening dominated plasma.

V. CONCLUSIONS

In summary, we have presented a general definition of the radiation temperature $T_r(\omega, k)$ in terms of the steady-state photon number density $N_0(\omega, k)$ solution of the photon master equation for systems whose optical depth $\tau \gg 1$. This master equation is a consequence of the application of the principle of detailed balance in terms of the Einstein's A and B coefficients for the emission and absorption processes in the random phase approximation. In the Fermi's golden-rule approximation of the time-dependent quantum perturbation theory, this definition reduces to the familiar definition of $T_r(\omega, k)$ as the temperature of an equivalent blackbody that has the same emission intensity as the observed emission from the system under study. By equipartition theorem, this is also a measure of the steady-state electromagnetic energy density $\epsilon_0(\omega, k) = \hbar\omega N_0(\omega, k)$ inside the system under study. In the classical limit $\epsilon_0(\omega, k) = kT_r(\omega, k)$. For turbulent systems it is not clear how one can generalize this definition of $T_r(\omega, k)$ so as to obtain the appropriate intensity-dependent non-linear radiation temperature $T_r[\omega, k, N_0(\omega, k)]$.

We have then applied these general results to study the cyclotron-radiation temperature of non-equilibrium plasmas, both for the fundamental O-mode and the second harmonic X-mode. First we have examined the case of a plasma which can be described by a simple

drifting anisotropic Maxwellian distribution function. Then we have examined the case of a bi-Maxwellian plasma. The effects of plasma streaming $\xi = v_d/v_t$ and the temperature anisotropy $T_\perp/T_\parallel$ comes into $T_r(\omega, k)$ only via the familiar cyclotron-overstability term $A_n = [(\omega - n\omega_c)/\omega + T_\perp(\omega - n\omega_c - k_\parallel v_d)/\omega T_\parallel]$. The radiation temperature $T_r(\omega, k)$ is fairly insensitive to the line broadening mechanism. We have also indicated the origin of the apparent paradox associated with the limit $k_\perp \rightarrow 0$ of the cyclotron-overstability term. In general, the streaming parameter ξ has a very weak effect on $T_r(\omega, k)$. The physical reason for this is that the streaming velocity v_d are usually very much less than the phase velocity of the electromagnetic cyclotron waves. The temperature anisotropy $T_\perp/T_\parallel$ does have a significant effect on $T_r(\omega, k)$. Further, this radiation temperature $T_r(\omega, k)$ is very sensitive to even a small fractional population of hot electrons [i.e., $\eta = n_h/(n_c + n_h) \ll 1$].

The measured electron cyclotron emission (ECE) temperature $T(\text{ECE})$ of an optically thick plasma is the frequency average of this radiation temperature $T_r(\omega, k)$ over the bandwidth $\Delta\omega$ of the ECE receiver of center frequency ω_0 [i.e., the average over the range $\omega_0 - \Delta\omega/2 \leq \omega \leq \omega_0 + \Delta\omega/2$]. Thus, even though the electron temperature profile and the cyclotron overstability corrections to $T_r(\omega, k)$ are fairly

large, their corresponding contributions to the measured $T(\text{ECE})$ are rather small. This is so because of the symmetry of the integral limits the $T(\text{ECE})$ -correction can only depend on even powers of $\Delta\omega/\omega$, while these leading corrections to $T_r(\omega, k)$ are linear in $\Delta\omega/\omega$. These leading corrections to $T_r(\omega, k)$ are mainly responsible for the uncertainty in the $T(\text{ECE})$ measurements via the uncertainty in ω in the range $\omega_0 - \Delta\omega/2 \leq \omega \leq \omega_0 + \Delta\omega/2$.

Both the radiation temperature $T_r(\omega, k)$ and the measured ECE-temperature $T(\text{ECE})$ are very sensitive functions of the fractional hot electron population η of a bi-Maxwellian plasma. This is because the leading η -correction to $T_r(\omega, k)$ is independent of $\Delta\omega/\omega$ [i.e., proportional to $(\Delta\omega/\omega)^0$], and hence the frequency averaging yields the same correction for $T(\text{ECE})$. However, in the non-relativistic Doppler regime [i.e., $\cos\theta = k_{\parallel}/k \gg v_t/2c = (2\mu)^{-1/2}$] the second harmonic X-mode $T(\text{ECE})$ is much more sensitive to this η -correction than the fundamental O-mode $T(\text{ECE})$. For example, for $\eta \approx 1\%$ and $T_H \approx 16T_D$, the second harmonic X-mode $T(\text{ECE}) \approx 1.58T_D$, while the fundamental O-mode $T(\text{ECE}) \approx 1.04T_D$. Indeed, this η -sensitivity of both $T_r(\omega, k)$ and $T(\text{ECE})$ will increase rapidly with increasing value of the harmonic number n . For $k_{\parallel} = 0$, if the relativistic Doppler broadening dominates over the collisional broadening, then the radiation temperatures near the second harmonic X-mode and the fundamental O-mode are given by

