NOVEMBER 1981

PPPL-1849
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**SHEAR-BALLOONING STABILITY
ANALYSIS OF LOW- β PLASMAS**

BY

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Shear-Ballooning Stability Analysis of Low- β Plasmas

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Abstract

A model of magnetic configurations that has both shear and variation of the field curvature along the magnetic field is constructed. The stability analysis of this system is performed by expanding the perturbation function to ϕ_0 which is constant along the field and ϕ_1 which is sinusoidally varied along the field line. The shear term is retained in the form of differential operations in ξ , the coordinate parallel to the pressure gradient.

After several simplifications, the equation reduces itself to the fourth order differential equation. The eigenvalue of this equation is numerically obtained. The calculated critical β plotted versus the well depth parameter, h , makes a smooth transition between $h < 0$ (maximum average B) and $h > 0$ (minimum average B). Using the same technique, the nondivergent solution to the localized shear mode (Suydam mode) is also obtained by retaining the inertia term.

I. Introduction

The magnetohydrodynamic stability problem of a complicated magnetic geometry such as stellarators is usually solved by the energy principle method. Unfortunately the energy principle is sufficiently complex so that a clear-cut physical picture is hard to obtain. Hence simplified concepts are often substituted to aid in understanding of the various aspects of the stability problem. For example, the shear criterion^{1,2} and ballooning mode analysis^{3,4} are very useful concepts in isolating the main components of the destabilizing and stabilizing effects. These concepts help people to come up with new magnetic configurations which may have higher stability β , β_c , for example. The approach of isolating components obviously has a fault; that is, the contribution of other components may be unjustifiably ignored. For example, in the shear calculation, one may neglect the contributions from ballooning or the finite β effect. On the other hand, the full energy principle treatment may not be practical in a number of magnetic configurations such as stellarators where the three-dimensional nature of the problem makes the calculation very difficult.

In this paper the author aims to combine the shear criterion and ballooning mode effect in a simplified geometry. Thus we may be able to assess the interplay between the shear length, L_s , and the connection length, L_c , both of which are important concepts in designing a stable magnetic confinement device. This model,

however, is not complete in that both the diamagnetic effect and the geodesic curvature effect [due to the nonzero contribution of $(\vec{B} \times \vec{\nabla} B) \times \vec{\nabla} p$] are not included. However, in the low β case, this assumption is probably valid.

II. Model

We introduce a rectangular coordinate system with $\partial/\partial y = 0$. The magnetic field $\vec{B} = (0, B_y, B_z)$ with $B_y = B_0(x/L_s)$, $B_0 = \text{constant}$, and $\partial B_z/\partial x = f(x)$. The perturbation is small enough so that $B_z \approx B_0 (>0)$ near $x = 0$, where B_0 is independent of the spatial coordinate. Here L_s is taken to be constant. The connection length L_c is defined as

$$f(z) = \frac{B_0}{R_c} \left[h + \cos\left(\frac{2\pi z}{L_c}\right) + a_2 \cos\left(\frac{4\pi z}{L_c}\right) + \dots \right], \quad (1)$$

$$R_c > 0, \quad B_0 > 0. \quad (2)$$

In this paper we assume $a_2 = \dots = 0$.

The starting equations are two-fluid equations plus Maxwell's equations. These can be reduced to

$$0 = \vec{\nabla}_\perp \cdot \vec{J}_\perp - \vec{\nabla}_\perp \cdot \frac{\vec{\nabla} p \times \vec{B}}{B^2}, \quad (3)$$

$$0 = \vec{E}_\perp = -\vec{\nabla}_\perp \phi - \frac{\partial \vec{A}_\perp}{\partial t}, \quad (4)$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}_n = -\nabla^2 \vec{A}_n = \mu_0 \vec{J}_n, \quad (5)$$

$$0 = \frac{\partial n}{\partial t} + \vec{v}_{n_0} \cdot \frac{\vec{E} \times \vec{B}}{B^2}, \quad (6)$$

$$\vec{P}_1 = -\vec{v}_1, \quad (7)$$

Here n_0 and $\vec{B}$ are unperturbed quantities. If we further assume that the temperatures are independent of position,

$$p = n(kT_0 + kT_1) = nkT. \quad (8)$$

The perturbed quantities i , etc., may be expressed as

$$i = i(x, z) \exp(ik_y y + st). \quad (9)$$

With the usual assumption that $\frac{\partial^2}{\partial x^2} \gg \frac{\partial^2}{\partial z^2}$, we can for eqs. (1) through (9), for the marginal stability case, write

$$i = -\frac{1}{2} \frac{\partial^2}{\partial x^2} i + \frac{1}{2} \frac{\partial^2}{\partial z^2} i + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\frac{1}{B^2} \right) k_y^2 i.$$

