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A SIMULTANEOUS ACCELERATED
RELAXATION METHOD
FOR ELLIPTIC EQUATIONS

BY

S. C. JARDIN AND A. MILLER

**PLASMA PHYSICS
LABORATORY**

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**PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY**

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A Simultaneous Accelerated Relaxation Method
For Elliptic Equations

S. C. Jardin and A. Miller

A simple physically motivated iterative method is presented for solving elliptic equations. The method is shown to converge 3 to 4 times as fast as the optimal SOR method when implemented on the CRAY-I computer.

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Iterative methods for solving elliptic equations are often desirable because of their simplicity, generality, and their ability to profit from a good initial "guess" to the solution. This note describes the Simultaneous Accelerated Relaxation Method (SAR), a simple, fast, physically intuitive iterative method for solving elliptic equations on vector processing computers. The method works well on nonorthogonal grids in irregular domains, and converges fast with either Neuman or Dirichlet boundary conditions. It is similar to the previously discussed Dynamic Relaxation Method.^{1,2}

We consider equations of the form

$$L \{ \phi \} = \rho(\vec{x}), \quad (1)$$

where L is an elliptic operator. The iteration is physically motivated by converting Eq. (1) into a damped wave equation,

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{2}{\tau} \frac{\partial \phi}{\partial t} = L \{ \phi \} - \rho(\vec{x}), \quad (2)$$

where t is the pseudo-time iteration variable and τ is a damping parameter. The steady state solution of Eq. (2) is a solution of Eq. (1). By finite-differencing Eq. (2) using the leap-frog method, a three level iteration scheme is obtained.

Applying this iteration to a two-dimensional $N \times N$ grid and using the maximum allowable stable time step δt and an optimal damping parameter τ , the transient in Eq. (2) will decay by e^{-1} in the order of N iterations. Since each iteration requires computing N^2 grid

points, the method requires on the order of N^3 operations to converge. This is the same scaling as for the familiar Successive Over-Relaxation (SOR) Method. However, the SAR method has the potentially enormous advantage of being vectorizable, so that for large enough vector size the method requires only N steps, each consisting of one vector calculation of length $N \times N$.

To illustrate the use of this method and to present a comparison of timing results we specialize Eqs. (1) and (2) to the Laplacian operator. We emphasize, however, that this method is of general applicability and that none of the special properties of the Laplacian operator have been utilized. For $L\{\phi\} \equiv \nabla^2 \phi$ on a uniformly spaced x-y grid, Eq. (2) takes the form

$$(1 + \epsilon)\phi_{i,j}^{n+1} = -(1 - \epsilon)\phi_{i,j}^{n-1} + 2\phi_{i,j}^n + s^2 \left[\left(\frac{\partial \phi}{\partial i} \right)_{i+\frac{1}{2},j}^n - \left(\frac{\partial \phi}{\partial i} \right)_{i-\frac{1}{2},j}^n + \left(\frac{\partial \phi}{\partial j} \right)_{i,j+\frac{1}{2}}^n - \left(\frac{\partial \phi}{\partial j} \right)_{i,j-\frac{1}{2}}^n - (\delta x)^2 \rho_{i,j} \right]. \quad (3)$$

Here $\epsilon \equiv \delta t/\tau$ and $s \equiv \delta t/\delta x$. The values $\epsilon = \pi/N$ and $s = 1/\sqrt{2}$ correspond to critical damping of the slowest decaying eigenmode of Eq. (2) and using the CFL value for the time step.

The centered difference operators in Eq. (3) have their usual meaning, for example,

$$\left(\frac{\partial \phi}{\partial i} \right)_{i+\frac{1}{2},j}^n = \phi_{i+1,j}^n - \phi_{i,j}^n, \quad (4a)$$

except at a boundary where Neuman boundary conditions are imposed. If the derivative boundary condition is $\frac{\partial \phi}{\partial n} = b(\ell)$, then at the boundary point $(I+\frac{1}{2}, j)$ we use the prescription

$$\left(\frac{\partial \phi}{\partial i}\right)_{I+\frac{1}{2}, j}^n = \frac{1}{2} \left(\phi_{i, j}^{n+1} + \phi_{i, j}^{n-1} \right) - \phi_{i, j}^n + \delta x \, b_{I+\frac{1}{2}, j} . \quad (4b)$$

Examination of the amplification matrix reveals that the insertion of Eq. (4b) into Eq. (3) will yield a stable iteration. A solution with Neuman boundary conditions, of course, exists only if $\rho(\vec{x})$ and $b(\ell)$ satisfy the consistency relation

$$\oint_{\partial R} b(\ell) \, d\ell = \int_R \rho(\vec{x}) \, ds . \quad (5)$$

A numerical experiment was performed using standard Fortran on the CRAY-I computer comparing the SAR method, Eq. (3), and the optimal SOR method in solving the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \sin x \sin y \quad (6)$$

on the domain $0 < x < \pi$, $0 < y < \pi$. Initial conditions were that $\phi_{i, j}^{n-1} = \phi_{i, j}^n = 0$. Both Neuman and Dirchlet boundary conditions were used for the SAR method, but the SOR method was not run with Neuman boundary conditions. The results of this study are presented in Table I. It is seen that although the SAR method requires more iterations than does the SOR method, the simultaneous nature of the former leads to faster running times. In computers with larger vector registers, this advantage of the SAR method will be considerably greater.

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References:

- (1) Otter, J. R. H., Nuclear Structural Engineering, Vol 1,
(1965) p. 61.
- (2) Rushton, K. R. and Laing, L. M., Aeronautical Quarterly,
Vol. 19, (1968) p. 375.

TABLE I

GRID DIMENSION	METHOD	NO. OF ITERATIONS	CPU TIME (SEC)	OVERRELAXATION PARAMETER ω	DAMPING PARAMETER ϵ
50	SOR(D)	138	.290	1.88	—
50	SAR(D)	266	.108	—	.054
50	SAR(N)	297	.140	—	.054
100	SOR(D)	286	2.47	1.94	—
100	SAR(D)	553	.867	—	.027
100	SAR(N)	484	.875	—	.027

Table I. A Comparison of the number of iterations and CPU Seconds to obtain an average absolute error of 10^{-8} in solving Laplace's equation. Both Dirichlet (D) and Neuman (N) boundary conditions are implemented for the SAR method.