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BLOCK RELAXATION STRATEGIES

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1. INTRODUCTION

In connection with finite difference approximations to the solution of the general elliptic equation $\nabla a \cdot u = -f$ on an arbitrary region in rectangular coordinates, B. Buzbee, Los Alamos Scientific Laboratory, has suggested investigating various block line relaxation strategies to determine which ones give the fastest rate of convergence.

More specifically, we wish to solve the linear system $Ax = b$, where

$$A = \begin{bmatrix} T_1 & U_1 & & & \\ S_2 & T_2 & U_2 & & \\ & S_3 & T_3 & U_3 & \\ & & \ddots & \ddots & \ddots \\ & & & S_n & T_n & U_n \end{bmatrix} \quad (1)$$

is $n \times n$ block tridiagonal with block size m . We plan to solve this system by block line relaxation when the largest block problem with block size m we can cheaply solve directly is $l \times l$.

II. STEPS

Each iteration of the strategies that we consider is composed of a sequence of steps. After the j^{th} step, we have the approximate solution vector $x^j = (x_1^j, x_2^j, \dots, x_n^j)$, where each x_i^j is an m -vector, and error vector $e^j = x - x^j$. We partition the matrix A as

$$A = \begin{bmatrix} B_1 & . & C_1 & . & 0 \\ . & . & . & . & . \\ A_2 & . & B_2 & . & C_2 \\ . & . & . & . & . \\ 0 & . & A_3 & . & B_3 \\ . & . & . & . & . \end{bmatrix} \quad (2)$$

where B_1 is $k \times k$ block tridiagonal; B_2 is $l \times l$ block tridiagonal; A_2 and A_3 have their only nonzero block, a_2 and a_3 , respectively, at the upper right; and C_1 and C_2 have their only nonzero block, c_1 and c_2 , respectively, at the lower left. We shall consider l to be fixed, but k and m can be any nonnegative integers, depending on j , that satisfy $k + l + m = n$. During this step we shall relax only the vector variables $x_{k+1}, x_{k+2}, \dots, x_{k+l}$. Let $z_1 = (x_1, \dots, x_k)$; $z_2 = (x_{k+1}, \dots, x_{k+l})$; $z_3 = (x_{k+l+1}, \dots, x_n)$ and similarly for z_1^j, z_2^j, z_3^j . Also let $\beta_1 = (b_1, \dots, b_k)$, $\beta_2 = (b_{k+1}, \dots, b_{k+l})$, and $\beta_3 = (b_{k+l+1}, \dots, b_n)$. The system we are solving in this notation is

$$\begin{aligned} B_1 z_1 + C_1 z_2 &= \beta_1 \\ A_2 z_1 + B_2 z_2 + C_2 z_3 &= \beta_2 \\ A_3 z_2 + B_3 z_3 &= \beta_3 \end{aligned} \quad (3)$$

At the j^{th} step we relax using the equations

$$\begin{aligned} z_1^{j+1} &= z_1^j \\ B_2 z_2^{j+1} &= \beta_2 - A_2 z_1^j - C_2 z_3^j \\ z_3^{j+1} &= z_3^j \end{aligned} \quad (4)$$

Thus e^{j+1} has vector components (define $e_0^j = e_{n+1}^j = 0$ for all j):

$$\begin{aligned} e_i^{j+1} &= e_i^j, \quad i \leq k \text{ or } i \geq l+1+k \\ e_i^{j+1} &= - (B_2^{-1})_{i-k,l} a_2 e_k^j - (B_2^{-1})_{i-k,l} c_2 e_{k+l+1}^j, \\ &\quad k < i < l+1+k. \end{aligned} \quad (5)$$

A single iteration of a 'periodic) strategy of length p , $(k_1, k_2, \dots, k_p)$, is defined as a Gauss-Seidel sequence of steps, (k_j, l_j, m_j) , $j = 1, 2, \dots, p$, where $k_j + l_j + m_j = n$ and, for the purposes of our discussion, all $l_j = l$. The convergence rate per step, θ , of the strategy $(k_1, k_2, \dots, k_p)$ is $\theta = -\frac{1}{p} \ln \rho$; where ρ is the spectral radius of the error iteration matrix of the strategy.

III. PARTICULAR STRATEGIES; TEST PROBLEMS

Certain strategies can be analytically evaluated when they are applied to the linear systems that arise from finite difference approximations to the solution of a Helmholtz-type equation. We shall call a strategy $(k_1, k_2, \dots, k_p)$ ordered if $0 = k_1 < k_2 < \dots < k_p = n - l$ (a left-to-right sweep); well ordered if it is ordered and $k_{j+1} < k_j + l + 1$ for every j (all variables are relaxed); and super ordered if it is well

ordered and $k_j + i + 1 \leq k_{j+2}$ (the variables corresponding to the beginning and the end of each step are relaxed exactly once in each iteration of the strategy).

The test problems that we consider have $U_i - S_i = 1$ and $T_i = T$ (for some fixed T) for all i in Eq. (1). Under these assumptions, the convergence rate per step of the strategy $(k_1, k_2, \dots, k_p)$ depends only on the ratios

$$\alpha_j = \frac{k_{j+1} - k_j}{i + 1}, \quad j = 1, 2, \dots, p-1.$$

In addition, any strategy that is a permutation of a super-ordered strategy has the same convergence rate per step. If $\alpha_j = \alpha$ for all $j = 1, 2, \dots, p-1$, the strategy is called regular.

IV. RESULTS

The results deal only with the test cases. The proofs will be published elsewhere.

Result 1. Analytic expressions can be found for the convergence rates of the regular strategies in terms of the spectral radius of the matrix T .

Result 2. Asymptotically, as $n/i \rightarrow \infty$, the regular strategy with the fastest convergence rate per step has $\alpha = 2/3$ (the overlap from one step to the next is one-third the length of the step).

Result 3. The super-ordered strategy with the fastest rate of convergence per step is never regular when more than three steps are required per iteration.

Result 4. For the regular strategy with $\alpha = 1/2$ (the

overlap from one step to the next is one-half the length of the step) applied to the 5-point difference approximation to the solution of Poisson's equation on the unit square with uniform grid spacing ($\frac{1}{n+1} = h = \frac{1}{m+1}$), the spectral radius is $\rho \approx 1 - \frac{1}{2} \pi^2 h^2 (l+1)^2 + \dots$.

V. CONCLUSIONS

The positive aspects of block relaxation schemes, namely, ease of implementation, generality, and the ability to exploit vector and parallel architecture, have traditionally been negated by the low convergence rate, high work rate, and large storage requirements of the schemes. We have shown, however, that the convergence rate of some block relaxation strategies is much faster than that of the strategies generally employed. For example, Parter and Steuervalt (1) have shown that the spectral radius of l -line relaxation applied to the 5-point difference approximation to the solution of Poisson's equation on the unit square with uniform grid spacing h is

$$\rho \approx 1 - 2l \pi^2 h^2 + \dots$$

So by the results given above, the regular strategy with $n = 1/2$ (which is far from optimal) converges faster by a factor of $l/4$. This improved convergence rate for these strategies warrants a deeper look into these block relaxation schemes.

REFERENCES

1. Seymour V. Parter and Michael Steuervalt, "Another look at Iterative Methods for Elliptic Difference Equations," Los Alamos Scientific Laboratory report LA-UR-80-202 (1980).