Eqs.(52) and (53), respectively. In this relativistic regime for $\eta \approx 1\%$ and $T_h \approx 16T_D$, both the second harmonic X-mode and the fundamental O-mode $T(ECE) \approx 1.15T_D$. Furthermore, the higher the value of the ratio T_h/T_D the smaller is the value of η that is needed to obtain the relation $T(ECE) \approx 1.15 T_D$. But, if the collisional broadening dominates over the relativistic Doppler broadening, then these are again given by Eqs.(52) and (53) [or Eqs.(36a) and (36b)], respectively, with $L_j(s,n,0)$ [or $\delta_j(\omega - n\omega_c)$] replaced by the factor $\{(v_j/2\pi) / [(\omega - n\omega_c)^2 + v_j^2/4]\}$, where the Coulomb collision frequency $v_j \propto T_j^{-3/2}$ for $j = b$ and h . Here again for $\eta \approx 1\%$ and $T_h \approx 16T_D$, we find that $T(ECE) \approx 14.7T_D$ for Coulomb collisional broadening. Further, for $[\eta (T_h/T_D)^{5/2}] \gg 1$, $T(ECE) \rightarrow T_h$ and is independent of T_D for Coulomb collisional broadening dominated plasmas. In the intermediate regime $0 < \cos\theta < (2\mu)^{-1/2}$, one has to seek numerical solutions to Eqs.(42) and (43), respectively, with the aid of computers. Nevertheless it is clear that, regardless of the nature of the line broadening mechanism, when the experimentally measured ratio $T(ECE)/T(\text{Thomson laser scattering})$ is larger than unity for $\tau(\omega,k) \gg 1$, it is an early warning of the presence of non-equilibrium electron distributions in the plasmas under study. Thomson laser scattering usually measures the bulk electron temperature T_D . Of course, under no conditions can $T(ECE)$ be less than $T(\text{Thomson laser scattering})$ for any plasma with $\tau(\omega,k) \gg 1$. However, for optically thin plasmas with $\tau(\omega,k) < 1$ and under

conditions where quenching of spontaneous emission coefficients occur,⁴⁴ one can encounter the unusual situation where $T(\text{ECE})$ is less than $T(\text{Thomson laser scattering})$.

Finally, we have not only presented closed form analytic expressions for the radiation temperatures near the second harmonic X-mode and the fundamental O-mode in regimes dominated by the non-relativistic Doppler broadening, relativistic Doppler broadening, and collisional broadening, but also have shown that our relativistic and non-relativistic results agree in their common limit. In particular, the $k_{\parallel} \rightarrow 0$ of the non-relativistic hot plasma theory result is identically equal to the limit $c \rightarrow \infty$ of our relativistic $k_{\parallel} = 0$ result. This reiterates the well-known view that the line broadening mechanisms in general do not alter the strength of the resonances.^{2,41} In the cases considered here, the strength of these resonances are only determined by the non-relativistic finite Larmor radius effect and relativity only enters as a line broadening mechanism when $\cos\theta \ll (2\mu)^{-1/2}$.

VI. ACKNOWLEDGMENTS

This work was supported by the U.S. Department of energy under Contract No. DE-AC02-76-CHO-3073. I thank Drs. P.C. Efthimion, F.W. Perkins, C. Karney, and professors T.H. Stix, I.B. Bernstein, A. Kritz, N. Fisch, and R.N. Sudan for useful comments.

