Macroscopic Stability and β Limits in the ELMO Bumpy Torus

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IN THE ELMO BUMPY TORUS

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ABSTRACT

Magnetohydrodynamic stability limits are determined for the ELMO Bumpy Torus. The relativistic hot electron annuli are considered to be rigid, modifying the magnetic field but not interacting with the instability. A modified energy principle is used, and the stability problem is reduced to determination of the eigenvalues of an ordinary differential equation along each field line. A threshold hot electron current is required for stability; its value agrees with experimental measurements. The calculations show that stable high β equilibria are easily created.

1. INTRODUCTION

In a fusion reactor based on the ELMO Bumpy Torus (EBT) [1], it is desirable that the plasma β (ratio of plasma pressure to magnetic pressure) be at least 10-20%. Plasma instabilities are one of the physical processes which can limit β , and among these the magnetohydrodynamic (MHD) instabilities are often the most dangerous, carrying the plasma rapidly to the wall. In this paper we determine the maximum β for which the EBT configuration is magnetohydrodynamically stable.

It is not possible to treat the entire plasma in the MHD approximation; the drift velocity of the hot electrons ($T_{eh} \sim 100$ keV) typically exceeds the ion thermal velocity ($T_i \sim 100$ eV), while for the MHD model to be valid all drift velocities must be small compared with the ion thermal velocity. However, it has been shown using the Vlasov-Maxwell equations in a slab geometry that for EBT conditions ($n_{eh} \ll n_i$, $T_{eh} \gg T_i$), a modified MHD model is valid, one in which the hot electrons are considered as a rigid noninteracting external current [2]. If we assume that these equations are valid for general geometry, then the modified energy principle used to study the stability of Astron is applicable to EBT [3].

In Section 2, we show that for the EBT geometry the stability limit is determined by the solution of a second order Sturm-Liouville ordinary differential equation on each field line. Because this solution is well approximated by its infinite aspect ratio limit (i.e., that of an axisymmetric bumpy cylinder), the computations to find the stability limit are performed in this geometry. Section 3 discusses the equilibrium calculation, which is done in the guiding center formulation so as to

include the anisotropic hot electron population in the equilibrium force balance. Section 4 presents the stability calculations and the stability limits that result from them. In general, it is found that up to a hot electron current strength corresponding to a hot electron β of about 10-15%, no ion plasma β is magnetohydrodynamically stable. As the electron current is increased beyond this value, a local magnetic well is formed and the ion β limit for stability to all MHD modes becomes comparable to the hot electron β . Because the hot electron β for the present EBT is 0.2 to 0.3, ion β of 10-20% should be magnetohydrodynamically stable. This result is very encouraging for the success of an EBT fusion reactor.

2. ENERGY PRINCIPLE AND STABILITY CRITERION

We assume that the hot electrons do not interact with the rest of the plasma on the MHD time scale, although they are in force balance with the plasma and the external magnetic field on the diffusion time scale. With this assumption, the equation of motion of the plasma can be written as [3]

$$\rho \frac{d\mathbf{y}}{dt} = (\nabla \times \mathbf{B} - \mathbf{j}_e) \times \mathbf{B} - \nabla p$$

with $\mathbf{j}_e$ the current produced by the diamagnetic drift of the hot electrons. (This differs from Astron, in which $\mathbf{j}_e$ is produced by the gyromotion of the electrons.) Both the hot electrons and the bulk plasma satisfy equilibrium equations:

$$\mathbf{j}_e \times \mathbf{B} = \nabla \cdot \mathbf{P}_e$$

$$\mathbf{j}_e + \mathbf{j}_p = \nabla \times \mathbf{B} = \mathbf{j}_{\text{total}}$$

$$\mathbf{j}_p \times \mathbf{B} = \nabla p$$

where $\underline{j}_p$ is the plasma current, $\underline{P}_e$ is the pressure tensor of the hot electrons, and p is the scalar pressure of the ions. A modified energy principle can be constructed for this problem; stability is determined by the sign of

$$\delta W = \frac{1}{2} \int d\tau [Q^2 + (\nabla \times \underline{B} - \underline{j}_e) \times \underline{\xi} \cdot \underline{Q} + \underline{\xi} \cdot \nabla p \nabla \cdot \underline{\xi} + \gamma p (\nabla \cdot \underline{\xi})^2] \quad (1)$$

