**Assessment of Effectiveness of
Geologic Isolation Systems**

Analytic Modeling of Flow in a Permeable Fissured Medium

February 1982

**Prepared for the Office of Nuclear Waste Isolation
with the U.S. Department of Energy
under Contract DE-AC06-76RLO 1830**

**Pacific Northwest Laboratory
Operated for the U.S. Department of Energy
by Battelle Memorial Institute**

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

PACIFIC NORTHWEST LABORATORY
operated by
BATTELLE
for the
UNITED STATES DEPARTMENT OF ENERGY
under Contract DE-AC06-76RLO 1830

Printed in the United States of America
Available from
National Technical Information Service
United States Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

NTIS Price Codes
Microfiche A01

Printed Copy	
Pages	Price Codes
001-025	A02
026-050	A03
051-075	A04
076-100	A05
101-125	A06
126-150	A07
151-175	A08
176-200	A09
201-225	A010
226-250	A011
251-275	A012
276-300	A013

Assessment of Effectiveness of
Geologic Isolation Systems

ANALYTIC MODELING OF FLOW IN A
PERMEABLE FISSURED MEDIUM

OTTO D. L. STRACK

with Foreword by
R.W.Nelson

Prepared for
the Office of Nuclear Waste Isolation
under its contract with the U.S. Department of Energy

Pacific Northwest Laboratory
Richland, Washington 99352

Foreword

R.W. Nelson*

The description of flow in fissured permeable media is a new and vitally important technical aspect of evaluation for flow and transport in dense rock systems. This area of developing technology is of central importance in the performance assessment of deep nuclear repositories in fractured rock. As contrasted to the flow in granular porous materials where over a century and a half of measurement and analysis experience exists, flow in fractured materials has been considered only during the last score of years. In this young emerging field of fracture flow analysis viable approaches to measurement and analysis are being sought that will provide feasible analytical approaches to enable better insight and understanding of such systems.

The analytical element method, a special boundary integral approach, developed and reported by Dr. Strack to analyze flow and transport in fractured materials is a unique approach to gaining much needed understanding and insight in the performance of such flow systems. Often in the past the entire numerical solution capacity of computer codes have been exhausted yet only a relatively small number of fractures could be simulated with limited accuracy. By contrast, the analytical element method, presented hereafter has the advantage of better analysis accuracy for systems made up of larger numbers of fractures. The greater accuracy and versatility for considering fractures is an important contribution to better describing flow in such systems.

The utility of the fracture flow analysis capability and associated computer code developed from the work reported here and from planned extensions is in four important areas, namely:

1. To evaluate the equivalent permeable porous media concepts for describing saturated flow in fractured rock.
2. To enable reducing statistical aperture, length and orientation data for fractures to tensorial values of equivalent hydraulic conductivity.
3. To enable studying contaminant transport through fractured materials when sorption properties differ in the fissures and in the surrounding rock.
4. To enable initial study of the effects of thermal mechanically induced changes in fracture apertures and length upon the equivalent hydraulic conductivity tensor.

* Staff Scientist, Hydrologic Systems Section, responsible for fracture flow research as Task Leader for Methodology Development for Fiscal 1981 AEGIS Program

These four areas suggest the range of evaluations that should be possible using the research evaluation capability that is derived and presented here after. The analysis capability and computer analysis tool provided by Dr. Strack when logically extended should materially extend our understanding of flow and contaminant transport in fractured permeable materials.

ACKNOWLEDGEMENTS

This research, supported by the Assessment of Effectiveness of Geologic Isolation Systems (AEGIS) Program, was conducted by Pacific Northwest Laboratory, managed by Battelle Memorial Institute for the Department of Energy under Contract DE-AC06-76RLO 1830. The AEGIS Program is sponsored by the Office of Nuclear Waste Isolation (ONWI). The research has been done for Pacific Northwest Laboratories under Consultant Agreement B-59308-A-L.

The organizational portion of the computer program developed as part of this research, including input, output, graphics, debugging facilities and organization of storage, was written by Henk Haitjema. An experimental version of the computer program was developed by Christine Detournay, who also implemented the exact solution.

SUMMARY

An analytic model has been developed for two dimensional steady flow through infinite fissured porous media, and is implemented in a computer program. The model is the first, and major, step toward the development of a model with finite boundaries, intended for use as a tool for 'numerical experiments'. These experiments may serve to verify some of the simplifying assumptions made in continuum models and to gain insight in the mechanics of the flow. The model is formulated in terms of complex variables and the analytic functions presented are closed-form expressions obtained from singular Cauchy integrals. An exact solution is given for the case of a single crack in an infinite porous medium. The exact solution is compared with the result obtained by the use of an independent method, which assumes Darcian flow in the crack and models the crack as an inhomogeneity in the permeability, in order to verify the simplifying assumptions. The approximate model is compared with solutions obtained from the above independent method for some cases of intersecting cracks. The agreement is good, provided that a sufficient number of elements are used to model the cracks.

CONTENTS

FOREWORD	iii
ACKNOWLEDGEMENT	v
SUMMARY	vii
1 INTRODUCTION	1
2 BASIC EQUATIONS	3
2.1 FLOW IN THE ROCK	3
Complex Variables	4
2.2 CUBIC LAW FOR FLOW IN THE FISSURES	5
2.3 LINEAR LAW FOR FLOW IN THE FISSURES	6
2.4 CONTINUITY OF FLOW IN THE FISSURES	6
2.5 COMBINED FLOW	6
Cubic Law	8
Linear Law	9
3 A SINGLE CRACK	10
3.1 EXACT SOLUTION	10
3.2 VERIFICATION FOR A SINGLE CRACK	14
4 MULTIPLE ELEMENT APPROXIMATION	23
4.1 DISTRIBUTED DIPOLES WITH AN ANALYTIC DENSITY DISTRIBUTION	23
The Behavior near the Element	26
4.2 TIP ELEMENTS	27
The Behavior along the Element	31
4.3 MODELING OF THE CRACK	33
Continuity at z_2 and z_n	36
Determination of the Unknowns	39
5 THE CROSSING	40
5.1 CONDITIONS AT THE CROSSING	40
5.2 THE ELEMENTS FOR THE CROSSING	42
The Main Element	42
Auxiliary Element	46
6 NUMERICAL MODEL	49
6.1 THE COMPLEX POTENTIAL FOR THE CRACKS	50
6.2 THE COMPLEX POTENTIAL FOR THE CROSSING	53
6.3 THE SYSTEM OF EQUATIONS	54
7 VERIFICATIONS	59

8 ALTERNATIVE APPROACH	76
REFERENCES	78
APPENDIX A EQUATION OF MOTION FOR FLOW IN THE FISSURES	A.1
APPENDIX B CONDITIONS OF CONTINUITY AT NODAL POINTS	B.1
APPENDIX C ASYMPTOTIC EXPANSIONS	C.1
APPENDIX D CONTINUITY OF $\partial\mu/\partial s$	D.1

FIGURES

1. A fissure with the local coordinates s and n	5
2. Fissures, as viewed for flow in the fissures (a) and in the rock (b)	7
3. The dimensionless variable Z	10
4. Branch cuts for θ_1 and θ_2	13
5. Elements along the crack	15
6. Piezometric contours obtained from the exact solution for cases 1 and 2	16
7. Comparison for case 1; the dashed curves correspond to the idealized crack	18
8. Comparison for case 2; the dashed curves correspond to the idealized crack	19
9. Comparison for case 1 with a rectangular crack; the dashed curves correspond to the idealized crack	20
10. Contourlines near the tip of a tapered crack; the dashed curves correspond to the idealized crack	21
11. Contour lines near the tip of a rectangular crack	22
12. Modeling of a crack by a string of elements	23
13. The branch cut for $\theta_j = \arg[(Z_j - 1)/(Z_j + 1)]$	26
14. The tip element of the crack in the z , χ_1 , and $\chi_1^{1/2}$ planes	28
15. The last element of the crack in the z , χ_n and $\chi_n^{1/2}$ planes	32
16. Continuity conditions at node z_n	38
17. Two intersecting cracks	40
18. The three paths of integration	41
19. Discontinuities at the intersection	43
20. The behavior of μ_s along the element	46
21. Distribution of control points on a single crack	55
22. The verification problems; cases A and B	60
23. Comparison for case A.1; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	62
24. Comparison for case A.2; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	63
25. Comparison for case A.3; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	64
26. Comparison for case A.4; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	65
27. Comparison for case B.1; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	66
28. Comparison for case B.2; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	67
29. Comparison for case B.3; the solid lines are obtained from	

SYLENS and the dashed ones from BACRACK	68
30. Comparison for case B.4; the solid lines are obtained from SYLENS and the dashed ones from BACRACK	69
31. Piezometric contours obtained from BACRACK for case B.3 using 4 elements per crack	70
32. Distribution of control points for case B.3 and three elements per crack	71
33. Plot of ϕ versus s for 3 elements (a) and 4 elements (b); the solid lines are obtained from SYLENS	72
34. Plot of ϕ versus s for 3 elements (a) and 4 elements (b); the solid lines are obtained from SYLENS	73
35. Piezometric contours for flow with 9 cracks and 11 intersections	74
36. Piezometric contours for flow with 9 cracks and 11 intersections; detail	75
A1. A crack	A.1
D1. Local coordinates at the crossing	D.1

TABLES

1. Data for cases 1 and 2	17
2. Values of the constant a_m^k	51
3. Values of the constant b_m^k	51
4. Expressions for the constants α_m^k and β_m^k	52
5. Expressions for the constants b_m^k and β_m^k	52
6. Data for case A	59
7. Data for case B	60

1 INTRODUCTION

The mechanics of flow through fissured porous media has received considerable attention in recent years. Questions regarding spreading patterns of waste in fissured media, for example, can be answered only when the mechanics of the flow are more fully understood and realistic models are available. Laboratory experimentation, the first step in gaining insight in the physical phenomena, suffers from the drawback that pressures in both the medium and the fissures are difficult to measure except at some isolated points. It is equally difficult to trace particles on their way through the medium. Most of the existing models therefore are based upon assumptions that have as yet not been validated.

