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MULTIPLE TIME SCALE METHODS IN TOKAMAK MAGNETOHYDRODYNAMICS

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ABSTRACT

Several methods are discussed for integrating the magnetohydrodynamic (MHD) equations in tokamak systems on other than the fastest time scale. The dynamical grid method for simulating ideal MHD instabilities utilizes a natural nonorthogonal time-dependent coordinate transformation based on the magnetic field lines. The coordinate transformation is chosen to be free of the fast time scale motion itself, and to yield a relatively simple scalar equation for the total pressure, $P = p + B^2/2\mu_0$, which can be integrated implicitly to average over the fast time scale oscillations. Two methods are described for the resistive time scale. The zero-mass method uses a reduced set of two-fluid transport equations obtained by expanding in the inverse magnetic Reynolds number, and in the small ratio of perpendicular to parallel mobilities and thermal conductivities. The momentum equation becomes a constraint equation that forces the pressure and magnetic fields and currents to remain in force balance equilibrium as they evolve, $\vec{\nabla} p = \vec{J} \times \vec{B}$. The large mass method artificially scales up the ion mass and viscosity, thereby reducing the severe time scale disparity between wavelike and diffusionlike phenomena, but not changing the resistive time scale behavior. Other methods addressing the intermediate time scales are discussed.

I. INTRODUCTION

A. Scope of work

A high temperature magnetized plasma is an exceedingly complicated medium with many different resonances and fundamental modes of vibration. Natural time scales for phenomena occurring in a tokamak plasma near reactor parameters range from the subpicosecond period of an electron gyrating about the magnetic field to the several seconds it takes for the toroidal current to penetrate into the discharge. It is evident that to make progress in understanding and describing the behavior of the plasma in a device such as a tokamak, one must be fully aware of the time scales involved and of the nature of the phenomena occurring on each time scale.

The most basic set of equations describing the interaction of an ionized plasma with an electromagnetic field are the combined Maxwell and Vlasov-Fokker-Planck equations for the electric and magnetic fields and for the six-dimensional phase space distribution function. These equations are necessary for describing high frequency phenomena involving particle orbits, wave-particle resonances, and the evolution of highly nonMaxwellian distribution functions. However, if we restrict our attention to relatively low frequency macroscopic phenomena which does not depend on the details of the velocity space distribution function, then by taking appropriate velocity moments of the kinetic equations we obtain a closed set of fluid and field equations, the magnetohydrodynamic (MHD) equations. The subject of this article is the numerical simulation of phenomena described by these MHD equations. We shall give particular emphasis to the application of the MHD equations to the description of phenomena occurring in tokamak fusion experiments.

There presently exists a substantial amount of accumulated literature describing methods for the numerical simulation of various phenomena occurring

in tokamak discharges. The present article will further restrict itself to exclude discussing numerical methods based on solving model equations obtained by performing an asymptotic expansion in the geometrical parameter a/R , the toroidal aspect ratio. A large amount of literature exists on this subject and the interested reader is referred to several excellent articles (Strauss, 1976; Potter, 1976; Strauss, 1983).

The remainder of this introductory section is devoted to describing the magnetohydrodynamic equations, and the time scales that are present when these equations are applied to describe the physics of the tokamak (Furth, 1975). We then state the general principles which are developed in the subsequent sections. Section II outlines a method for describing the fastest time scale of interest in MHD, the transverse ideal time scale. In Sec. III we describe two methods for computing the slowest time scale of interest in MHD, the resistive or transport time scale. The final section briefly discusses ways of treating the intermediate MHD time scales and reviews the literature concerning certain techniques not discussed here.

B. Fundamental equations

The basic equations describing the time evolution of a magnetically confined plasma are comprised of field equations (Maxwell's equations ignoring the displacement current)

$$\partial \vec{B} / \partial t = - \nabla \times \vec{E} \quad , \quad (1a)$$

$$\nabla \times \vec{E} = \mu_0 \vec{j} \quad , \quad (1b)$$

$$\nabla \cdot \vec{B} = 0 \quad , \quad (1c)$$

and fluid equations which are appropriate velocity moments of the Vlasov-Fokker-Planck equations for electrons and ions. The fluid equations are distinguished by their parity in $\vec{v}$ (the microscopic particle velocity). The even parity moment equations represent the conservation of particles and energy:

$$\partial n_j / \partial t + \nabla \cdot (n_j \vec{u}_j) = S_{nj} \quad , \quad (2a)$$

$$(3/2) \partial p / \partial t + \nabla \cdot \left(\vec{q}_j + (5/2) p_j \vec{u}_j \right) = \vec{J} \cdot \vec{E} + S_p + \pi \quad , \quad (2b)$$

$$\begin{aligned} (3/2) \partial p_e / \partial t + \nabla \cdot (\vec{q}_e + (5/2) p_e \vec{u}_e) \\ = \vec{J} \cdot \vec{E} + Q_{\Delta e} + S_e + \pi - \vec{u}_i \cdot \nabla p_i + \vec{\pi}_i : \nabla \vec{u}_i \quad . \end{aligned} \quad (2c)$$

Here, n_j is the particle density, $\vec{u}_j$ is the conductive heat flux, $p_j = n_j T_j$ is the pressure, $p = p_e + p_i$, S_{nj} , S_p , S_e are particle and pressure sources,

$$\pi = - \sum_{e,i} \nabla \cdot (\vec{u}_j \cdot \vec{\pi}_j) \quad (2d)$$

is the viscous heating term,

$$Q_{\Delta e} = 3(m_e/m_i) n_e / \tau_{ei} (T_i - T_e) \quad (2e)$$

is the temperature equipartition term, $\vec{\pi}_j$ is the viscous stress tensor, and τ_{ei} is the electron-ion collision time. The relation $p_j \gg (1/2) m_j n_j u_j^2$ has been assumed for the subsonic flows of interest.

The odd parity moments yield the force balance equations for each

species:

$$\vec{E} + \vec{u}_j \times \vec{B} = -\vec{R}_j / (n_j e_j) + m_j \vec{T}(u_j) / e_j, \quad (3a)$$

where

$$\vec{T}(u) = \partial \vec{u} / \partial t + \vec{u} \cdot \nabla \vec{u} \quad (3b)$$

is the inertial term,

$$\vec{R}_j = -\nabla p_j - \nabla \cdot \vec{\pi}_j + \vec{F}_j \quad (3c)$$

represents the departure from ideal $(\vec{E} \times \vec{B})$ drift motion,

$$\vec{F}_j = \int m_j \vec{v} c_j d\vec{v} \quad (3d)$$

is the collisional friction between electrons and ions, and c_j is the collision operator of species j . Equation (3a) for electrons is the general form of Ohm's law. Summing $e_j n_j \times$ Eq. (3a) and using quasineutrality, $\sum_{e,i} n_j e_j = 0$, yields the net force balance

$$\rho \vec{T}(\vec{u}) + \nabla \cdot (\vec{\pi}_i + \vec{\pi}_e) = -\nabla p + \vec{J} \times \vec{B} \quad (3a)'$$

where $\rho = \sum_{e,i} m_j n_j$ is the mass density, $\vec{u} = \sum_{e,i} m_j n_j \vec{u}_j / \rho$ is the mean mass flow velocity, and $\vec{J} = \sum_{e,i} n_j e_j \vec{u}_j$ is the electrical current density.

To close this set of equations, one must specify the remaining transport quantities $\vec{d}_i$, $\vec{d}_e$, $\vec{\pi}_i$, $\vec{\pi}_e$, and $\vec{F}_e = -\vec{F}_i$ that appear in Eqs. (2) through (3).

The standard collisional transport model is given in an article by Braginskii (1965), in which these are evaluated in the limit of large fields ($\omega_{cj}\tau_j \gg$

1). The collisional friction term is

$$\mathbf{R}_e = -\mathbf{R}_i = ne(\hat{\mathbf{j}}_{\perp}/\sigma_{\perp} + \hat{\mathbf{j}}_{\parallel}/\sigma_{\parallel}) - 0.7 nk_b \nabla_{\parallel} T_e - (3/2)(nk_b/\omega_{ce}\tau_e)\hat{\mathbf{b}} \times \nabla T_e \quad (4a)$$

where $\sigma = 2.0$, $\sigma_{\perp} = 2.0 ne^2\tau_e/m_e$ are the electrical conductivities, $\parallel$ and $\perp$ refer to the direction of the magnetic field, $\hat{\mathbf{B}} = |\hat{\mathbf{B}}|\hat{\mathbf{b}}$, ω_{ce} and ω_{ci} are the electron and ion gyrofrequencies, and electron and ion collisional times are

$$\tau_j = \frac{3\sqrt{m_j}(k_b T_j)^{3/2}}{4\sqrt{2}\pi n_j e^4 \ln \Lambda} \quad (4b)$$

for $j = e, i$, with $\ln \Lambda$ being the Coulomb logarithm and k_b being Boltzman's constant. The ion and electron heat fluxes are given by

$$\hat{\mathbf{q}}_i = -K_{\parallel}^i \nabla_{\parallel} k_b T_i - K_{\perp}^i \nabla_{\perp} k_b T_i - K_{\wedge}^i \hat{\mathbf{b}} \times \nabla_{\perp} k_b T_i, \quad (4c)$$

$$\begin{aligned} \hat{\mathbf{q}}_e = & -K_{\parallel}^e \nabla_{\parallel} k_b T_e - K_{\perp}^e \nabla_{\perp} k_b T_e - K_{\wedge}^e \hat{\mathbf{b}} \times \nabla_{\perp} k_b T_e \\ & - 0.71 \frac{k_b T_e}{e} \hat{\mathbf{j}}_{\parallel} + \frac{3}{2} \frac{k_b T_e}{\omega_{ce}\tau_e} \hat{\mathbf{j}} \times \hat{\mathbf{b}}, \end{aligned} \quad (4d)$$

with the thermal conductivities being given by

$$K_{\parallel}^i = 3.9 \frac{nk_b T_i \tau_i}{m_i}, \quad K_{\perp}^i = 2 \frac{nk_b T_i}{m_i \omega_{ci} \tau_i}, \quad K_{\wedge}^i = \frac{5}{2} \frac{nk_b T_i}{m_i \omega_{ci}} \quad (4e)$$

$$K_{\parallel}^e = 3.2 \frac{nk_b T_e \tau_e}{m_e}, \quad K_{\perp}^e = 4.7 \frac{nk_b T_e}{m_e \omega_{ce} \tau_e}, \quad K_{\lambda}^e = \frac{5}{2} \frac{nk_b T_e}{m_e \omega_{ce}}.$$

The reader is referred to Braginskii (1965) for the forms of the viscous stress tensors $\vec{\pi}_e$ and $\vec{\pi}_i$. The forms of the transport coefficients valid in the more collisionless regimes are found in Hazeltine and Hinton (1973).

C. MHD time scales

We consider the application of the MHD Eqs. (1) through (4) to the toroidal geometry illustrated in Fig. 1. A tokamak has a large externally produced toroidal field $\vec{B}_T$ and a smaller poloidal magnetic field $\vec{B}_P$ produced primarily by the toroidal plasma current. If R and a are the approximate major and minor radii of the toroidal plasma, we denote the ratio

$$q = B_T a / B_P R$$

the "safety factor" of the device. (A more precise definition of the safety factor is given in Sec. II.) Typically, in tokamaks, q is in the range of 2-4 at the plasma edge. The inverse of the safety factor, $\kappa = 2\pi/q$ measures the number of radians in poloidal angle a magnetic field line travels in transiting once the long way around the device.

There are several spatial lengths and directions of importance in the tokamak. In the most idealized case, the magnetic field lines can be thought to lie in two-dimensional surfaces, called magnetic surfaces. We denote these nested surfaces by a "flux function" $\psi = \text{constant}$, so that the magnetic field satisfies $\vec{B} \cdot \nabla \psi = 0$. Clearly, the relevant spatial length in the normal direction $\hat{n} \equiv \vec{\nabla} \psi / |\nabla \psi|$ across the magnetic surfaces and perpendicular to the

magnetic field is the minor radius a .

For particles or waves traveling parallel to the magnetic field, in the direction $\hat{b} \equiv \vec{B}/|\vec{B}|$, the periodicity distance in the toroidal angle ϕ is $2\pi R$, but the periodicity distance in the poloidal angle θ is $2\pi Rq$ (see Fig. 1). It is this latter distance which is physically most relevant since the equilibrium tokamak is normally symmetric in the ϕ direction but not in the θ direction.

The MHD equations contain both wavelike (hyperbolic) and diffusionlike (parabolic) characteristics. The time scales associated with the wave phenomena are the time it takes for a fast compressible wave to traverse the minor radius

$$\tau_f \sim \mu_0 a (\rho_0)^{1/2} / B_T, \quad (5a)$$

the time it takes a shear-Alfvén (transverse) wave, which propagates only along magnetic field lines, to travel a connection length qR along a magnetic surface

$$\tau_t \sim \mu_0 Rq (\rho_0)^{1/2} / B_T, \quad (5b)$$

and the time it takes a slow compressible (sound) wave, which also propagates only along the field, to travel a connection length

$$\tau_s \sim Rq (\rho_0 / \beta)^{1/2} / B_T. \quad (5c)$$

Here $\beta = \mu_0 p_0 / B_T^2$ is the ratio of plasma to field pressure and ρ_0 is the mass density. We note that it is the connection length qR and not the minor radius

a that enters into Eqs. (5b) and (5c), since both the shear-Alfvén and the slow-compressible waves propagate essentially in one dimension along the direction of the magnetic field.

The fundamental cross field diffusion time scale, the plasma skin time, is the time it takes for the toroidal current (or poloidal field) to penetrate the minor radius a,

$$\tau_s \sim \mu_0 a^2 / \eta_{\parallel} \quad (5d)$$

where $\eta_{\parallel}$ is the parallel plasma resistivity, $\eta_{\parallel} = \sigma_{\parallel}^{-1}$. Other, related diffusion time scales contained in Eqs. (1) through (4) are the neoclassical (Hazeltine and Hinton, 1973) cross field heat conduction time

$$\begin{aligned} \tau_h &\sim \beta^{-1} (m_e/m_i)^{1/2} (1 + 2q^2)^{-1} \tau_s && \text{(high collisionality regime)} \\ &\sim (R/a)^{-3/2} q^{-2} \beta^{-1} (m_e/m_i)^{1/2} \tau_s && \text{(low collisionality regime)} \end{aligned}$$

(5e)

and the neoclassical cross field particle transport time

$$\tau_p \sim (m_i/m_e)^{1/2} \tau_h \quad (5f)$$

The equilibration times parallel to the magnetic field are much shorter. Since both the electron and ion transit times around the device are much shorter than their collision times, the characteristic times for

equilibration within the surfaces are given by

$$\tau_{et} \sim \frac{qR}{v_{te}}, \quad (\text{electrons}) \quad (5g)$$

and

$$\tau_{it} \sim \frac{qR}{v_{ti}}, \quad (\text{ions}) \quad (5h)$$

We illustrate the multiplicity of time scales that exist in a present generation tokamak fusion experiment in Fig. 2 where we have evaluated these expressions for $B_T = 5$ tesla, $I_p = 4$ Ma, $R = 3$ m, $a = 1$ m, $n_0 = 10^{20} \text{ m}^{-3}$, and $T = 5$ keV. For comparison we have also indicated the fundamental kinetic time scales, the electron and ion plasma oscillation periods and cyclotron periods. We see that even neglecting the kinetic phenomena, the MHD phenomena span almost 10 orders of magnitude. Clearly, straightforward time integration of the equations is unfeasible on all but the fastest time scales.

