192
6-10-80
JAL

LA-8340-MS

Informal Report

DR. 1289

MASTER

**Growth Rates of Interchange Modes in the
Dense Z-Pinch and Gas Blanket Effects**

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GROWTH RATES OF INTERCHANGE MODES IN THE DENSE Z-PINCH AND GAS BLANKET EFFECTS

by

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ABSTRACT

We study a class of models of the dense z-pinch in which the bulk of the plasma is neutrally stable against interchange. At a radius a the plasma is abruptly terminated and surrounded by a gas blanket. Instability results from the surface current necessary to maintain pressure balance at the plasma-gas interface. A single dispersion relation applies to all models in our class and shows two effects of the gas blanket in reducing growth rate.

1. Partial support of plasma pressure by external gas pressure reduces the driving force for the instability, and
 2. Mass loading of the relatively dense neutral gas reduces growth rate.
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Introduction

In [1] we studied the stabilizing effect of the neutral gas blanket on the dense z-pinch, using the energy principle. We found that the $m = 0$ (sausage or interchange) modes are stable if and only if the plasma pressure is exactly balanced by the gas pressure at the plasma-gas interface, and also the plasma pressure profile is such that

$$\frac{\gamma p - B^2/4\pi}{\gamma p + B^2/4\pi} \geq \frac{rB'}{B}.$$

In the case of equality the plasma is neutrally stable against interchange and the exact form of the pressure profile has been given by Kadomtsev [2].

The stabilizing effect of the gas blanket on kink ($m = 1$) modes was studied by Manheimer, Lampe, and Boris [3] in a perfectly diamagnetic sharp

boundary model. Our work [1], applying the energy principle to a diffuse plasma model, shows that short wavelength internal kink modes are completely unaffected by the gas blanket. Manheimer et al. [3] found some stabilization (i.e. growth rate reduction) under all circumstances but their model does not permit the study of internal kink modes.

The present work extends that of Ref. 1 as regards the $m = 0$, or interchange modes. If the gas pressure does not exactly balance that of the plasma then destabilizing forces are generated. In this paper we study the effect of the neutral gas blanket on the growth rate of the resulting instability. For mathematical clarity and simplicity, we limit our attention to the fastest growing short wavelength modes.

Two models are studied, a perfectly diamagnetic plasma ($B_{p\theta} = 0$), and a plasma distributed according to the Kadomtsev profile [2]. In both cases the bulk of the plasma is neutrally stable against internal interchange, but the plasma as a whole is unstable because of the surface current necessary to compensate for inequality of gas and plasma pressure. We find that for the short wavelength modes, one dispersion relation describes both models. The growth rate of the instability is reduced, as expected, by two effects:

- 1) Partial balance of plasma pressure by external gas pressure reduces the driving force for the instability.
- 2) The relatively cold and dense gas blanket reduces the growth rate through mass loading.

We make no estimate of such nonlinear effects as shock formation in the neutral gas. Clearly this will have a further inhibiting effect on the growth.

1. The Perturbed Equation of Motion

In the approximation which treats a plasma as a perfectly conducting ideal gas the linearized equations of motion for a small perturbation may be written as

$$(1.1) \quad \rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = -\nabla(\delta p) + \frac{1}{4\pi} \left[(\text{curl } \vec{B}) \times \vec{Q} + (\text{curl } \vec{Q}) \times \vec{B} \right],$$

where $\vec{\xi}$ is the displacement from equilibrium and

$$(1.2) \quad \begin{cases} \delta p \equiv -\gamma p \nabla \cdot \vec{\xi} - \vec{\xi} \cdot \nabla p, \\ \vec{Q} \equiv \text{curl} (\vec{\xi} \times \vec{B}). \end{cases}$$

We consider a pure z-pinch so that, in cylinder coordinates r, θ, z , we have

$$(1.3) \quad \vec{B} = (0, B, 0), \quad B = 2I/r$$

in the equilibrium state, and

$$(1.4) \quad p' = -(B/4\pi)(B' + B/r),$$

where here and hereafter primes mean d/dr . We can Fourier analyse, writing

$$(1.5) \quad \vec{\xi}(r, \theta, z, t) = \vec{\xi}(r) \exp\{i(\omega t + m\theta + kz)\}.$$

