Radiation Damping of Betatron Oscillations

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ABSTRACT

The emission of synchrotron radiation damps the incoherent betatron oscillations of a pinched beam, causing its radius to shrink. However, the rate of shrinkage is small compared with the rate of expansion caused by scattering for typical propagation parameters.

I. Introduction

It has been occasionally suggested that the emission of incoherent synchrotron radiation would strongly damp the betatron oscillations of single particles in a pinched beam. This idea probably results from a confusion of the beam radius with the actual radius of curvature of a particle orbit, which is of order k_{β}^{-1} . To settle this point, a simple example of radiation induced shrinkage is worked through here. It is found that the radius (a) decreases according to the formula

$$\frac{1}{a^2} \frac{da^2}{dz} = - (34500 \text{ km})^{-1} \left(\frac{.3 \text{ cm}}{a_0} \right)^2 \left(\frac{\gamma_0}{100} \right)^2 \left(\frac{I}{10 \text{ kA}} \right)^2, \quad (1)$$

where a_0 and γ_0 are initial values of a and the relativistic factor γ , and I is net current. This scale length should be compared with the Nordisck length for expansion due to scattering

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$$L_N = (.0203 \text{ km}) \left(\frac{\lambda_R}{.3 \text{ km}} \right) \left(\frac{E}{50 \text{ MeV}} \right) \left(\frac{I}{10 \text{ kA}} \right). \quad (2)$$

II. Model Equations

A single particle subject only to an imposed electromagnetic force (F_{ex}) satisfies the "Dirac equation of motion." (1)

$$\frac{dp^\mu}{d\tau} = F_{ex}^\mu + F_{rad}^\mu, \quad (3)$$

where

$$(p^\mu) = (mc, \underline{p}) \quad (4)$$

is the four-momentum,

$$F_{ex}^\mu = \gamma q \left(\frac{\underline{v} \cdot \underline{E}}{c}, \underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right) \quad (5)$$

is the Lorentz four-force, τ is the proper time:

$$d\tau = dt / \gamma(t), \quad (6)$$

and

$$F_{rad}^\mu = \frac{2}{3} \frac{q^2}{mc^3} \left(\frac{d^2 p^\mu}{d\tau^2} + \frac{p^\mu}{m^2 c^2} \frac{dp^\nu}{d\tau} \frac{dp_\nu}{d\tau} \right) \quad (7)$$

is the radiation reaction four-force. Here we use the convention

$$p_\nu = (\gamma mc, -\underline{p}), \quad (8)$$

so we have the momentum invariant

$$P^{\nu} P_{\nu} = (\gamma mc)^2 - |\underline{P}|^2 = (mc)^2. \quad (9)$$

Equation (3) conserves this invariant (in time as well as under Lorentz transformation). The quantities q , m , and c have their usual meaning of charge, mass and speed of light. Velocity and momentum are related by

$$\underline{P} = \gamma m \underline{v}, \quad (10)$$

so Eq. (9) yields

$$= (1 - |\underline{v}|^2/c^2)^{-1/2}. \quad (11)$$

Equation (3), written in explicit terms is

$$\begin{aligned} \gamma \frac{d}{dt} (\gamma mc, \underline{P}) = \gamma q \left(\frac{\underline{v} \cdot \underline{E}}{c}, \underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right) \\ + \frac{2}{3} \frac{q^2}{mc} \left\{ \gamma \frac{d}{dt} \gamma \frac{d}{dt} (\gamma mc, \underline{P}) + \frac{(\gamma mc, \underline{P})}{m^2 c^2} \right. \\ \left. \times \gamma^2 \left[\left(\frac{d\gamma mc}{dt} \right)^2 - \left| \frac{d\underline{P}}{dt} \right|^2 \right] \right\}. \end{aligned} \quad (12)$$

The simplest beam model which is self consistent and contains the radiation phenomena is an infinitely long rod of constant current (I) and variable radius $a(t)$. There is no variation of any quantity with the longitudinal coordinate (z) or azimuth (θ). The only field components are $B_{\theta}(r,t)$ and $E_z(r,t)$, which are related by Maxwell's equations:

$$c \frac{\partial E_z}{\partial r} = \frac{\partial B_\theta}{\partial t} \quad (13)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r B_\theta = \frac{4\pi}{c} J_z + \frac{1}{c} \frac{\partial E_z}{\partial t} \quad (14)$$

The assumption that I is constant and that J_z vanish outside radius a implies that there are small secondary (plasma) currents flowing in the r and z directions to compensate for changes in the beam current. This is incorporated into J_z . There is no problem with E_r and net J_r vanishing.

The dynamics of a single particle which resides at the beam edge are studied. In the absence of any radiation reaction, its orbit would be a helix of radius a , with centrifugal force balancing the pinch force from the current. Due to radiation reaction this particle has a radial velocity

$$v_r = \frac{da}{dt} \quad (15)$$

which we compute here. It can be assumed that the particle of interest continues to define the beam edge despite the radial motions of other particles, e.g., the beam may be comprised entirely of helical orbits which shrink in a laminar pattern.