If $\nabla \cdot \underline{j}_e = 0$ (this will be true for all the computations in Section 4), Eq. (1) can be integrated by parts to give

$$\delta W = \frac{1}{2} \int d\tau [(Q + \underline{F}X)^2 - AX^2 + \gamma p (\nabla \cdot \underline{\xi})^2]$$

where

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}), \quad X = \underline{\xi} \cdot \nabla \psi, \quad \underline{F} = \underline{j}_p \times \underline{e}$$

$$\underline{e} = \nabla \psi / |\nabla \psi|^2, \quad A = 2\underline{F} \cdot (\underline{B} \cdot \nabla) \underline{e} - \underline{j}_p \cdot \underline{j}_e / |\nabla \psi|^2$$

and $2\pi\psi$ is the magnetic flux enclosed by a surface of constant pressure. Apart from the appearance of $\underline{j}_p$ instead of $\underline{j}_{\text{total}}$ in $\underline{F}$ and the extra term in A involving $\underline{j}_e$, this expression is identical to that considered in Ref. 4. The methods used there can be applied to derive a sufficient condition for stability of equilibria with external, divergence-free currents in devices such as EBT. These equilibria have closed magnetic field lines and vanishing current density at the magnetic axis. They are stable to all MHD modes if the ordinary integrodifferential equation along each field line

$$\underline{B} \cdot \nabla (\underline{B} \cdot \nabla X / |\nabla \psi|^2) + (A + \Lambda)X = A \langle \Lambda X \rangle / \alpha^2 \quad (2)$$

has only positive eigenvalues Λ for which the solution X is periodic.

Here

$$\alpha^2 = \langle F^2 \rangle + (p')^2 / \gamma p, \quad p' = dp/d\psi$$

$$\langle f \rangle = (\oint f d\ell / B) / q, \quad q = \oint d\ell / B, \quad B = |B|$$

Equation (2) is a nonstandard integrodifferential equation. However, its lowest eigenvalue, Λ_1 , can be found from the eigenvalues of the reduced equation, that is, the ordinary differential equation which results from setting the right-hand side to zero. One uses the following theorem: let λ_1 and λ_2 be the two smallest eigenvalues of the reduced equation. Then $\lambda_1 \leq \Lambda_1 \leq \lambda_2$, and the following cases arise:

- (1) If $\lambda_1 > 0$, then $\Lambda_1 > 0$ (stable).
- (2) If $\lambda_2 < 0$, then $\Lambda_1 < 0$ (unstable).
- (3) If $\lambda_1 < 0$ and $\lambda_2 > 0$, then $\Lambda_1 > 0$ (stable) if and only if $p'(p'q + \gamma p q') < 0$.

Note that in a region where $q' < 0$ and $p' < 0$ the algebraic condition of case (3) cannot be satisfied, so stability is determined by the sign of λ_1 . On the other hand, if $q' > 0$ and $p' < 0$, then one can show that λ_1 is negative and stability is determined by case (2) or (3). The equilibrium thus breaks naturally into two regions, depending on the sign of q' . Constant pressure ($p' = 0$) is always stable, and $p' < 0$ may be stable in either region, but the edge of the plasma (where $p \rightarrow 0$) can only lie in a region where $q' < 0$. In the EBT it is precisely the function of the hot electrons to create such a stabilizing region of $q' < 0$. The threshold strength of j_e required for containing a low β bulk plasma can be determined by expanding Eq. (2) in the limit of low β

but arbitrary j_e [4]. One finds that the threshold is reached when j_e is strong enough to satisfy the well-known interchange condition [5]

$$q'(p'q + \gamma p q') \geq 0$$

Because Eq. (2) can be integrated numerically, the stability of a closed line toroidal equilibrium can be determined directly. However, in EBT these equilibria are intrinsically three-dimensional, requiring considerable amounts of computer time to calculate. For a survey of the stability boundary, it is preferable to use the fact that EBT is a large aspect ratio device, so that the equilibrium is a perturbation about a periodic bumpy cylinder with vanishing $\underline{j} \cdot \underline{B}$. Perturbation theory can then be used to find the eigenvalues of Eq. (2). In lowest order, they are those of the equivalent bumpy cylinder; the corrections are proportional to the aspect ratio. Expansion of the coefficient A is facilitated by writing it as

$$A = 2\underline{e} \cdot \nabla p \underline{e} \cdot \underline{\kappa} + 2B^{-2}(\underline{j}_p \cdot \underline{B})\underline{B} \times \underline{e} \cdot (\underline{B} \cdot \nabla)\underline{e} - \underline{j}_p \cdot \underline{j}_e / |\nabla\psi|^2$$