We shall consider only the $m = 0$ modes. For these modes

$$(1.6) \quad \begin{cases} \vec{Q} = (0, Q, 0), \\ Q \equiv -(B\xi_r)' - ikB\xi_z. \end{cases}$$

If we substitute Eqs (1.2), (1.3), (1.4) and (1.6) into Eq (1.1), we find that the θ and the z equations of motion give

$$(1.7) \quad \begin{cases} \xi_\theta = 0, \\ \xi_z = (ik/\Delta) \left\{ (\gamma p + B^2/4\pi)\xi_r' + (\gamma p - B^2/4\pi)(\xi_r/r) \right\}, \\ \Delta \equiv k^2(\gamma p + B^2/4\pi) - \rho\omega^2. \end{cases}$$

We can use Eq (1.7) to eliminate ξ_z and ξ_θ to get

$$(1.8) \quad \begin{cases} \nabla \cdot \vec{\xi} = -(1/\Delta) \left\{ \rho\omega^2\xi_r' + (\rho\omega^2 - 2k^2B^2/4\pi)(\xi_r/r) \right\}, \\ Q = (1/\Delta) \left\{ \rho\omega^2(B\xi)' - k^2 \left[(\gamma p + B^2/4\pi)B' - (\gamma p - B^2/4\pi)(B/r) \right] \xi \right\}. \end{cases}$$

Our equation of motion is

$$(1.9) \quad -\rho\omega^2\xi = -\partial_r \left[\delta p + BQ/4\pi \right] - 2BQ/4\pi r.$$

As we have eliminated ξ_θ and ξ_z , we use ξ in place of ξ_r in Eqs (1.8), (1.9) and in all that follows. Using Eqs (1.8), for δp and Q , Eq (1.9) takes the form

$$(1.10) \quad \begin{cases} \rho\omega^2\xi = \frac{d}{dr} \left\{ (\rho\omega^2/\Delta) \left[(\gamma p + B^2/4\pi)\xi_r' + (\gamma p - B^2/4\pi)(\xi/r) \right] \right\} \\ \quad + (2\rho\omega^2B/4\pi r\Delta)(B\xi)' - (2k^2B/4\pi r\Delta) \left[(\gamma p + B^2/4\pi)B' - (\gamma p - B^2/4\pi)(B/r) \right] \xi \\ \Delta \equiv k^2(\gamma p + B^2/4\pi) - \rho\omega^2. \end{cases}$$

The boundary conditions are that ξ and the total pressure be continuous. The latter implies that $\delta[p + B^2/8\pi]$ evaluated on the perturbed plasma surface be continuous. Now on the perturbed surface Σ , we have

$$(1.11) \quad [\delta p]_{\Sigma} = -\gamma p(\nabla \cdot \xi), \quad [\delta(B^2/8\pi)]_{\Sigma} = BQ/4\pi + \xi_{BB'}/4\pi$$

whence, using Eqs (1.8)

$$(1.12) \quad [\delta(p + B^2/8\pi)]_{\Sigma} = (1/\Delta) \left\{ \rho\omega^2(\gamma p + B^2/4\pi)\xi' + [\rho\omega^2\gamma p - (k^2 B^2/4\pi)(\gamma p + B^2/4\pi)](\xi/r) \right\}$$

on the plasma side of the surface.

Outside the plasma we visualize a neutral gas of uniform pressure and density plus a vacuum magnetic field. The perturbed equation of motion for the gas is simply Eq (1.10) with p , ρ constant and $B = 0$, namely

$$(1.13) \quad \begin{cases} \xi'' + (1/r)\xi' - [\lambda^2 + 1/r^2]\xi = 0, \\ \lambda^2 \equiv k^2 - \rho\omega^2/\gamma p. \end{cases}$$