The existing models may be divided into two groups:

- a) continuum models
- b) discrete models

The simplest continuum model rests on the assumption that the flow can be approximated as Darcian flow through an anisotropic porous medium. Although this approach may give an acceptable approximation for a dense mesh of connected fissures, it is not known just how dense such a mesh must be. Modern continuum models for flow through fissured media are based upon a double-porosity model developed by Barenblatt et al. [1960] and Barenblatt and Zheltov [1960], modified by Bear and Braaster [1972] and further developed into a multi-porosity theory by Aifantis ([1977], [1979], [1980]). The main characteristics of these models are that two porosities, two pressures, and two velocity vectors are defined at each point of the medium; one for the rock and one for the fissures. Critical in these models is the assumption that the flow from the fractures into the rock can be accounted for by taking it to be proportional to the difference in the two pressures.

A variety of discrete models is presented, all numerical, and mostly based on the assumption that the rock is impermeable (e.g., Louis [1974], Wittke [1970]). Permeability of the rock has been considered in a finite element formulation presented by Wilson and Witherspoon, [1970].

Discrete models are useful for modeling real cases primarily when the medium is too discrete to be approximated as a continuum. A continuum model would be preferable over a discrete one when the mesh of fissures is dense and the area to be modeled contains a large number of fissures; a discrete model then would become cumbersome and excessive in both storage requirements and computation time.

An accurate discrete model can be used with advantage to gain insight in the mechanisms of flow, e.g. spreading patterns, travel times, and pressure distributions, with a minimum of simplifying assumptions. It will be useful in validating the basic assumptions made in double porosity models, for example the interaction between flow in the rock and flow in the fissures, and may lead

to improvements of existing continuum models, or the development of new ones. An example of such a use of a discrete model is given by De Josselin de Jong and Way [1972], who used a model of an impermeable medium with a uniform system of cracks to validate the dispersion theory developed by De Josselin de Jong.

The discrete analytical model presented in this report is intended for such use. The model is presented for two dimensional steady flow in an infinite fractured porous medium, where the fissures may either be isolated or may intersect one another. Pressures and velocities are produced as analytic functions and it is possible to generalize the model so that streamlines can be traced through the medium and the fissures, and arbitrary boundary conditions can be applied. The model is based on the use of singular Cauchy integrals and may be considered to belong to the family of boundary element models, although it differs from these in many respects.

The model as presented in this report and implemented in the computer program BACRACK is capable of handling an assembly of cracks imbedded in an infinite porous medium with a free-field of uniform flow. Conditions along finite boundaries have as yet not been implemented. The program is written for the use of computer interactive graphics; a mesh of cracks may be generated by the use of a cross-hair on a graphics screen. The cursor may be used to specify points where heads or velocities are to be calculated. The program is capable of producing plots of piezometric contours. BACRACK has been verified by comparing results for a few selected problems with those obtained from the computer program SYLENS, developed by Strack and Haitjema [1981b]. The latter program is capable of modeling inhomogeneities in the permeability, and has been used to model the cracks as discrete anomalies of finite width. It appears that BACRACK is indeed realistic, provided that the cracks are modeled by a sufficient number of elements.

The emphasis of this report is on the development of the analytical model and contains a complete description of the functions used. Comparison both with an exact solution and results obtained from SYLENS are given, along with an example.

It is important to note that the model is intended to be used ultimately with finite boundaries, and the present model is to be considered as the first step toward a tool suitable for numerical experimentation.

2 BASIC EQUATIONS

The validity of the cubic law for laminar flow of fluids through open fractures consisting of parallel planar plates has been established by many researchers over a wide range of conditions. Witherspoon et al. [1980] have shown that the cubic law is valid whether the fractures are held open or are allowed to close under stress. Governing equations for transient flow through a fissured porous medium have been presented, among others, by Duguid [1973]. The equations for steady two dimensional flow through a fissured porous medium, although well-established, will be presented below for the sake of completeness. The equation of motion will be derived first adopting the cubic law and second for the linear law, which is used for the purpose of verification. For both cases the flow in the primary pores of the medium is assumed to be governed by Darcy's law. In order to prevent ambiguity, the multiply-connected domain consisting of the porous material outside the cracks or fissures will be referred to as the region R.

2.1 FLOW IN THE ROCK

The flow in the rock is taken to be governed by Darcy's law. A Cartesian x, y coordinate system is introduced, where x is horizontal and y points vertically upward. The permeability of the rock, K [L/T], is homogeneous and isotropic, so that Darcy's law becomes:

$$q_x = -K \frac{\partial \phi}{\partial x}, \quad q_y = -K \frac{\partial \phi}{\partial y} \quad (2.1)$$

where q_x and q_y represent the components of the specific discharge vector in the directions x and y. Introducing the potential Φ [L²/T] as the permeability times the head,

$$\Phi = K\phi \quad (2.2)$$

(2.1) becomes

$$q_x = -\frac{\partial \Phi}{\partial x}, \quad q_y = -\frac{\partial \Phi}{\partial y} \quad (2.3)$$

Continuity of flow requires, for the present case of steady flow, that the divergence of the specific discharge vector vanishes throughout R, i.e.,

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \quad (2.4)$$

Combination of (2.1) and (2.4) leads to the differential equation of Laplace for the potential,

$$\nabla^2 \Phi = 0 \quad (2.5)$$

Complex Variables

Use of complex variables will be made throughout this report and some definitions of these will be given below. The potential Φ is represented as the real part of a complex potential Ω ,

$$\Omega = \Phi + i\psi \quad (2.6)$$

whose imaginary part is the stream function ψ (see, e.g., Verruijt [1970]). The complex variable z is defined in the physical plane as

$$z = x + iy \quad (2.7)$$

Since Φ is harmonic, Ω must be an analytic function of z ,

$$\Omega = \Omega(z) \quad (2.8)$$

and Φ and ψ fulfill the Cauchy-Riemann conditions,

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = - \frac{\partial \psi}{\partial x} \quad (2.9)$$

so that Darcy's law (2.3) may be expressed in terms of the gradients of both Φ and ψ as follows:

$$q_x = - \frac{\partial \Phi}{\partial x} = - \frac{\partial \psi}{\partial y} \quad (2.10)$$

$$q_y = - \frac{\partial \Phi}{\partial y} = + \frac{\partial \psi}{\partial x} \quad (2.11)$$

The hodograph, q , is the complex representation of the specific discharge vector,

$$q = q_x + iq_y \quad (2.12)$$

and the specific discharge function w is defined as

$$w = - \frac{d\Omega}{dz} \quad (2.13)$$

It follows from (2.13) and the Cauchy-Riemann conditions (2.10) and (2.11) that

$$w = - \frac{\partial \Phi}{\partial x} - i \frac{\partial \psi}{\partial x} = q_x - iq_y \quad (2.14)$$

2.2 CUBIC LAW FOR FLOW IN THE FISSURES

The approximate equations obtained by assuming laminar flow in the fissures will be given below in terms of local Cartesian coordinates. These coordinates are denoted as s and n , where s and n are parallel and normal to the axis of

Figure 1. A fissure with the local coordinates s and n .

the fissure, respectively, see Figure 1. The origin of the system of local coordinates is at the center of the fissure. The width of the crack may vary with position and is represented as b , and the length of the crack is L , see Figure 1. It is noted that the fissure extends to infinity in the directions normal to the plane of flow.

For the sake of completeness, the equations of motion are derived for the three dimensional case in Appendix A, assuming that the width b of the fissure is small with respect to its length, so that the head across the fissure may be replaced by an average value. For the present two dimensional case, these equations (see (A.16) and (A.17)) become

$$Q_s = - \beta^* b^3 \frac{\partial \tilde{\phi}}{\partial s} \quad (2.15)$$

where $\tilde{\phi}$ represents the average piezometric head across the fissure, Q_s [L^2/T], the discharge flowing through the fissure as a function of s , and β^*

is defined as

$$\beta^* = \frac{\rho g}{12\mu} \quad (2.16)$$

where ρ is the unit mass of water, g the acceleration of gravity, and μ the dynamic viscosity.

2.3 LINEAR LAW FOR FLOW IN THE FISSURES

The linear law would hold in case the fissures were filled with some porous material with permeability k^* . The expression for the discharge Q_s in the s direction is analogous to (2.15), except that the crack width b appears linearly:

$$Q_s = -k^* b \frac{\partial \tilde{\phi}}{\partial s} \quad (2.17)$$

where, again, $\tilde{\phi}$ represents the average head across the fissure.

2.4 CONTINUITY OF FLOW IN THE FISSURES

Continuity of flow requires that the discharge Q_s in the fissure at any location $-L/2 \leq s \leq L/2$, $n=0$, is equal to the total inflow into the crack, measured from the starting point of the fissure, $s = -L/2$, $n=0$,

$$Q_s(s) = \int_{-\frac{1}{2}L}^s \sigma d\tau, \quad -\frac{1}{2}L \leq s \leq \frac{1}{2}L, \quad n=0 \quad (2.18)$$

where σ [L/T] represents the sum of the inflows from both sides, per unit length of fissure.