Then the lowest order of Eq. (2) is

$$\frac{d}{d\ell} \left(\frac{1}{r_B^2} \frac{dX}{d\ell} \right) + \frac{\Lambda - p'D}{B} X = D \langle DX \rangle \frac{\gamma P}{1 + \gamma p I} \quad (3)$$

where

$$D = -2\underline{e}_\psi \cdot \underline{\kappa} / r_B + j_e / r_B^2 \quad (4)$$

$$I = \oint d\ell / B^3$$

and $\tilde{e}_\psi = \tilde{e}/|\tilde{e}|$. Using the identity $\tilde{e}_\psi \cdot \tilde{\kappa} = j_{\text{total}}/B + r\partial B/\partial\psi$, we can also write D as

$$D = -p'/B^2 - j_{\text{total}}/rB^2 - 2\partial \ln B/\partial\psi \quad (5)$$

Equation (3) is identical to that derived ab initio for the bumpy cylinder equilibrium, in which case positivity of the lowest eigenvalue Λ_1 is necessary as well as sufficient for stability [3]. Although higher order corrections to Eq. (3) can be obtained from Eq. (2), they are very small for a large aspect ratio device such as EBT. Hence, for the remainder of this paper only solutions of Eq. (3) will be discussed.

Equation (3) is the Euler equation for the form of δW in an axisymmetric equilibrium which remains after algebraic minimization on the components of the displacement lying in the surface:

$$\delta W = \int \delta W(\psi) d\psi$$

with

$$\delta W(\psi) = \int \frac{d\ell}{B} \left[\left(\frac{1}{r^2 B} \frac{dX}{d\ell} \right)^2 + p' DX^2 \right] + \frac{q^2}{I + q/\gamma p} \langle DX \rangle^2 \quad (6)$$

The stabilizing effect of the hot electron currents in EBT can be further discerned in the large aspect ratio limit from Eqs (4)-(6). For contained equilibria, p' is negative. Hence, if D is negative everywhere, δW is positive. In the absence of external currents D is predominantly positive, and a careful calculation shows such a configuration to be always unstable. With external currents in EBT, there is a region near the outer edge of the plasma where j_e is negative; in this region the sign of D can become negative. However, because j_{total} is negative,

Eq. (4) shows that D cannot be negative unless $\partial B/\partial \psi$ is positive, i.e., a local magnetic well must be created. More information about requirements for stability is gained by inserting the test function $X = \text{const}$ into Eq. (6). One then finds that on each flux surface the relation

$$p'q\langle D \rangle + q^2\langle D \rangle^2/(I + q/\gamma p) \geq 0 \quad (7)$$

is necessary for stability. However, because

$$q\langle D \rangle = q' - p'I$$

the condition (7) is equivalent to

$$(q' - p'I)(p'q + \gamma p q') \geq 0$$

This condition assumes one of two forms, depending on whether q' is positive or negative:

$$\begin{aligned} p'q + \gamma p q' &\geq 0 & \text{if } q' &\geq 0 \\ q' - p'I &< 0 & \text{if } q' &\leq 0 \end{aligned} \quad (8)$$

The first of these forms is just the interchange criterion for the region where q' is positive. The second condition is more stringent than the interchange condition; it shows that except at low β it is not sufficient for stability merely to have a region where q' is negative. The stricter condition, $\langle D \rangle$ negative, is also required in this region. While positivity of the lowest eigenvalue of Eq. (3) is still more stringent (except at low β), the condition (8) is a good initial guide for finding stable equilibria.