This has the solution, regular at $r = \infty$,

$$(1.14) \quad \xi = AK_1(\lambda r),$$

where K_1 is the first order Bessel function of imaginary argument. On the boundary Σ in the gas

$$(1.15) \quad [\delta p]_{\Sigma} = -\gamma p \nabla \cdot \xi = (\rho\omega^2/\lambda^2)(\xi' + \xi/r) = (\rho\omega^2/\lambda)AK_0(\lambda r)$$

and, from $B = 2I/r$, we get in the gas

$$(1.16) \quad \delta(B^2/8\pi) = -B^2\xi/4\pi r.$$

Thus, if our boundary is at the point $r = a$, combining Eqs (1.12) with (1.15) and (1.16) gives

$$(1.17) \quad \left[1/\Delta \left\{ \rho\omega^2(\gamma p + B^2/4\pi)\xi' + [\rho\omega^2\gamma p - (k^2 B^2/4\pi)(\gamma p + B^2/4\pi)](\xi/a) \right\} \right]_p \\ = (\rho_g \omega^2/\lambda)AR_0(\lambda a) - [B^2/4\pi]_g(\xi/a)$$

together with

$$(1.18) \quad \xi(a) = AK_1(\lambda a).$$

The subscripts p, g refer respectively to the plasma side and the gas side of the interface.

Our model is now characterized by the plasma equation of motion Eq (1.10) together with the boundary conditions, Eqs (1.17) and (1.18). We shall discuss two simple cases of this model.

2. The Perfectly Diamagnetic Plasma

The simplest of all models is to take p, ρ constant and $B = 0$ inside the plasma. Then Eq (1.10) reduces, as we saw, to Eq (1.13). The solution which is regular at the origin is

$$(2.1) \quad \xi = I_1(\lambda_p r) .$$

As λ depends on ρ , which may differ in the gas and in the plasma, we give it a medium suffix. On the plasma surface, plasma side

$$(2.2) \quad \begin{cases} [\delta p]_\Sigma = -\gamma_p \nabla \cdot \vec{\xi} = (\rho_p \omega^2 / \lambda_p^2) (\xi' + \xi/a) \\ \quad \quad \quad = (\rho_p \omega^2 / \lambda_p^2) I_0(\lambda_p a) , \end{cases}$$

and of course $\delta(B^2/8\pi) = 0$ on the plasma side. Using Eq (1.18), our boundary condition, Eq (1.17) becomes

$$(2.3) \quad \omega^2 \frac{I_0(\lambda_p a)}{I_1(\lambda_p a)} = \omega^2 \left(\frac{\rho_g}{\rho_p} \right) \left(\frac{\lambda_p}{\lambda_g} \right) \frac{K_0(\lambda_g a)}{K_1(\lambda_g a)} - \frac{\lambda_p B_g^2}{4\pi a \rho_p} .$$

Our notation for the Bessel functions I_0 , I_1 , K_0 , K_1 is that of Whittaker and Watson [4]. Pressure balance for the unperturbed state gives

$$(2.4) \quad B_g^2 = 8\pi(p_p - p_g)$$

so we see that $p_g = p_p$ gives $B_g^2 = 0$ and is neutrally stable.

As we normally have instability, it is convenient to work with the growth rate v , defined as

$$(2.5) \quad v^2 = -\omega^2$$

and further to define the acceleration g by

$$(2.6) \quad g \equiv B_g^2 / 4\pi a \rho_p = 2(p_p - p_g) / a \rho_p .$$

We also define the two sound speeds

$$(2.7) \quad c_i^2 = \gamma p_i / \rho_i, \quad i = p, g,$$

and the density ratio

$$(2.8) \quad R \equiv \rho_g / \rho_p = (p_g / p_p) (T_p / T_g) .$$

Clearly, as $T_p \gg T_g$ normally, $\lambda_g > \lambda_p$. Thus if we limit our attention to wavelengths such that λ_p is so large that I_0/I_1 takes on its asymptotic value 1 to good approximation, then $K_0/K_1 = -1$ is even a better approximation. Under this condition Eq (2.3) simplifies to

$$(2.9) \quad v^2 - \lambda_p^2 g = -R v^2 (\lambda_p / \lambda_g) .$$

Now consider first the case of a naked plasma, i.e. set $\rho_g = 0$. Then Eq (2.9) becomes

$$(2.10) \quad v^2 = g \left[k^2 + v^2 / c_p^2 \right]^{1/2} ,$$

which readily solves to yield

$$(2.11) \quad v^2 = kg \left\{ \left[1 + (g/2kc_p^2)^2 \right]^{1/2} + g/2kc_p^2 \right\} .$$