It will be seen that displacement current may be ignored and Eq. (14) yields for the magnetic field experienced by the particle at radius a

$$B_\theta(a) = \frac{2I}{a} \quad (16)$$

For radii greater than a we have

$$B_\theta(r > a) = \frac{2I}{r} \quad (17)$$

which is constant in time. Hence

$$E_z(a) = \int_{-\infty}^{\infty} dr' \frac{1}{c} \frac{\partial}{\partial t} B_{\theta}(r', t) = 0. \quad (18)$$

The electric field does not vanish in the beam interior and needs to be taken into account if we studied the dynamics of an interior particle. However, the displacement current can still be neglected in computing B_{θ} .

III. Quasi Static Approximation

The general equation of motion [Eq. (12)] is too complicated for analytical treatment, but we can greatly simplify it by assuming that the radiation effects are small on the time scale of a betatron oscillation (in the present case, this is a single helical revolution of the bounding particle). Formally, an expansion in the small quantity

$$\epsilon \propto \frac{2}{3} \frac{q^2}{mc^3} \quad (19)$$

is made. In the absence of radiation reaction we would have $P_r = 0$, and $(\gamma, P_{\theta}, P_z, a)$ constant. Hence with radiation included we order

$$P_r = O(\epsilon), \quad (20a)$$

$$\frac{d}{dt} (\gamma, P_{\theta}, P_z, a) = O(\epsilon), \quad (20b)$$

$$\frac{dP_r}{dt} = O(\epsilon^2). \quad (20c)$$

Since E_z is induced by changes in the current profile (in the present model), it is of order ϵ and the displacement current is of order ϵ^2 ; this justifies the use of the relation (16). Note that

$$\left(\frac{d\tilde{p}}{dt}\right)_r = \frac{dp_r}{dt} - \frac{p_\theta v_\theta}{a}, \quad (21a)$$

i.e., it is the sum of $O(\epsilon^2)$ and $O(1)$ terms.

Similarly, the second derivative contains an $O(1)$ term:

$$\left(\frac{d^2\tilde{p}}{dt^2}\right)_\theta = -p_\theta \left(\frac{v_\theta}{a}\right)^2 + O(\epsilon^2). \quad (21b)$$

Dropping terms which are small of $O(\epsilon^2)$, Eq. (12) becomes

$$\gamma \frac{d}{dt} (\gamma mc) = -\frac{2}{3} \frac{q^2}{mc^3} \left(\gamma mc \frac{\gamma^2}{m^2 c^2} \left| \frac{d\tilde{p}}{dt} \right|^2 \right), \quad (22a)$$

$$-\gamma \frac{p_\theta v_\theta}{a} = -\gamma q \frac{v_z}{c} B_\theta, \quad (22b)$$

$$\gamma \left(\frac{dp_\theta}{dt} + p_r \frac{v_\theta}{a} \right) = -\frac{2}{3} \frac{q^2}{mc^3} \left[\gamma^2 p_\theta \left(\frac{v_\theta}{a} \right)^2 + p_\theta \frac{\gamma^2}{m^2 c^2} \left| \frac{d\tilde{p}}{dt} \right|^2 \right], \quad (22c)$$

$$\gamma \frac{dp_z}{dt} = \gamma q \frac{v_r}{c} B_\theta - \frac{2}{3} \frac{q^2}{mc^3} \left(p_z \frac{\gamma^2}{m^2 c^2} \left| \frac{d\tilde{p}}{dt} \right|^2 \right), \quad (22d)$$

with

$$\left| \frac{d\tilde{p}}{dt} \right|^2 = \left(\frac{p_\theta v_\theta}{a} \right)^2. \quad (23)$$

The momentum invariant becomes

$$(\gamma mc)^2 - p_\theta^2 - p_z^2 = m^2 c^2, \quad (24)$$

which is conserved by Eqs. (22a - d) (verification is left to the reader).

IV. Solution in Terms of Dimensionless Variables

Equations (22a) - (22d) still look complicated, but they are "cleaned up" by defining dimensionless quantities as follows. Let a_0 be the radius at time $t = 0$; then there is a dimensional time associated with radiation

$$T_0 = \left(\frac{2}{3} \frac{c}{a_0^2} \frac{q^2}{mc^2} \right)^{-1}, \quad (25)$$

and we let

$$t/T_0 \rightarrow t, \quad (26a)$$

$$a/a_0 \rightarrow a. \quad (26b)$$

Similarly let

$$\underline{p}/mc \rightarrow \underline{p} \quad (27)$$

and define the constant

$$K^2 = \frac{q I}{mc^3}. \quad (28)$$

We have from Eq. (22)

$$\frac{d\gamma}{dt} = -\frac{p_{\theta}^4}{a^2} \quad , \quad (29a)$$

$$p_{\theta}^2 = 2K^2 p_z \quad , \quad (29b)$$

$$\gamma \left(\frac{dp_{\theta}}{dt} + \frac{p_{\theta}}{a} \frac{da}{dt} \right) = -\frac{p_{\theta}^3 (1+p_{\theta}^2)}{a^2} \quad , \quad (29c)$$

$$\gamma \left(\frac{dp_z}{dt} - \frac{2K^2}{a} \frac{da}{dt} \right) = -\frac{p_z p_{\theta}^4}{a^2} \quad . \quad (29d)$$