D. General principles

We discuss here some of the principles that are explored in the subsequent sections of this chapter to enable time integration of the MHD equations on the different time scales. The most obvious but most important principle is this: Only attempt to model accurately transient phenomena on a single time scale in a given simulation. Average over the phenomena occurring on the faster time scales and ignore the variations occurring on the slower time scales.

Averaging of the fast time scale phenomena is justified by the fact that there are no unstable modes on this faster time scale, but only stable

oscillations. If there were unstable modes on a faster time scale, it would make no sense in simulating the slower time scale. For example, we would not bother trying to simulate the diffusion time scale evolution of a plasma that is unstable to MHD instabilities that grow on the time of the transverse Alfvén wave (kink modes) or the intermediate time scale (tearing modes).

A second principle that we will utilize is that natural coordinates aligned with the magnetic field are essential in separating out anisotropic wave motion and anisotropic diffusion. Since the magnetic field in general will change its direction during the course of a calculation, this necessitates utilizing a time dependent coordinate transformation that continually aligns itself with the field. We will see how this formalism can greatly elucidate the separate physics occurring when modeling both the wave (ideal) and the diffusion (resistive) time scales.

A third technique that is utilized is to use a combination stream function-potential representation for a two-dimensional velocity field. We will find that doing this aids in separating out the fast compressible MHD wave from the slower incompressible transverse wave. In fact, in Sec. II we will describe the dynamical grid method which combines this with the previous technique so that the incompressible part of the velocity is treated as Lagrangian and evolves the coordinate transformation, or equivalently the computational grid. The compressible part of the velocity field is treated as Eulerian, or as flow relative to this moving grid.

A final technique we will utilize in Sec. III is to take some freedom with modifying the phenomena occurring on the faster time scales. If the slower time scale motion depends only on the long time asymptotic response of the faster time scale motion, we are free to modify the dynamics of the faster time scale phenomena, as long as we do not affect its long time behavior.

describe two methods in Sec. III which are able to get at the slow (resistive) time scale by modifying the fast (ideal) time scale waves in two very different ways. The first method speeds up these waves to infinite velocity and thus changes these characteristics from hyperbolic to elliptic, thereby eliminating the wave time scales from the problem. The second method slows down these waves so they can be integrated along with the resistive phenomena.

II. IDEAL TIME SCALE MHD SIMULATIONS

A. Introduction

There is a large class of MHD instabilities in tokamaks that takes place on a time scale short compared to the dissipative times. In describing this motion we can utilize a simplified fluid description in which the dissipative terms are ignored and the differences in the electron and ion properties are ignored. This simplified set of equations describing the evolution of the magnetic field, the density, the velocity, and the pressure is referred to as the ideal MHD equations:

$$\partial \vec{B} / \partial t = \nabla \times (\vec{v} \times \vec{B}) \quad , \quad (6a)$$

$$\partial \rho / \partial t + \nabla \cdot (\rho \vec{v}) = 0 \quad , \quad (6b)$$

$$d/dt (p/\rho^\gamma) = 0 \quad , \quad (6c)$$

$$\rho \partial \vec{v} / \partial t + \rho \vec{v} \cdot \nabla \vec{v} + \nabla p = 1/\mu_0 (\nabla \times \vec{B}) \times \vec{B} \quad . \quad (6d)$$

We note that neither the electric field $\vec{E}$ nor the current density $\vec{J}$ explicitly appear in Eqs. (6a) through (6d), although they may be obtained from the

infinite conductivity limit of Ohm's law and from Ampere's law ignoring the displacement current

$$\vec{E} + \vec{v} \times \vec{B} = 0 \quad , \quad (7a)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad . \quad (7b)$$

The transient solutions of Eqs. (6a) through (6d) are dominated by the wave solutions. We have seen in Eqs. (5a) and (5b) that the time scale τ_t for the nearly incompressible transverse modes associated with MHD stability phenomena is longer than the time scale for the fast compressible modes by the factor Rq/a , i.e.,

$$\tau_f \sim Rq/a \tau_t \quad .$$

This factor Rq/a is typically 10 to 30 in tokamaks of interest.

There are several different ways to deal with this disparity. One (Strauss, 1976; Potter, 1976; Strauss, 1983) is to use the limiting form of the MHD equations as Rq/a approaches infinity. This forces incompressibility and thus annihilates the fast waves altogether leaving only the transverse time scale in the equations. However, this approximation can be inadequate for Rq/a finite. Another (Wesson and Sykes, 1975; Schneider and Bateman, 1975; Bateman *et al.*, 1974; Brackbill, 1976) approach has been to integrate the primitive compressible equations without taking special care to isolate phenomena occurring on different time scales. This approach is useful when simulating tokamaks with small Rq/a and large $\beta \sim 2\mu_0 p/B^2$ so that the transverse and the fast time scales come together. Fully implicit methods

have been considered but have not been perceived as practical because of the large number of variables. Alternating direction implicit methods have not been demonstrated to be useful in such a highly anisotropic medium.

We describe here the dynamical grid method (Jardin *et al.*, 1978), a technique for solving the ideal MHD equations on the transverse time scale by averaging over the fast time scale phenomena numerically using a partially implicit method. This is made possible by introducing a time-dependent nonorthogonal magnetic flux coordinate transformation that determines the grid used in the computation. The coordinate system is chosen to conform to the field in such a way that the characteristics of the partial differential equations can be represented accurately. In particular, the transverse wave which rotates the magnetic field as it propagates will be represented in the time dependent coordinate transformation. The fast compressible wave, on the other hand, compresses both the field and the fluid as it propagates. It is removed from the coordinate transformation by forcing the grid velocity to be incompressible, i.e., by making the Jacobian constant in time. This constrains the fast wave motion to appear in the slippage flow, i.e., in the Eulerian motion of the plasma flowing through the grid. We will see this leads to a relatively simple scalar equation which can be solved implicitly, effectively averaging over the fast wave phenomena.

B. Magnetic flux coordinates

We introduce the coordinates (Kruskal and Kulsrud, 1958) (ψ, θ, ϕ) which, along with the time t , serve as the independent variables. The coordinates θ and ϕ are periodic anglelike variables with period 2π . The angle θ increases when going the short way around the torus (poloidal direction) while the angle ϕ increases when going the long way around the torus (toroidal direction).

The coordinate ψ is chosen to be constant along a magnetic field line, i.e.,
 $\vec{B} \cdot \nabla \psi = 0$.

A general form for the three-dimensional magnetic field $\vec{B}$, a vector function satisfying $\vec{B} \cdot \nabla \psi = 0$ and $\nabla \cdot \vec{B} = 0$, is given by

$$\vec{B} = [\chi_\psi(\psi, t) + h_\phi(\psi, \theta, \phi, t)] \nabla \phi \times \nabla \psi + [\Psi_\psi(\psi, t) - h_\theta(\psi, \theta, \phi, t)] \nabla \psi \times \nabla \theta \quad (8)$$

where here, and elsewhere, we use subscripts to denote partial differentiation. The two one-dimensional scalar functions $\chi(\psi, t)$ and $\Psi(\psi, t)$ and the three-dimensional scalar function $h(\psi, \theta, \phi, t)$ completely determine the magnetic field once the coordinate basic vectors $(\nabla \psi, \nabla \theta, \nabla \phi)$ are known.

We note here that the magnetic field as represented by Eq. (8) can vary in several ways. It can change its magnitude and its orientation within a magnetic surface by changing the functions $\chi(\psi, t)$, $\Psi(\psi, t)$, and $h(\psi, \theta, \phi, t)$ within the brackets. Alternatively, it can change its orientation by changing the directions of the basis vectors $\nabla \psi \times \nabla \psi$ and $\nabla \psi \times \nabla \theta$.

It is assumed throughout that the surfaces $\psi = \text{constant}$ form nested toroidal surfaces. If this is true initially at time $t = 0$, then it is a property of the ideal MHD equations that this topology will remain invariant for all time. The functions $\chi(\psi, t)$ and $\Psi(\psi, t)$ introduced in Eq. (8) are physically $1/2\pi$ times the amounts of poloidal and toroidal magnetic flux contained within a magnetic surface with label ψ . Thus, by direct computation, the poloidal flux is

$$2\pi\chi(\psi, t) \equiv \frac{1}{(2\pi)} \int \vec{B} \cdot \nabla \theta \, d\tau = 2\pi \int_0^\psi \chi_{\psi'}(\psi', t) d\psi' \quad (9a)$$

and the toroidal flux is

$$2\pi\psi(\psi, t) \equiv \frac{1}{(2\pi)} \int \vec{B} \cdot \nabla \phi \, d\tau = 2\pi \int_0^\psi \Psi_{\psi'}(\psi', t) d\psi' \quad (9b)$$

We note here that the "safety factor" q can be expressed as the ratio $q \equiv d\psi/d\chi$. The volume element used in Eq. (9) is

$$d\tau \equiv J \, d\psi d\theta d\phi \quad (10a)$$

where

$$J \equiv [\nabla\psi \times \nabla\theta \cdot \nabla\phi]^{-1} = \left[\frac{\partial \vec{x}}{\partial \psi} \times \frac{\partial \vec{x}}{\partial \theta} \cdot \frac{\partial \vec{x}}{\partial \phi} \right] \quad (10b)$$

is the Jacobian of the transformation from Cartesian to magnetic flux coordinates.

It is notationally convenient to introduce the natural basis vectors for the ψ, θ, ϕ coordinate system

$$\hat{e}^1 \equiv \nabla\psi, \quad \hat{e}^2 \equiv \nabla\theta, \quad \hat{e}^3 \equiv \nabla\phi, \quad (11)$$

and the reciprocal basis

$$\hat{e}_1 \equiv \partial \vec{x} / \partial \psi = x_\psi \hat{x} + y_\psi \hat{y} + z_\psi \hat{z}, \quad (12a)$$

$$\hat{e}_2 \equiv \partial \vec{x} / \partial \theta = x_\theta \hat{x} + y_\theta \hat{y} + z_\theta \hat{z}, \quad (12b)$$

$$\hat{e}_3 \equiv \partial \vec{x} / \partial \phi = x_\phi \hat{x} + y_\phi \hat{y} + z_\phi \hat{z}, \quad (12c)$$

or

$$\hat{e}_i = \partial x^j / \partial u^i \hat{x}^j \quad (12d)$$

Here $(x^1 \equiv x, x^2 \equiv y, x^3 \equiv z)$ form a standard Cartesian coordinate system, $(\hat{x}^1 \equiv \hat{x}, \hat{x}^2 \equiv \hat{y}, \hat{x}^3 \equiv \hat{z})$ are unit vectors along the Cartesian axis, and for notational convenience we have let $u^1 \equiv \psi, u^2 \equiv \theta, u^3 \equiv \phi$. The basis vectors are related by way of the formula

$$\begin{aligned} \hat{e}^i &= \frac{1}{2J} \hat{e}_j \times \hat{e}_k \epsilon_{ijk}, \quad \hat{e}_i = \frac{1}{2} J \hat{e}^j \times \hat{e}^k \epsilon_{ijk}, \quad \hat{e}^i \cdot \hat{e}_j = \delta_{ij}, \\ \hat{e}^i &= g^{ij} \hat{e}_j, \quad \hat{e}_i = g_{ij} \hat{e}^j, \quad g^{ij} = \hat{e}^i \cdot \hat{e}^j, \quad g_{ij} = \hat{e}_i \cdot \hat{e}_j, \\ \frac{\partial \hat{e}_i}{\partial u^j} &= \{i^k j\} \hat{e}_k, \quad \frac{\partial \hat{e}^i}{\partial u^j} = -\{j^i k\} \hat{e}^k, \end{aligned} \quad (13a)$$

where (ijk) run from 1 to 3, δ_{ij} is the Kronocker delta, ϵ_{ijk} is the usual permutation symbol, and summation of repeated indices is implied.

It is readily verified that the metric tensor, the Jacobian, and the Christoffel symbols can be expressed as the following derivatives of the Cartesian coordinates

$$g_{ij} = (\partial x^k / \partial u^i) (\partial x^k / \partial u^j) \quad (13b)$$

$$g^{ij} = \frac{1}{2J^2} \epsilon_{ikl} \epsilon_{jmn} g_{km} g_{ln} \quad (13c)$$

$$J = \epsilon_{ijk} (\partial x^i / \partial \psi) (\partial x^j / \partial \theta) (\partial x^k / \partial \phi) \quad (13d)$$

$$\{\beta^\alpha \gamma\} = \frac{1}{2J} \epsilon_{ijk} \epsilon_{\alpha mn} (\partial^2 x^i / \partial u^\beta \partial u^\gamma) (\partial x^j / \partial u^m) (\partial x^k / \partial u^n) \quad (13e)$$

Thus, for example

$$g_{12} = (x_\psi x_\theta + y_\psi y_\theta + z_\psi z_\theta)$$

$$g^{13} = [(y_\theta z_\phi - y_\phi z_\theta)(y_\psi z_\theta - y_\theta z_\psi) + (x_\phi z_\theta - x_\theta z_\phi)(x_\theta z_\psi - z_\theta x_\psi) \\ + (x_\theta y_\phi - y_\theta x_\phi)(x_\psi y_\theta - y_\psi x_\theta)]/J^2$$

$$\{2\ 3\}^1 = [x_{\theta\phi}(y_\theta z_\phi - y_\phi z_\theta) + y_{\theta\phi}(x_\phi z_\theta - x_\theta z_\phi) + z_{\theta\phi}(y_\phi x_\theta - y_\theta x_\phi)]/J$$

C. Velocity decomposition

A "coordinate velocity" may be defined by

$$\vec{v}_g \equiv \partial \vec{x} / \partial t \quad (14a)$$

where time derivatives, unless explicitly indicated, are taken with (ψ, θ, ϕ) held fixed. The time rate of change of the Jacobian and the grid velocity are related by the identity (in the flux surface frame)

$$\partial J / \partial t = J \nabla \cdot \vec{v}_g \quad (14b)$$

The coordinate velocity enters naturally when transforming time derivatives from a stationary system to the ψ, θ, ϕ system. If Q is an arbitrary scalar quantity, then the chain rule takes the form

$$\frac{\partial \Omega}{\partial t} \Big|_{\psi, \theta, \phi} = \frac{\partial \Omega}{\partial t} \Big|_{\vec{x}} + \vec{v}_g \cdot \nabla \Omega \quad . \quad (15)$$

The velocity of a fluid element is thus divided into two parts, the velocity of the coordinates and a velocity relative to the coordinates

$$\vec{v} = \vec{v}_g + \vec{v}_r \quad . \quad (16)$$

It is clear that only the combination $\vec{v}_g + \vec{v}_r$ can be determined from the physical evolution equations.