This reduces to the Rayleigh-Taylor result for an incompressible fluid supported against gravity g by a fluid of zero density in the incompressible limit $g/2kc_p^2 \rightarrow 0$, or equivalently in the limit

$$(2.12) \quad k \gg g/2c_p^2 .$$

In our case g is given by Eq (2.6) and this condition can be written as

$$(2.12)' \quad ka \gg (p_p - p_g) / \gamma p_p .$$

As the right hand member of this expression is always less than 3/5 compressibility effects in the plasma are certainly negligible whenever ka is so large that the Bessel functions are well approximated by their asymptotic expressions. In this latter limit the effects of cylindrical curvature disappear, except that the cylinder radius a enters into the expression for g . In this approximation then

$$(2.13) \quad v = v_0 \equiv \left[2k p_p / (a \rho_p) \right]^{1/2}$$

is the growth rate for short wave perturbations of a perfectly diamagnetic naked plasma.

As we saw above, whenever Eq (2.9) is valid, we can also neglect plasma compressibility and set $\lambda_p = k$ to a good approximation. Doing this and introducing the dimensionless quantities n, κ defined by

$$(2.14) \quad n^2 = v^2 / k g, \quad \kappa = k c_g^2 / g,$$

Eq (2.9) takes the form

$$(2.15) \quad 1 - n^2 = R n^2 \left[1 + n^2 / \kappa \right]^{-1/2},$$

from which it clearly follows that $n^2 \leq 1$ in all cases. We can write

$$(2.16) \quad \kappa = (\gamma/2) k a (T_g / T_p) [p_p / (p_p - p_g)] .$$

Normally $p_p / (p_p - p_g)$ is of order unity and $T_g \ll T_p$ so that even though ka be quite large we may yet well have

$$(2.17) \quad \kappa \ll 1 .$$

If into Eq (2.15) we set $n^2 = 1 - \epsilon$ and then neglect κ as compared with unity, Eq (2.15) simplifies to

$$(2.18) \quad \epsilon^2 = \kappa R^2 (1 - \epsilon)$$

or

$$(2.19) \quad \begin{cases} \epsilon = \frac{1}{2} \kappa R^2 \left\{ \left[1 + 4 / (\kappa R^2) \right]^{1/2} - 1 \right\} , \\ n = [1 - \epsilon]^{1/2} \end{cases}$$

to a good approximation.

Consider now an example. Let I = one meg amp, or in our units $I = 10^5$ abamperes. Then, if we choose $a = 0.1$ cm.,

$$(2.20) \quad p_p = B^2 / 8\pi = I^2 / 2\pi a^2 = 1.6 \times 10^5 \text{ bars} .$$

Thus $p_g \ll p_p$ for any reasonable gas pressure and $p_p/(p_p - p_g) = 1$ to high precision. If $T_p = 5$ KeV, as is necessary for thermonuclear burn and T_g is of the order of room temperature, then T_p/T_g is of order 10^5 so κ is of order $10^{-5}(ka)$ which well satisfies condition (2.17). For these conditions R is of order unity so $R^2\kappa$ is very small and Eq (2.19) gives

$$(2.21) \quad n = \left[1 - R\kappa^{1/2}/2 \right]^{1/2},$$

which with $R\kappa^{1/2}$ of order 10^{-2} , is essentially indistinguishable from unity. For this model the gas blanket has no measurable effect on the stability. The growth rate is essentially that of a naked plasma, namely

$$(2.22) \quad \nu \doteq \nu_0 = 9.79 \times 10^8 (ka)^{1/2} (\text{sec})^{-1}.$$