Combination of (2.18) with either one of the two possible equations of motion (2.15) or (2.17), and with the governing equations for flow in R yields the complete set of equations to be solved.

2.5 COMBINED FLOW

The problem of combined flow will be solved by viewing the conditions along the fissures as internal boundary conditions. The effect of the width of the fissures is incorporated in an approximate fashion in the equations for the

flow in the fissures. The volume of the fissures is small as compared to the volume of rock and therefore the flow through the rock may be modeled ignoring the width of the cracks.

The variation of the piezometric head across the cracks is neglected and the fissures are saturated. Therefore, the piezometric head in the fissures, $\tilde{\phi}$, must be equal to that in the rock along the crack boundaries, i.e.,

$$\phi = \tilde{\phi}, \quad -\frac{1}{2}L \leq s \leq \frac{1}{2}L, \quad n=0 \quad (2.19)$$

The inflow into the crack, σ , may be expressed in terms of the component q_n of the specific discharge vector as follows, see Figure 2,

$$\sigma = -[q_n^+ - q_n^-] \quad (2.20)$$

Figure 2. Fissures, as viewed for flow in the fissures (a) and in the rock (b).

where the $+$ and $-$ signs refer to the sides of the fissures facing the positive and negative n directions, respectively. Writing the Cauchy-Riemann equations (2.10) and (2.11) in terms of the local coordinates s and n ,

$$q_s = -\frac{\partial \Phi}{\partial s} = -\frac{\partial \tilde{\Phi}}{\partial n} \quad (2.21)$$

and

$$q_n = -\frac{\partial \Phi}{\partial n} = +\frac{\partial \tilde{\Phi}}{\partial s} \quad (2.22)$$

(2.20) becomes:

$$\sigma = - \left[\frac{\partial \bar{x}^+}{\partial s} - \frac{\partial \bar{x}^-}{\partial s} \right] = - \frac{\partial [\bar{x}^+ - \bar{x}^-]}{\partial s} \quad (2.23)$$

and the condition for continuity of flow (2.18) yields, with (2.23),

$$Q_s(s) = - \int_{-\frac{1}{2}L}^s \frac{\partial [\bar{x}^+ - \bar{x}^-]}{\partial \tau} d\tau = - [\bar{x}^+ - \bar{x}^-] \Big|_{-\frac{1}{2}L}^s, \quad (2.24)$$

$$-\frac{1}{2}L \leq s \leq \frac{1}{2}L, \quad n=0$$

The width of the crack is taken to be zero at the tips, so that both Q_s and $\bar{x}^+ - \bar{x}^-$ vanish there, and (2.24) may be simplified to

$$Q_s = - [\bar{x}^+ - \bar{x}^-], \quad -\frac{1}{2}L \leq s \leq \frac{1}{2}L, \quad n=0 \quad (2.25)$$

The internal boundary condition along the crack may be formulated for the cubic and the linear laws as is shown below.

Cubic Law

The cubic law is given by (2.15) and combination with (2.19) yields

$$Q_s = - \beta^* b^3 \frac{\partial \phi}{\partial s}, \quad -\frac{1}{2}L \leq s \leq \frac{1}{2}L, \quad n=0 \quad (2.26)$$

or, with (2.2),

$$Q_s = - \frac{\beta^* b^3}{k} \frac{\partial \bar{x}}{\partial s} \quad (2.27)$$

The condition (2.25) can now be formulated in terms of the stream function and the potential along the crack,

$$\frac{\beta^* b^3}{k} \frac{\partial \bar{x}}{\partial s} = \bar{x}^+ - \bar{x}^- \quad (2.28)$$

The problem of describing the flow in the fissured medium thus consists of determining a complex potential $\Omega = \Omega(z)$ which is analytic in the slotted domain R and fullfills the conditions (2.28) as internal boundary conditions along the crack.

Linear Law

The equations for the case that a linear law is adopted differ from those above merely in the coefficient of $\partial\Phi/\partial s$ in (2.28),

$$\frac{k^*b}{k} \frac{\partial\Phi}{\partial s} = \Phi^+ - \Phi^- \quad (2.29)$$

Apart from the differences in magnitudes of the coefficients β^*b^3/k for the cubic and k^*b/k for the linear law, the behavior of the two models will be different only if the width of the crack varies with position.

As far as the author is aware, no exact solutions to the above equations are available in the literature. An approximate model will be presented in this report, that is sufficiently accurate to gain insight in the various flow phenomena by the use of inter-active computer graphics. The approximate model can be used equally well for the cubic law as for the linear law. The linear law is used strictly for the purpose of verification.

Verification for the case of the linear law is possible by comparison with the computer program SYLENS (see Strack and Haitjema [1981a] and [1981b]) which is capable of modeling inhomogeneities in the permeability accurately. Comparisons of results obtained from BACRACK and SYLENS for several single and crossing cracks will be presented in Sec. 7.

A single exact solution has been determined for the case of one crack of a special shape in a flow field that reduces to uniform flow near infinity. The exact solution will be compared to the solution obtained from SYLENS where the crack is modeled as having a finite width. The comparison serves to estimate the inaccuracies introduced by averaging the head across the crack and by neglecting the crack width for the flow in the rock. This exact solution will be discussed below.

3 A SINGLE CRACK

The solution for a single crack presented below is determined only for a free-field of uniform flow and for a particular shape of the crack. (The term free-field is used to denote the field that would occur in the absence of cracks, and in the present report is always a field of uniform flow.) For the case of the linear law the shape of the crack corresponds to that of a pressurized crack in an infinite elastic body, see, e.g., Sneddon [1951]. For the case of the cubic law the shape of the fissure is modified in order that the boundary conditions are met.

The solution for the linear law is used primarily to verify the assumptions that the crack width can be neglected for the flow in the rock and that the head can be averaged across the crack. The solutions are derived in Sec. 3.1 and the verifications are discussed in Sec. 3.2.

3.1 EXACT SOLUTION

The endpoints of the crack will be represented as z_1 and z_2 , chosen such that a vector drawn from z_1 to z_2 points in the positive s direction, see

Figure 3. The dimensionless variable Z .

Figure 3 (a). The dimensionless variable Z is introduced as

$$Z = X + iY = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)} \quad (3.1)$$

so that the crack in the Z plane is defined by, see Figure 3(b),

$$-1 \leq X \leq 1, \quad Y=0 \quad (3.2)$$

The boundary condition (2.29) becomes in terms of X ,

$$\frac{k^*b}{k} \frac{\partial \Phi}{\partial s} = \frac{2k^*b}{kL} \frac{\partial \Phi}{\partial X} = \Phi^+ - \Phi^- \quad (3.3)$$

where

$$L = |z_2 - z_1| \quad (3.4)$$

It is assumed that the shape of the crack corresponds to that obtained from the normal displacements of the sides of a pressurized crack in an elastic medium. This gives

$$b = b_{\max} \sqrt{1 - X^2} \quad (3.5)$$

where $b_{\max}$ is the width of the crack at the center, $X=0$.

The free-field is one of uniform flow, with a specific discharge q_0 ($\text{Im } q_0 = 0$) pointing in a direction that is inclined to the x axis at an angle β . One obtains from (2.12) and (2.14), using the subscript f to denote the free-field

$$w_f = q_0 e^{-i\beta} \quad (3.6)$$

A function W is introduced as

$$W = -\Omega' = -\frac{d\Omega}{dZ} = -\frac{d\Omega}{dz} \frac{dz}{dZ} = \frac{1}{2} w L e^{i\alpha} \quad (3.7)$$

see (3.1), where the prime denotes differentiation with respect to Z , and

$$L e^{i\alpha} = z_2 - z_1 \quad (3.8)$$

It follows from (3.6) and (3.7) that

$$\Omega_f = -\frac{1}{2} q_0 L e^{i(\alpha - \beta)} Z + C \quad (3.9)$$

where C is a constant of integration.

The boundary value problem is defined by the condition (3.3) and the

requirement that the complex potential Ω must approach Ω_f near infinity. The solution has been obtained by trial and error and has the form

$$\Omega = -A \sqrt{(Z-1)(Z+1)}' + AZ - \frac{1}{2}q_0 L e^{i(\alpha-\beta)} Z + C \quad (3.10)$$

where

$$A = \frac{\frac{1}{2}k^* b_{\max}}{kL + k^* b_{\max}} q_0 L \cos(\alpha - \beta) \quad (3.11)$$

The function (3.10) meets the conditions listed above, as will be shown below. The first two terms combine to an expansion of the form $\frac{1}{2}AZ^{-1} + O(Z^{-3})$ (O stands for 'order of') near infinity so that the complex potential approaches Ω_f for $Z \rightarrow \infty$, as asserted. The behavior along the element will be examined next. The arguments θ_1 and θ_2 , defined as

$$\theta_1 = \arg(Z-1), \quad \theta_2 = \arg(Z+1) \quad (3.12)$$

are taken such that

$$Y \geq 0; \quad 0 \leq \theta_1 \leq \pi, \quad 0 \leq \theta_2 \leq \pi \quad (3.13)$$

and

$$Y \leq 0; \quad -\pi \leq \theta_1 \leq 0, \quad -\pi \leq \theta_2 \leq 0 \quad (3.14)$$

see Figure 4. Hence the behavior of the function

$$\Omega^*(Z) = \Phi^* + i\mathcal{F}^* = \sqrt{(Z-1)(Z+1)}' \quad (3.15)$$

along the real axis is as follows:

$$-1 \leq X \leq 1, \quad Y=0^+; \quad \Phi^*=0, \quad \mathcal{F}^*=(\mathcal{F}^*)^+ = \sqrt{1-X^2} \quad (3.16)$$

$$-1 \leq X \leq 1, \quad Y=0^-; \quad \Phi^*=0, \quad \mathcal{F}^*=(\mathcal{F}^*)^- = -\sqrt{1-X^2} \quad (3.17)$$

$$-\infty < X \leq -1, \quad Y=0; \quad \Phi^*=-\sqrt{1-X^2}, \quad \mathcal{F}^*=0 \quad (3.18)$$

$$1 \leq X < \infty, \quad Y=0; \quad \Phi^*=+\sqrt{1-X^2}, \quad \mathcal{F}^*=0 \quad (3.19)$$

Both Φ^* and $\mathcal{F}^*$ are continuous in the cut plane. The real part, Φ^* , is continuous, but the imaginary part, $\mathcal{F}^*$, is discontinuous across the element.