For definiteness, we introduce the six scalar variables $(\alpha, \beta, \gamma), (u, v, w)$ so that the grid velocity and the relative velocity are

$$\vec{v}_g \cdot \nabla \psi = \alpha(\psi, \theta, \phi, t) \quad , \quad \vec{v}_r \cdot \nabla \psi = u(\psi, \theta, \phi, t) \quad , \quad (16a)$$

$$\vec{v}_g \cdot \nabla \theta = \beta(\psi, \theta, \phi, t) \quad , \quad \vec{v}_r \cdot \nabla \theta = v(\psi, \theta, \phi, t) \quad , \quad (16b)$$

$$\vec{v}_g \cdot \nabla \phi = \gamma(\psi, \theta, \phi, t) \quad , \quad \vec{v}_r \cdot \nabla \phi = w(\psi, \theta, \phi, t) \quad . \quad (16c)$$

D. The transformed ideal MHD equations

By transforming to the coordinate system given by Eqs. (10) - (13) and using the representation for the magnetic field given by Eq. (8) and for the velocity field given by Eqs. (14) - (16), the magnetic flux conservation equations, (6a) become

$$(\chi_\psi)_t + (\chi_\psi u)_\psi = 0 \quad , \quad (17a)$$

$$(\Psi_\psi)_t + (\Psi_\psi u)_\psi = 0 \quad , \quad (17b)$$

$$u_{\theta} = u_{\phi} = 0 \quad , \quad (17c)$$

$$h_t + (hu)_{\psi} + \omega(\chi_{\psi} + h_{\phi}) - v(\Psi_{\psi} - h_{\theta}) = 0 \quad , \quad (17d)$$

the conservation of mass and entropy, Eqs. (6b) and (6c) become

$$(\rho J)_t + (\rho Ju)_{\psi} + (\rho Jv)_{\theta} + (\rho Jw)_{\phi} = 0 \quad , \quad (17e)$$

$$(p^{1/\gamma J})_t + (p^{1/\gamma Ju})_{\psi} + (p^{1/\gamma Jv})_{\theta} + (p^{1/\gamma Jw})_{\phi} = 0 \quad , \quad (17f)$$

and the grid evolution and momentum equations assume the compact form

$$x_t^i = (\vec{v}_g \cdot \nabla u_j) \partial x^i / \partial u_j \quad , \quad (17g)$$

$$(\vec{v} \cdot \nabla u^i)_t + (1/\rho)(\nabla u_i \cdot \nabla u_j) \partial P / \partial u^j + R^i = 0 \quad , \quad (17h)$$

where we have defined the total pressure

$$P \equiv p + \frac{1}{2\mu_0} v_g^2 \quad (18a)$$

with

$$B^2 = (\chi_{\psi} + h_{\phi})^2 g_{22} + 2(\chi_{\psi} + h_{\phi})(\Psi_{\psi} - h_{\theta})g_{23} + (\Psi_{\psi} - h_{\theta})^2 g_{33} \quad (18b)$$

being the square of the magnetic field strength, and

$$\begin{aligned}
 R^i \equiv & \vec{v} \cdot \nabla (\vec{v}_g \cdot \nabla u^i) + \vec{v}_r \cdot \nabla (\vec{v} \cdot \nabla u^i) - \frac{\partial}{\partial u^j} [J(\vec{B} \cdot \nabla u^i)(\vec{B} \cdot \nabla u^j)] / \rho \mu_0 J \\
 & + [(\vec{v} \cdot \nabla u^j)(\vec{v} \cdot \nabla u^k) - (\vec{B} \cdot \nabla u^j)(\vec{B} \cdot \nabla u^k) / \rho \mu_0] [j^{ik}] .
 \end{aligned} \quad (18c)$$

E. 2-d dynamical grid method

To illustrate the dynamical grid method, we consider a 2-d axisymmetric subsystem of the equations presented in Sec. II. D obtained by taking

$$\begin{aligned}
 x &= R(\psi, \theta) \sin \phi \\
 y &= R(\psi, \theta) \cos \phi \\
 z &= Z(\psi, \theta)
 \end{aligned} \quad (19)$$

and by letting $R(\psi, \theta)$ and $Z(\psi, \theta)$ be the coordinate functions (see Fig. 3). We consider only motions which have no ϕ variation other than that explicitly indicated in Eq. (19), i.e., have $\partial/\partial\phi \equiv 0$. This corresponds to considering the axisymmetric motion of a toroidal plasma column with no ϕ variation.

The form of the Jacobian is arbitrary so far, although it is related to the coordinate velocity through Eq. (14b). To force the moving (ψ, θ) mesh to be nearly incompressible and thus free of the fast time scale motion, we take the Jacobian to be of the form

$$J = T(t) R^m \psi^n \quad (20)$$

where m, n are integers to be specified later and $T(t)$ is a time dependent normalization constant which we allow to change on the slow time scale. It then follows from Eqs. (14b) and (20) that if $T_t = 0$,

$$\nabla \cdot (R^{-m} \vec{v}_g) = 0 \quad (21)$$

and both the Jacobian and hence the coordinate functions $R(\psi, \theta, t)$ and $Z(\psi, \theta, t)$ are free of the fast time scale motion.

From Eq. (21) it follows that a stream function ξ exists for part of the two-dimensional coordinate velocity. For $T_t \neq 0$ we generalize the concept of stream function and write

$$\vec{v}_g = R^m \nabla \times (\xi \nabla \phi) + (T_t/T)(\vec{Z} - \vec{Z}_0) \quad (22)$$

The second term in Eq. (22) is to accommodate the coordinate system expansion implied by allowing a time dependent constant $T(t)$ in the expression for the Jacobian, Eq. (20). Thus, in Eq. (16) we have

$$\alpha \equiv \vec{v}_g \cdot \nabla \psi = \frac{1}{J} \left\{ R^m \xi_{,\theta} - \frac{T_t}{T} R \xi_{,\theta} (Z - Z_0) \right\} \quad (23a)$$

$$\beta \equiv \vec{v}_g \cdot \nabla \theta = \frac{1}{J} \left\{ -R^m \xi_{,\psi} + \frac{T_t}{T} R R_{,\psi} (Z - Z_0) \right\} \quad (23b)$$

$$\gamma \equiv \vec{v}_g \cdot \nabla \phi = 0 \quad (23c)$$

In this 2-d axisymmetric limit, the metric tensor connecting (ψ, θ, ϕ) and (R, Z, ϕ) simplifies so that $g_{13} = g_{31} = g_{23} = g_{32} = 0$, $g^{13} = g^{31} = g^{23} = g^{32} = 0$, $g_{33} = R^2$, $g^{33} = R^{-2}$. Other nonzero metric terms are given by

$$J \equiv [\nabla \psi \times \nabla \theta \cdot \nabla \phi]^{-1} = R(R_{,\psi} Z_{,\theta} - Z_{,\psi} R_{,\theta}) \quad (24)$$

$$g^{11} \equiv |\nabla \psi|^2 = \left(\frac{R^2}{J^2}\right)(R_{,\theta}^2 + Z_{,\theta}^2) \quad g^{22} \equiv |\nabla \theta|^2 = \left(\frac{R^2}{J^2}\right)(R_{,\psi}^2 + Z_{,\psi}^2)$$

$$g^{12} = g^{21} \equiv \nabla\theta \cdot \nabla\psi = -\left(\frac{R^2}{J^2}\right)(R_\theta R_\psi + Z_\theta Z_\psi)$$

$$\{1^1 1\} = \left(\frac{R}{J}\right)(Z_\theta R_{\psi\psi} - R_\theta Z_{\psi\psi}) \quad \{2^1 2\} = \left(\frac{R}{J}\right)(Z_\theta R_{\theta\theta} - R_\theta Z_{\theta\theta})$$

$$\{1^2 1\} = \left(\frac{R}{J}\right)(R_\psi Z_{\psi\psi} - Z_\psi R_{\psi\psi}) \quad \{2^2 2\} = \left(\frac{R}{J}\right)(R_\psi Z_{\theta\theta} - Z_\psi R_{\theta\theta})$$

$$\{2^1 1\} = \{1^1 2\} = \left(\frac{R}{J}\right)(Z_\theta R_{\psi\theta} - R_\theta Z_{\psi\theta})$$

$$\{2^2 1\} = \{1^2 2\} = \left(\frac{R}{J}\right)(R_\psi Z_{\psi\theta} - Z_\psi R_{\psi\theta})$$

$$\{1^3 3\} = \{3^3 1\} = \frac{R_\psi}{R} \quad \{2^3 3\} = \{3^3 2\} = \frac{R_\theta}{R}$$

$$\{3^1 3\} = \frac{-R^2 Z_\theta}{J} \quad \{3^2 3\} = \frac{R^2 Z_\psi}{J} \quad .$$

It is natural and convenient to introduce a new variable for the toroidal magnetic field,

$$F(\psi, \theta) \equiv \left(\frac{R^2}{J}\right)[\Psi' - n_\theta]$$

so that the axisymmetric magnetic field, a specialization of the general form given by Eq. (8), is given by

$$\vec{B} = \chi_\psi(\psi, t) \nabla\psi \times \nabla\psi + F(\psi, \theta, t) \nabla\psi \quad . \quad (25)$$

We note that in axisymmetric equilibrium, when $\vec{J} \times \vec{B} = \nabla p$, it follows from Eq. (25) that $F_\theta = 0$. The analogue of the 3-d Eqs. (17) in this 2-d geometry

becomes

$$(\chi_\psi)_t + (\chi_\psi u)_\psi = 0 \quad , \quad (26a)$$

$$(FJ/R^2)_t + (uFJ/R^2)_\psi + (vFJ/R^2 - \omega\chi_\psi)_\theta = 0 \quad , \quad (26b)$$

$$u_\theta = 0 \quad , \quad (26c)$$

$$(\rho J)_t + (\rho J u)_\psi + (\rho J v)_\theta = 0 \quad , \quad (26d)$$

$$(\rho^{1/\gamma J})_t + (\rho^{1/\gamma J} u)_\psi + (\rho^{1/\gamma J} v)_\theta = 0 \quad , \quad (26e)$$

$$R_t = R_\psi \alpha + R_\theta \beta \quad , \quad (26f)$$

$$Z_t = Z_\psi \alpha + Z_\theta \beta \quad , \quad (26g)$$

$$(u + \alpha)_t + \frac{1}{\rho} |\nabla \psi|^2 P_\psi + \frac{1}{\rho} (\nabla \theta \cdot \nabla \psi) P_\theta + R = 0 \quad , \quad (26h)$$

$$(v + \beta)_t + \frac{1}{\rho} (\nabla \theta \cdot \nabla \psi) P_\psi + \frac{1}{\rho} |\nabla \theta|^2 P_\theta + S = 0 \quad , \quad (26i)$$

$$\omega_t + Q = 0 \quad . \quad (26j)$$

Here, the total pressure, Eq. (18a) and (18b) takes the simplified form

$$P = p + \frac{1}{2\mu_0} [\chi_\psi^2 |\nabla \psi|^2 + F^2] / R^2 \quad , \quad (27a)$$

and we have defined

$$\begin{aligned}
R \equiv & v(u + \alpha)_{\theta} + u(u + \alpha)_{\psi} + (u + \alpha)_{\alpha} + (v + \beta)_{\alpha\theta} \\
& + (u + \alpha)^2 \{1^1 1\} + (v + \beta)^2 \{2^1 2\} + 2(u + \alpha)(v + \beta) \{2^1 1\} \\
& - \frac{1}{\rho\mu_0} \left(\frac{\chi_{\psi}}{J}\right)^2 \{2^1 2\} + (\omega^2 - F^2/\rho\mu_0) \{3^1 3\} \quad , \quad (27b)
\end{aligned}$$

$$\begin{aligned}
S \equiv & v(v + \beta)_{\theta} + u(v + \beta)_{\psi} + (u + \alpha)_{\beta} + (v + \beta)_{\beta\theta} \\
& + (u + \alpha)^2 \{1^2 1\} + (v + \beta)^2 \{2^2 2\} + 2(u + \alpha)(v + \beta) \{1^2 2\} \\
& - \frac{1}{\rho\mu_0} \left(\frac{\chi_{\psi}}{J}\right)^2 \{2^2 2\} + (\omega^2 - \frac{F^2}{\rho\mu_0}) \{3^2 3\} \quad , \quad (27c)
\end{aligned}$$

$$\begin{aligned}
U = & u\omega_{\psi} + v\omega_{\theta} + 2\omega(u + \alpha) \{1^3 3\} \\
& + [2\omega(v + \beta) - \chi'F/JR^2 \rho\mu_0] \{2^3 3\} \\
& - \frac{\partial}{\partial\theta} [\chi'F/R^2] / \rho\mu_0 J \quad . \quad (27d)
\end{aligned}$$

We obtain two auxiliary equations for the fast wave by adding the ψ derivative of (26h) to the θ derivative of (26i) and by time differentiating Eq. (27a), thus

$$\begin{aligned}
\Delta_t + \left[\frac{1}{\rho} |\nabla\psi|^2 p_{\psi} + \frac{1}{\rho} (\nabla\theta \cdot \nabla\psi) p_{\theta} \right]_{\psi} \\
+ \left[\frac{1}{\rho} (\nabla\theta \cdot \nabla\psi) p_{\psi} + \frac{1}{\rho} |\nabla\theta|^2 p_{\theta} \right]_{\theta} + Q = 0 \quad (28a)
\end{aligned}$$

$$P_t + (\gamma P + F^2/\mu_o R^2)\Delta + N = 0 \quad . \quad (28b)$$

Here

$$\Delta \equiv \nabla \cdot v - \frac{1}{J} \vec{v} \cdot \nabla J = (u + \alpha)_\psi + (v + \beta)_\theta \quad (29a)$$

is the part of the divergence of the velocity field not associated with spatial variation of the Jacobian.

$$Q \equiv R_\psi + S_\theta \quad , \quad (29b)$$

$$\begin{aligned} N \equiv & (u + \alpha) [\gamma P J_\psi + (1/\mu_o R) F^2 (J/R)_\psi] / J \\ & + (v + \beta) [\gamma P J_\theta + (1/\mu_o R) F^2 (J/R)_\theta] / J \\ & + u [p + (1/2) F^2 / \mu_o R^2]_\psi + v [p + (1/2) F^2 / \mu_o R^2]_\theta \\ & - F \omega_\theta \chi_\psi / \mu_o J + \chi_\psi (|\nabla \psi|^2 / \mu_o R^2) (u \chi_\psi)_\psi \\ & - (1/2) \chi_\psi^2 (|\nabla \psi|^2 / \mu_o R^2)_t \quad . \quad (29c) \end{aligned}$$

F. 2-d vacuum equations

The ideal MHD boundary conditions between a plasma region and a vacuum region are that the tangential electric field be continuous and that the total pressure P [Eq. (27a)] be continuous. The first condition is equivalent to the statement that the poloidal flux χ be continuous and that the plasma/vacuum interface remain a flux surface. This is built into the present

formulation. The second condition requires the calculation of B^2 on the vacuum side of the plasma-vacuum interface. This is obtained by solving an elliptic magnetostatic problem each time step in the irregularly shaped and changing vacuum region. We describe here a Green's function solution method which allows for the existence of external currents in the vacuum region.