3. Plasma with the Kadomtsev Profile

The perfectly diamagnetic plasma is a special case of a plasma which is neutrally stable against internal interchanges and is $m = 0$ unstable only because of the surface current necessary when the plasma is terminated at some finite radius $r = a$. A more general plasma distribution with this property was given by Kadomtsev [2] and is discussed in [1]. For this distribution of plasma and field, which we shall call the Kadomtsev profile, in addition to pressure balance the relationship

$$(3.1) \quad \frac{\gamma p - B^2/4\pi}{\gamma p + B^2/4\pi} = \frac{rB'}{B}$$

is satisfied and in [1] the resulting pressure and magnetic field profile are given analytically. We can eliminate the pressure, writing

$$(3.2) \quad \gamma p = \frac{B^2}{4\pi} \frac{B + rB'}{B - rB'},$$

and define

$$(3.3) \quad Y \equiv \frac{2\pi\rho\omega^2}{k^2 B^2} (1 - rB'/B).$$

Equation (1.10) then takes the form

$$(3.4) \quad \rho\omega^2 k^2 \xi = \frac{d}{dr} \left[\frac{\rho\omega^2}{1-Y} \left(\xi' + \frac{B'}{B} \xi \right) \right] + \frac{\rho\omega^2(1-rB'/B)}{r(1-Y)} \left(\xi' + \frac{B'}{B} \xi \right).$$

As $\omega^2 = 0$ satisfies Eq (3.4), identically, this profile is neutrally stable against interchange if extended to $r = \infty$. If, on the other hand, $\omega \neq 0$, then we can define

$$(3.5) \quad \xi = u/B, \quad Q = \rho/(1-Y)B, \quad ,$$

and the equation of motion assumes the apparently simpler form

$$(3.6) \quad u'' + (1/r + Q'/Q - B'/B) u' - k^2(1-Y) u = 0.$$

The boundary conditions are continuity of ξ and the condition

$$(3.7) \quad \left[k^{-2}(1-Y)^{-1} \left\{ \rho \omega^2 (\xi' + \xi/r) - [\rho \omega^2 (1-rB'/B) + k^2 B^2/4\pi] (\xi/r) \right\} \right]_{p\ell} \\ = (\rho_g \omega^2 / \lambda_g) A K_0(\lambda_g a) - (B_g^2/4\pi) (\xi/a).$$

We shall now make a simple model by confining our attention to wavelengths sufficiently short that

$$(3.8) \quad |Y| < < 1.$$

As in fact

$$(3.9) \quad Y = \frac{\rho \omega^2}{k^2(\gamma_p + B^2/4\pi)},$$

and as $|\omega^2| \leq kg$, Eq (3.8) is guaranteed if

$$(3.10) \quad k \gg g/[C_p^2 + C_A^2], \quad C_A^2 \equiv B_p^2/4\pi\rho_p, \quad C_p^2 \equiv \gamma_p\rho_p.$$

Thus condition (3.8) is just the condition that the perturbation behave incompressibly. Our equations of motion reduce to

$$(3.11) \quad \begin{cases} u'' + Pu' - k^2u = 0, \\ P \equiv 1/r + \rho'/\rho - 2B'/B. \end{cases}$$

For $\gamma = 5/3$ the Kadomtsev profile is given in Ref. 2 (typographic error corrected in Ref. 1) in the parametric form

$$(3.12) \quad \begin{cases} p = p_0 \beta^{5/2} (\beta+4/5)^{-5/2} , & r = r_0 \beta^{-3/4} (\beta+4/5)^{1/4} \\ B = (8\pi p_0)^{1/2} \beta^{3/4} (\beta+4/5)^{-5/4} , & \beta \equiv 8\pi p/B^2 , \end{cases}$$

where p_0 and r_0 are integration constants. As $\beta \rightarrow \infty$ when $r \rightarrow 0$, p_0 is the central value of the pressure. There are various ways of constructing temperature, and hence also density profiles. We shall do so by demanding that at each point in the plasma the ohmic heating rate exactly balance the bremsstrahlung loss. If E is in esu and J is in e.m.u. then the ohmic heating rate is $c\vec{E} \cdot \vec{J}$ ergs $(\text{cm})^{-3} \text{ sec}^{-1}$. In our case, we readily compute