VII. REFERENCES

- 1 M.N. Rosenbluth, Nucl. Fusion 10, 340 (1970); W.E. Drummond and M.N. Rosenbluth, Phys. Fluids 3, 45 (1960); *ibid* 6, 276 (1963); B.A. Trubnikov, Phys. Fluids 4, 195 (1961). Here the disagreement between the Drummond-Rosenbluth results and the Trubnikov results for the angular dependence of the absorption coefficient of cyclotron radiation is discussed and clarified. Further, these papers also show that for $\omega^2 \approx c^2 k^2 \approx (n\omega_c)^2 \gg \omega_p^2$ the refractive properties of the medium can be neglected and the individual electrons do indeed radiate as though they were in vacuum. The physical reason for this is that the radius of the transverse shielding current of the radiating test charge is of the order of (c/ω_p) and thus will not materially affect radiations at wavelengths λ_0 that are much less than (c/ω_p) . Further, they show that the off-diagonal elements of the dielectric tensor K are at most of order (ω_p^2/ω^2) . Thus, for $k^2 \gg (\omega_p^2/c^2)$ the off-diagonal elements can be neglected. This simply means that for wavelengths λ_0 short compared to (c/ω_p) , the longitudinal (Bernstein) and the transverse waves are not coupled. It may also be noted that the gap in the spectrum for propagation near resonance for the electromagnetic cyclotron harmonic O-modes always occurs for $|\omega|$ slightly larger than $|n\omega_c|$, whereas the converse is true for the electrostatic Bernstein modes. Thus the

transverse O-modes cannot convert easily to the electrostatic longitudinal Bernstein modes.

- 2 G. Bekefi, *Radiation Processes in Plasmas*, (Wiley, New York, 1966).
- 3 L.D. Landau and E.M. Lifshitz, *Statistical Physics*, (Addison-Wesley, Reading, Massachusetts, 1958).
- 4 G. Bekefi, J.L. Hirshfield, and S.C. Brown, *Phys. Fluids* **4**, 173 (1961).
- 5 F. Engelmann and M. Curatoto, *Nucl. Fusion* **13**, 497 (1973).
- 6 H. Dreicer, *Phys. Rev.* **115**, 238 (1959).
- 7 B. Coppi, F. Pegoraro, R. Pozzoli, and G. Rewoldt, *Nucl. Fusion* **16**, 309 (1976); B.H. Hui, N.K. Windsor, and B. Coppi, *Phys. Fluids* **20**, 1275 (1977).
- 8 V. Arunasalam, E.B. Meservey, M.N. Gurnee, and R.C. Davidson, *Phys. Fluids* **11**, 1076 (1968); A.E. Costley, R.J. Hastie, J.W.M. Paul, and J. Chamberlain, *Phys. Rev. Lett.* **33**, 758 (1974); J. Hosea, V. Arunasalam, and R. Cano, *Phys. Rev. Lett.* **39**, 408 (1977); P.C. Erthimion, V. Arunasalam, and J.C. Hosea, *Phys. Rev. Lett.* **44**, 396 (1980); D.A. Boyd, F.J. Stauffer, and A.W. Trivelpiece, *Phys. Rev.*

- Lett. 37, 98 (1976); I.H. Hutchinson and D.S. Komm, Nucl. Fusion 17, 1077 (1977).
- 9 L. Pieroni and S.E. Segre, Phys. Rev. Lett. 34, 928 (1975).
 - 10 R.D. Hazeltine, The University of Texas Report, FRCR#326 (1988).
 - 11 J.E. Rice, K. Molvig, and H.I. Helava, Phys. Rev. A25, 1645 (1982).
 - 12 R. Bartiromo, M. Brusati, P. Buratti, *et.al.* Proceedings of the Eighth International Conference on Plasma Physics and Controlled Nuclear Fusion Research, Brussels, 1980 (International Atomic Energy Agency, Vienna, 1981) Vol.1, p43.
 - 13 G. Taylor, C.W. Barnes, A. Cavallo, *et. al.*, Plasma Phys. and Controlled Fusion 31, 1957 (1989).
 - 14 M.N. Rosenbluth, R.D. Hazeltine, and F.L. Hinton, Phys. Fluids 15, 116 (1972); F.L. Hinton, and R.D. Hazeltine, Rev. Mod. Phys. 48, 239 (1976); S.P. Hirshman and D.J. Sigmar, Nucl. Fusion 21, 1079 (1981).
 - 15 B. Coppi and R. Pozzoli, Plasma Phys. 16, 223 (1974).
 - 16 V. Arunasalam, M. Okabayashi, R.J. Hawryluk, and S. Suckewer, Phys. Fluids 20, 95 (1977).