3. EQUILIBRIUM

The EBT configuration is shown conceptually in Fig. 1, and the equivalent bumpy cylinder is shown in Fig. 2. In the present experiment there are 24 sectors, each formed by a pair of mirror coils. The hot electron plasma, formed by electron cyclotron resonant heating, is anisotropic and mirror-confined in an annular region between each pair of coils. The bulk plasma threads these annuli to form an isotropic torus. Both the anisotropic hot electrons and the isotropic bulk plasma are included in the equilibrium calculation through use of the guiding center formulation in an axisymmetric bumpy cylinder. The fundamental equilibrium equations are

$$\nabla \times \underline{\underline{B}} = \underline{\underline{j}}_e + \underline{\underline{j}}_p$$

$$(\underline{\underline{j}}_e + \underline{\underline{j}}_p) \times \underline{\underline{B}} = \nabla \cdot (\underline{\underline{P}}_e + p\underline{\underline{I}})$$

$$\nabla \cdot \underline{\underline{B}} = 0$$

For axisymmetric systems with $j_{||} = 0$, these equations reduce to

$$\underline{\underline{B}} \cdot \nabla p_{||} = \frac{p_{\perp} - p_{||}}{B} \underline{\underline{B}} \cdot \nabla B \quad (9)$$

$$\Delta^* \psi + \nabla \ln \sigma \cdot \nabla \psi = - \frac{r^2}{\sigma} \partial p_{||} / \partial \psi \quad (10)$$

where

$$\Delta^* \psi \equiv \frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right)$$

Here ψ is the flux function introduced as a coordinate in Section 2 and

$$\vec{B} = \nabla\psi \times \nabla\theta$$

Ellipticity of the equilibrium requires that σ and τ be positive, where

$$\sigma = 1 + (p_{\perp} - p_{\parallel})/B^2$$

$$\tau = 1 + B^{-1} \partial p_{\perp} / \partial B$$

For numerical purposes, it is convenient to write

$$\psi = \psi_c + \psi_p$$

where ψ_c is produced by external coils and ψ_p is due to hot electron currents and bulk plasma currents. Equation (9) then takes the form

$$\Delta^* \psi_p + \nabla \ln \sigma \cdot \nabla \psi_p = S$$

where

$$S \equiv - \frac{r^2}{\sigma} \partial p_{\parallel} / \partial \psi \quad \nabla \ln \sigma \cdot \nabla \psi_c$$

To solve Eq. (10), the functions $p_{\perp}(B, \psi)$ and $p_{\parallel}(B, \psi)$ are specified, subject to Eq. (9), and an iteration procedure is used to determine ψ_p numerically. One begins by using the flux due to the coils alone to determine S and $\nabla \ln \sigma$. One then solves Eq. (10) for ψ_p using standard finite difference techniques for elliptic partial differential equations. From this ψ_p , one uses $\psi = \psi_c + \psi_p$ to redetermine S and $\nabla \ln \sigma$. One then solves for a new ψ_p . This procedure is repeated until the desired degree of convergence is obtained. At each step a boundary condition on ψ_p must be imposed at the edge of the computational grid. The one employed here,

regularity at infinity, is appropriate for steady-state plasmas in a bumpy cylinder, that is, a bumpy torus with infinite aspect ratio. To compute the boundary values of ψ_p , we assume periodicity in z and impose periodic boundary conditions at $z = 0$ and $z = L/2$ (i.e., at a midplane and at an adjacent coil plane). On the radial boundary, $r = r_{\max}$, we use the law of Biot and Savart

$$\vec{A} = \frac{1}{4\pi} \int \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|} \vec{j}$$

and $\psi = rA_\theta$, as well as the expansion of $|\vec{x} - \vec{x}'|^{-1}$ in spherical harmonics, to obtain

$$\begin{aligned} \psi_p = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{r}{n(n+1)} \frac{P_n^1(\cos \phi_m)}{R_m^{n+1}} \\ \times \int dz' \int r' dr' (R'_0)^n P_n^1(\cos \phi'_0) j_\theta \end{aligned} \quad (12)$$

where

$$\cos \phi_m = (z - mL)/R_m^2$$

$$R_m = \sqrt{(z - mL)^2 + r^2}$$

For the results reported here, m was truncated at $m = \pm 6$ so that only the six sectors on each side of the central sector contribute to the radial boundary values of the central sector. Because of the assumed symmetry

$$j_\theta(r, -z) = j_\theta(r, z)$$

$$\psi(r, -z) = \psi(r, z)$$

only odd n occurs in the summation appearing in Eq. (12). Then for each iteration the boundary values of ψ_p on the radial boundary $r = r_{\max}$ are computed from Eq. (12) using the previous iterate of j_θ . In most cases $r_{\max}$ is about twice the maximum plasma radius.