The expressions for $\partial\Phi/\partial X$ and $\mathcal{F}^+ - \mathcal{F}^-$ along the element become, see (3.10)

$$-1 \leq X \leq 1, \quad Y=0; \quad \frac{\partial\Phi}{\partial X} = A - \frac{1}{2}q_0 L \cos(\alpha - \beta) \quad (3.20)$$

Figure 4. Branchcuts for θ_1 and θ_2 .

$$-1 \leq X \leq 1, Y = 0; \quad \Phi^+ - \Phi^- = -2A \sqrt{1-X^2} \quad (3.21)$$

The complex potential matches the boundary condition (3.3), as may be seen by the use of expressions (3.5) for b and (3.11) for A .

The solution is applicable to the case that the cubic law is adopted, provided that the following equation is used to define the width of the crack.

$$b = b_{\max} (1-X^2)^{1/6} \quad (3.22)$$

The crack width increases more rapidly near the tip than for the previous case (see (3.5)), namely as the $\frac{1}{6}$ th power of the distance from the tip. Replacing k^*b by β^*b^3 in (3.3), compare (2.27), and replacing $k^*b_{\max}$ by $\beta b_{\max}^3$ in (3.11) it is seen that the solution fulfills the boundary condition.

It follows from the above solution that the complex potential near the tip of the crack behaves as a square root. It may be inferred from this that for any problem where the widths of the cracks increase as the square root of the distance from the tip for the linear law and as the $\frac{1}{6}$ th power of this distance for the cubic law, the complex potential can be expanded as follows about the crack tip,

$$\Omega(z) = \sqrt{z-z_j} \left[d_0 + d_1(z-z_j) + d_2(z-z_j)^2 + \dots \right] \quad (3.23)$$

where z_j is the complex coordinate of the tip and d_m , $m=0,1,2,\dots$ are constants. It will be assumed throughout this report that $\Omega(z)$ has the behavior (3.23) near the crack tips.

3.2 VERIFICATION FOR A SINGLE CRACK

The solution presented in Sec. 3.1 for a single porous crack in a field of uniform flow is verified below against the solution obtained from the computer program SYLENS. The latter program has been developed for the modeling of flow in aquifers with inhomogeneities in the permeability (Strack and Haitjema, [1981b]). Piezometric contour lines have been obtained from SYLENS for a crack in a field of uniform flow, see Figure 6(a) and Figure 6(b). The shape of the crack is given by (3.5) where

$$b_{\max}/L = 0.05 \quad (3.24)$$

and the ratio of the permeabilities, K , is

$$K = k^*/k = 200 \quad (3.25)$$

where k^* and k are the permeabilities of the porous crack and the rock, respectively.

The approach used in the program SYLENS to model inhomogeneities in the permeability will be summarized briefly below. For a more detailed description of this method, the reader is referred to Strack and Haitjema [1981a], [1981b]. The potential Φ jumps across the crack as a result of the discontinuity in the permeability. If Φ and Φ^* represent the potentials just outside and inside the boundary of the crack, respectively, then the jump in Φ is given by λ , with

$$\lambda = \Phi - \Phi^* = k\phi - k^*\phi = \frac{k-k^*}{k} k\phi = \frac{k-k^*}{k} \Phi \quad (3.26)$$

where use is made of the continuity of the head, ϕ , across the crack boundary. Equation (3.26) represents one of the two conditions that must be met across the boundary of an inhomogeneity in the permeability. The other one is a continuity condition; the component of the specific discharge vector normal to the boundary must be continuous across the boundary. The potential for distributed line-doublets has the property that it is discontinuous across the line-doublet, whereas its normal derivative is continuous. Thus, line-doublets may be used to model the boundaries of inhomogeneities, where the distribution strength λ is chosen such that (3.26) is met. The line doublets are distributed in SYLENS along straight lines, and the jump λ is

varied parabolically along the element. The computer program SYLENS has been verified against exact solutions for flow with inhomogeneities and the method outlined above appears to be sufficiently accurate to serve for a verification.

The crack is modeled by the use of 28 elements, 14 on either side of the crack. The axis of the crack is parallel to the x axis and the origin $x=y=0$ is taken at the center of the crack. The width b of the crack is given by

$$\frac{b}{L} = \frac{b_{\max}}{L} \sqrt{\left[\frac{2x}{L}\right]^2 - 1} \quad (3.27)$$

The elements along the crack boundaries are shown in Figure 5. The elements are short near the tips, where the curvature of the crack boundary is large, and increases in length toward the center of the element. Two verification runs have been made with the various programs. For case 1, the dimensionless specific discharge of the free-field flow, q_0/k , is 1 and the free-field flow is directed parallel to the axis of the element. The value for q_0/k is the

Figure 5. Elements along the crack.

same for case 2, but the free-field flow is oriented at 45 degrees to the crack, i.e., the angle β in (3.6) is $\frac{1}{4}\pi$.

The piezometric contours obtained from the exact solution for a crack, idealized as discussed in this report, are shown for the two cases in Figure 6(a) and Figure 6(b). The head is taken to be zero at the center of the crack and the difference in value for ϕ/L between two neighboring contours, $\Delta\phi/L$, is 0.05. The data for the two cases are summarized in Table 1.

The piezometric contour lines obtained by the use of SYLENS and the exact solution for an idealized crack are compared in Figure 7 and Figure 8 for cases 1 and 2, respectively. The solid lines are obtained from SYLENS and the dashed

CASE 1

CASE 2

Figure 6. Piezometric contours obtained from the exact solution for cases 1 and 2.

TABLE 1. Data for cases 1 and 2

	k^*/k	$b_{\max}/L$	q_0/k	β	$\Delta\phi/L$
case 1	200	0.05	1	0	0.05
case 2	200	0.05	1	45	0.05

lines from the exact solution for an idealized crack. The effect of the shape of the crack is investigated by modeling a rectangular crack with SYLENS and comparing the results with the solution for an idealized crack for case 1. The contour lines are reproduced in Figure 9, where, again, the solid lines are obtained from SYLENS. The effect of the shape of the crack is felt most profoundly near the crack tips; details of the contour-plots near the two different crack tips are reproduced, for case 1, in Figure 10 and Figure 11.

Figure 7. Comparison for case 1; the dashed curves correspond to the idealized crack.

Figure 8. Comparison for case 2; the dashed curves correspond to the idealized crack.

Figure 9. Comparison for case 1 with a rectangular crack; the dashed curves correspond to the idealized crack.

Figure 10. Contour lines near the tip of a tapered crack;
the dashed curves correspond to the idealized crack.

Figure 11. Contour lines near the tip of a rectangular crack.

4 MULTIPLE ELEMENT APPROXIMATION

An approximate model for porous cracks is obtained by dividing the crack into a number of separate elements as shown in Figure 12. The elements are

Figure 12. Modeling of a crack by a string of elements.

represented by various forms of singular Cauchy integrals, consisting of distributed dipoles. The inner elements are modeled by distributed dipoles with an analytic density distribution. The elements at the tips of the cracks have a density distribution that is chosen such that the behavior of the complex potential near the tips is of the form (3.23). The complex potentials for the various elements are derived below.