- 17 L. Spitzer, Jr., and R. Harm, Phys. Rev. **89**, 977 (1953).
- 18 J. Busnardo-Neto, J. Dawson, T. Kamimura, and A.T. Lin, Phys. Rev. Lett. **36**, 28 (1976).
- 19 V. Arunasalam, Phys. Rev. Lett. **37**, 747 (1976).
- 20 I. Fidone, G. Giruzzi, and G. Chiozzi, Plasma Phys. and Contr. Fusion **31**, 2003 (1989); I. Fidone, and G. Giruzzi, Nucl. Fusion **30**, 803 (1990); I. Fidone, G. Granata, R.L. Meyer, E.H. Jornada, R.S. Schneider, and L.F. Zeibell, Phys. Fluids **23**, 1336 (1980).
- 21 M. Bornatici, U. Ruffina, and E. Westerhof, Plasma Phys. and Contr. Fusion, **28**, 629 (1986); *ibid* **31**, 221 (1989).
- 22 P.H. Yoon, and T. Chang, J. Plasma Phys. **42**, 193 (1989); K. Matsuda, and J.Y. Hsu, General Atomics Report, GA-A19627 (September, 1989); S.T. Tsai, C.S. Wu, Y.D. Yang, and S.W. Kang, Phys. Fluids **24**, 2186 (1981); R.W. Harvey, M.R. O'Brian, V.V. Rozhdestvensky, and T.C. Luce, General Atomics Report, GA-A19643 (December, 1990).
- 23 B.A. Trubnikov, Magnetic Emission of High Temperature Plasma, Dissertation, Moscow (1958), USAEC Tech. Information Service AEC-tr-4073 (1960).

- 24 C. Kittel, *Elementary Statistical Physics* (Wiley, New York, 1958).
- 25 L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Reading, Massachusetts, 1958).
- 26 V. Arunasalam, J. Math. Phys. 10, 1305 (1969); V. Arunasalam, Phys. Rev. 149, No.1, 102 (1966).
- 27 D. Pines and J.R. Schrieffer, Phys. Rev. 125, 804 (1962); V.N. Tsytovich, *An Introduction to the Theory of Plasma Turbulence* (Pergamon, New York, 1972).
- 28 W. Heitler, *The Quantum Theory of Radiation*, (Clarendon Press, Oxford, England, 1954).
- 29 V. Arunasalam, Phys.Rev. A7, 1353 (1973); V. Arunasalam, Phys. Fluids 16, 718 (1973).
- 30 N.A. Krall and A.W. Trivelpiece, *Principles Of Plasma Physics* (McGraw-Hill, New York, 1973) Secs. 8.14 and 11.19; D.C. Montgomery and D.A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964) Secs. 10.6 and 14.1.
- 31 E.G. Harris, *A Pedestrian Approach to Quantum Field Theory*, (John Wiley & sons, New York, 1972), pp.42-43; V. Arunasalam, J.

Phys.A7, 2211(1974).

- 32 W.B. Thompson, "Kinetic Theory of Plasma", in *Advanced Plasma Theory*, M.N. Rosenbluth, Ed. (Academic Press, New York, 1964).
- 33 I. Fidone, G. Granata, G. Ramponi, and R.L. Meyer, Phys. Fluids 21, 645 (1978); M. Bornatici, U. Ruffina, and E. Westerhoff, Comments Plasma Phys. Controlled Fusion 9, 73 (1985); I. Fidone, G. Giruzzi, G. Granata, V. Krivenski, R.L. Meyer, M. Bornatici, and E. Mazzucato, Phys. Fluids B1, 1937 (1989). These papers and those of references 20 and 21 also include the contribution from the X-mode component $E_X \propto E_Z$ in the O-mode emission. However, their factor $\{[|E_X - iE_Y + (ck_\perp/n\omega_c) v_\parallel E_Z| J_n(\lambda)]^2 / (E^* \cdot E)\}$ do not seem equal to the Trubnikov factor $\{[(c \cos\theta - v_\parallel)/\sin\theta] J_n(\lambda)\}^2 / c$ for $\omega^2 \gg \omega_p^2$, since for weakly relativistic case with $\gamma \approx 1$, this Trubnikov factor is double humped with a minimum around $(c \cos\theta) \approx (\omega - \omega_c)/k_\parallel$. See Fig.3 of S. Tamor, Nucl. Fusion 18, 229 (1978). According to references 1, 2, 23, and 30 the Trubnikov's result is the correct result for $\omega^2 \gg \omega_p^2$. Since the work of Fidone, Bornatici, *et. al.* is essentially a numerical solution of the dispersion theory analysis of $\text{Im}k_\perp$, it is extremely difficult for me to point to the source of this discrepancy between their factor and the Trubnikov factor. Nevertheless, it is their duty to at least discuss the source of this discrepancy. Perhaps the footnote of

Ref.36 may have something to do with this, since one usually takes the $k_{\parallel} = 0$ solution as the zero-order solution to the case of small but finite $k_{\parallel}$ problems. In the literature a somewhat similar discrepancy in the angular dependence of the cyclotron absorption which existed between the Drummond-Rosenbluth result and the Trubnikov's result has been resolved in favor of Trubnikov, see ref.1.