The magnetic field in the vacuum region is represented as

$$\vec{B} = \nabla\phi \times \nabla\chi + F_0 \nabla\phi \quad (30)$$

where F_0 is a spatial constant related to the total external current flowing through the center (hole) of the torus. We allow for a singular current distribution in the vacuum region due to M current carrying rings of negligible cross section located at $R_m, Z_m; m = 1, M$,

$$J = \sum_{m=1}^M I_m \delta(\vec{x} - \vec{x}_m) \quad .$$

Away from these singularities, the magnetic field in the vacuum satisfies the equation $\nabla \times \vec{B} = 0$, or from Eq. (30), $R^2 \nabla \cdot R^{-2} \nabla \chi = 0$. The infinite medium toroidal Green's function is given by

$$G(\vec{x}; \vec{x}') = [(RR')^{1/2}/k] [(2 - k^2)K(k^2) - E(k^2)]$$

where

$$k^2 \equiv \frac{4RR'}{[(R+R')^2 + (Z-Z')^2]^{1/2}}$$

and $K(k^2)$ and $E(k^2)$ are elliptic integrals of the first and second kind. The

poloidal flux in the vacuum region then satisfies the integral equation

$$\chi(\vec{x}) = \sum_{m=1}^M I_m G(\vec{x}; \vec{x}_m) + \oint \frac{d\ell'}{R'} G(\vec{x}; \vec{x}') \frac{\partial \chi}{\partial n'} \quad (31)$$

where the line integral is evaluated over the plasma-vacuum interface. Equation (31) is solved by approximating the line integral by many line segments $\Delta \ell_s$ over each of which $\partial \chi / \partial n$ is taken as a constant. Evaluating Eq. (30) at each line segment transforms the integral equation into a matrix equation: thus for each interface segment $\Delta \ell_i$ centered at $\vec{x}_i$ with poloidal flux $x_i = \text{constant}$, we have

$$\begin{aligned} \chi_i = & \sum_{m=1}^M I_m G(\vec{x}_i; \vec{x}_m) + \sum_{\substack{j=1, N \\ j \neq i}} \frac{\Delta \ell_j}{R_j} G(\vec{x}_i; \vec{x}_j) \left(\frac{\partial \chi}{\partial n} \right)_j \\ & + \int_{\Delta \ell_i} \frac{d\ell'}{R'} G(\vec{x}_i; \vec{x}') \frac{\partial \chi}{\partial n'} \quad . \end{aligned} \quad (32)$$

Special care is necessary in evaluating the last "self field" term since $|G_{ij}|$ diverges as $\vec{x}_i$ approaches $\vec{x}_j$. This is approximated by

$$\int_{\Delta \ell_i} \frac{d\ell'}{R'} G(\vec{x}_i; \vec{x}') \frac{\partial \chi}{\partial n'} \approx \Delta \ell_i \left[\ln \left(\frac{\Delta \ell_i}{16 R_i} \right) + 1 \right] \left(\frac{\partial \chi}{\partial n} \right)_i \quad . \quad (33)$$

The integral Eq. (31) is thus reduced to the matrix equation

$$\vec{A} \cdot \vec{B} = \vec{C} \quad (34)$$

where

$$c_i = x_i - \sum_{m=1}^M I_m G_{im} \quad , \quad (35a)$$

$$b_i = \left(\frac{\partial x}{\partial n} \right)_i \quad , \quad (35b)$$

$$a_{ij} = \frac{\Delta l_j}{R_j} G(\vec{x}_i; \vec{x}_j) \quad \text{for } i \neq j \quad , \quad (35c)$$

$$a_{ii} = \Delta l_i \left[2n \left(\frac{\Delta l_i}{16R_i} \right) + 1 \right] \quad . \quad (35d)$$

Equation (34) is a well-conditioned matrix equation solved by standard techniques. The total pressure on the vacuum side of the plasma boundary is then given by

$$P = \frac{1}{2\mu_0 R^2} \left[\left(\frac{\partial x}{\partial n} \right)^2 + F_0^2 \right] \quad . \quad (36)$$

G. Finite-difference equations

We describe here a partially implicit finite difference method in which the compressible wave motion is treated implicitly, but the incompressible motion and the material convection are treated explicitly. The feasibility of this treatment lies in the fact that the only fast time-scale motion is associated with the fast compressible magnetoacoustic wave which propagates isotropically and has been isolated in the variables P and Δ through Eqs. (28a) and (28b). The coefficients in Eq. (28) (the metric terms, Q , N , $\gamma P + P^2/R^2$, etc.) can be held constant during the implicit iteration. By time centering Eqs. (28a) and (28b) we obtain the following equation for the advanced time total pressure P^{n+1}

$$P^{n+1} - \frac{1}{2} a \delta t [(bP_{\psi}^{n+1} + cP_{\theta}^{n+1})_{\psi} + (cP_{\psi}^{n+1} + dP_{\theta}^{n+1})_{\theta}] + D = 0 \quad (37)$$

Here we have defined the variables

$$a \equiv \delta t (\gamma_P + F^2 / \mu_0 R^2)^{n+1/2}, \quad b \equiv [(1/\rho) |\nabla \psi|^2]^{n+1/2},$$

$$c \equiv [(1/\rho) \nabla \theta \cdot \nabla \psi]^{n+1/2}, \quad d \equiv [(1/\rho) |\nabla \theta|^2]^{n+1/2},$$

$$D \equiv \delta t N^{n+1/2} + 2a\Delta^n - P^n - a\delta t Q^{n+1/2}$$

$$- \frac{1}{2} a \delta t [(bP_{\psi}^n + cP_{\theta}^n)_{\psi} + (cP_{\psi}^n + dP_{\theta}^n)_{\theta}] \quad (38)$$

We now summarize the partially implicit time advancement scheme to take the solution from time level (n) to time level $(n + 1)$:

- (1) Advance the coordinates to a predicted value at time level $n + 1$ using Eqs. (26f) and (26g) with the velocities at time level n

$$\frac{\tilde{R}^{n+1} - R^n}{\delta t} = (R_{\psi} \alpha + R_{\theta} \beta)^n$$

$$\frac{\tilde{Z}^{n+1} - Z^n}{\delta t} = (Z_{\psi} \alpha + Z_{\theta} \beta)^n.$$

- (2) Using the predicted values of the coordinates at time level $(n + 1)$ to define the plasma-vacuum interface, invert the vacuum matrix, Eq. (34), to define the total pressure P^{n+1} on the vacuum side of the interface, using Eqs. (35b) and (36).

(3) Iterate Eq. (37) for P^{n+1} throughout the plasma interior until the convergence criterion is met, using the boundary values from step (2).

(4) Advance the velocity equations, Eq. (26h) and (26i) using P^{n+1} obtained from step 3 to time center the spatial derivatives. For example

$$\begin{aligned} \frac{(u+\alpha)^{n+1} - (u+\alpha)^n}{\delta t} + \left(\frac{|\Delta\psi|^2}{\rho} \right)^n \frac{1}{2} (P_\psi^{n+1} + P_\psi^n) \\ + \left(\frac{\nabla\theta \cdot \nabla\psi}{\rho} \right)^n \frac{1}{2} (P_\theta^{n+1} + P_\theta^n) + R^{n+1/2} \end{aligned}$$

(5) Separate out the compressible velocity components u and v from the coordinate stream function ξ and expansion parameter T to be consistent with Eqs (16), (23), and (26c). The quantity $(u + \alpha)^{n+1}$ is surface averaged to give

$$u(\psi) = \frac{1}{2\pi} \int_0^{2\pi} (u + \alpha) d\theta + \frac{\dot{T}}{T^2} \frac{1}{2\pi} \int_0^{2\pi} RR_\theta (Z - Z_0)/R^m d\theta$$

which is used to determine $u(\psi)$ except on the last flux surface (plasma-vacuum boundary) where $u(\psi) = 0$ and the surface average equation is used to determine the expansion parameter $\dot{T}(t)$. The surface varying part of $(u + \alpha)^{n+1}$ gives the coordinate stream function

$$\xi_\theta = T\psi^n [(u + \alpha) - u(\psi)] + \frac{\dot{T}}{T} [RR_\theta (Z - Z_0)/R^m]$$

and $\xi_\psi = (\partial/\partial\psi) \int_0^\theta \xi_\theta d\theta$. The quantities α and β are then obtained from their definitions, Eq. (23).

- (6) Advance the conservation Eqs. (26a) and (26e) using the velocity variables at level $(n + 1)$ to time center the spatial derivatives.

Physically, the compressible wave motion is being treated implicitly but the incompressible motion and the material convection are being treated explicitly. Since the incompressible motion is still treated by explicit differencing, the Courant stability criteria for the incompressible (transverse) waves

$$\delta t \lesssim (\delta x / c_T)$$

must be obeyed. Since $c_T \sim c_F(Rq/a)^{-1}$. The maximum allowable time step is greatly increased over that for a fully explicit method.

H. Application

As an initial application of the dynamical grid method, we demonstrate the ability of this code to calculate the correct time evolution of a large aspect ratio tokamak plasma of circular cross section initially prepared close to an exact eigenmode, calculated by a linear normal modes, or δW code (Chance *et al.*, 1977). We show in Fig. 4 some of the normalized eigenvalues $\tilde{\omega}^2 = \omega^2 a^2 \rho_0 / B_0^2$ with poloidal mode number $m = 4$ for a cylindrical equilibrium with minor radius $a = 1$, wall radius $b = 1.1055$, toroidal periodicity length $R = 2\pi$, constant density and current with $\rho_0 = 1$, $q = 2.5$, and pressure profile $\mu_0 p_0 = 0.01[1 - (r/a)^2]B_0^2$. The modes divide into three groupings, associated with the three types of waves in ideal MHD.

Figure 5 shows the velocity vectors at time $t = 0$ and $t = 48\tau_f$ on the equal area finite-difference mesh used, initially prepared in the eigenmode

labeled A in Fig. 4. There are 18 zones in the radial direction and 50 in the θ direction. The calculation is run for 600 cycles with the average time step being 10 times larger than that allowed by the Courant criterion based on the fast waves, $0.08\tau_F$.

The theoretical period of this oscillation is $\tau = 13.8\tau_F$. Figures 6a and 6c present an interesting comparison. They represent the time histories of the $\nabla\psi$ component of the velocity, $\nabla\psi \cdot \vec{v} = u + \alpha$, the metric tensor component $|\nabla\psi|^2$, the plasma density at a given location, the eighth flux surface from the center, and the fifth theta surface from the horizontal. From Fig. 6a we see that the velocity consists of two components, one rapidly varying and the other slowly varying. The slowly varying component is largest in amplitude and corresponds to the slow nearly incompressible eigenmode we are trying to compute. The smaller amplitude high-frequency component arises primarily from the inexactness of the finite-difference initial conditions. Examination of Fig. 6b shows that there is no high-frequency component in the metric tensor element $|\nabla\psi|^2$. Figure 6c similarly shows that there is little or no low-frequency component in the time variation of the density. Thus, the separation of the velocity into a slowly varying incompressible part and a rapidly varying compressible part has been successful.

As a second example of the dynamical grid method, we describe its application to the computation of an ideal MHD timescale instability in a real tokamak experiment (Meade *et al.*, 1975; Jardin, 1978). Figure 7 illustrates the location of the coil currents (pluses and minuses), plasma magnetic flux surfaces (solid lines), and vacuum magnetic flux surfaces (dotted lines) in the Princeton Divertor Experiment, PDX. The original top bottom symmetric plasma is given a small asymmetrical perturbation and its time evolution is followed. It is found that the plasma is unstable to an axisymmetric motion

attracting it into one of the coils. In Fig. 8 we illustrate the magnetic configuration at time $t = 73 \tau_f$, after the instability has had a chance to grow. It is seen that the instability is not merely a rigid displacement but is larger on the inside of the torus and significantly distorts the plasma cross section. Further investigation (Jardin, 1978) not shown reveals that this instability can be stabilized by the presence of nearby conductors.

III. RESISTIVE TIME SCALE MHD SIMULATIONS

A. Introduction

The resistive time scale in a tokamak is that in which the particles, energy, and electrical current penetrate into and diffuse out of the entire discharge. It includes the time it takes for the current to build up to its steady-state value, the plasma heating time, the resistive skin time, and the particle and energy confinement times. The resistive time scale is very slow compared to all three wave transit times τ_f , τ_t , τ_s [Eq. (5)] and to the time it takes for particles and energy to diffuse along (parallel to) the magnetic field.

We consider only axisymmetric configurations so that a flux function $\chi(R,Z)$ exists, defined by Eq. (25). In this chapter we describe two very different methods for performing calculations on this time scale. The "zero mass" method (Hirschman and Jardin, 1979) uses an asymptotically valid reduced set of equations obtained by expanding in the small ratios of the Alfvén wave transit time to the resistive diffusion time, and of the perpendicular to parallel mobilities and thermal conductivities. It obtains as solubility constraints one-dimensional time evolution equations for the surface-averaged thermodynamic variables and magnetic fluxes. The condition that force balance be satisfied, $\nabla p = \vec{J} \times \vec{B}$, provides a constraint on the allowed motion. A

time-dependent coordinate transformation is used to describe the diffusion of plasma quantities through magnetic surfaces of changing shape.