We note that if $B_z = B_y$, i_n is

$$i_n = \frac{\partial^2}{\partial z^2} + ik_y \frac{x}{L_s}. \quad (11)$$

Thus

$$0 = \left(\frac{\partial}{\partial z} + ik_y \frac{x}{L_s} \right) \left(\frac{\partial^2}{\partial x^2} - k_y^2 \right) \left(\frac{\partial}{\partial z} + ik_y \frac{x}{L_s} \right) \phi$$

$$- 2\mu_o \frac{dp_o}{dx} \frac{1}{B_o^2} \frac{1}{R_c} \left(h + \cos \frac{2\pi z}{L_c} \right) k_y^2 \phi . \quad (12)$$

We define

$$\xi = k_y x , \quad (13)$$

$$\frac{\beta_c}{a} = - \frac{2\mu_o}{B_o^2} \frac{dp_o}{dx} . \quad (14)$$

Then

$$0 = \left(\frac{\partial}{\partial z} + i \frac{\xi}{L_s} \right) \left(\frac{\partial^2}{\partial \xi^2} - 1 \right) \left(\frac{\partial}{\partial z} + i \frac{\xi}{L_s} \right) \phi$$

$$+ \frac{\beta_c}{a R_c} \left(h + \cos \frac{2\pi z}{L_c} \right) \phi . \quad (15)$$

To simplify Eq. (15) we express

$$\phi = \phi_o(x) + \phi_{+1}(x) \exp i \frac{2\pi z}{L_c} + \phi_{-1}(x) \exp -i \frac{2\pi z}{L_c}$$

$$+ \phi_{+2}(x) \exp i \frac{4\pi z}{L_c} + \phi_{-2}(x) \exp -i \frac{4\pi z}{L_c} + \dots \quad (16)$$

Then Eq. (15) can be simplified to give

$$0 = -\frac{\tau}{L_S} \left(\frac{\eta^2}{2} - 1 \right) \frac{\xi}{L_S} \phi_0 + \frac{\beta_C h}{aR_C} \phi_0 + \frac{1}{2} \frac{\rho_C}{aR_C} \phi_{+1} + \frac{1}{2} \frac{\rho_C}{aR_C} \phi_{-1}, \quad (17)$$

$$0 = -\left(\frac{2\tau}{L_C} + \frac{\tau}{L_S} \right) \left(\frac{\eta^2}{2} - 1 \right) \left(\frac{2\tau}{L_C} + \frac{\tau}{L_S} \right) \phi_{+1} + \beta_C \frac{h}{aR_C} \phi_{+1} + \frac{1}{2} \frac{\rho_C}{aR_C} \phi_0 + \frac{1}{2} \frac{\rho_C}{aR_C} \phi_{+2}, \quad (18)$$

and so on.

III. Solution

If $2\tau/L_C \ll 1/L_S$, we may approximate $2\tau/L_C + \tau/L_S$ by τ/L_S . Then Eq. (18) is identical to Eq. (17) except for a difference in the suffix of ϕ . Approximately we let $\phi_{+2} = \phi_{+1}$. Then either $\phi_0 = \phi_{+1}$ or $\phi_0 = -\phi_{+1}$ will satisfy Eqs. (17) and (18). The former yields no convergent solution at $\eta \rightarrow \infty$, but the latter leads to the shear stability equation.¹⁻²

$$0 = \left(\tau^2 \frac{\eta^2}{2} + 2\tau \frac{d}{d\eta} + \frac{\beta_C L_S^2}{aR_C} (1-h) - \tau^2 \right) \phi. \quad (19)$$

The existence of $\phi(\xi)$ localized near $\eta = 0$ requires (Suydam criterion)

$$\beta_C = \frac{aR_C}{4(1-h)L_S^2}. \quad (20)$$

As is the case with Suydam, the solution of Eq. (19) at $\xi = 0$ is singular. For β_c given in Eq. (20), ϕ is

$$\phi = \frac{1}{\xi^{1/2}} K_0(\xi) , \quad (21)$$

where K_0 is the Bessel function with an imaginary argument. Thus ϕ is large in $|\xi| \sim 0(1)$. Hence if $2\pi/L_c \ll 1/L_s$, we are justified in letting

$$\frac{2\pi}{L_c} \ll \frac{|\xi|}{L_s} . \quad (22)$$

In the opposite limit where $2\pi/L_c \gg |\xi|/L_s$, Eq. (18) can be rewritten as

$$\begin{aligned} 0 = & -\frac{4\pi^2}{L_c^2} \left(\frac{\gamma^2}{\partial \xi^2} - 1 \right) \phi_{+1} + \beta_c \frac{h}{aR_c} \phi_{+1} + \frac{1}{2} \frac{\beta_c}{aR_c} \phi_0 \\ & + \frac{1}{2} \frac{\beta_c}{aR_c} \phi_{+2} . \end{aligned} \quad (23)$$