- 34 V. Arunasalam, P.C. Efthimion, J.C. Hosea, H. Hsuan, and G. Taylor, Phys. Rev. **A36**, 3909 (1987) and **A37**, 2063 (1988); I. Fidone, R.L. Meyer, and G. Granata, Phys. Fluids **26**, 3292 (1983); I. Fidone, G. Granata, R.L. Meyer, and G. Ramponi, Plasma Phys. **22**, 203 (1980). All these papers neglect the $c \cos\theta$ in comparison to $v_{\parallel}$ and replace the factor $[(c \cos\theta - v_{\parallel})J_n(\lambda)]^2$ by the factor $[v_{\parallel}J_n(\lambda)]^2$ in the emission formula of Eq.(18). This replacement is only valid for $k_{\parallel} = 0$ and is not valid for $k_{\parallel} \approx 0$, where the contribution from $c \cos\theta$ is comparable to that of $v_{\parallel}$. In the usual dispersion theory analysis of $\text{Im}k_{\perp}$, the factor $[v_{\parallel} J_n(\lambda)]^2$ comes from the diagonal dielectric tensor element K_{zz} , while the remaining terms $[(c \cos\theta)J_n(\lambda)]^2$ and $\{-2v_{\parallel} c \cos\theta J_n^2(\lambda)\}$ come from the off-diagonal elements K_{xz} , K_{yz} , K_{zx} , and K_{zy} . See also the appendix. This same approximation of neglecting the $c \cos\theta$ in comparison to $v_{\parallel}$ is also found in Eq.(8.14.2) of Krall and Trivelpiece of reference 30.

Note, in the non-relativistic limit, the Trubnikov's O-mode emission line shape is double humped with zero emission at $(c \cos \theta) = (\omega - \omega_c)/k_{\parallel}$. In the weakly relativistic case where $\gamma \approx 1$, this emission is not zero but is a minimum around this value. See Fig. 3b of S. Tamor, Nucl. Fusion 18, 229 (1978).

- 35 T.H. Stix, *The Theory Of Plasma Waves*, (McGraw-Hill, New York, 1962).
- 36 The dielectric tensor elements K_{ij} have an essential singularity in the neighborhood of the point $\omega \rightarrow n\omega_c$ and $k_{\parallel} \rightarrow 0$, simultaneously. This essential singularity in K is due to the behavior of the plasma dispersion function $Z[\alpha_n = (\omega - n\omega_c)/k_{\parallel}v_t]$ when $\omega \rightarrow n\omega_c$ and $k_{\parallel} \rightarrow 0$. There are two distinct paths to approach this dual limit: (a) Let $k_{\parallel} \rightarrow 0$ first and $\omega \rightarrow n\omega_c$ later, then $\alpha_n \rightarrow \infty$ and one can use the large argument ($\alpha_n \gg 1$) expansion of the Z-function, i.e., $\text{Re } Z \approx 1/\alpha_n$. In this limit the real (Re) and imaginary (Im) parts of K_{ij} will always satisfy the Kramers-Kronig causality relations (since in this region the group velocity is identical to the velocity of energy transport). This is also the path of the usual cold plasma limit. (b) Let $\omega \rightarrow n\omega_c$ first and $k_{\parallel} \rightarrow 0$ later, then $\alpha_n \rightarrow 0$ and one may use the small argument ($\alpha_n \ll 1$) expansion of the Z-function, i.e., $\text{Re } Z \approx -2\alpha_n$. This is the path of the anomalous dispersion region

of the resonance. In this anomalous region the group velocity, the velocity of energy transport, and the signal velocity are all different distinct functions of ω . This path does not satisfy the usual causality with respect to the group velocity but it does satisfy the causality with respect to the velocity of energy transport, and in some cases yield non-physical results. See L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960). Thus, these two paths lead to two distinct results for this same dual limit. See also A.A. Rukhadze and V.P. Silin, Sov. Phys.- Tech. Phys. 7, 307 (1962) [Z.T.F. 32, 423 (1962)].