The second method, the "large mass" method, artificially slows down and damps the Alfvén waves by enhancing the ion mass and viscosity. Since the physical results are known to be asymptotically independent of these parameters, we can increase their values by several orders of magnitude without modifying the physical solution. This allows us to bring together the Alfvén and transport time scales. Of course, one must verify *a posteriori* that the results obtained are indeed independent of the enhancement factors.

B. The zero-mass method

A formal ordering scheme based on the smallness of the quantity

$$\delta \equiv \tau_A / \tau_R \approx (\rho_0 / \mu_0)^{1/2} \eta / aB \quad (39)$$

shows that the inertial terms in the equation of motion, Eq. (4), can be neglected provided there are no Alfvén time scale instabilities. This equation is then replaced by the equilibrium constraint

$$\nabla p = \mathbf{j} \times \mathbf{B} \quad (4')$$

We again utilize the magnetic flux coordinate system introduced in Sec. IIB. The three component equations obtained by operating on Eq. (4') with $\mathbf{B} \cdot$, $\nabla \psi \times$, and $\nabla \theta \times \nabla \phi$ are

$$p(\vec{x}, t) = p(\psi, t) \quad , \quad (40a)$$

$$F(\vec{x}, t) = F(\psi, t) \quad , \quad (40b)$$

$$\Delta^* \chi + \frac{FF}{\chi} + \mu_0 R^2 p_\chi = 0 \quad , \quad (40c)$$

Here we have introduced the operator

$$\begin{aligned} \Delta^* a(\vec{x}) &\equiv R^2 \nabla \cdot R^{-2} \nabla a \\ &= R^2 J^{-1} \{ [(J/R^2) |\nabla \psi|^2 a_\psi + (J/R^2) \nabla \psi \cdot \nabla \theta a_\theta]_\psi \\ &\quad + [(J/R^2) \nabla \theta \cdot \nabla \psi a_\psi + (J/R^2) |\nabla \theta|^2 a_\theta]_\theta \} \quad . \end{aligned} \quad (41)$$

Equation (40c) is a two-dimensional, nonlinear differential constraint relating the surface quantities $p(\psi, t)$, $F(\psi, t)$, $\chi(\psi, t)$ to the magnetic surface geometry at all times. The incorporation of this constraint into the time evolution equations for the differential particle number, entropy, and magnetic fluxes is the central problem to be addressed in this section. We shall proceed to time differentiate Eq. (40c) and use the resulting linear equation to determine the velocity of the constant ψ coordinate surfaces, and the velocities of the magnetic fluxes relative to the constant ψ surfaces.

(i) One-dimensional transport equations

In a confined tokamak plasma, the motion of particles and heat along the magnetic field $\vec{B}$ is rapid compared with the resistive motion across magnetic surfaces. This leads to density and temperature profiles which are nearly uniform on a flux surface. Formally, expanding in the small parameter $\Delta \equiv \rho_e/a$, it can be shown that

$$p_j(\vec{x}, t) = p_j(\psi, t) + \Delta \tilde{p}(\psi, \theta, t) \quad , \quad (42a)$$

$$T_j(\vec{x}, t) = T_j(\psi, t) + \Delta \tilde{T}(\psi, \theta, t) \quad , \quad (42b)$$

where

$$p_j(\psi, t) \equiv \langle p_j(\vec{x}, t) \rangle \quad , \quad (42c)$$

$$T_j(\psi, t) \equiv \langle T_j(\vec{x}, t) \rangle \quad , \quad (42d)$$

and we have introduced the flux surface average operator

$$\langle a \rangle \equiv \int_0^{2\pi} d\theta J a / \int_0^{2\pi} d\theta J \quad . \quad (43)$$

In view of Eqs. (42a) and (42b), the local conservation laws contain more information than is necessary to advance the dominant surface averaged part of the thermodynamic variables. The extraneous information can be annihilated by flux surface averaging the conservation equations. The one-dimensional set of transport equations so obtained can be summarized as follows:

$$(N_j')_t + (N_j' u)_\psi = -(V' \Gamma_j)_\psi + V' \langle S_{nj} \rangle \quad , \quad (44a)$$

$$\sigma_t' + (\sigma' u)_\psi = \frac{-2}{5} \left(\frac{\sigma_e'}{p} \right) S \quad , \quad (44b)$$

$$(\sigma_e')_t + (\sigma' u)_\psi = \frac{-2}{5} \left(\frac{\sigma_e'}{p_e} \right) S_e \quad , \quad (44c)$$

$$\chi'_t + (\chi'u)_\psi = (E'_\parallel)^* \psi \quad , \quad (44d)$$

$$\Psi'_t + (\Psi'u)_\psi = 0 \quad . \quad (44e)$$

The variables advanced in time in Eqs. (44a) through (44e) are the differential particle number for each species $N'_j(\psi, t) \equiv n_j(\psi, t)V'$, the differential plasma entropy $\sigma'(\psi, t) \equiv [p(\psi, t)]^{3/5}V'$, the differential electron entropy $\sigma'_e(\psi, t) \equiv [p_e(\psi, t)]^{3/5}V'$, the poloidal flux density per radian [from Eq. (25)]

$$\chi'(\psi, t) = (2\pi)^{-2} \partial \left(\int_0^\psi d\vec{x} \cdot \vec{B} \cdot \nabla \phi \right) / \partial \psi = J \vec{B} \cdot \nabla \theta \quad , \quad (45)$$

and the toroidal flux density per radian

$$\Psi'(\psi, t) \equiv (2\pi)^{-2} \partial \left(\int_0^\psi d\vec{x} \cdot \vec{B} \cdot \nabla \phi \right) / \partial \psi = (2\pi)^{-2} F \langle R^{-2} \rangle V' \quad , \quad (46)$$

with the differential volume being defined as

$$V'(\psi, t) \equiv \partial \left(\int_0^\psi d\vec{x} \right) / \partial \psi = 2\pi \int_0^{2\pi} J d\theta \quad . \quad (47)$$

Here n_j is the particle density of species j , p_j is the pressure of species j , and p is the combined electron and ion pressure $p = p_e + p_i$. The surface averaged entropy and electron entropy terms appearing in Eqs. (44b) and (44c) are defined as

$$S \equiv -\langle \vec{J} \cdot \nabla \phi \rangle E'_\parallel + \frac{1}{V'} \sum_{j=i,e} [Q_j]_\psi - \langle S_p \rangle \quad , \quad (48a)$$

$$S_e \equiv -\langle \vec{j} \cdot \nabla \phi \rangle E_{\parallel}^* + \frac{i}{V'} [Q_e]_{\psi} + (p_i)_{\psi} \Gamma_i / n_i$$

$$+ \langle u_i \cdot \nabla \cdot \pi_i \rangle - Q_{\Delta e} - \langle S_{pe} \rangle \quad (48b)$$

Besides the electron-ion equipartition term $Q_{\Delta e}$ there are five transport model dependent quantities in Eqs. (44) describing relative slippage in a two-fluid resistive high temperature plasma; the parallel electric field $\alpha \equiv E_{\parallel}^* = \langle \vec{E} \cdot \vec{B} \rangle / \langle \vec{B} \cdot \nabla \phi \rangle$, the relative particle flux $\beta \equiv \Gamma_i / n_i = \Gamma_e / n_e$, the ion heat flux $\gamma \equiv Q_i = V' (\langle \vec{q}_i \cdot \nabla \phi \rangle + 5/2 \Gamma_i T_i)$, the electron heat flux $\delta \equiv Q_e = V' (\langle \vec{q}_e \cdot \nabla \phi \rangle + 5/2 \Gamma_e T_e)$, and the viscous heating term $\varepsilon \equiv V' \langle u_i \cdot \nabla \cdot \pi_i \rangle$. It is the objective of transport theory to relate these fluxes to the vector of forces consisting of derivatives of the fundamental thermodynamic variables. It is convenient to define the force vector as

$$\vec{F} \equiv [1, \langle n' / V' \rangle_{\psi}, \langle \sigma' / V' \rangle_{\psi}, \langle \sigma_e' / V' \rangle_{\psi}, \langle \chi' V' \langle |\nabla \phi|^2 / R^2 \rangle / F \rangle_{\psi}]$$

To specify a particular transport model, it is then sufficient to supply the 25 functions $\langle \alpha^j, \beta^j, \gamma^j, \delta^j, \varepsilon^j \rangle$, $j = 0, 4$, according to the format

$$\alpha = \sum_{j=0}^4 \alpha^j F^j, \quad \beta = \sum_{j=0}^4 \beta^j F^j, \quad \text{etc.} \quad (50)$$

These functions are tabulated in Appendix C of Jardin (1981) for several transport regimes.

(ii) Incorporation of the equilibrium constraint

To complete the specification of the transport problem, we must provide an equation to obtain the one-dimensional toroidal flux velocity $u(\psi, t)$ that

appears in all of Eqs. (44a) through (44e), and to obtain the two-dimensional coordinate velocity $\vec{v}_g$, Eqs. (22) and (23), so that the coordinates of the magnetic geometry can be advanced through Eqs. (26f) and (26g). An equation to determine both of these quantities is obtained by time differentiating the plasma equilibrium equation, Eq. (40c), and substituting for the time derivatives of σ' , χ' , and Ψ' from Eq. (44). Using these and the convective derivative Eq. (15), one obtains a linear equation for the normal component of the toroidal flux velocity Ω

$$\left[\frac{1}{\mu_0 R^2} \chi' \Delta^* + L_0 + L_1 \right] \Omega = 2 \vec{J} \cdot \nabla \phi (E_{\parallel}^*)_{\psi} + \frac{1}{2\mu_0} B_p^2 \left[(E_{\parallel}^*)_{\psi} / \chi' \right]_{\psi} - \frac{2}{3} S_{\psi} \quad (51)$$

Here $B_p = \chi' |\nabla \phi| / R$ is the magnitude of the poloidal field, $E_{\parallel}^*$ and S appear in Eqs. (44), and Δ^* is defined in Eq. (41). The new operators appearing in Eq. (51) are defined as

$$L_0(a) \equiv \frac{1}{\mu_0 R^2} \{ F^2 (V' \langle R^{-2} \rangle)^{-1} [(V' / \chi') \langle R^{-2} a \rangle]_{\psi} \}_{\psi} \quad (52a)$$

$$L_1(a) \equiv [(p_{\psi} / \chi')_{\psi} + (1 / \mu_0 R^2) (FF_{\psi} / \chi')_{\psi}] a + 5/3 \{ (p/V') [(V' / \chi') \langle a \rangle]_{\psi} \}_{\psi} \quad (52b)$$

We note that $L_0/L_1 \sim B_t^2/p$ and $\mu_0 L_0 / \chi' \Delta^* \sim B_t^2/B_p^2$ where $B_t \sim R \vec{B} \cdot \nabla \phi$ is the magnitude of the toroidal field. Thus for a tokamak plasma with $\beta \sim B_p^2/B_t^2 \ll 1$, L_0 is the dominant operator in Eq. (51).

We have used the symbol Ω in Eq. (51) to denote $(\chi' / \Psi') \equiv q^{-1}$ times the absolute time derivative of the toroidal flux function. Thus

$$\Omega(\psi, \theta, t) \equiv q^{-1} \frac{d\psi}{dt} \Big|_{\chi} = -\chi'(\psi, t) [u(\psi, t) + \alpha(\psi, \theta, t)] \quad (53)$$

Here the second equality follows from Eqs. (15), (16a), and (44e). If we impose the natural boundary conditions that $u = 0$ at $\psi = \psi_{\max} = 1$, then Eq. (53) can be unraveled to obtain T_t , ξ_θ , ξ_ψ , α , β , and u from Ω by using Eq. (23). Thus, at $\psi = 1$ where $u = 0$, we have

$$T_t/T = \langle R^{-m} \Omega \rangle / \langle RR_\theta (Z - Z_0) / JR^m \rangle \chi' \quad (54a)$$

With T_t/T given, we compute $u(\psi, t)$, $\xi_\theta(\psi, \theta, t)$, and $\xi_\psi(\psi, \theta, t)$ elsewhere as

$$u(\psi, t) = [\langle R^{-m} \Omega \rangle / \chi' - T_t/T \langle RR_\theta (Z - Z_0) / JR^m \rangle] / \langle R^{-m} \rangle \quad (54b)$$

$$\xi_\theta(\psi, \theta, t) = (T_t/T) RR_\theta (Z - Z_0) / R^m - (u + \Omega / \chi') J / R^m \quad (54c)$$

$$\xi_\psi(\psi, \theta, t) = \partial / \partial \psi \int_0^\theta \xi_\theta d\theta \quad (54d)$$

Eqs. (23a) and (23b) may then be used to compute the coordinate velocities $\alpha(\psi, \theta, t)$ and $\beta(\psi, \theta, t)$.

The description employed here is partly Lagrangian in that the (R, Z) coordinates of the ψ, θ grid evolve in time with the velocity $\vec{v}_g = (\alpha, \beta)$ according to Eqs. (26f) and (26g). Thus the two-dimensional variable $\Omega(\psi, \theta)$ which is obtained from Eq. (51) and describes how the toroidal flux surfaces change in time, gives several pieces of information. Its flux surface average, Eqs. (54a) and (54b), gives the uniform expansion of the coordinate system, $T_t(t)$, and the one-dimensional velocity $u(\psi, t)$ with respect to an Eulerian grid equally spaced in the geometrical coordinate ψ . Its surface

varying part gives the Lagrangian coordinate stream function ξ which describes how the constant ψ surfaces change their shape.

Physically we have replaced the equation of motion, Eq. (3a)', which evolves the velocity field in time, with a global constraint equation, Eq. (51), which determines the toroidal flux velocity as an instantaneous boundary value problem. This velocity field so determined is free of the fast time scale motion associated with wave phenomena, but is such that it will evolve one equilibrium configuration into another. The one-dimensional transport Eqs. (44) then describe how the other thermodynamic and field variables evolve relative to the toroidal flux velocity. Other, similar formulations have been given previously (Byrne and Klein, 1978; Grad, 1970; Nelson and Grad, 1977; Todd and Grimm, 1975; Helton et al., 1977; Hogan 1978) in which the plasma equilibrium equation is not time differentiated, but is resolved each time step. While being conceptually more straightforward, these methods must solve a nonlinear elliptic equation each time step, which can be very time consuming.