Truncating at ϕ_{+1} , thus ignoring ϕ_{+2} , we obtain a closed set of equations, Eq. (17) and Eq. (23), to determine ϕ_0 and ϕ_{+1} . Except for Eq. (17) where ξ^2 appears, the set does not contain ξ . Thus eliminating ϕ_{+1} , we obtain

$$\begin{aligned} 0 = & \left[\left(\frac{2\pi L_s}{L_c} \right)^2 \nabla^2 - \frac{h}{4} \frac{\beta_c}{\beta_0} \right] \xi \nabla^2 \xi \phi_0 - \frac{1}{32} \frac{\beta_c^2}{\beta_0^2} \phi_0 \\ & - \frac{h}{4} \frac{\beta_c}{\beta_0} \left[\left(\frac{2\pi L_s}{L_c} \right)^2 \nabla^2 - \frac{h}{4} \frac{\beta_c}{\beta_0} \right] \phi_0 , \end{aligned} \quad (24)$$

$$\nabla^2 = \frac{\beta^2}{\beta_1^2} - 1, \quad (25)$$

$$\beta_0 = \frac{aR_c}{4L_s^2}. \quad (26)$$

The condition that ϕ_0 which satisfies Eq. (24) must be bounded at $r = \infty$ determines β_c/β_0 in terms of $2\pi L_s/L_c$ (21) and h . The numerical solution is given in Appendix I.

It is instructive to derive limiting cases. If $2\pi L_s/L_c \rightarrow \infty$ and β_c remains finite,

$$0 = -\frac{1}{2} \frac{\beta_c^2}{\beta_1^2} \phi_0 - h \frac{\beta_c}{\beta_1} \nabla^2 \phi_0 + \left(\frac{L_c}{2\pi L_s} \right)^2 \nabla^2 \nabla^2 \phi_0, \quad (27)$$

or

$$0 = -\frac{1}{2} \frac{\beta_c^2}{\beta_1^2} \phi_0 - h \frac{\beta_c}{\beta_1} \nabla^2 \phi_0 = 0 \left(\frac{L_c}{2\pi L_s} \right)^2, \quad (28)$$

with

$$\beta_1 = \frac{aR_c}{(L_c/2\pi)^2}. \quad (29)$$

Since $(1/\beta_1)^2$ can be made arbitrarily small [provided inequality (22) is satisfied], we obtain

$$\beta_c (\equiv \beta_b) = \frac{2haR_c}{(L_c/2\pi)^2}, \quad (30)$$

which is the critical β for the ballooning mode.

In the limit of $h = 0$, Eq. (24) becomes

$$0 = \left(\frac{2\pi L_s}{L_c} \right)^2 \psi^2 \phi_0^2 - \frac{1}{32} \frac{\beta_c^2}{\beta_0^2} \phi_0^2. \quad (31)$$

The numerical solution yields (see Appendix I)

$$\beta_c = A \frac{2\pi R_c}{L_c L_s}, \quad (32)$$

$$A = 1.35. \quad (33)$$

Eigenfunctions ϕ_0 and ϕ_1 are shown in Fig. 1.

IV. Conclusions and Discussion

In Table 1, the numerical results of β_c from Eq. (24) are shown. As expected, β_c is greater than β_b given in Eq. (30). Also, β_c is less than the critical β_s computed assuming the variation in the curvature along the magnetic field line is averaged out. It is important to note that if the averaged field curvature or second derivative of the volume within a magnetic surface is used as the stability criterion, the critical β thus estimated is higher than the actual stability β . The criterion becomes meaningless especially when the average curvature approaches zero. This is demonstrated in Fig. 2 for $L_c/2\pi = L_s/3$. Three β 's (β_c , β_s , β_b) are plotted versus h , the well parameter. The stability β_c smoothly bridges the two regions, $h > 0$ and $h < 0$.

The technique used to solve Eq. (24) can be applied to the shear stability equation with the electron inertia effect included. This is appended in Appendix II.