The components of the pressure are chosen to have the functional form

$$p_{\parallel} = \hat{p}_{\parallel}(B)g(\psi) + p(\psi)$$

$$p_{\perp} = \hat{p}_{\perp}(B)g(\psi) + p(\psi)$$

The first term in these expressions describes the hot electrons; the second describes the bulk plasma. In order to simulate the annular hot electron plasma observed in EBT, the function $g(\psi)$ was taken to be

$$g(\psi) = \begin{cases} (\psi - \psi_1)^2(\psi - \psi_a)^2/(\psi_2 - \psi_1)^2(\psi_2 - \psi_a)^2, & \psi_1 \leq \psi \leq \psi_2 \\ (\psi - \psi_b)^2(\psi - \psi_3)^2/(\psi_2 - \psi_b)^2(\psi_2 - \psi_3)^2, & \psi_2 \leq \psi \leq \psi_3 \\ 0 & , \text{ otherwise} \end{cases} \quad (13)$$

where

$$\psi_a \equiv 2\psi_2 - \psi_1$$

$$\psi_b \equiv 2\psi_2 - \psi_3$$

Note that the maximum value of $g(\psi)$, which occurs at $\psi = \psi_2$, is one. In order to control the radial position of the plasma during the iteration process, the coefficients ψ_1 , ψ_2 , and ψ_3 which appear in Eq. (13) were periodically adjusted so that the peak and zeros of the function occur at specified radii in the midplane ($z = 0$).

Several functional forms for $\hat{p}_{\parallel}(B)$ and $\hat{p}_{\perp}(B)$ were investigated, corresponding to different forms of the hot electron distribution function. One of these is

$$\hat{p}_{\perp}(B) = \frac{1}{2} s(B_c^2 - B^2), \quad B \leq B_c$$

$$\hat{p}_{\parallel}(B) = \frac{1}{2} s(B_c - B)^2, \quad B \leq B_c$$

and zero for $B > B_c$. This choice is particularly convenient because

$$\tau = 1 - \text{sg}(\psi)$$

and

$$\sigma = 1 + \text{sg}(\psi)(B_c/B - 1)$$

so that one need only keep $s < 1$ to guarantee that $\tau > 0$ and $\sigma > 0$. We have also employed the functional form introduced by Taylor [6]:

$$\hat{p}_{\perp}(B) = mCB^2(B_c - B)^{m-1}$$

$$\hat{p}_{\parallel}(B) = CB(B_c - B)^m$$

For the scalar pressure term, we have used a model in which the pressure is constant from the magnetic axis out to a specified value of ψ and then falls to zero:

$$p(\psi) = \begin{cases} p_o, & \psi \leq \psi_4 \\ p_o(\psi - \psi_c)^2(\psi - \psi_5)^2/(\psi_4 - \psi_c)^2(\psi_4 - \psi_5)^2, & \psi_4 \leq \psi \leq \psi_5 \\ 0, & \psi_5 \leq \psi \end{cases}$$

where

$$\psi_c = 2\psi_4 - \psi_5$$

4. CALCULATIONS AND RESULTS

An example of the equilibrium calculation is shown in Figs 3 and 4. Figure 3 gives level contours of ψ_p and indicates clearly the local minimum in B created by the annular diamagnetic current of the hot electrons. This minimum is also shown in Fig. 4, which gives B as a function of radius on the midplane where $z = 0$. This figure also shows $q = \oint d\ell/B$ as a function of radius. As indicated by the discussion in Section 2, q' is negative near the outer edge of the hot electron annulus, so that in this region a bulk plasma with $p' < 0$ can be stably contained. Therefore, Fig. 4 shows a bulk plasma pressure which might be stably contained, having $p' = 0$ in the unfavorable region where $q' > 0$ and $p' < 0$ in the favorable region where $q' < 0$.

The details of calculating the eigenvalues and determining the stability boundary are presented in the Appendix. One set of results is presented in Fig. 5. This graph shows the marginal β_p as a function of β_l where

$$\beta_p = 2p_{\max}/B_o^2$$

$$\beta_l = 2p_{le\max}/B_o^2$$

p_{le} is the perpendicular pressure of the hot electrons alone, and B_o is the vacuum magnetic field on axis in the midplane. Until a region

where $q' < 0$ is formed by j_e , no bulk plasma can be stably contained. The threshold occurs at β_1 of 10-15%. Above this threshold Fig. 5 shows that the marginally stable β_p rapidly becomes comparable to β_1 . At a fixed β_1 , if β_p exceeds the marginally stable value the plasma is unstable to a ballooning interchange mode, whereas below this value the plasma is stable to all ideal magnetohydrodynamic modes. Among the factors which influence the marginal point is the shape of the hot electron annulus. Figure 5 shows that if the annulus is longer, extending deeper into the mirror throat, the marginal β_p is increased because the local magnetic well extends further along the field line. Since the results shown in Fig. 5 are extremely encouraging for an EBT fusion reactor, we have not attempted to optimize the function $\beta_p(\beta_1)$.