(iii) Numerical method

Once the transport coefficients are given in terms of the plasma variables and their derivatives through Eq. (50), the system of Eqs. (44), (26f), (26g), and (51), together with the necessary definitions in those sections, provide a closed system of equations, needing only the source functions $\langle S_{nj} \rangle$, $\langle S_p \rangle$, $\langle S_{pe} \rangle$, and the boundary conditions to specify a problem completely. The numerical method utilized is described here. If we denote by $\vec{Y}$ the state vector consisting of the one-dimensional variables being advanced in time in Eq. (44), i.e., N_j' , σ' , σ_e' , χ' , and Ψ' by $A(\vec{Y}, \vec{X})$, the vector of right-hand sides of Eq. (44) by $L(\vec{Y}, \vec{X})$, the operator on the left-hand side of

Eq. (51) operating on Q , and by $B(\vec{Y}, \vec{X})$ the right-hand side of Eq. (51), then the evolution equations take the compact form

$$\vec{x}_t = \vec{v}_g, \quad (55a)$$

$$\vec{Y}_t + (\vec{Y}u)_\phi = A(\vec{Y}, \vec{X}), \quad (55b)$$

$$L(\vec{Y}, \vec{X})\{Q\} = B(\vec{Y}, \vec{X}). \quad (55c)$$

Here, of course, $\vec{v}_g$ and u are obtainable from Q through Eqs. (54) and the definitions, Eqs. (16) and (23). The notation used in Eq. (55) underscores the fact the operator L is an explicit function of $\vec{Y}$ and $\vec{X}$ and acts on the function Q .

As discussed in Hirshman and Jardin (1979), the formulation adopted here ensures that the toroidal flux velocity u is "small" compared to the other diffusive velocities, i.e., $u \sim B_p/B_T \ll 1$. This justifies adopting the following prescription for advancing the solution from time level n to time level $(n+1)$:

$$\vec{x}^{(n+1)} - \vec{x}^{(n)} = \Delta t \vec{v}_g^{(n)}, \quad (55a)'$$

$$\vec{Y}^{(n+1)} - \vec{Y}^{(n)} + \Delta t [\vec{Y}^{(n+1/2)} u^{(n)}]_\phi = \Delta t A[\vec{Y}^{(n+1/2)}, \vec{x}^{(n+1/2)}], \quad (55b)'$$

$$L[\vec{Y}^{(n+1)}, \vec{x}^{(n+1)}]\{Q^{(n+1)}\} = B[\vec{Y}^{(n+1)}, \vec{x}^{(n+1)}]. \quad (55c)'$$

Since only the old time level of u , $u^{(n)}$, appears in Eq. (55b)', Eqs. (55b)' and (55c)' decouple and their finite difference solution can be considered

separately.

Let us consider first the solution of Eq. (55b) for the vector Y . Since this equation contains all of the transport time scales, a time centered implicit unconditionally stable (Crank-Nicolson) method is called for. Since the right-hand side of Eq. (44e) for Ψ' is zero, this equation splits off from the others and is solved simply by applying centered spatial difference operators to obtain a tridiagonal system. For the remaining 4-vector, we introduce the notation that a subscript j corresponds to location $\phi = [j - 1/2]\Delta\phi$ and a superscript n corresponds to time level $t = (n-1)\Delta t$. The finite difference form of the surface averaged transport equations, Eqs. (44a) - (44d), assumes the form

$$\vec{A}_j^n \cdot \vec{\Phi}_{j+1}^{n+1} - \vec{B}_j^n \cdot \vec{\Phi}_j^{n+1} - \vec{C}_j^n \cdot \vec{\Phi}_{j-1}^{n+1} + \vec{D}_j^n = 0 \quad (56)$$

Here $\vec{\Phi}_j^n$ is the solution vector at position j and time level n

$$\vec{\Phi}_j^n = [N_j'^n, \sigma_j'^n, \sigma_{ej}'^n, \chi_j'^n]$$

Each of the tridiagonal elements $\vec{A}_j^n, \vec{B}_j^n, \vec{C}_j^n$ is a full 4×4 matrix with elements given in terms of the α^i, β^i , etc. of Eq. (50). These elements and those of the 4 vectors $\vec{D}_j^n$ can be found in Appendix A of Jardin (1981). The matrix tridiagonal system Eq. (56) is inverted the usual way with fixed value or derivative boundary conditions easily being incorporated at the outer boundary.

We next consider equations of the form

$$\begin{aligned}
 (\Omega_\theta^a + \Omega_\psi^b)_\psi + (\Omega_\theta^c + \Omega_\psi^a)_\theta + d\Omega + JH[P(V' \langle H\Omega \rangle)_\psi]_\psi \\
 + JK[G(V' \langle K\Omega \rangle)_\psi]_\psi + e = 0 \quad .
 \end{aligned}
 \tag{57}$$

Subscripts here denote differentiation and brackets denote the usual flux surface average operator, defined in Eq. (43). Equation (57) is identical to Eq. (51) or (55c) if we make the identification

$$a(\psi, \theta) = -(R_\theta R_\psi + Z_\theta Z_\psi)/J \quad , \tag{58a}$$

$$b(\psi, \theta) = (R_\theta^2 + Z_\theta^2)/J \quad , \tag{58b}$$

$$c(\psi, \theta) = (R_\psi^2 + Z_\psi^2)/J \quad , \tag{58c}$$

$$d(\psi, \theta) = J[\mu_o(p_\psi/\chi')_\psi + R^{-2}(FF_\psi/\chi')_\psi]/\chi' \quad . \tag{58d}$$

$$e(\psi, \theta) = -\mu_o J\{2\vec{J} \cdot \nabla \phi(E_\parallel^*)\}_\psi + (1/2\mu_o)B_p^2[(E_\parallel^*)_\psi/\chi']_\psi - (2/3)S_\psi \quad , \tag{58e}$$

$$K(\psi, \theta) = 1/R^2 \chi' \quad , \tag{58f}$$

$$H(\psi, \theta) = 1/\chi' \quad , \tag{58g}$$

$$G(\psi) = F^2/(V' \langle R^{-2} \rangle) \quad , \tag{58h}$$

$$P(\psi) = (5/3)\mu_o p/V' \quad . \tag{58i}$$

Equation (57) is a linear nonlocal two-dimensional equation for Ω to be inverted each time step determining the time evolution of the toroidal flux surfaces by determining the velocity variables u , α , and β through Eq. (54). Mathematically, it is a generalized differential equation requiring boundary conditions for $\Omega(\psi_{\max}, \theta)$ and a single boundary condition for $\Omega(0, \theta)$. The θ integrals (or flux surface averages) in Eq. (21) make it nonlocal and thus impractical to solve by iteration techniques. Instead we solve this equation directly by Fourier decomposing in the angle coordinate θ and by finite differencing in the surface coordinate ψ . Fourier techniques are natural in that the θ integrals and θ derivatives are treated on an equal footing. The efficiency of this method lies in the fact that for most problems, only 5-10 Fourier harmonics need be kept at each ψ surface to obtain accurate solutions.

For simplicity we restrict discussion to systems which are symmetric about the plasma midplane. Then Ω has a discrete Fourier representation of the form $\Omega(\psi, \theta) = \Omega^0(\psi) + \sum_{m=1}^M \Omega^m(\psi) \cos m\theta$, and the coefficients can likewise be expanded as $b(\psi, \theta) = b^0(\psi) + \sum_{m=1}^M b^m(\psi) \cos m\theta$, etc. By substituting these expansions into Eq. (57), evaluating the product terms, and finite differencing in ψ with the notation $\psi_j = j\Delta\psi$, we obtain the canonical system

$$\vec{A}_j \cdot \vec{\Omega}_{j+1} - \vec{B}_j \cdot \vec{\Omega}_j + \vec{C}_j \cdot \vec{\Omega}_{j-1} + \vec{D}_j = 0 \quad (59)$$

Here $\vec{\Omega}_j \equiv \{\Omega_j^0, \Omega_j^2, \dots, \Omega_j^M\}$ is a vector containing the coefficients of the $M+1$ Fourier harmonics of Ω at location ψ_j . The elements of the $(M+1) \times (M+1)$ matrices A_j , B_j , C_j , and the length $(M+1)$ vectors D_j are defined in terms of the coefficient harmonic amplitudes b^m , c^m , etc. in Appendix B of Jardin (1981).

The system of Eqs. (59), identical in form to Eq. (56), is solved according to the same algorithm. We note that a single boundary value, for Q_0^0 , need be supplied at the origin. As discussed in Jardin (1981), setting $Q_0^0 = 0$ corresponds to no toroidal flux being created there. Boundary values for all the $(Q_j^m; m = 0, M)$ must be supplied at the outer boundary, $j = J$. These either are prescribed or are calculated by Green's function techniques to be consistent with currents in external coils.

As an initial condition, the equilibrium (Grad-Shafranov) equation, Eq. (40c), is solved at time $t = 0$ using a flux coordinate equilibrium code (DeLucia et al. 1980). In subsequent time steps, it is generally not necessary to solve the equilibrium equation since the solution of Eq. (51) describes how the equilibrium flux surfaces change from one time step to the next. Some truncation error will accumulate in this process so that, as time progresses, the equilibrium equation will not be exactly satisfied. One can constantly monitor this error, and when it builds up to a noticeable value, the flux coordinate equilibrium code can be called in and run in an adiabatic mode to reduce this error to zero. In practice, this readjustment of the equilibrium is seldom, if ever, needed to maintain the accumulated error in the equilibrium equation to a few percent over the course of a complete problem.

(iv) Application

As an example of the numerical solution procedure described, we analyze the performance of a proposed fusion experiment. A deuterium-tritium tokamak plasma is modeled as it is heated to near ignition conditions by intense neutral particle injection, and then is subjected to major radius compression causing the plasma to ignite. Standard (Jardin, 1981) anomalous loss terms,

neutral gas refilling models, thermonuclear fusion source terms, and neutral beam source terms have been included to increase the realism of the model.

Using these source and transport models, the calculation proceeds as follows. An initial plasma equilibrium is computed with $R = 200$ cm, $a = 60$ cm, toroidal magnetic field at axis $B_T = 61$ kG, and with plasma current $I_p = 2.47$ MA. The calculation proceeds for 0.5 sec with no external heating source, at which time the 14 MW neutral beam source term is turned on and left on until $t = 1.6$ sec. The major radius compression takes place in the interval $1.5 \text{ sec} < t < 1.6 \text{ sec}$, and the calculation is halted at $t = 2.0$ sec.

The boundary conditions used in this calculation are as follows. The values of n , T_e , T_i , and χ' were held at fixed values at the plasma-vacuum boundary (pedestal boundary conditions) so that $n_b = 2 \times 10^{13} \text{ (cm)}^{-3}$, $T_{ib} = T_{eb} = 30 \text{ eV}$, $q_b = 2.68$. The toroidal field in the vacuum is $\vec{B}_T = g_v \Delta\phi$, with g_v a constant which gives the value $|B_T| = 61 \text{ kG}$ at $R = 200$ cm. The surface averaged part of the toroidal flux velocity at the boundary, u_b , is determined self-consistently so that the equilibrium equation remains satisfied across the plasma-vacuum boundary. The surface varying part of the toroidal flux velocity is prescribed to be zero except during the compression phase at which time it is given a value corresponding to a uniform, radially inward velocity.

The results of this calculation are illustrated in Figs. 9 through 11. Figure 9 shows the magnetic surfaces at $t = 0$ (before compression) and at $t = 2$ sec (after compression). During the compression, the position of the magnetic axis has decreased from $R = 207$ cm to 140 cm, and the minor radius has decreased from $a = 61$ cm to 48 cm. Figure 10 shows the midplane values of T_e , T_i , n , and J_ϕ at various times during the calculation, and Fig. 11 shows the central values of the temperatures and density as a function of the time t .

C. The large mass method

The method described in Sec. III B for integrating the MHD equations on the resistive time scale consisted of taking the asymptotic limit of the equations as the plasma mass goes to zero, or equivalently, as the Alfvén velocity $V_A \sim B/(\mu_0 \rho_0)^{1/2}$ goes to infinity. This has the effect of removing the wave transit time scales completely from the problem. The penalty paid for doing so is that now an elliptic equation, Eq. (51), must be solved each time step, reflecting the fact that there is an infinite propagation velocity.

The method described in this section (Jardin and Park, 1981) is the opposite of the zero mass method in that it artificially increases the plasma mass to slow down rather than speed up the Alfvén waves. It is shown that these waves can be slowed down by several orders of magnitude, to almost the transport time scale, without affecting the time averaged physical results. This result follows from the fact that there are two distinct time scales in the equations, ideal (fast) and resistive (slow), with no intermediate time scales present. The situation becomes quite different when we go to three dimensions and intermediate time scales arise associated with resistive tearing and interchange modes.