4.1 DISTRIBUTED DIPOLES WITH AN ANALYTIC DENSITY DISTRIBUTION

Consider the following singular Cauchy integral,

$$\Omega_j(z) = -\frac{1}{2\pi} \int_{z_j}^{z_{j+1}} \frac{\mu_j(\delta)}{z-\delta} d\delta \quad (4.1)$$

where $\mu_j(\delta)$ is the density distribution, and where the path of integration is a straight line from z_j to z_{j+1} . The index j in Ω_j refers to the number of the element. The integral represents dipoles oriented with their axes along the path of integration, and is of a type discussed extensively by Muskhelishvili [1958]. It will be shown below that the integral can be written in a form suitable for numerical evaluation, whereby numerical integration is avoided. The complex variable Z_j is introduced as

$$Z_j = X_j + iY_j = \frac{z - \frac{1}{2}(z_j + z_{j+1})}{\frac{1}{2}(z_{j+1} - z_j)} \quad (4.2)$$

so that the element may be characterized as follows,

$$-1 \leq X_j \leq 1, \quad Y_j = 0 \quad (4.3)$$

The orientation and length of the element are represented as α and L_j , i.e.,

$$z_{j+1} - z_j = L_j e^{i\alpha} \quad (4.4)$$

Similarly, let Δ be introduced as

$$\Delta = \text{Re}\Delta + i\text{Im}\Delta = \frac{\delta - \frac{1}{2}(z_j + z_{j+1})}{\frac{1}{2}(z_{j+1} - z_j)} \quad (4.5)$$

Introduction of these new variables in (4.1) yields

$$\Omega_j(Z_j) = -\frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu_j(\Delta)}{Z_j - \Delta} d\Delta \quad (4.6)$$

Restriction will be made, at first, to the case that the real density function $\mu_j(\Delta)$ may be continued analytically. Thus, the function $\mu_j(Z_j)$ is defined such that $\mu_j(Z_j)$ is analytic for all finite Z_j and such that

$$-1 \leq X_j \leq 1, \quad Y_j = 0; \quad \text{Im } \mu_j(Z_j) = 0 \quad (4.7)$$

The function (4.6) may be written as

$$\Omega_j(Z_j) = +\frac{1}{2\pi} \int_{-1}^{+1} \frac{\mu_j(Z_j) - \mu_j(\Delta)}{Z_j - \Delta} d\Delta - \frac{\mu_j(Z_j)}{2\pi} \int_{-1}^{+1} \frac{d\Delta}{Z_j - \Delta} \quad (4.8)$$

Since $\mu_j(Z_j)$ is analytic for all finite Z_j , the first integral is analytic for all finite Z_j . Representing this integral as $p_j(Z_j)$, (4.8) may be written

as

$$\Omega_j(Z_j) = p_j(Z_j) + \frac{\mu_j(Z_j)}{2\pi} \int_{-1}^{+1} d \ln(Z_j - \Delta) \quad (4.9)$$

or

$$\Omega_j(Z_j) = p_j(Z_j) + \frac{\mu_j(Z_j)}{2\pi} \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] \quad (4.10)$$

The function $\mu_j(Z_j)$ may be expanded in a Taylor's series about $Z_j = 0$. Approximating $\mu_j(Z_j)$ by the first m terms of the series, the expression for $\mu_j(Z_j)$ becomes

$$\mu_j(Z_j) = \sum_{r=0}^m A_r Z_j^r \quad (4.11)$$

It follows from (4.7) that the $m+1$ coefficients A_r must be real, i.e.,

$$\text{Im } A_r = 0, \quad r = 0, 1, 2, \dots, m \quad (4.12)$$

The function $p_j(Z_j)$ may be expressed in terms of $\mu_j(Z_j)$, using that the behavior of the integral (4.6) near $Z_j = \infty$ is known. It follows from (4.6) that the expansion of $\Omega_j(Z_j)$ about infinity is of the form

$$\Omega_j(Z_j) = - \frac{1}{2\pi} \left[\frac{1}{Z_j} \int_{-1}^{+1} \mu_j(\Delta) d\Delta + O(Z_j^{-2}) \right] \quad (4.13)$$

Hence, $\Omega_j(Z_j)$ is analytic at infinity. Expansion of (4.10) about infinity yields

$$\Omega_j(Z_j) = - \frac{1}{\pi} \sum_{r=0}^m A_r Z_j^r \left[\sum_{j=1}^{\infty} \frac{1}{2j-1} Z_j^{-(2j-1)} \right] + p_j(Z_j) \quad (4.14)$$

The function $p_j(Z_j)$ thus is found from (4.13) and (4.14),

$$p_j(Z_j) = \sum_{r=0}^{m-1} c_r Z_j^r \quad (\text{Im } c_r = 0, \quad r = 0, 1, 2, \dots, m-1) \quad (4.15)$$

where the constants c_r are such that all terms with positive exponents in the expansion (4.14) vanish. It follows from (4.12), (4.13) and (4.14) that the c_r are real.

The Behavior near the Element.

The function (4.10) contains the many-valued imaginary part of the logarithm $\ln[(Z_j - 1)/(Z_j + 1)]$. With

$$\ln \left[\frac{Z_j - 1}{Z_j + 1} \right] = \ln \left| \frac{Z_j - 1}{Z_j + 1} \right| + i\theta_j \quad (4.16)$$

(4.10) becomes

$$\Omega_j = \frac{\mu_j(Z_j)}{2\pi} \ln \left| \frac{Z_j - 1}{Z_j + 1} \right| + i\mu_j(Z_j) \frac{\theta_j}{2\pi} + p_j(Z_j) \quad (4.17)$$

The angle θ_j represents the difference in arguments of $Z_j - 1$ and $Z_j + 1$, which will be taken such that, see Figure 13,

$$Y_j \geq 0 ; 0 \leq \arg(Z_j - 1) \leq \pi, 0 \leq \arg(Z_j + 1) \leq \pi \quad (4.18)$$

$$Y_j \leq 0 ; -\pi \leq \arg(Z_j - 1) \leq 0, -\pi \leq \arg(Z_j + 1) \leq 0 \quad (4.19)$$

Figure 13. The branch cut for $\theta_j = \arg[(Z_j - 1)/(Z_j + 1)]$.

Thus, the range of values for θ_j becomes

$$Y_j \geq 0 ; 0 \leq \theta_j \leq \pi \quad (4.20)$$

$$Y_j \leq 0 ; -\pi \leq \theta_j \leq 0 \quad (4.21)$$

Note that θ_j is truly double-valued along the element,

$$\theta_j^+ = \pi, \quad \theta_j^- = -\pi \quad (4.22)$$

where superscripts $+$ and $-$ refer to superscripts side of element approached by Z_j . Using that both $\mu_j(Z_j)$ and $p_j(Z_j)$ are real along the element, the expressions for Φ and $\mathcal{F}$ become, for the $+$ and $-$ sides,

$$\begin{aligned} -1 \leq X_j \leq 1, \quad Y_j = 0^+; \quad \Phi &= \frac{\mu_j(X_j)}{2\pi} \ln \left| \frac{X_j - 1}{X_j + 1} \right| + p_j(X_j) \\ -1 \leq X_j \leq 1, \quad Y_j = 0^+; \quad \mathcal{F} &= \frac{1}{2} \mu_j(X_j) \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} -1 \leq X_j \leq 1, \quad Y_j = 0^-; \quad \Phi &= \frac{\mu_j(X_j)}{2\pi} \ln \left| \frac{X_j - 1}{X_j + 1} \right| + p_j(X_j) \\ -1 \leq X_j \leq 1, \quad Y_j = 0^-; \quad \mathcal{F} &= -\frac{1}{2} \mu_j(X_j) \end{aligned} \quad (4.24)$$

Thus, it follows that the potential Φ is continuous across the element whereas $\mathcal{F}$ jumps,

$$-1 \leq X_j \leq 1, \quad Y_j = 0; \quad \Phi^+ = \Phi^-, \quad \mathcal{F}^+ - \mathcal{F}^- = \mu_j(X_j) \quad (4.25)$$

Hence, functions of the above type have just the behavior required along the cracks, c.f., (2.19) and (2.29), and are suitable to model the crack, with the exception of the tips where the density function is not analytic. Elements suitable for modeling the tips of the cracks are discussed below.

4.2 TIP ELEMENTS

Complex potentials used to model the tips of a crack are determined in such a way that their expansion about the endpoints of the crack is of the form (3.23). The functions are determined by constructing integrals of the form discussed in Sec. 4.1 in a transformed plane as follows, see Figure 14. The complex potential for the first element of the crack, to be represented as Ω_1 will be presented first. The element extends from z_1 to z_2 , has an orientation α , and a length L_1 ,

$$z_2 - z_1 = L_1 e^{i\alpha} \quad (4.26)$$

The transformation

Figure 14. The tip element of the crack in the z , χ_1 , and $\chi_1^{1/2}$ planes.

$$\lambda_1 = K_1 + i\Lambda_1 = \frac{z - z_1}{z_2 - z_1} \quad (4.27)$$

maps the element onto the slot

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0 \quad (4.28)$$

in the λ_1 plane. The transformation

$$\zeta = \xi + i\eta = \lambda_1^{1/2} \quad (4.29)$$

maps the slotted λ_1 plane on the upper half plane $\text{Im } \zeta \geq 0$, provided that the argument of λ_1 is taken within the range from 0 to 2π ,

$$0 \leq \arg \lambda_1 \leq 2\pi \quad (4.30)$$

The desired singularity at $z = z_1$ is obtained by distributing dipoles in the ζ plane with an analytic density distribution μ . The element is defined in $\text{Im } \zeta \geq 0$ by

$$-1 \leq \xi \leq 1, \quad \eta = 0 \quad (4.31)$$

The range of definition of $\arg \zeta$ is obtained from (4.29) and (4.30),

$$0 \leq \arg \zeta \leq \pi \quad (4.32)$$

Application of (4.10) gives, replacing the index j by 1 and Z_j by ζ ,

$$\Omega_1(\zeta) = \frac{\mu_1(\zeta)}{2\pi} \ln \left[\frac{\zeta - 1}{\zeta + 1} \right] + p_1(\zeta) \quad (4.33)$$

where $\mu_1(\zeta)$ and $p_1(\zeta)$ are polynomials in terms of ζ and the coefficients of $p_1(\zeta)$ are expressed in terms of those of $\mu_1(\zeta)$ similarly as shown in Sec. 4.1. Both functions are real along the element, i.e.,

$$-1 \leq \xi \leq 1, \quad \eta = 0, \quad \text{Im } \mu_1(\zeta) = 0, \quad \text{Im } p_1(\zeta) = 0 \quad (4.34)$$