$$(3.13) \quad J_z = (1/4\pi r) \partial_r (rB) = -(1/r_0) (p_0/2\pi)^{1/2} \beta^{5/2} (\beta+4/5)^{-3/2} (\beta+6/5)^{-1}$$

by differentiation of the second and third members of Eqs (3.12). The bremsstrahlung loss rate is proportional to $\rho^2/T^{1/2}$ so equating this loss with the ohmic heating rate gives

$$(3.14) \quad \rho^2 T^{1/2} = A \beta^{5/2} (\beta+4/5)^{-3/2} (\beta+6/5)^{-1} ,$$

and the perfect gas law gives

$$(3.15) \quad \rho T = C \beta^{5/2} (\beta+4/5)^{-5/2}$$

where A, C are constants. From these two equations we readily find

$$(3.16) \quad \begin{cases} \rho = \rho_0 \beta^{5/6} (\beta+4/5)^{-1/6} (\beta+6/5)^{-2/3} \\ T = T_0 \beta^{5/3} (\beta+6/5)^{2/3} (\beta+4/5)^{-7/3} , \end{cases}$$

where clearly ρ_0, T_0 are the central values of density and temperature. Near $r = 0$ β is large and Eqs (3.12), (3.16) expand to give

$$(3.17) \quad \begin{cases} \beta \doteq (r_0/r)^2 [1 + (2/5)(r/r_0)^2] , & p \doteq p_0 [1 - 2(r/r_0)^2] \\ \rho \doteq \rho_0 [1 - (14/15)(r/r_0)^2] , & T \doteq T_0 [1 - (16/15)(r/r_0)^2] \\ B = (8\pi p_0)^{-1/2} (r/r_0) [1 - (6/5)(r/r_0)^2] \end{cases}$$

for small values of r/r_0 . For large r , β is small and Eqs (3.12), (3.16) become

$$(3.18) \left\{ \begin{array}{ll} \beta \sim (4/5)^{1/3} (r_0/r)^{4/3} & p \sim p_0 (5/4)^{5/3} (r_0/r)^{10/3} \\ \rho \sim \rho_0 (4/5)^{1/9} (5/6)^{2/3} (r_0/r)^{10/9} & T \sim T_0 (5/4)^{16/9} (6/5)^{2/3} (r_0/r)^{20/9} \\ & B \sim (8\pi p_0)^{1/2} (5/4) (r_0/r) \end{array} \right.$$

asymptotically, as r/r_0 becomes very large. From Eqs (3.12) and (3.16) we readily calculate

$$(3.19) \quad P = \frac{1}{3r} \left[\frac{-3\beta^2 + (8/5)\beta + 204/25}{(\beta + 6/5)^2} \right],$$

whence

$$(3.20) \quad P \doteq -(1/r) [1 - (8/5)(r/r_0)^2]$$

for small r/r_0 and

$$(3.21) \quad P \sim (17/9r) [1 - 1.3652(r_0/r)^{4/3}]$$

for large r/r_0 . Actually the two limiting expressions, Eqs (3.20) and (3.21) are all we shall need.

Near $r = 0$, Eq (3.11) has one solution which remains bounded; let us call it $u = \bar{U}$. We find

$$(3.22) \quad \bar{U} = br^2 \left\{ 1 + (k^2 r_0^2 / 8 - 2/5)(r/r_0)^2 + \dots \right\},$$

where b is a normalizing constant which we choose to be positive. It follows that for sufficiently small values of r $\bar{U}, \bar{U}'$ are both positive. From this and Eq (3.11) we can conclude that $\bar{U}'$ is positive for all r . For if not, then $\bar{U}'$ would have to vanish at some point $r = r_c$ and at this point $\bar{U}$ would be a maximum. Thus $\bar{U}''$ would be negative at $r = r_c$, whereas Eq (3.11) requires $\bar{U}''$ to be positive where $\bar{U}'$ vanishes. Thus $\bar{U}'$ cannot change sign but must be everywhere positive; $\bar{U}(r)$ increases monotonely with r .