(i) Reduced compressible magnetohydrodynamic equations

For simplicity, we restrict consideration to a specific subset of the resistive MHD equations (1) through (4) which describe a single temperature ($T_e = T_i \equiv T$) collisional regime plasma so that the viscous heating term π is 0 and a simplified Ohm's law applies. These one fluid equations are

$$\frac{\partial}{\partial t} \vec{m} + \nabla \cdot \left(\frac{1}{\rho} \vec{m} \vec{m} \right) + \nabla p = \vec{J} \times \vec{B} \quad , \quad (60a)$$

$$\frac{3}{2} \frac{\partial}{\partial t} p + \nabla \cdot \left(\vec{q} + \frac{5}{2} \frac{p}{\rho} \vec{m} \right) = \vec{J} \cdot \vec{E} + s_e \quad , \quad (60b)$$

$$\frac{\partial p}{\partial t} + \nabla \cdot \vec{m} = s_\rho \quad , \quad (60c)$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \quad , \quad (60d)$$

$$\vec{E} + \frac{1}{\rho} \vec{m} \times \vec{B} = \eta \vec{J} \quad , \quad (60e)$$

$$\mu_0 \vec{J} = \nabla \times \vec{B} \quad , \quad (60f)$$

$$k_B T = \left(\frac{M_i}{2\rho} \right) p \text{ (ev)} \quad , \quad (60g)$$

$$\eta = 0.002 (k_B T)^{-3/2} \text{ (Ohm-m)} \quad . \quad (60h)$$

In an axisymmetric geometry with symmetry angle ϕ , the magnetic field is written as in Eq. (25) in terms of the poloidal flux per radian χ and the toroidal field function F

$$\vec{B} = \nabla \phi \times \nabla \chi + F \nabla \phi \quad . \quad (61)$$

The momentum density $\vec{m}$ is written in terms of a stream function Λ , a toroidal component w , and a potential Ω , thus

$$\vec{m} = \nabla \phi \times \nabla \Lambda + w \nabla \phi + \nabla \Omega \quad . \quad (62)$$

We are looking for solutions with slowly changing boundary conditions and small mass flow so that the ordering in the reciprocal magnetic Reynolds number

$$(\mathbf{m}^2/\rho) \sim \epsilon^2 B^2/\mu_0 \rho \sim \epsilon^2 p \quad (63)$$

with $\epsilon^2 \ll 1$ is appropriate. Here we have taken the plasma β to be order unity in the ϵ expansion. The reduced equations are then obtained by using Eq. (63) to justify dropping the convective derivative term in Eq. (60a) and adding a viscous term. Equation (60a) becomes

$$\frac{\partial \mathbf{m}}{\partial t} - \nu \nabla^2 \mathbf{m} = \mathbf{j} \times \mathbf{B} - \nabla p \quad (60a)'$$

Substitution of the definitions, Eq. (61) and (62) into Eqs. (60a)', (60b) through (60g), and taking the curl and divergence of Eq. (60a) and performing the necessary vector manipulations yields the set of scalar evolution equations solved numerically,

$$\frac{\partial}{\partial t} \nabla^2 \Omega + \nabla \cdot \left(\frac{\Delta^* \chi}{\mu_0 R^2} \nabla \chi + \frac{F}{\mu_0 R^2} \nabla F + \nabla p - \nu \nabla (\nabla^2 \Omega) \right) = 0 \quad (64a)$$

$$\frac{\partial}{\partial t} \Delta^* A + R^2 \nabla \cdot \left(\frac{\Delta^* \chi}{\mu_0 R^2} \nabla \chi \times \nabla \phi + \frac{F}{\mu_0 R^2} \nabla F \times \nabla \phi - \frac{\nu}{R^2} \nabla (\Delta^* A) \right) = 0 \quad (64b)$$

$$\frac{\partial}{\partial t} w + \frac{1}{\mu_0} (\nabla \phi \times \nabla g) \cdot \nabla \psi - \nu \Delta^* w = 0 \quad (64c)$$

$$\frac{\partial p}{\partial t} + \nabla^2 \Omega + \vec{\nabla} \cdot \vec{\Gamma}_\parallel = s_p \quad (64d)$$

$$\frac{\partial}{\partial t} \chi + \frac{1}{\rho} (\nabla \phi \times \nabla A \cdot \nabla \chi + \nabla \Omega \cdot \nabla \chi) = \frac{\eta}{\mu_0} \Delta^* \chi \quad (64e)$$

$$\frac{\partial}{\partial t} F + R^2 \nabla \cdot \left(\frac{F}{\rho R^2} (\nabla \phi \times \nabla A + \nabla \Omega) - \frac{\omega}{\rho R^2} \nabla \phi \times \nabla \chi \right) = \frac{R^2}{\mu_0} \nabla \cdot \frac{\eta}{R^2} \nabla F, \quad (64f)$$

$$\frac{\partial}{\partial t} p^{3/5} + \nabla \cdot \left(\frac{p^{3/5}}{\rho} (\nabla \phi \times \nabla A + \nabla \Omega) \right) = \frac{2}{5} p^{-2/5} (\eta^2 - \nabla \cdot \vec{q} + S_p). \quad (64g)$$

Here

$$\vec{\Gamma}_{\parallel} = \frac{-\vec{B}\vec{B}}{B^2} \cdot D_{\parallel} \eta_{\perp} \nabla p \quad (65a)$$

is the flux of particles parallel to the magnetic field

$$\vec{q} = \frac{-\vec{B}\vec{B}}{B^2} \cdot D_{\parallel} \eta_{\perp} \nabla p - (15.2) \frac{p}{\mu_0 B^2} \rho \eta_{\perp} \nabla(p/\rho) \quad (65b)$$

is the heat flux vector, $\Delta^* \equiv R^2 \nabla \cdot R^{-2} \nabla$ is the toroidal elliptic operator, and $R = |\nabla \phi|^{-1}$ measures the distance from the axis of symmetry.

The set of Eqs. (64) offers several advantages over more primitive forms. By taking the divergence and the curl of the momentum equation, the shear Alfvén wave motion has been removed from the velocity potential variable Ω and the fast compressible wave motion has been removed from the variables A and u . This partial isolation of the different characteristics can lead to dramatic improvements in the accuracy and the stability of the numerical solutions.

A physically interesting subset of Eqs. (64) is that obtained by dropping Eq. (64a), (64d), and (64g) and setting $\Omega = 0$ and $\rho = \text{const.}$ in Eqs. (64b), (64c), (64e), and (64f). This is the familiar incompressible approximation. The fact that the incompressible equations are a subset of our full set of equations can be exploited in developing and debugging a numerical solution

method for the full set of equations.

(ii) Physical interpretation

Here we discuss the replacement of the momentum Eq. (60a) with the reduced momentum equation (60a)' by interpreting the equations using multiple time scale formalism. The reduced Eqs. (64) expressed in vector form are

$$\frac{\partial \mathbf{m}}{\partial t} - \nabla^2 \mathbf{m} = \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p \quad (66a)$$

$$\frac{\partial}{\partial t} \mathbf{B} = \nabla \times \left(\frac{1}{\rho} \mathbf{m} \times \mathbf{B} \right) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) \quad (66b)$$

$$\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{m} + \nabla \cdot \mathbf{j}_{\parallel} = S_p \quad (66c)$$

$$\frac{\partial}{\partial t} p^{3/5} + \nabla \cdot \left(\frac{p^{3/5} \mathbf{m}}{\rho} \right) = \frac{2}{5} p^{-2/5} (\eta^2 - \nabla \cdot \mathbf{q} + S_p) \quad (66d)$$

Equations (66) contain phenomena occurring on two distinct time scales, the ideal time scale

$$\tau_0 \sim \tau_f \sim (a/B)(\mu_0 \rho_0)^{1/2}$$

and the resistive time scale

$$t_1 \sim \tau_R \sim a^2 \mu_0 / \eta \sim \epsilon^{-1} \tau_0 \quad ,$$

where $\epsilon = (\eta/aB)(\rho_0/\mu_0)^{1/2} \ll 1$.

For boundary conditions changing on the (slow) t_1 time scale, we look for an ordering that describes a configuration which is always near ($\sim \epsilon$) equilibrium but in which the equilibrium evolves on the t_1 time scale. Such a

description is given formally by the multiple time scale method with the ordering

$$\eta = \varepsilon \eta, \quad \vec{q} = \varepsilon \vec{q}, \quad \vec{l} = \varepsilon \vec{l} \quad , \quad (67a)$$

$$\vec{B} = \vec{B}_0(t_1) + \varepsilon \vec{B}_1(t_0, t_1) + \dots \quad , \quad (67b)$$

$$\rho = \rho_0(t_1) + \varepsilon \rho_1(t_0, t_1) + \dots \quad , \quad (67c)$$

$$p = p_0(t_1) + \varepsilon p_1(t_0, t_1) + \dots \quad , \quad (67d)$$

$$\vec{m} = \varepsilon \vec{m}_1(t_0, t_1) + \dots \quad , \quad (67e)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \dots \quad , \quad (67f)$$

with t_0 and t_1 being treated as independent variables. Substitution of the ordering, Eq. (67), into Eq. (66) yields, to order ε^0

$$(\nabla \times \vec{B}_0) \times \vec{B}_0 = \mu_0 \nabla p_0 \quad . \quad (68)$$

and to order ε^1

$$\frac{\partial \vec{m}_1}{\partial t_0} - \nabla^2 \vec{m}_1 = \mu_0^{-1} [(\nabla \times \vec{B}_1) \times \vec{B}_0 + (\nabla \times \vec{B}_0) \times \vec{B}_1] - \nabla p_1 \quad . \quad (69)$$

Thus, the zero-order magnetic field and pressure are in equilibrium without zero-order mass flow, and the first-order mass flow evolves on the fast time scale, being driven by the small departure of p and $\vec{B}$ from an equilibrium

configuration.

Substitution of Eq. (67) into Eq. (66b) gives, to order ε

$$\frac{\partial \vec{B}_0}{\partial t_1} + \frac{\partial \vec{B}_1}{\partial t_0} = \nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) . \quad (70)$$

Averaging Eq. (70) over the t_0 time scale and subtracting that average from Eq. (70) gives an equation for the slow time scale evolution of the equilibrium magnetic field $\vec{B}_0$

$$\frac{\partial \vec{B}_0}{\partial t_1} = \nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) \quad (71)$$

where $\langle \vec{m}_1 \rangle(t_1) \equiv \tau^{-1} \int \tau m_1(t_0, t_1) dt_0$ denotes the average of $\vec{m}_1$ over the t_0 time scale, and an equation for the fast time scale evolution of $\vec{B}_1$

$$\frac{\partial \vec{B}_1}{\partial t_0} = \nabla \times \left[\frac{(\vec{m}_1 - \langle \vec{m}_1 \rangle)}{\rho_0} \times \vec{B}_0 \right] . \quad (72)$$

Note that we used the fact that $\langle \vec{B}_1 \rangle(t_1) = 0$, which is implied by the two time scale ordering of Eq. (67). Similar operations on the density and pressure Eqs. (66c) and (66d) yield separate evolution equations for ρ_0 , ρ_1 , p_0 , and p_1

$$\frac{\partial \rho_0}{\partial t_1} + \nabla \cdot [\langle \vec{m}_1 \rangle + \langle \vec{\Gamma}_1 \rangle] = 0 , \quad (73)$$

$$\frac{\partial \rho_1}{\partial t_0} + \nabla \cdot [\vec{m}_1 - \langle \vec{m}_1 \rangle] = 0 , \quad (74)$$

$$\frac{\partial p_0^{3/5}}{\partial t_1} + \nabla \cdot \left\langle \frac{p_0^{3/5} \vec{m}}{\rho_0} \right\rangle = \frac{2}{5} \frac{p_0^{-2/5}}{p_0} (\eta p^2 - \nabla \cdot \vec{q}) , \quad (75)$$

$$\frac{\partial p_1^{3/5}}{\partial t_0} + \nabla \cdot \left[\frac{p_0^{3/5} \vec{m}}{\rho_0} - \left\langle \frac{p_0^{3/5} \vec{m}}{\rho_0} \right\rangle \right] = 0 \quad (76)$$

A description of the fast t_0 time scale physics is given by Eqs. (69), (72), (74), and (76). Taking the t_0 derivative of Eq. (69) and substituting from Eqs. (72) and (76) gives dispersion relation for the fast time scale mass flow

$$\begin{aligned} \frac{\partial^2 \vec{m}_1}{\partial t_0^2} - \nu \nabla^2 \frac{\partial \vec{m}_1}{\partial t_0} = \mu_0^{-1} \{ \nabla \times [\nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 \right)] \times \vec{B}_0 \\ + (\nabla \times \vec{B}_0) \times [\nabla \times \left(\frac{\vec{m}_1}{\rho_0} \times \vec{B}_0 \right)] \} + \nabla \left(\frac{\vec{m}_1}{\rho_0} \cdot \nabla p_0 + \frac{5}{3} p_0 \nabla \cdot \frac{\vec{m}_1}{\rho_0} \right). \quad (77) \end{aligned}$$

With $\nu = 0$, Eq. (77) is just the ideal MHD equation of motion, containing both the transverse and the longitudinal wave branches. Since there are no unstable modes in two dimensions, the fast time scale behavior is simply that of small amplitude oscillations about the slowly varying equilibrium fields. With $\nu > 0$, these oscillations are damped. Setting $\nu = 2aB_0(\mu_0\rho_0)^{-1/2}$ makes the longest wavelength modes critically damped, decaying on the τ_f time scale.

A description of the slow t_1 time scale physics is given by Eqs. (68), (71), (73), and (75). Taking the t_1 derivatives of Eq. (63) and substituting from Eq. (71) and (75) yields the constraint equation determining the slowly varying momentum density $\langle m_1 \rangle \langle t_1 \rangle$

$$\begin{aligned}
& \left\{ \nabla \times \left[\nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) \right] \right\} \times \vec{B}_0 + (\nabla \times \vec{B}_0) \\
& \times \left[\nabla \times \left(\frac{\langle \vec{m}_1 \rangle}{\rho_0} \times \vec{B}_0 - \frac{\eta}{\mu_0} \nabla \times \vec{B}_0 \right) \right] \\
& + \nabla \left[\frac{\langle \vec{m}_1 \rangle}{\rho_0} \cdot \nabla p_0 + \frac{5}{3} p_0 \nabla \cdot \frac{\langle \vec{m}_1 \rangle}{\rho_0} + \frac{2}{3} (\nabla \cdot \vec{q} - \eta \nabla^2) \right] = 0 \quad (78)
\end{aligned}$$

Equation (78) is the well-known constraint equation for the velocity field associated with resistive diffusion, the analogue of Eq. (51). Other forms of this equation have been derived by Pao (1978), Hazeltine *et al.* (1973), Turnbull and Storer (1983), and Hirshman and Jardin (1979). The fundamental point to note here is that the resistive component of the velocity $\langle \vec{m}_1 \rangle(t_1)$ is not really determined from a time advancement equation as is the rapidly varying part $\vec{m}_1(t_0, t_1)$. It is determined in fact from the constraint equation that the system evolve through a series of near equilibrium states $[B_0(t_1), p_0(t_1)]$ satisfying the equilibrium Eq. (68). However, this occurs automatically when the system of Eqs. (64) is integrated in time.

(iii) Numerical method

The Eqs. (64) are advanced in time using the leap-frog method for the convective and the wave terms, and the DuFort-Frankel (1953) method for the dissipative terms. Thus, for example, Eq. (64e) which is of the form (in one dimension)

$$\frac{\partial}{\partial t} \chi + c \frac{\partial \chi}{\partial x} = D \frac{\partial^2 \chi}{\partial x^2} \quad (79)$$

would be advanced in time as

$$\begin{aligned}
 & (\chi_j^{n+1} - \chi_j^{n-1})/(2\Delta t) + c(\chi_{j+1}^n - \chi_{j-1}^n)/(2\Delta x) \\
 & = D[\chi_{j+1}^n + \chi_{j-1}^n - (\chi_j^{n+1} + \chi_j^{n-1})]/\Delta x^2.
 \end{aligned}
 \tag{80}$$

with Eq. (80) being solved algebraically for χ_j^{n+1} . The condition for stability of the method in Eq. (80) is that

$$\Delta t < \frac{\Delta x}{c}, \tag{81}$$

independent of the diffusion coefficient D . However, the numerical solution will become inaccurate for $\Delta t > \Delta x^2/D$. Since D can become very large in the low temperature vacuum region where the pressure is near zero and thus the convective term C in Eq. (79) is small, we utilize sub-stepping where the diffusive terms (right-hand sides) in Eqs. (64e) and (64f) are evaluated many times (typically 10-20) each time step used by the rest of the problem.