Most of these results can be understood qualitatively without computation. The threshold for confining a bulk plasma occurs when $dq/d\psi < 0$, with

$$\frac{dq}{d\psi} = - \int \frac{d\ell}{B^2} \left(\frac{\tilde{e}_\psi \cdot \tilde{\kappa}}{r} + \frac{\partial B}{\partial \psi} \right)$$

Although the curvature, $\tilde{e}_\psi \cdot \tilde{\kappa}$, varies approximately linearly from negative (destabilizing) on the midplane to positive at the coil plane, the factor $1/rB^2$ emphasizes the negative contribution. Thus, $\partial B/\partial \psi$ must be positive along most of the line to balance the curvature term, and the threshold occurs approximately when $\partial B/\partial \psi$ becomes positive in the midplane. Using the equilibrium relation

$$\nabla_\perp (p_\perp + B^2/2) = \sigma B^2 \tilde{\kappa}$$

and replacing derivatives by scale lengths, one finds that $\partial B / \partial \psi$ vanishes when

$$\beta_1 \approx 2\sigma r_p / R_c$$

where r_p is the half width of the annulus and R_c is the radius of curvature in the midplane. Because in the equilibria computed for Fig. 5, $r_p / R_c \approx 0.05-0.10$, the threshold β_1 given by this estimate is 10-20%, in reasonable agreement with the computed threshold. The scaling $\beta_1 \sim r_p / R_c$ given by this estimate is borne out by computations which show that a thinner annulus (smaller r_p) or one located closer to the magnetic axis (large R_c) leads to a lower threshold β_1 .

The rapid rise in the marginal β_p above this threshold can also be understood without computation. Note that in general the coefficient D [defined in Eq. (4)] is positive near the midplane (destabilizing) and negative near the coil plane. As β_1 is increased (making j_e more negative), the positive region becomes smaller. However, as β_p is increased the field lines bulge out, increasing the bad curvature near the midplane and making the positive region larger. The competition between these two effects can be seen most easily by using the equilibrium relation

$$\tilde{j}_{\text{total}} \times \tilde{B} = (p_{\parallel} - p_{\perp}) \tilde{\kappa} + \nabla_{\perp} p_{\perp}$$

to write D as

$$D = \frac{1}{rB^3} \left[-(1 + \sigma) B^2 \tilde{e}_{\psi} \cdot \tilde{\kappa} + \tilde{e}_{\psi} \cdot \nabla p_{\perp e} \right]$$

where $p_{\perp e}$ is the perpendicular pressure of the hot electrons alone. Raising β_p makes the first term larger by increasing the curvature. However, because the curvature increases slowly with β_p , the marginal β_p goes up rapidly with increased $\nabla p_{\perp e}$. Since $\beta_{\perp}$ is the integral of $\nabla p_{\perp e}$, Fig. 5 reflects this behavior.

It is not possible to verify these predictions in detail in the present EBT experiment, which is transport-limited to $\beta_p \sim 0.5\%$. However, the threshold value of $\beta_{\perp}$ at about 10-15% does agree with an abrupt change in experimental operating conditions; as $\beta_{\perp}$ is raised through this value, the plasma density fluctuations and rf noise decrease rapidly and bulk plasma containment is considerably improved. It is also noted experimentally that the region near the wall where $q' > 0$ is noisy under all operating conditions. This indicates an unstable surface plasma, in agreement with the theory presented here.