The density distribution $\mu_1(\zeta)$ must be real along the element and must be chosen in such a way that Φ is continuous and $\mathcal{F}$ is discontinuous across the element. Separation of real and imaginary parts gives

$$\Phi_1(\zeta) + i\mathcal{F}_1(\zeta) = \frac{\mu_1(\zeta)}{2\pi} \left[\ln \left| \frac{\zeta - 1}{\zeta + 1} \right| + i\theta_1 \right] + p_1(\zeta) \quad (4.35)$$

where

$$\theta_1 = \arg(\zeta - 1) - \arg(\zeta + 1) \quad (4.36)$$

The parameter ζ is restricted to $\text{Im } \zeta \geq 0$ and the argument θ_1 is defined accordingly,

$$0 \leq \theta_1 \leq \pi \quad (4.37)$$

Along the element, θ_1 equals π and (4.35) becomes

$$\Phi_1(\xi) + i\bar{\Phi}_1(\xi) = \frac{\mu_1(\xi)}{2\pi} \ln \left| \frac{\xi-1}{\xi+1} \right| + p_1(\xi) + i\frac{1}{2}\mu_1(\xi) \quad (4.38)$$

Points of the element that are located opposite to each other in the physical plane correspond to points of the ξ axis that are located symmetrically with respect to the origin, see Figure 14, i.e.,

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0; \quad \xi^- = -\xi^+, \quad \eta^+ = \eta^- = 0 \quad (4.39)$$

The condition that Φ be continuous across the element thus implies that

$$\begin{aligned} \Phi^+ - \Phi^- = 0 = & \frac{\mu_1(\xi^+)}{2\pi} \ln \left| \frac{\xi^+ - 1}{\xi^+ + 1} \right| - \frac{\mu_1(-\xi^+)}{2\pi} \ln \left| \frac{-\xi^+ - 1}{-\xi^+ + 1} \right| + \\ & p_1(\xi^+) - p_1(-\xi^+) \end{aligned} \quad (4.40)$$

The functions $\mu_1(\zeta)$ and $p_1(\zeta)$ therefore must fulfill the conditions that

$$\mu_1(\xi) = -\mu_1(-\xi), \quad p_1(\xi) = p_1(-\xi) \quad (4.41)$$

These conditions are met if $\mu_1(\zeta)$ and $p_1(\zeta)$ are represented by polynomials of the following form:

$$\mu_1(\zeta) = \sum_{m=0}^n B_m \zeta^{2m+1}, \quad p_1(\zeta) = \sum_{r=0}^n d_r \zeta^{2r} \quad (4.42)$$

where the coefficients B_m and d_m are real,

$$\text{Im } B_m = 0, \quad \text{Im } d_m = 0; \quad m = 0, 1, 2, \dots, n \quad (4.43)$$

With (4.29), this becomes

$$\mu_1(\chi_1) = \sum_{m=0}^n B_m \chi_1^{m+1/2}, \quad p_1(\chi_1) = \sum_{r=0}^N d_r \chi_1^r \quad (4.44)$$

and (4.33) may be written in the form:

$$\Omega_1(\chi_1) = \frac{\mu_1^*(\chi_1)}{2\pi} \sqrt{\chi_1} \ln \left[\frac{\sqrt{\chi_1} - 1}{\sqrt{\chi_1} + 1} \right] + p_1(\chi_1) \quad (4.45)$$

where

$$\mu_1^*(\chi_1) = \sum_{m=0}^n B_m \chi_1^m \quad (4.46)$$

The Behavior along the Element

Expressions for Φ and $\mathcal{F}$ along the element are obtained from (4.28), (4.32) and (4.45) as follows. $\chi_1^{1/2}$ and θ_1 along the element are given by

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^+; \quad \sqrt{\chi_1'} = \sqrt{K_1'}, \quad \theta_1 = \pi \quad (4.47)$$

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^-; \quad \sqrt{\chi_1'} = -\sqrt{K_1'}, \quad \theta_1 = \pi \quad (4.48)$$

and (4.45) yields

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^+; \quad \Phi = \frac{\mu_1^*(K_1)}{2\pi} \sqrt{K_1'} \ln \left| \frac{\sqrt{K_1'} - 1}{\sqrt{K_1'} + 1} \right| + p_1(K_1) \quad (4.49)$$

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^+; \quad \mathcal{F} = \frac{1}{2} \sqrt{K_1'} \mu_1^+(K_1)$$

and

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^-; \quad \Phi = -\frac{\mu_1^*(K_1)}{2\pi} \sqrt{K_1'} \ln \left| \frac{-\sqrt{K_1'} - 1}{-\sqrt{K_1'} + 1} \right| + p_1(K_1) \quad (4.50)$$

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0^-; \quad \mathcal{F} = -\frac{1}{2} \sqrt{K_1'} \mu_1^*(K_1)$$

so that Φ is continuous across the element whereas $\mathcal{F}$ jumps,

$$0 \leq K_1 \leq 1, \quad \Lambda_1 = 0; \quad \Phi^+ - \Phi^- = 0, \quad \mathcal{F}^+ - \mathcal{F}^- =$$

$$\mathcal{F}^+ - \mathcal{F} = \sqrt{K_1'} \mu_1^*(K_1) = \mu_1(K_1) \quad (4.51)$$

Both Φ and $\mathcal{F}$ are continuous across the remainder of the real axis ($\Lambda_1 = 0$), as may be verified by the use of (4.45) and by noticing that $\theta_1 = 0$ there.

The coefficients d_r of the polynomial $p_1(\chi_1)$, see (4.42), are obtained from an expansion about infinity similarly as for the case discussed in Sec. 4.1. Expressions for the coefficients in these expansions will be presented in

Sec. 6.

The complex potential Ω_{n-1} for the last element of the crack, extending from z_{n-1} to z_n , is determined in a similar fashion, and will be given below without derivation. The variable χ_n is defined as follows, see Figure 15.

$$\chi_n = \frac{z - z_{n+1}}{z_n - z_{n+1}} = - (z - z_{n+1}) \frac{e^{-i\alpha}}{L_n} \quad (4.52)$$

where L_n and α represent the length and the orientation of the element, respectively. The complex potential is given by

$$\Omega_n = - \frac{\mu_n(\chi_n)}{2\pi} \ln \left[\frac{\chi_n^{1/2} - 1}{\chi_n^{1/2} + 1} \right] - p_n(\chi_n) \quad (4.53)$$

The minus signs are a result of the definition of χ_n ; the + side of the crack now corresponds to $\Lambda_n = 0^-$ and the - side to $\Lambda_n = 0^+$, see Figure 15.

Figure 15. The last element of the crack in the z , χ_n and $\chi_n^{1/2}$ planes.

4.3 MODELING OF THE CRACK

The crack is modeled by stringing elements together. If there are n elements with complex potentials Ω_j , $j=1,2,\dots,n$, then the complex potential representing the crack is obtained by superposition,

$$\Omega = \sum_{j=1}^n \Omega_j \quad (4.54)$$

There are $n+1$ nodal points $z_1, z_2, \dots, z_{n+1}$; z_1 and z_{n+1} represent the two endpoints of the crack.

Each of the n complex potentials contains as unknowns the coefficients in the n polynomials $\mu_j(Z_j)$. Conditions available for the determination of these unknowns are

- 1) the internal boundary conditions
- 2) continuity requirements at the nodes

The internal boundary conditions, for both the linear and the cubic law may be written in the form (2.29),

$$D \frac{\partial \Phi}{\partial s} = \Phi^+ - \Phi^- \quad (4.55)$$

where D is a function of the width b , and is given for the cubic law by

$$D = \frac{\beta^* b^3}{k} \quad (4.56)$$

and for the linear law by

$$D = \frac{k^* b}{k} \quad (4.57)$$

Using the definitions of λ_1 , Z_j and λ_n , see (4.27), (4.2) and (4.52), this boundary condition may be written as follows for the first element,

$$\frac{D}{L} \frac{\partial \Phi}{\partial K_1} = \Phi^+ - \Phi^-, \quad 0 \leq K_1 \leq 1, \quad \Lambda_1 = 0 \quad (4.58)$$

for any element between the first and the last,

$$\frac{2D}{L} \frac{\partial \Phi}{\partial X_j} = \Phi^+ - \Phi^-, \quad -1 \leq X_j \leq 1, \quad Y_j = 0 \quad j=2,3,\dots,n \quad (4.59)$$

and for the last element,

$$-\frac{D}{L} \frac{\partial \Phi}{\partial K_n} = \Phi^+ - \Phi^-, \quad 0 \leq K_n \leq 1, \quad \Lambda_n = 0 \quad (4.60)$$

The internal boundary conditions will be applied to a number of discrete points of the crack, each leading to a single real equation involving the unknown coefficients in the complex potentials Ω_j .

The solution is approximate; the behavior near the $n-1$ internal nodes, in particular, is governed by a logarithmic singularity that would not occur in the exact solution. The approximation may be improved, however, by requiring that the complex potential along with one or more of its derivatives be finite and continuous at the nodal points. These requirements represent constraints that are added to the internal boundary conditions. The continuity requirements at the nodes chosen for the present model are that Ω and its first and second order derivatives are continuous at the nodes. This choice guarantees that the boundary conditions, which involve the derivative of Ω , can be applied at nodal points, if so desired.