For very large r , $P \rightarrow 0$ and there are two solutions which behave roughly as $\exp[kr]$ and as $\exp[-kr]$ respectively. As $\bar{U}$ increases everywhere with r its asymptotic form at large r must be the exponentially increasing solution. We now proceed to find this asymptotic expression. For large r we use Eq (3.21) for P and u satisfies, to the order in r^{-1} carried, the equation

$$(3.23) \quad \begin{cases} u'' + [(1+2\alpha)/r - \delta/r^{7/3}] u' - k^2 u = 0, \\ \alpha = +4/9, \quad \delta = 2.5787r_0^{4/3}. \end{cases}$$

If we temporarily ignore the term $\delta r^{-7/3}$, we would have the exponentially increasing solution $(kr)^{-\alpha} I_\alpha(kr)$, so we shall try

$$(3.24) \quad u = w u_0, \quad u_0 \equiv (kr)^{-\alpha} I_\alpha(kr).$$

Then w satisfies the equation

$$(3.25) \quad w'' + [2u'_0/u_0 + (1+2\alpha r)/r - \delta/r^{7/3}] w' - (\delta/r^{7/3})(u'_0/u_0)w = 0.$$

Note that $u'_0/u_0 \sim k$ for large r . Then we can calculate w to lowest non-vanishing order in r^{-1} and obtain

$$(3.26) \quad w \sim 1 - (3\delta/8)r^{-4/3}.$$

Combining this with Eqs (3.24) and (3.5) we find for the displacement

$$(3.27) \quad \xi \sim (1/B)(kr)^{-\alpha} I_\alpha(kr) [1 - 0.9670(r_0/r)^{4/3}],$$

which is valid for large r/r_0 .

We shall suppose that a/r_0 is so large that Eq (3.27) is indeed a reasonably good approximation so long as r is near a . Then we can substitute Eq (3.27) into the boundary condition, Eq (3.7), obtaining

$$(3.28) \quad \begin{cases} \omega^2 \left\{ \frac{I_{\alpha+1}(ka)}{I_\alpha(ka)} - \frac{1.289}{ka} \left(\frac{r_0}{a} \right)^{4/3} \right\} = R \omega^2 (k/\lambda_g) \frac{K_0(\lambda_g a)}{K_1(\lambda_g a)} - k g, \\ g \equiv \frac{B^2 - B_p^2}{4\pi a \rho_p} = \frac{2(p_p - p_g)}{a \rho_p}, \quad R \equiv \rho_g / \rho_p, \\ \lambda_g \equiv \sqrt{k^2 - \frac{\rho_g \omega^2}{\gamma p_g}}. \end{cases}$$

In Eq (3.28) we have kept only the lowest order correction term in (r_0/a) . If now we do as we did in Section 2, and restrict our attention to wavelengths short enough that ka is appreciably larger than one, we may replace the Bessel functions by their asymptotic expansion. When we do this Eq (3.28) goes over into Eq (2.9) whence, retracing our previous arguments, we recover Eq (2.15) as our dispersion relation for the short wavelength modes.

Let us now consider some numerical examples. If we use the relation $B = 2I/r$ with Eqs (3.18) and let $r \rightarrow \infty$, we obtain

$$(3.29) \quad p_0 = [8/(25\pi)][I/r_0]^2,$$

where I is the total current. If I is in abamperes, r_0 in cm., then p_0 is in dynes (cm) $^{-2}$. For our examples we shall set

$$(3.30) \quad \begin{aligned} I &= 1.1 \times 10^5 \text{ abamps,} \\ T_0 &= 5 \text{ KeV.} \end{aligned}$$

The first is the Pease Current [5,6] required for equilibrium and the second is chosen to make thermonuclear burn possible. In the absence of a gas blanket the growth rate ν_0 is given by Eq (2.13). Using Eqs (3.8) to evaluate plasma quantities at the surface, $r = a$, we compute the growth rates ν_0 for several values of r_0 and of a , always choosing the wavelength of the perturbation such that $kr_0 = 1$. The results are shown in Table 1.