Acknowledgment

This work was supported by US Department of Energy Contract No. DC-AC02-76-CHO-3073.

Table 1. β_c calculated from Eq. (24).

(Shear critical β , β_s , and ballooning critical β , β_b , are also shown.)

$\frac{1}{2\pi} \frac{L_c}{L_s}$	h	$\frac{L_{sc}^2}{aR_c}$	$\frac{L_{ss}^2}{aR_c}$	$\frac{L_{sb}^2}{aR_c}$
0.5	-0.2	0.94	1.25	
	0	2.7		
	0.2	5.21		1.6
1/3	-0.5	0.40	0.5	
	-0.2	0.98	1.25	
	-0.1	1.798	2.5	
	-0.04	2.99	6.25	
	0	4.05		
	0.04	4.08		0.72
	0.1	6.33		1.8
	0.2	9.18		3.6
	0.3	12.84		5.4
	0.5	18.78		9.0
0.1	-0.2	1.01	2.5	
	0	13.5		
	0.05	25.33		10
	0.1	36.7		20
	0.2	49.96		40

APPENDIX I

In solving Eq. (24) numerically, we use a modified form of the set of Eqs. (17) and (23). Thus,

$$0 = -\epsilon^2 \psi_0 + h \psi_0 + \psi_1, \quad (I-1)$$

$$0 = -\epsilon^2 \psi_1 + h \psi_1 + \frac{1}{2} \psi_0. \quad (I-2)$$

We let

$$\psi_1 = \psi_{+1} = \psi_{-1}, \quad \psi_2 = 0, \quad (I-3)$$

and

$$\epsilon = \left(\frac{L_c}{3-L_s} \right)^2, \quad (I-4)$$

$$h = \frac{L_s^2}{a R_c}, \quad (I-5)$$

and ϵ^2 is defined in Eq. (25).

Near the origin Eqs. (I-1) and (I-2) admit four independent solutions. They are (for $h \neq 0$)

$$\psi_{01} = 1 + A_1 \epsilon^2 + \dots, \quad (I-6)$$

$$\phi_{02} = \xi + A_2 \xi^3 + \dots, \quad (I-7)$$

$$\phi_{03} = \xi^{v_1} (1 + A_3 \xi^2 + \dots), \quad (I-8)$$

$$\phi_{04} = \xi^{v_2} (1 + A_4 \xi^2 + \dots), \quad (I-9)$$

$$v_1 = -\frac{1}{2} + \frac{1}{2} (1 + 4h)^{1/2}, \quad (I-10)$$

$$v_2 = -\frac{1}{2} - \frac{1}{2} (1 + 4h)^{1/2}. \quad (I-11)$$

The singularity at $\xi = 0$ originates from the assumption that electrons are massless (see Appendix II). Thus we suppose that in general the singularity can be removed by the full complement of equations. Thus ϕ_0 , which is symmetric with respect to τ , must be ϕ_{01} and a linear combination,

$$C_3 \phi_{03} + C_4 \phi_{04}.$$

In the regime $h < 0$, we impose the condition that $1 + 4h > 0$. If we can find an eigenfunction with this constraint (there exists such an eigenfunction), then the constraint is justified. Then everywhere v_1 and v_2 are real and $v_2 < v_1$. Therefore if C_4 is nonzero near $\tau = 0$, ϕ_{04} dominates. Thus the eigenfunction is searched first with $\phi_0 = \phi_{04}$, and only when ϕ cannot be found is $\phi = \phi_{03}$ used. It turns out that $\phi_0 \approx \phi_{03}$ at $\xi \sim 0$ is necessary for $h \leq 0$.

The numerical procedure is then straightforward. First ϕ_0 and ϕ_1 [through Eq. (I-1) and (I-2)] are calculated near the origin for two independent functions, and then two numerical

solutions are obtained. At a suitable choice of $\xi_{\max}$ (10 was chosen), the linear combination of these two solutions is made to vanish. Then if λ is changed to let the first derivative go to zero, it corresponds to the eigenvalue λ . For $h > 0$, except near $h = 0$, the choice of ϕ_{01} and ϕ_{04} yields an eigenfunction with eigenvalues tabulated in Table I. In the $h < 0$ region the choice of ϕ_{01} and ϕ_{04} yields the eigenfunction which is zero everywhere ($\lambda = 0$). This is discarded. The combination of ϕ_{01} and ϕ_{03} , on the other hand, yields a smooth eigenfunction as shown in Fig. 3a. Also shown in Fig. 3b is the case where $\alpha = 1/3$, $h = 0.1$.