The model is formulated in such a way that the continuity conditions at the nodal points are analytically satisfied; the corresponding unknowns are eliminated before numerical evaluation takes place. This will be shown below, first for the nodal points $z_3, z_4, \dots, z_{n-1}$ between interior elements. It is shown in Appendix B that Ω along with its first two derivatives is continuous, provided that the crack is straight, if the following conditions are met at z_j :

$$\mu_{j-1}(1) = \mu_j(-1) = \mu_j^0 \quad (4.61)$$

$$\frac{2}{L_{j-1}} \mu'_{j-1}(1) = \frac{2}{L_j} \mu'_j(-1) = \mu_j^1 \quad (4.62)$$

$$\frac{4}{L_{j-1}^2} \mu''_{j-1}(1) = \frac{4}{L_j^2} \mu''_j(-1) = \mu_j^2 \quad (4.63)$$

where the prime stands for differentiation with respect to the local complex variables. The μ_j^0 , μ_j^1 and μ_j^2 are real constants that represent the values at node z_j of the distribution strength, its first, and its second derivatives with respect to the coordinate s , defined in Figure 1.

The distribution strength $\mu_j(Z_j)$ is written in the following form

$$\mu_j(Z_j) = \sum_{k=0}^2 \mu_j^k f(Z_j) \{L_j\}^k + \sum_{k=0}^2 \mu_{j+1}^k g(Z_j) \{L_j\}^k \quad (4.64)$$

where $f(Z_j)$ and $g(Z_j)$ ($k=0,1,2$) represents six 5th order polynomials each of which meets 6 conditions as indicated below.

$$Z_j = -1: \quad \mu_j^0 = 1, \quad \mu_j^0' = \mu_j^0'' = 0, \quad \mu_j^1 = 0, \quad \mu_j^1' = \frac{1}{2}, \quad \mu_j^1'' = 0, \quad \mu_j^2 = \mu_j^2' = 0, \quad \mu_j^2'' = \frac{1}{4} \quad (4.65)$$

$$Z_j = 1: \quad \overset{k}{f} = \overset{k}{f}' = \overset{k}{f}'' = 0 \quad (k = 0, 1, 2) \quad (4.66)$$

and

$$Z_j = -1: \quad \overset{k}{g} = \overset{k}{g}' = \overset{k}{g}'' = 0 \quad (k = 0, 1, 2) \quad (4.67)$$

$$Z_j = 1: \quad \overset{0}{g} = 1, \quad \overset{0}{g}' = \overset{0}{g}'' = 0, \quad \overset{1}{g} = 0, \quad \overset{1}{g}' = \frac{1}{2}, \quad \overset{1}{g}'' = 0, \quad \overset{2}{g} = \overset{2}{g}' = 0, \quad \overset{2}{g}'' = \frac{1}{4} \quad (4.68)$$

where the primes stand for differentiation with respect to Z_j . It may be seen from (4.64) through (4.68) that the continuity requirements (4.61) through (4.63) are indeed met. The coefficients of $\overset{k}{f}$ and $\overset{k}{g}$, $k = 0, 1, 2$, are obtained from the six conditions that each must meet.

The complex potential Ω_j , $j = 2, 3, \dots, n-1$, given by (4.10) is written in a somewhat different form as follows:

$$\Omega_j = \frac{1}{2\pi} \sum_{k=0}^2 \left[\overset{k}{\mu}_j \overset{k}{F}(Z_j) + \overset{k}{\mu}_{j+1} \overset{k}{G}(Z_j) \right] L_j^k \quad (4.69)$$

where

$$\overset{k}{F}(Z_j) = \overset{k}{f}(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] + \overset{k}{P}(Z_j) \quad (4.70)$$

and

$$\overset{k}{G}(Z_j) = \overset{k}{g}(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] + \overset{k}{Q}(Z_j) \quad (4.71)$$

where

$$\overset{k}{f}(Z_j) = \sum_{m=0}^s \overset{k}{a}_m Z_j^m, \quad \overset{k}{g}(Z_j) = \sum_{m=0}^s \overset{k}{a}_m (-1)^{m+k} Z_j^m \quad (4.72)$$

and

$$\overset{k}{P}(Z_j) = \sum_{m=0}^4 \overset{k}{a}_m Z_j^m, \quad \overset{k}{Q}(Z_j) = \sum_{m=0}^4 \overset{k}{a}_m (-1)^{m+k+1} Z_j^m \quad (4.73)$$

The coefficients $\overset{k}{a}_m$ are obtained in a straightforward manner from algebraic equations derived from (4.65) through (4.68), and are listed in Sec. 6, where

the numerical model is discussed in some more detail. The coefficients α_m^k are found from an expansion about infinity and are also given in Sec. 6. It may be noted that all coefficients α_m^k , $k=0,1,2$, $m=0,1,2,\dots,5$, and α_m^k , $k=0,1,2$, $m=0,1,2,4$, are numbers that can be calculated once and for all.

The continuity of Ω , $d\Omega/dz$ and $d^2\Omega/dz^2$ at the nodes z_j , $j=3,4,\dots,n-1$, is thus achieved by adopting the formulation (4.69). The unknowns are now organized per node and consist of the three μ_j^k ($k=0,1,2$) for node j .

Continuity at z_2 and z_n

The continuity conditions imposed at nodes connecting tip elements to interior elements (nodes z_2 and z_n) are the same as those discussed above. The analysis is somewhat different, however, due to the special forms of Ω_1 and Ω_n . The conditions at node 2 will be discussed first. As is shown in Appendix B, Ω , $d\Omega/dz$ and $d^2\Omega/dz^2$ are continuous at z_2 if the following conditions are met:

$$\mu_1(1) = \mu_2(-1) = \mu_2^0 \quad (4.74)$$

$$\frac{1}{L_1} \mu_1'(1) = \frac{2}{L_2} \mu_2'(-1) = \mu_2^1 \quad (4.75)$$

$$\frac{1}{L_1^2} \mu_1''(1) = \frac{4}{L_2^2} \mu_2''(-1) = \mu_2^2 \quad (4.76)$$

The density distribution $\mu_1(\chi_1)$ for the first element will be represented as follows:

$$\mu_1(\chi_1) = \mu_1^0 f_1(\chi_1) + \sum_{k=0}^2 \mu_2^k g_1^k(\chi_1) \{L_1\}^k \quad (4.77)$$

where

$$\chi_1 = \frac{z - z_1}{z_2 - z_1} = (z - z_1) \frac{e^{ia}}{L_1} \quad (4.78)$$

Only one degree of freedom is accommodated for the behavior at the tip: the coefficient of the singularity (coefficient d_0 in (3.23)). The distribution strength $\mu_1(\chi_1)$ thus has four degrees of freedom: one at the tip of the crack and three at node z_2 . The polynomial f_1 and the three polynomials g_1^k are functions of $\chi_1^{1/2}$ (see (4.44)), each satisfying four conditions as is indicated below

$$\lambda_1 = 0 : f_1 = 1 \quad (4.79)$$

$$\lambda_1 = 1 : f_1 = f'_1 = f''_1 = 0 \quad (4.80)$$

$$\lambda_1 = 0 : g_1^k = g_1'^k = g_1''^k = 0, \quad k = 0, 1, 2 \quad (4.81)$$

$$\lambda_1 = 1 : g_1^0 = 1, g_1'^0 = g_1''^0 = 0; g_1^1 = 0, g_1'^1 = 1, g_1''^1 = 0, g_1^2 = 0, g_1'^2 = 0, g_1''^2 = 1 \quad (4.82)$$

where the primes stand for differentiation with respect to λ_1 . It may be seen from (4.77) and (4.79) through (4.82) that the conditions (4.74) through (4.76) are indeed met.

The complex potential Ω_1 is written in the following form:

$$\Omega_1 = \frac{1}{2\pi} \left[\mu_1^0 F_1(\lambda_1) + \sum_{k=0}^2 \mu_2^k G_1^k(\lambda_1) \{L_1\}^k \right] \quad (4.83)$$

where

$$F_1(\lambda_1) = f_1(\lambda_1) \ln \left[\frac{\sqrt{\lambda_1'} - 1}{\sqrt{\lambda_1'} + 1} \right] + P_1(\lambda_1) \quad (4.84)$$

and

$$G_1^k(\lambda_1) = g_1^k(\lambda_1) \ln \left[\frac{\sqrt{\lambda_1'} - 1}{\sqrt{\lambda_1'} + 1} \right] + Q_1^k(\lambda_1) \quad (4.85)$$

where

$$f_1(\lambda_1) = \sum_{m=0}^3 b_m \lambda_1^{m+1/2}, \quad g_1^k(\lambda_1) = \sum_{m=1}^3 \frac{k}{b_m} \lambda_1^{m+1/2} \quad (4.86)$$

and

$$P_1(\lambda_1) = \sum_{m=0}^3 \beta_m \lambda_1^m, \quad Q_1^k(\lambda_1) = \sum_{m=0}^3 \frac{k}{\beta_m} \lambda_1^m \quad (4.87)$$

The coefficient b_m and $\frac{k}{b_m}$ are found by solving linear equations obtained from (4.79) through (4.82), and are listed in Sec. 6. The coefficients β_m and $\frac{k}{\beta_m}$ are obtained from conditions at infinity as shown in Sec. 6.