The invariant, Eq. (24), becomes

$$\gamma^2 - p_{\theta}^2 - p_z^2 = 1 \quad . \quad (30)$$

Equations (29a - d) can be solved by direct integration (note that Eqs. (29b) and (30) can be used to eliminate p_{θ} and γ immediately). However, the solution is unnecessarily complicated for the situation of interest, which is the highly relativistic, nearly paraxial limit. We make use of the further ordering

$$\gamma \gg 1 \quad , \quad (31a)$$

$$K = O(1) \quad . \quad (31b)$$

In lowest order of γ^{-1} , Eqs. (29) and (30) yield

$$p_z = \gamma \quad , \quad (32)$$

$$p_{\theta}^2 = 2K^2 \gamma \quad . \quad (33)$$

$$\frac{da^2}{dt} = -4K^4 \gamma \quad , \quad (34)$$

$$\frac{d\gamma}{dt} = -\frac{4K^4}{a^2} \gamma^2 \quad . \quad (35)$$

Equation (34) is obtained from either Eq. (29c) or (29d) but some care must be taken in eliminating low order terms.

Taking the ratio of Eqs. (34) and (35) we obtain

$$\frac{da^2}{d\gamma} = \frac{a^2}{\gamma} \quad , \quad (36)$$

or

$$\frac{d \log a^2}{d \log \gamma} = 1 \quad , \quad (37)$$

which has solution

$$\frac{a^2}{\gamma} = \text{constant} = \frac{a_0^2}{\gamma_0} \quad . \quad (38)$$

We note that the normalized emittance varies with a and γ as

$$\epsilon_N^2 \propto p_\theta^2 a^2 \propto \gamma a^2 = \gamma^2 \frac{a_0^2}{\gamma_0} \quad . \quad (39)$$

The usual (unnormalized) emittance is

$$\epsilon^2 = \frac{\epsilon_N^2}{\gamma^2} = \text{constant} \quad . \quad (40)$$

A characteristic time scale for radiation effects (τ_r) is defined as

$$\tau_r^{-1} = -\frac{1}{\gamma} \frac{d\gamma}{dt} = -\frac{1}{a^2} \frac{da^2}{dt} = \frac{4K^4 \gamma}{a^2} \quad . \quad (41)$$

In real units this time is given by

$$\tau_r^{-1} = \frac{4K^4 \gamma_0 T_0^{-1}}{(a/a_0)^2} = 4 K^4 \gamma_0 T_0^{-1}, \quad (42)$$

where we have used Eq. (38) to eliminate a^2/γ . Hence τ_r is independent of t and the solution of the model equations is

$$\frac{a^2}{a_0^2} = \frac{\gamma}{\gamma_0} = \exp\left(-\frac{t}{\tau_r}\right). \quad (43)$$

The actual value of τ_r is found for typical ATA parameters ($a_0 = .3$ cm, $\gamma_0 = 100$, $I = 10$ kA) to be

$$\begin{aligned} \tau_r &= \frac{T_0}{4K^4 \gamma_0} = \left(\frac{3}{2} \frac{a_0^2}{c} \frac{mc^2}{q^2}\right) \left(\frac{1}{4\gamma_0}\right) \left(\frac{mc^3}{qI}\right)^2 \\ &= \left(\frac{3}{2}\right) \left(\frac{.3 \text{ cm}}{3 \times 10^{10} \text{ cm/s}}\right)^2 \left(2.82 \times 10^{-13} \text{ cm}\right)^{-1} \left(\frac{1}{4 \times 100}\right) \left(\frac{17 \text{ kA}}{10 \text{ kA}}\right)^2 \\ &= .115 \text{ s} \end{aligned} \quad (44a)$$

This corresponds to the length

$$c\tau_r = 34500 \text{ km} \quad (44b)$$

More generally

$$c\tau_r = (34500 \text{ km}) \left(\frac{a_0}{.3 \text{ cm}}\right)^2 \left(\frac{100}{\gamma_0}\right) \left(\frac{10 \text{ kA}}{I}\right)^2, \quad (45)$$

which is probably small for any parameter range of interest.

Reference

1. D. Jackson, Classical Electrodynamics 2nd Ed., 1975, Pg. 808.

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