- 37 R.C. Davidson and P.H. Yoon, Phys. Fluids B1, 195 (1989); P.H. Yoon and R.C. Davidson, Phys. Rev. A35, 2619 (1987); P.L. Pritchett, Phys. Fluids 29, 2919 (1986).
- 38 J.L. Synge, *The Relativistic Gas* (Interscience, New York, 1957).
- 39 K.R. Chu and B. Hui, Phys. Fluids 26, 69 (1983).
- 40 B.D. Fried and S.D. Conte, *The Plasma Dispersion Function* (Academic Press, New York, 1961).
- 41 I.H. Hutchinson, *Principles of Plasma Diagnostics* (Cambridge University Press, Cambridge, 1987).

- 42 I.P. Shkarofsky, Phys. Fluids 9, 561 (1966) and J. Plasma Phys. 35, 319 (1986); M. Bornatici, R. Cano, O. DeBarbieri, and F. Engelmann, Nucl. Fusion 23, 1153 (1983); I. Fidone, G. Granata, and R.L. Meyer, Phys. Fluids 25, 2249 (1982); P. Misra, J. Plasma Phys. 14, 529 (1975); I.B. Bernstein and D.C. Baxter, Phys. Fluids 24, 108 (1981); A. Nassri and M. Heindler, Phys. Fluids 29, 3275 (1986); D.B. Batchelor, R.C. Goldfinger, and H. Weitzner, Phys. Fluids 27, 2835 (1984).
- 43 W.M. Tang, Nucl. Fusion 26, 1605 (1986), and Comments Plasma Phys. Controlled Fusion 10, 57 (1986); V. Arunasalam, N.L. Bretz, P.C. Efthimion, R.J. Goldston, *et. al.* Nucl. Fusion 30, 2111 (1990).
- 44 G. Gabrielse and H. Dehmelt, Phys. Rev. Lett. 55, 67 (1985); Y. Chung, P. Lemaire, and S. Suckewer, *ibid* 60, 1122 (1988); E. Yablonovitch, *ibid* 58, 2059 (1987); W. Jhe, A. Anderson, E.A. Hinds, *et. al.*, *ibid* 58, 666 (1987); R.G. Hulet, E.S. Hilfer, and D. Kleppner, *ibid* 55, 2137 (1985); D. Kleppner, *ibid* 47, 233 (1981); E. M. Purcell, Phys. Rev. 69, 681 (1946).

V111. FIGURE CAPTIONS

Fig.1. The top half shows $[(T_r/T_D) - 1]$ vs (T_h/T_D) for a fixed value of $\eta \approx 10^{-2}$. Here, the non-relativistic Doppler broadened $n = 2$ X-mode and the $n = 1$ O-mode are from Eqs.(60) and (61), respectively. The broadening due to the relativistic variation of mass with speed both for the $n = 2$ X-mode and the $n = 1$ O-mode is from Eq.(62). The bottom half shows the similar variation of $[(T_r/T_D) - 1]$ vs η for a fixed value of $(T_h/T_D) \approx 16$.

Fig.2. A similar plot of $[(T_r/T_D) - 1]$ vs (T_h/T_D) for $\eta \approx 10^{-2}$ (top half), and $[(T_r/T_D) - 1]$ vs η for $(T_h/T_D) \approx 16$ (bottom half) for the Coulomb collisional broadening case of Eq.(63).

APPENDIX: ALTERNATIVE DERIVATION OF THE TRUBNIKOV O-MODE FACTOR

In the text we have shown that for $\omega^2 \gg \omega_p^2$ the conventional Dirac's quantum theory of emission and absorption of radiation yield the familiar Trubnikov's angular factor $F_T = [(\cos\theta - \beta_{||})/\sin\theta]^2$ for the O-mode processes, where $\beta_{||} = (\omega - \omega_c)/ck_{||}$ in the non-relativistic limit. This Trubnikov factor is for the emission and absorption by a "bare particle" in vacuum. The corresponding emission and absorption by a "dressed test particle" in a medium of dielectric tensor K is then proportional to $C^{-1}F_T$, where the "dressing factor" $C = [E^* \cdot (1/2\omega) (\partial/\partial\omega)(\omega^2 K_H) \cdot E / (E^* \cdot E)]$, see reference 31. For O-mode $K_H \approx K_{ZZ} \approx (1 - \omega_p^2/\omega^2)$, and hence $C \approx 1$ for $\omega^2 \gg \omega_p^2$. This Trubnikov factor F_T is obviously equal to zero when $\cos\theta = \beta_{||}$, and is positive definite both for $\cos\theta > \beta_{||}$ and $\cos\theta < \beta_{||}$. That is, this factor is double-humped in the non-relativistic limit with a minimum value of zero when $\cos\theta = \beta_{||}$. In the relativistic case $\beta_{||} = (\omega - \omega_c/\gamma)/ck_{||}$, where $\gamma = (1 - v^2/c^2)^{-1/2} = (1 + p^2/m^2c^2)^{1/2}$. Hence, in the weakly relativistic case where $\gamma \approx 1$, this factor is not zero but is a minimum around $\cos\theta \approx (\omega - \omega_c)/ck_{||}$. Thus, this double-humped Trubnikov result for O-mode seems to disagree with those of some authors; see reference 33. Hence in this appendix we