The case for $h = 0$ is anomalous. In this case $\psi_1 = 0$ and $\psi_2 = 1$. The root $\gamma_2 = -1$ yields an antisymmetric solution. Thus the solution near the origin is

$$\psi = C_1 + C_2 \log |\xi| . \quad (I-12)$$

The solution is then constructed as discussed previously.

The results are plotted in Fig. 2.

APPENDIX II

If the inertia term is included in the two-fluid equations and the equations are linearized, we have instead of Eq. (12) of Ref. 2 (n is replaced by ϕ)

$$0 = \xi \nabla^2 \left(1 + \frac{C^2 k y^2}{2 \omega_{pe}^2} \nabla^2 \right) \xi \phi + M^2 \phi, \quad (\text{II-1})$$

where ∇^2 is defined in Eq. (25) of the main text and M^2 in the present notation is

$$M^2 = \frac{\beta C^2 s^2}{a R_C}. \quad (\text{II-2})$$

If we introduce $\phi_0 = \phi$, and ϕ_1 as

$$0 = \frac{C^2 k y^2}{2 \omega_{pe}^2} \nabla^2 \xi \phi_0 + \xi \phi_0 - M \phi_1, \quad (\text{II-3})$$

Eq. (II-1) can be written as

$$0 = \xi \nabla^2 \phi_1 + M \xi \phi_0. \quad (\text{II-4})$$

The coefficient $C^2 k y^2 / \omega_{pe}^2 \ll 1$, as the classical skin depth of the plasma, C / ω_{pe} is much less than the ion Larmor radius which, in turn, is smaller than the wavelength $1 / ky$.

Thus the coupled Eqs. (II-3) and (II-4) are solved near $r = 0$ where ϕ , in the absence of $C^2 k_y^2 / \omega_{pe}^2$, has singularity. Near the origin four solutions are

$$\phi_1 = 1 + A_1 \xi^2 \dots, \quad (\text{II-5})$$

$$\phi_1 = \xi + A_2 \xi^3 \dots, \quad (\text{II-6})$$

$$\phi_1 = M \log |\xi| (1 + A_3 \xi^2 + \dots), \quad (\text{II-7})$$

$$\phi_1 = -M \log |\xi| (\xi + A_4 \xi^3 + \dots). \quad (\text{II-8})$$

We use this starting solution to solve the shear equation for finite ξ , that is

$$\xi \nabla^2 \xi \phi_0 + M^2 \phi_0 = 0. \quad (\text{II-9})$$

Equation (II-5) admits $\phi_0 \sim \xi$. This is anti-symmetric in ξ , so it is not the solution for the lowest eigenfunction.

Equation (II-6) allows $\phi_0 \sim \xi^2$. Since both the initial value and the initial derivative are zero, this is not the correct solution for Eq. (II-9). Equation (II-7) has $\phi_0 \sim 1/\xi$ which is divergent at the origin and is thus spurious. Finally, Eq. (II-8) has the right kind of behavior near the origin and a nonvanishing initial value. To the first two significant terms near $\xi = 0$,

$$\phi_0 = 1 - \frac{C^2 k_y^2}{\omega_{pe}^2} \frac{M^2}{6} \xi^2 \log |\xi| . \quad (\text{II-10})$$

Numerical integration starting at $\xi = (1/2) (|Ck_y/\omega_{pe}|)$ and imposing the condition that $\phi_0 \rightarrow 0$ as $\xi \rightarrow \infty$, yields the following values.

$\frac{Ck_y}{\omega_{pe}}$	M
0.1	0.876
0.05	0.79
0.02	0.7242
0 (Suydam criterion)	0.5

The eigenfunction for $Ck_y/\omega_{pe} = 0.02$ is shown in Fig. 4.

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Figure Captions

Fig. 1. Eigenfunctions ϕ_0 and ϕ_1 for $h = 0$.

Fig. 2. Calculated β plotted versus h , well depth. Also plotted are the critical β from shear, β_s , and from the ballooning mode, β_b . $L_C/2\pi L_S = 1/3$.

Fig. 3a. Eigenfunction for $L_C/2\pi L_S = 1/3$ and $h = -0.1$.

Fig. 3b. The same as Fig. 3a except that $h = 0.1$.

Fig. 4. Eigenfunction of Suydam-type mode. $Ck_y/\omega_{pe} = 0.02$.

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Fig. 1

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Fig. 2

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Fig. 3

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Fig. 4