Table 1. Growth Rates for a naked plasma column for $kr_0 = 1$

a/r_0	ν_0 in $(\mu \text{ sec})^{-1}$		
	$r_0 = 0.1 \text{ cm}$	$r_0 = 0.03 \text{ cm}$	$r_0 = 0.01 \text{ cm}$
5	94.11	313.7	941.1
10	30.81	102.7	308.1
15	16.03	53.43	160.3
20	10.08	36.13	100.8
30	5.25	15.74	52.47

Next we assume a gas blanket with

$$(3.31) \quad \begin{aligned} p_g &= 1 \text{ bar,} \\ T_g &= 0.025 \text{ e.V.} \end{aligned}$$

With these parameters $\kappa \ll 1$ for all cases in Table 1, so we can use Eqs (2.19) to compute the normalized growth rate n . The results are shown in Table 2.

Table 2. Normalized growth rates n showing stabilizing effect of gas blanket

a/r_0	$n = v[\text{kg}]^{-1/2} \text{ for } \kappa r_0 = 1$		
	$r_0 = 0.1 \text{ cm}$	$r_0 = 0.03 \text{ cm}$	$r_0 = 0.01 \text{ cm}$
5	0.7509	0.9167	0.9714
10	0.2929	0.6360	0.3560
15	0.1050	0.3182	0.6345
20	0.0484	0.1578	0.4056
30	0.0221	0.0732	0.2111

For all entries in Tables 1 and 2 $p_p \gg p_g$ so the actual growth rate v is given by

$$(3.32) \quad v = n v_0$$

to good precision.

The first columns of Figs. 1 and 2, labeled $r_0 = 0.1 \text{ cm}$, apply to a pinch of effective radius nearly identical with that of the example given at the end of Section 2. If we put $\kappa a = 1$ in Eq (2.22) (corresponding with $\kappa r_0 = 1$ in our tables) we note that the naked growth rates of Table 1 are much less than the $v_0 = 979 (\mu \text{ sec})^{-1}$ of the diamagnetic model. The reason for this is that the truncated Kadomtsev profile results in a surface pressure very much less than the central pressure. If now we turn to the corresponding column of Table 2 we note an additional reduction of growth rate by a factor ranging from 1.3 to 45 (or even higher) depending on the ratio a/r_0 . As $(p_p - p_g)/p_p$ is

very nearly unity in all cases listed, this additional reduction arises from mass loading of the plasma column by the much more dense gas blanket.

Conclusion

We have studied the interchange mode in a z-pinch configuration and find that the bulk of the plasma is neutrally stable against interchange. Instability results from the surface current necessary to support the finite pressure jump from plasma to surrounding gas. For the short wavelength modes the dispersion relation covers all models of this particular class.

We have found that growth rates, relative to that of the perfectly diamagnetic model, are greatly suppressed in pinches with the Kadomtsev profile provided the plasma radius a is much greater than the characteristic, or effective radius r_0 . Two effects contribute to this. First, the driving acceleration

$$a = (p_p - p_g)/(\rho_p a)$$

is greatly reduced when $a \gg r_0$, and secondly the mass loading of the gas blanket has a strong effect under the same circumstance. How much a happy state of affairs might come about has not been shown.

References

1. R. E. Dwyer, "Effects of the Gas Blanket on the Stability of the Z-Pinch", Los Alamos Scientific Laboratory report LA-7809-MS (May 1973).
2. L. D. Kadomtsev, "Hydromagnetic Stability of a Plasma", in Reviews of Plasma Physics, Ed. M. A. Leontovitch, Vol. 2, pp 165-166, Consultants Bureau, New York, 1966.
3. W. M. Manheimer, M. Lampe, and J. F. Boris, "Effect of a Surrounding Gas on Magnetohydrodynamic Instabilities in Z-Pinch", Phys. of Fluids 16 No. 7, pp 1126-1134 (July 1973).
4. Modern Analysis by E. T. Whittaker and G. N. Watson, Cambridge at the University Press, 1927.
5. F. C. Pease, "Equilibrium Characteristics of a Pinched Gas Discharge Cooled by Bremsstrahlung Radiation", Phys. Soc. of London, Proc. B, 70, pp 11-23 (1957).
6. J. E. Hammel, "An Ohmically Heated High-Density Z-Pinch", Los Alamos Scientific Laboratory report LA-6203-MS (Jan. 1976).