wish to give an alternative non-relativistic derivation of this Trubnikov factor F_T .

We consider a transverse electromagnetic wave incident on the plasma with E mainly parallel to B and with the wave vector k making an angle θ with the direction of $B = B i_z$. Without any loss of generality we will choose k to lie in the x - z plane. Then the transversality condition ($k \cdot E = 0$) for this O-mode yields

$$E_x = -(\cos\theta/\sin\theta) E_z, \quad (A1)$$

and since E_y being independent of E_z can be taken to be zero for the incident wave. That is, $E = E_x i_x + E_z i_z = E_z [i_z - (\cos\theta/\sin\theta) i_x]$. This result of Eq. (A1) is valid for $\sin\theta \neq 0$. For $\sin\theta = 0$, $E_z = 0$ and E_x is not determined by E_z . Inside the plasma this $E_x i_x$ component of this incident E field will rotate in the x - y plane so as to develop a nonzero $E_y i_y$ component of order $(E_x \omega_p^2/\omega^2)$. But the self-consistent electric field inside the plasma is elliptically polarized mainly in the left-handed sense for the O-mode, while the electrons rotate in the right-handed sense. For nearly perpendicular propagation (i.e., for $k_{\parallel} \ll k_{\perp}$) and for weak damping, since $N_r^2 \approx (1 - \omega_p^2/\omega^2)$ for the O-mode and since the Poynting power flux density must be conserved, the right-handed component $(E_x - i E_y)$ inside the plasma must be

equal to $(-\cos\theta/\sin\theta)E_z [1 + O(\omega_p^2/\omega^2)]$, while the dominant left-handed component $(E_x + iE_y) \approx (1 + \omega/\omega_c) E_x$ (i.e., $E_y \approx -i\omega E_x/\omega_c$).

The time-averaged rate of power absorption $\langle P_{abs} \rangle$ is then obtained from the anti-Hermitian part $K_a = (1/2i)[K - K^*]$ of the dielectric tensor K , where K^* is the adjoint matrix. That is, using the incident fields we may write

$$\begin{aligned} \langle P_{abs} \rangle &\propto E^* \cdot K_a \cdot E = |E_z|^2 \{i_z - (\cos\theta/\sin\theta)i_x\} \cdot K_a \cdot \{i_z - (\cos\theta/\sin\theta)i_x\} \\ &= |E_z|^2 [\text{Im } K_{zz} - (2\cos\theta/\sin\theta)\text{Im } K_{zx} + (\cos\theta/\sin\theta)^2 \text{Im } K_{xx}], \quad (A2) \end{aligned}$$

where Im stands for the imaginary part, and we have made use of Eq. (A1). Equivalently, in terms of the wave fields inside the plasma we may write $\langle P_{abs} \rangle \propto (K_a)_{xx} |E_x - iE_y|^2 + 2(K_a)_{xz} \text{Re}\{(E_x - iE_y)E_z^*\} + (K_a)_{zz} |E_z|^2$, where (neglecting terms of order ω_p^2/ω^2 in comparison to unity) $(E_x - iE_y) \approx (-\cos\theta/\sin\theta)E_z$. The incident wave energy density

$$U \propto E^* \cdot E = |E_z|^2 [i_z - (\cos\theta/\sin\theta)i_x]^2 = |E_z|^2 / (\sin\theta)^2 \quad (A3)$$

Hence from Eqs. (A2) and (A3), the absorption coefficient η of the O-mode is

$$\eta \propto \langle P_{\text{abs}} \rangle / U$$

$$\propto (\sin\theta)^2 [\text{Im } K_{ZZ} - (2\cos\theta/\sin\theta) \text{Im } K_{ZX} + (\cos\theta/\sin\theta)^2 \text{Im } K_{XX}] \quad (\text{A4})$$