APPENDIX

CALCULATING EIGENVALUES OF THE REDUCED EQUATION

Since the equilibrium is solved on an r - z grid, it is convenient to change the variable in Eq. (3) from ℓ to z using $d\ell = Bd_z/B_z$ (but of course evaluating the coefficients on the right field line) with z scaled so the distance between coil planes is unity. The reduced equation corresponding to Eq. (3) is then

$$\frac{d}{dz} \left(f \frac{dX}{dz} \right) - gX = 0; \quad X(z + 24) = X(z) \quad (A1)$$

with $f = B_z/r^2 B^2$ and $g = (p'D - \lambda)/B_z$. This is a Sturm-Liouville equation, so the number of zeros of the solution increases monotonically with λ . One expects the smallest eigenvalue, λ_1 , to correspond to a solution with no zeros. The simplest equation of this type arises if f and g are constant. Then if $g > 0$ there are no periodic solutions; they all blow up. The lowest eigenvalue occurs when $g = 0$, and the corresponding solution is $X = \text{const.}$ The other eigenvalues occur when $g = (2n\pi/24)^2$, leading to $X = \cos(2n\pi z/24)$ or $\sin(2n\pi z/24)$.

In the case of present interest, f and g are not constant; they are periodic with period one (each bump) and symmetric about each midplane. It is convenient to write Eq. (A1) as a system of two first order equations by defining $Y = fdX/dz$. For the moment we drop the periodicity condition on X ; then for each λ there are two linearly independent solutions of this system (corresponding to choice of initial conditions). It is helpful to write this system as a matrix differential equation for the 2×2 fundamental matrix [7]:

$$\underline{U}_z = \underline{A}(z)\underline{U}; \quad \underline{A}(z+n) = \underline{A}(z); \quad \underline{A}(-z) = \underline{A}(z)$$

where

$$\underline{U}(z) = \begin{pmatrix} X_1(z) & X_2(z) \\ Y_1(z) & Y_2(z) \end{pmatrix}, \quad \underline{A}(z) = \begin{pmatrix} 0 & 1/f \\ g & 0 \end{pmatrix}$$

Eq. (A2) has the solution

$$\underline{U} = \underline{P}(z)e^{\underline{Z} \underline{L}}$$

where $\tilde{P}$ is a periodic matrix, $\tilde{P}(z + n) = \tilde{P}(z)$, and $\tilde{\Gamma}$ is a constant matrix. Take as initial conditions $\tilde{U}(0) = \tilde{I}$, where $\tilde{I}$ is the identity matrix. Then $\tilde{U}(n) = e^{n\tilde{\Gamma}} \equiv \tilde{C}^n$. We want to know if, for a given λ , there is a vector solution, $\tilde{u} = (X, Y)$, which is periodic with period 24 (the number of sectors in the present EBT). To determine this, consider eigenvalues of $\tilde{C}$. If one of them is a 24th root of unity (lying on the unit circle), then there is such a solution. To prove this, let $\tilde{u}_0$ be an eigenvector of $\tilde{C}$ such that $\tilde{C}\tilde{u}_0 = \sqrt[24]{1} \tilde{u}_0$. Then the desired solution is $\tilde{U}(z)\tilde{u}_0$, since $\tilde{U}(24)\tilde{u}_0 = \tilde{C}^{24}\tilde{u}_0 = (\sqrt[24]{1})^{24}\tilde{u}_0 = \tilde{u}_0$ and hence is periodic.

To find the eigenvalues of $\tilde{C}$ we only need to integrate Eq. (A2) from $z = 0$ (a midplane) to $z = \frac{1}{2}$ (a coil plane). Using Eq. (A3) and noting $\tilde{C} = e^{\tilde{\Gamma}}$, we have $\tilde{U}(\frac{1}{2}) = \tilde{U}(-\frac{1}{2})\tilde{C}$, or

$$\tilde{C} = \tilde{U}^{-1}\left(-\frac{1}{2}\right)\tilde{U}\left(\frac{1}{2}\right)$$

$\tilde{U}\left(-\frac{1}{2}\right)$ can be determined from $\tilde{U}\left(\frac{1}{2}\right)$ by using the symmetry of $\tilde{A}$. If $\tilde{U}\left(\frac{1}{2}\right)$ has been computed (e.g., by Runge-Kutta techniques), starting with initial conditions $\tilde{U}(0) = \tilde{I}$, then studying Eq. (A2) shows that $\tilde{U}\left(\frac{1}{2}\right)$ and $\tilde{U}\left(-\frac{1}{2}\right)$ are related as follows:

$$\tilde{U}\left(\frac{1}{2}\right) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \quad \tilde{U}\left(-\frac{1}{2}\right) = \begin{pmatrix} x_1 & -y_1 \\ -x_2 & y_2 \end{pmatrix}$$

with the numbers x_1 , x_2 , y_1 , and y_2 determined numerically. To calculate $\tilde{U}^{-1}\left(\frac{1}{2}\right)$ we use the theorem

$$\det \tilde{U}(z) = \det \tilde{U}(0) \exp \left[\int_0^z \text{tr} \tilde{A}(x) dx \right]$$