The utility of the large mass method lies in the fact that the ion mass can be greatly enhanced without affecting the physical results. We typically choose the enhancement factor to be such that the fast wave transit time is 1/100 of the problem time. Rerunning the problem with double or half the mass enhancement factor will verify that the physical results are indeed independent of this quantity.

(iv) Application

As an application illustrating the versatility and the usefulness of this method, we will discuss the computation of the current build up and cross-sectional shaping phases of a tokamak discharge in the Princeton Beta Experiment (PBX), (Bol et al.), (1983). In this calculation, each

computational grid point is one of three types: a plasma point, a vacuum point, or a coil point. For the plasma points, the full set of resistive MHD Eqs. (64) is advanced in time. For the vacuum points, these same equations are solved except we add explicit source-sink terms of mass and energy, S_p and S_p , into Eqs. (64d) and (64g), to force the vacuum density and temperatures to be constants, ρ_v and T_v . Actually, the value of ρ_v is unimportant, but T_v must be chosen to be much less than the plasma temperature $T_v \ll T_p$, and such that the magnetic diffusion time across the vacuum region is short compared to the time scales of physical interest, i.e., the time scales over which the boundary conditions change or the physical instability growth times.

The coil points are treated as vacuum points except that the velocity is set to zero and an explicit electric field driving term RE_ϕ is added to the poloidal flux Eq. (64e). Thus Eq. (64e) takes the form

$$\frac{\partial \Phi}{\partial t} + rI = V \quad (82)$$

where $\Phi \equiv -2\pi\chi = LI + \sum_{j \neq i} M_{ij}I_j$ is the actual poloidal flux at gridpoint i , $r \equiv 2\pi\eta R/(\Delta R \Delta Z)$ is the effective resistance, $I = [(\Delta R \Delta Z)/\mu_0 R] \Delta^* \chi$ is the current associated with that gridpoint, and $V = 2\pi RE_\phi$ is the applied voltage. Thus, the actual circuitry of the external coils can be incorporated into the calculation, including active feedback systems.

In Fig. (12) we illustrate the current build up and bean-shaping stage of the discharge as computed by the time-dependent resistive MHD evolution code STARTUP. Figures (12a) through (12d) illustrate the evolution of the magnetic flux surfaces when perfectly conducting plates are present (nonshaded boxes). The top-bottom plates are connected in anti-series so that they do not couple to the symmetric Ohmic Heating (OH) or Equilibrium Field (EF)

systems, but they do provide stabilization for up-down motions of the plasma.

The plasma current I_p , equilibrium field current I_{EF} , and shaping field current I_{SF} take on the values illustrated in Fig. (13). The plasma size is limited by contact with the midplane limiter at $R = 1.1$ m.

We see from examining Figs. (12a) through (12d) that the plasma evolves smoothly from a circular cross section into a strong bean shape without exhibiting any tendency to bifurcate or to lose equilibrium. These calculations used classical transport in the plasma with additional enhanced resistivity inside the $q = 1$ surface to flatten the current density.

To illustrate the necessity of the conducting plates, we show in Figs. (12e) through (12h) the same calculation, but with high resistivity in the stabilizing plates. We see that the plasma becomes positionally unstable and given a small initial downward perturbation will move downward into the bottom plate on the resistive L/r time scale of the plates.

The validity of the large mass method rests on the fact that the physical results obtained are insensitive to variations in the nonphysical parameters such as the enhancement factors for the plasma mass and viscosity, and to the temperature used in the vacuum region. We have verified this by rerunning the calculation of Fig. (12a) through (12d) three additional times, varying each of these three quantities by a factor of two. The results of these calculations look identical to the eye, the plots of the magnetic axis position vs time differing by at most 2%.

IV. DISCUSSION

The computational methods discussed in the preceding three sections are illustrative of multiple time scale techniques used in the simulation of magnetohydrodynamic phenomena occurring in tokamak fusion experiments.

Needless to say, the specific topics covered are by no means comprehensive. Perhaps the most glaring omission has to do with the discussion of phenomena occurring on the time scales intermediate between ideal and resistive diffusive MHD (see Fig. 2). Resistive instabilities such as tearing modes, resistive interchange modes, and resistive ballooning modes grow on these intermediate time scales.

The great majority of simulation work performed to date on resistive instabilities has utilized the "reduced" MHD equations obtained by performing an asymptotic expansion in the geometric inverse aspect ratio of the tokamak, a/R . As mentioned in the introduction, since this expansion effectively annihilates the faster time scales from the equations, we have chosen to exclude these methods from discussion.

Recently, some progress has been made by Melucia (1983) in simulating intermediate time scale resistive instabilities using the full (nonreduced) MHD equations. This work utilizes a stream function/potential representation of the magnetic field and the velocity field which is a generalization of Eqs. (61) and (62) to arbitrary helical symmetry geometry. Fourier decomposition in the angle and implicit time differencing of the linear terms is shown to yield a trackable method for addressing the intermediate time scales.

Fully three-dimensional simulations have also not been emphasized in the present article. The reason for this is primarily due to the fact that relatively little work has been done in the area of multiple time scale three-dimensional simulations of tokamak magnetohydrodynamics. However, noteworthy efforts in this area include the work of Brackbill and Barnes (1980) which utilizes a generalized moving mesh and implicit time integration scheme based on the ICED-ALE technique of Hirt, Amsden, and Cook (1974) and of Finan (1978) who utilizes alternating direction implicit techniques in the numerical

solution of the time-dependent, three-dimensional, single fluid, resistive magnetohydrodynamic equations.

The complexity of the present subject matter undoubtedly makes it difficult for one to distinguish the numerical techniques from the underlying physical processes. There are several central themes, however, which should be evident in the methods described here. One is the use of magnetic coordinates, which implies the use of a time dependent coordinate transformation since the magnetic field evolves in time.

Aligning one coordinate with the magnetic field is useful in ideal MHD to represent accurately MHD waves and the plasma/vacuum interfaces. The fluid and the magnetic field are frozen together and thus one could devise a purely Lagrangian method that utilized magnetic coordinates, i.e., that had no relative fluid flow crossing the coordinate lines. Such a Lagrangian method would in general be compressible and so the positions of the grid points would exhibit the fast time scale motion. Since the equations of motion for the grid points of a Lagrangian grid are very complicated, a fully implicit Lagrangian formulation is impractical. The dynamical grid method described in Sec. II gets around this by allowing some slippage between the fluid flow and the magnetic coordinate lines so that the fast time scale compressible motion of the fluid shows up as Eulerian flow through the moving grid. We remark that this slippage does not represent slippage between the fluid and the magnetic field, which can only occur physically with the introduction of resistivity. Rather, it exploits the freedom inherent in defining a magnetic coordinate system consistent with a given magnetic field to force the coordinate transformation to be free of the fast time scale motion.

On the resistive time scale, the fluid and the field are no longer frozen together. In Sec. IIIB we describe a resistive time scale method based on the

same coordinate transformation as that utilized in Sec. II. However, here the motivation for using this is somewhat different. Again, the fundamental physics issue is that one coordinate be aligned with the magnetic field, this time to represent the long time effects of parallel transport accurately. The freedom associated with defining the remaining properties of the magnetic coordinate system are used to divide the physical magnetic flux velocity into two parts. The incompressible part associated with a change of shape, but not volume, is used to evolve the Lagrangian coordinates of the magnetic surfaces. The remaining compressible part of the velocity field is constant on each magnetic surface. It is represented as the Eulerian flow of toroidal flux across flux surfaces.

The final method in Sec. IIIC evolves the magnetic flux ψ in time, but does not explicitly use magnetic coordinates. This approach has some advantages in that it is conceptually straightforward, and that it can easily treat situations in which the magnetic topology changes, i.e., in which new minima or maxima appear in the poloidal flux ψ .

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REFERENCES

- G. Bateman, W. Schneider, and W. Grossmann, Nucl. Fusion 14, 669 (1974).
- K. Bol, et al. (1983) "PBX: The Princeton Beta Experiment," Princeton Plasma Physics Laboratory Report No. PPPL-2032, 1983.
- J. Brackbill in Methods in Computational Physics, (Academic Press, New York, 1976), Vol. 16, p. 1.
- J. V. Brackbill and D. C. Barnes, J. Comput. Phys. 35, 426 (1980).
- S. I. Braginskii in Reviews of Plasma Physics, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
- N. Byrne and H. Klein, J. Comput. Phys. 26, 352 (1978).
- M. S. Chance, J. M. Greene, R. C. Grimm, and J. L. Johnson, Nucl. Fusion 17, 65 (1977).
- J. DeLucia, "Linear and 2D Nonlinear Studies of Resistive Instabilities in the Cylindrical Spheromak," Ph.D. thesis, Princeton University, 1983.
- J. DeLucia, S. C. Jardin, and A. M. M. Todd, J. Comput. Phys. 37 183 (1980).
- E. C. DuFort and S. P. Frankel, Math. Tables Aids Comput. 7, 135 (1953).
- C. H. Finan, III, "The Alternating-direction implicit numerical solution of the time-dependent, three-dimensional, single fluid, resistive magnetohydrodynamic equations," Ph.D. thesis, University of California, Davis, 1978.
- H. P. Furth, Nucl. Fusion 15, 487 (1975).
- H. Grad et al., "Notes on Magnetohydrodynamics," Courant Institute of Mathematical Sciences, New York University Report No. NYU-6486, 1958.
- H. Grad in Proceedings Congress of Mathematicians, Nice (Gauthier-Villars, Paris, 1970), Vol. 3, p. 105.
- R. D. Hazeltine and F. L. Hinton, Phys. Fluids 16, 1883 (1973).

- R. D. Hazeltine, F. L. Hinton, and M. N. Rosenbluth, *Phys. Fluids* 16, 1645 (1973).
- F. J. Helton, R. L. Miller, and J. R. Rawls, *J. Comput. Phys.* 24, 117 (1977).
- S. P. Hirshman and S. C. Jardin, *Phys. Fluids* 22, 731 (1979).
- C. W. Hirt, A. A. Amsden, and J. L. Cook, *J. Comput. Phys.* 14 (1974).
- J. T. Hogan, Oak Ridge National Laboratory Report ORNL/TM-6049, 1978.
- S. C. Jardin, J. L. Johnson, J. M. Greene, and R. C. Grimm, *J. Comput. Phys.* 29, 101 (1978).
- S. C. Jardin, *J. Comput. Phys.* 43, 31 (1981).
- S. C. Jardin and W. Park, *Phys. Fluids* 24, 679 (1981).
- S. C. Jardin, *Phys. Fluids* 21, 1851 (1978).
- M. Kruskal and R. Kulsrud, *Phys. Fluids* 1, 285 (1958).
- D. M. Meade, H. P. Furth, P. H. Rutherford, and F. G. P. Seidel, in Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Vienna, 1975), Vol. I, p. 605.
- D. B. Nelson and H. Grad, Oak Ridge National Laboratory Report ORNL/TM-6094, 1977.
- Y. Pan, *Phys. Fluids* 21, 1120 (1978).
- D. Potter, in Methods in Computational Physics (Academic Press, New York, 1976), Vol. 16, p. 43.
- W. Schneider and G. Bateman, in Plasma Physics and Controlled Nuclear Fusion Research, 1974 (IAEA, Vienna, 1975), Vol. I, p. 429.
- H. R. Strauss, *Phys. Fluids* 19, 134 (1976).
- H. R. Strauss, *Nucl. Fusion* 23, 649 (1983).
- A. M. M. Todd and R. C. Grimm, *Bull. Amer. Phys. Soc.* 20, 1279 (1975).
- A. D. Turnbull and R. G. Storer, *J. Comput. Phys.* 50, 409 (1983).
- J. A. Wesson and A. Sykes in Plasma Physics and Controlled Nuclear Fusion Research, 1974 (IAEA, Vienna, 1975), Vol. I, p. 449.

FIGURE CAPTIONS

- FIG. 1 Axisymmetric toroidal geometry. Magnetic field spirals around two-dimensional magnetic surfaces.
- FIG. 2 Some of the fundamental time scales in a tokamak fusion experiment with $B_T = 5$ T, $I_p = 5$ Ma, $R = 3$ m, $a = 1$ m, $n_0 = 10^{20} \text{ m}^{-3}$, $T = 5$ keV.
- FIG. 3 A two-dimensional axisymmetric subsystem uses (R, Z, ϕ) as the fixed coordinates and (ψ, θ, ϕ) as the magnetic flux coordinates, with ϕ being ignorable.
- FIG. 4 Spectrum of axisymmetric eigenmodes with azimuthal wave number 4. Mode denoted by A will be examined closer.
- FIG. 5 Transverse eigenmode simulation showing the velocity vectors at time 0 and time $48\tau_E$.
- FIG. 6 Some time history's at location $I = 5$, $J = 8$. (a) $\nabla\psi$ component of velocity, (b) perturbed metric element $|\nabla\psi|^2$, (c) perturbed density.
- FIG. 7 Magnetic surfaces at time $t = 0$. Dashed surfaces lie in vacuum. Plus and minus indicate polarity of current carrying coils.
- FIG. 8 Magnetic surfaces at time $t = 73\tau_E$, after instability has distorted plasma.

FIG. 9 Constant (ψ, θ) coordinate surfaces at (a) $t = 0$ (before compression) and (b) $t = 2$ sec after compression. Magnetic axis has decreased from $R = 207$ cm to $R = 140$ cm.

FIG. 10 Profiles of the midplane values of T_e , T_i , n , and the toroidal current J_ψ at times $t = 0.00, 1.44, 1.51, 1.53, 1.54, 1.56, 1.57, 1.58, 1.60$, and 2.0 sec.

FIG. 11 Time history of the central electron and ion temperatures, $T_e^{(0)}$ and $T_i^{(i)}$ and the central density $n^{(0)}$.

FIG. 12 Two sequences of bean formation calculated by the code STARTUP. The left sequence is totally up/down symmetric. The right sequence has a small perturbation which grows in time.

FIG. 13 Time histories of the plasma current (I_p) , the equilibrium field circuit current (I_{EF}) , and the shaping field circuit current (I_{SF}) . The (a), (b), (c), and (d) refer to the snapshots in Fig. 12. Neutral beam auxiliary heating is on from 300 ms to 600 ms.

83T0405

Fig. 1

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Fig. 2

Fig. 3

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Fig. 4

Fig. 5

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Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

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