From Eq. (9) of page 189, Eq. (11) of page 190, and Eq. (15) of page 192 of Stix's book of reference 35 with $T_{\parallel} = T_{\perp} = T$ and $V_Z = 0$, we get

$$\text{Im } K = (\omega_p^2 / \omega \omega_c) \text{Re } M, \quad (\text{A5})$$

where the real part Re of the relevant elements of the mobility tensor M are

$$\text{Re } M_{ZZ} = D \beta_{\parallel}^2, \quad (\text{A6})$$

$$\text{Re } M_{ZX} = D \beta_{\parallel} (\omega_c / ck_X) = D \beta_{\parallel} (\omega_c / \omega \sin\theta), \quad (\text{A7})$$

$$\text{Re } M_{XX} = D (\omega_c / ck_X)^2 = D (\omega_c / \omega \sin\theta)^2, \quad (\text{A8})$$

and

$$D = -(m/2\pi T)^{3/2} (\omega_c c^2 / k_Z) \lambda \text{Re } F_0(\alpha_{-1}), \quad (\text{A9})$$

$$\beta_{\parallel} = (\omega - \omega_c)/ck_z = (\omega - \omega_c)/\omega \cos\theta \quad (\text{A10})$$

and

$$\lambda = (k_x^2 x_T / m \omega_c^2). \quad (\text{A11})$$

Thus, using Eqs. (A5) to (A11), Eq. (A4) becomes

$$\eta \propto D (\omega_p^2 / \omega \omega_c) (\sin\theta)^2 [\beta_{\parallel} - \omega_c \cos\theta / \omega (\sin\theta)^2]^2. \quad (\text{A12})$$

From Eq. (A10), we get

$$(\omega_c / \omega) = 1 - \beta_{\parallel} \cos\theta. \quad (\text{A13})$$

On making use of Eq. (A13), Eq. (A12) becomes

$$\eta \propto D (\omega_p^2 / \omega \omega_c) [(\cos\theta - \beta_{\parallel}) / \sin\theta]^2. \quad (\text{A14})$$

Thus we see that the absorption of the O-mode is proportional to the Trubnikov factor $F_T = [(\cos\theta - \beta_{\parallel}) / \sin\theta]^2$. By Kirchhoff's law the O-mode emission is also proportional to this same Trubnikov factor.

Finally, I have not been able to obtain this Trubnikov O-mode factor by solving for the imaginary part of k from the secular

determinant equal to zero of the plasma wave equation

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} + (\omega^2/c^2) \mathbf{K} \cdot \mathbf{E} = 0 \quad (\text{A15})$$

However, if one is clever enough, one must be able to obtain this O-mode Trubnikov factor F_T also from this plasma dispersion relation for cases where $\omega^2 \gg \omega_p^2$, since all these four methods: (1) Dirac's quantum theory of emission and absorption of radiation, (2) Classical electromagnetic theory of emission and absorption, (3) Time-averaged power absorption from the anti-Hermitian part K_a of the dielectric tensor divided by the energy density in the wave, and (4) The solution for $\text{Im } k$ from the plasma dispersion relation, must all yield the same result for the O-mode emission and absorption via Kirchhoff's law. It is interesting to note that Drummond and Rosenbluth in Phys. Fluids 6, 276 (1963), give the approximate expression for the O-mode absorption as

$$\begin{aligned} \eta &\propto \text{Re} \{ (M_{yy} + M_{zz}) - \{ (M_{yy} - M_{zz})^2 + 4 M_{yz}M_{zy} \}^{1/2} \} \\ &\approx 2 \text{Re} [M_{zz} - \{ M_{yz}M_{zy} / (M_{yy} - M_{zz}) \}], \end{aligned} \quad (\text{A16})$$

whereas Stix in a private communication gives this approximate expression as

$$\eta \propto \text{Im} [K_{ZZ} - (K_{XZ}K_{ZX} / K_{XX})]. \quad (\text{A17})$$

These two results are not the same, since the approximations used at arriving at them are quite different. Further, neither of these results will yield the Trubnikov factor even for situations where $\omega^2 \gg \omega_p^2$. This may imply that the approximations used both by Drummond and Rosenbluth and by Stix are too crude for situations where $\omega^2 \gg \omega_p^2$.

FIGURE 1

FIGURE 2