Because here $\text{tr} \tilde{A} = 0$ and $\det \tilde{U}(0) = 1$, we see that $\det \tilde{U}(z) = 1$. Thus $\tilde{U}^{-1}(\frac{1}{2})$ is very simply calculated, and

$$\tilde{C} = \begin{pmatrix} x_1 y_2 + x_2 y_1 & 2x_2 y_2 \\ 2x_1 y_1 & x_1 y_2 + x_2 y_1 \end{pmatrix}.$$

Note also that since $\det \tilde{U}(z) = 1$, $x_1 y_2 = 1 + x_2 y_1$, and either of these can be eliminated from $\tilde{C}$. The fact that $\tilde{U}(z)$ has a unit determinant is a good check on the accuracy of the numerical integration.

The eigenvalues of $\tilde{C}$ are found by solving

$$\det(\tilde{C} - \nu \tilde{I}) = 0$$

or

$$\nu^2 - \nu \text{tr} \tilde{C} + \det \tilde{C} = 0$$

Now $\text{tr} \tilde{C} = 4x_2 y_1 + 2$, and $\det \tilde{C} = 1$, so the eigenvalues of $\tilde{C}$ are

$$\nu_{\pm} = 1 + 2x_2 y_1 \pm \sqrt{(1 + 2x_2 y_1)^2 - 1}$$

Because $\det \tilde{C} = 1$, $\nu_+ \nu_- = 1$. The behavior of $\nu_{\pm}$ as functions of λ is easily determined. For very negative λ , g is positive, and all solutions blow up. Hence x_2 and y_1 are large and positive, and $\nu_{\pm}$ is real and positive ($\nu_+ > 1$, $\nu_- < 1$). As λ increases, x_2 and y_1 decrease, and $\nu_{\pm}$ moves toward unity, coalescing there when x_2 or y_1 first vanishes. Hence the first eigenvalue, λ_1 , occurs when $\nu = 1$. In this problem y_1 vanishes before x_2 does. Because g first becomes negative at the midplane, $X_1(z)$ begins to dip down before blowing up. $Y_1(z)$ is then negative for small

z , becoming positive for larger z . Thus, an easy test for the positivity of λ is the following: if $Y_1(z)$ has a zero before reaching $z = \frac{1}{2}$ when $\lambda = 0$ is used in g , then λ_1 is positive. The zero in Y_1 is monotonic in λ , and λ_1 corresponds to the zero occurring at $z = \frac{1}{2}$. As λ is further increased, y_1 becomes negative, the argument of the radical is negative, and $v_{\pm}$ becomes complex. They are complex conjugates because they are roots of a quadratic with real coefficients, and they lie on the unit circle because their product is always unity. When the phase takes on the values $2n\pi/24$, the corresponding λ_n has been reached.

Thus, the solutions of Eq. (A3) are very like sines and cosines, but with $P(z)$ adding bumpiness. The lowest eigenfunction is approximately constant and the next has two nodes over the whole 24-section cylinder. There is even the same degeneracy of eigenfunctions, two for each eigenvalue, as with sines and cosines. Because the behavior of $v_{\pm}$ with λ is easy to follow, a trial approach for determining the λ_i has no danger of missing eigenvalues.

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FIGURE CAPTIONS

- Fig. 1. EBT geometry.
- Fig. 2. Equivalent bumpy cylinder.
- Fig. 3. Magnetic field lines and $|B|$ contours of equilibrium.
- Fig. 4. $\int d\ell/B$, B in midplane, $\beta_{\perp}$, and β of toroidal core plotted as functions of radius in midplane.
- Fig. 5. Maximum stable toroidal β as a function of β_{annulus} using Taylor's form for the hot electron pressure with $m = 2$ and $B_c = 1.6$ for the long annulus; $B_c = 1.2$ for the short annulus.

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Fig. 1.

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Fig. 2.

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BUMPY CYLINDER EQUILIBRIUM

Fig. 3.

Fig. 4.

Fig. 5.

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