The continuity conditions at node z_n are given in Appendix B and are, see

Figure 16,

$$\mu_n(1) = \mu_{n-1}(1) = \mu_n^0 \quad (4.88)$$

$$\frac{-1}{L_n} \mu_n'(1) = \frac{2}{L_{n-1}} \mu_{n-1}'(1) = \mu_n^1 \quad (4.89)$$

$$\frac{+1}{L_n^2} \mu_n''(1) = \frac{4}{L_{n-1}^2} \mu_{n-1}''(1) = \mu_n^2 \quad (4.90)$$

Figure 16. Continuity conditions at node z_n .

where it is noted that both $Z_{n-1}(z_n)$ and $\chi_n(z_n)$ are 1. These equations have the same form as those for node 2, see (4.74) through (4.76), with the exception of the minus sign in (4.89). The conditions are met if Ω_n is written in the form

$$\Omega_n = -\frac{1}{2\pi} \left[\mu_{n+1}^0 F_1(\chi_n) + \sum_{k=0}^2 \mu_n^k G_1(\chi_n) (-L_n)^k \right] \quad (4.91)$$

where F_1 and G_1 are defined by (4.84) and (4.85), and where it is recalled that

$$\chi_n = \frac{z - z_{n+1}}{z_n - z_{n+1}} = -(z - z_{n+1}) \frac{e^{-i\alpha}}{L_n} \quad (4.92)$$

see (4.52). The - sign in the term $(-L_n)^k$ in (4.91) is a result of the minus sign occurring in (4.89), and the - sign in front of the entire expression (4.91) is a result of the definition of Ω_n , see (4.53).

Determination of the Unknowns

There are three unknown constants μ_j^k , $k=0,1,2$, for each internal node j with $j=2,3,\dots,n$ and one unknown for each tip, μ_1^0 and μ_{n+1}^0 . Thus, the total number of unknowns, N_n , is

$$N_n = 3 * (n-1) + 2 = 3n-1 \quad (4.93)$$

where n is the number of elements in the crack. Therefore, the appropriate internal boundary condition, (4.58), (4.59), or (4.60), may be applied at $3n-1$ discrete points of the crack. These points could be, for example, all $n-1$ internal nodes, and two points equally spaced between the nodes on each of the elements.

5 THE CROSSING

Two different strings of elements intersect at each crossing, where the water flowing through the two cracks is redistributed. The conditions applied for each crossing are discussed below, along with a combination of two special elements, suitable for the approximate modeling of the intersection.

5.1 CONDITIONS AT THE CROSSING

A typical crossing is depicted in Figure 17. Local coordinates s and t are

Figure 17. Two intersecting cracks.

introduced along the cracks as shown in the Figure. Points ξ_1 , ξ_2 , ξ_3 and ξ_4 are located between the intersection, z_0 , and the nearest control points. There is a redistribution of flow at the crossing and therefore the four discharges occurring in the cracks at four points at infinitesimal distances from z_0 (z_{01} , z_{02} , z_{03} and z_{04}) are different, see Figure 17(b). These four discharges must meet the following condition of continuity,

$$Q_{s2} - Q_{s1} + Q_{t4} - Q_{t3} = 0 \quad (5.1)$$

The amount of water that flows from crack s into crack t will be referred to as Q_0 , with

$$Q_0 = Q_{s2} - Q_{s1} \quad (5.2)$$

The continuity condition along with the internal boundary condition (2.29) fully defines the problem of flow near the crossing.

An exact solution to this flow problem does as yet not exist, except for the limiting case that the permeability of the cracks is infinite. For the latter case, four different singularities exist at the four sides of the intersection, each one of the form $w = A(z - z_0)^\alpha$ where $\alpha > 0$ and with w defined by (2.14). It is possible to incorporate this behavior by constructing special elements allowing for four types of singularities. In order to avoid the excessive memory requirements and calculation times associated with such elements, an alternative function is developed that models the behavior at the intersection with good approximation. Because this element is not capable of meeting the conditions at the point of intersection, the boundary conditions are applied by integrating them along the cracks near the intersection. Three paths of integration, labeled as 1, 2, and 3 in Figure 18, are defined. Path 1

Figure 18. The three path of integration.

extends from ξ_1 via z_0 to ξ_4 , path 2 from ξ_1 via z_0 to ξ_2 and path 3 from ξ_1 via z_0 to ξ_3 . The boundary condition (2.29),

$$\Phi^+ - \Phi^- = D \frac{\partial \Phi}{\partial s} \quad (5.3)$$

is written as

$$\frac{\partial \Phi}{\partial s} = \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) \quad (5.4)$$

and then integrated along the three paths:

$$\int_{\xi_1}^{z_0} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) ds + \int_{z_0}^{\xi_4} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) dt = \Phi(\xi_4) - \Phi(\xi_1) \quad (5.5)$$

$$\int_{\xi_1}^{z_0} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) ds + \int_{z_0}^{\xi_2} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) ds = \Phi(\xi_2) - \Phi(\xi_1) \quad (5.6)$$

$$\int_{\xi_1}^{z_0} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) ds + \int_{z_0}^{\xi_3} \left(\frac{1}{D} \right) (\mathcal{F}^+ - \mathcal{F}^-) dt = \Phi(\xi_3) - \Phi(\xi_1) \quad (5.7)$$

Two of the above conditions replace the two of the form (5.3) that otherwise would have been applied at two control points at z_0 ; one for each crack. The third one is used to determine the additional variable Q_0 , accounted for by means of the special element discussed below.

5.2 THE ELEMENTS FOR THE CROSSING

The element used to model the redistribution of flow, Q_0 , at the crossing consists of two functions. The first, referred to as the main element, creates a discontinuity in $\mu = \mathcal{F}^+ - \mathcal{F}^-$ caused by the redistribution of flow. The second, the auxiliary element, is added solely to increase efficiency of the numerical procedure.

The Main Element

The main element consists of two branches, one for each of the crossing cracks. The element for the s-crack, see Figure 17, will be considered first. It extends over the entire crack, which runs from z_{s1} to z_{se} . The local variable Z_s is defined as a function of z as

$$Z_s(z) = x_s(z) + iy_s(z) = \frac{z - \frac{1}{2}(z_{se} - z_{s1})}{\frac{1}{2}(z_{se} - z_{s1})} \quad (5.8)$$

and Z_{s0} represents the value of Z_s at the intersection z_0 ,

$$Z_{s0} = Z_s(z_0) = X_{s0} \quad (Y_{s0} = 0) \quad (5.9)$$

The element will be defined such that it accounts for the jump at Z_{s0} in the discharge flowing through the crack. This jump results in a discontinuity for the distribution strength $\mu = \mathfrak{F}^+ - \mathfrak{F}^-$, as is seen from (2.25),

$$Q_s = -(\mathfrak{F}_s^+ - \mathfrak{F}_s^-) = -\mu_s \quad (5.10)$$

where the index refers to crack s , and from (5.2),

$$Q_{s2} - Q_{s1} = Q_0 = -(\mu_{s2} - \mu_{s1}) \quad (5.11)$$

It is to be noted that even though the distribution strength μ_s jumps at Z_{s0} , the derivative $\partial\mu_s/\partial s$ is continuous. This may be seen as follows, see Figure 19. The two cracks divide the infinitesimal neighborhood of the point of intersection, z_0 , into four different zones labeled 1 through 4 in Figure 19. Within each of these zones the complex potential is analytic and each zone has its own specific discharge vector. The projections on the cracks of the

Figure 19. Discontinuities at the intersection.

specific discharge vectors at points on either side of the cracks (but just outside the cracks) are equal, because the head is continuous across the cracks. These components, however, are different on the two sides of the

intersection as is indicated by the arrows labeled q_{s1} and q_{s2} , and q_{t3} and q_{t4} in Figure 19. It is shown in Appendix D that the continuity of q_t across crack t implies that the difference of the components of (q_x, q_y) normal to crack s on the $+$ and $-$ sides, $q_{sn}^+ - q_{sn}^-$, is continuous across crack t . This is true for either side of crack s . Since

$$\sigma_s = - [q_{sn}^+ - q_{sn}^-] = - \frac{\partial [\chi^+ - \chi^-]}{\partial s} = - \frac{\partial \mu_s}{\partial s} \quad (5.12)$$

(see (2.20) and (2.23)), it follows that $\partial \mu_s / \partial s$ is continuous at the intersection. The same is true for crack t ; $\partial \mu_t / \partial t$ is continuous at the intersection.

The conditions that the element for crack s must fulfill may be summarized:

- 1) μ_s jumps at z_0 : $\mu_{s2} - \mu_{s1} = -Q_0$
- 2) $\partial \mu_s / \partial s$ is continuous at z_0
- 3) The complex potential must not have any singularity other than that of (3.23) at the crack tips.
- 4) The potential is continuous across the element, and the stream function jumps.

The above conditions leave a considerable amount of freedom in choosing the complex potential for element s , Ω_s . The choice given below was made after several others had been tried in BACRACK, and the preference is solely based upon numerical accuracy and efficiency. The function is the complex potential for flow toward a canal of finite length, plus that for a well located at Z_{s0} and has the form

$$\Omega_s = \frac{Q_0}{2\pi} \ln \left[\frac{Z_s + \sqrt{(Z_s - 1)(Z_s + 1)'}}{2(Z_s - Z_{s0})} \right] \quad (5.13)$$

The arguments of $(Z_s - 1)$ and $(Z_s + 1)$ are again (cf. (3.12) and (3.13)) chosen in the range:

$$-\pi \leq \arg(Z_s - 1) \leq \pi, \quad -\pi \leq \arg(Z_s + 1) \leq \pi \quad (5.14)$$

Expressions for Φ and χ along the $+$ and $-$ sides of the element are:

$$\begin{aligned} -1 \leq X_s \leq 1, \quad Y_s = 0^+; \quad \Phi &= -\frac{Q_0}{2\pi} \ln |2(X_s - X_{s0})| \\ -1 \leq X_s \leq 1, \quad Y_s = 0^+; \quad \chi &= \frac{Q_0}{2\pi} (\theta_c - \theta_w) \end{aligned} \quad (5.15)$$

and

$$-1 \leq X_s \leq 1, \quad Y_s = 0^-; \quad \Phi = \frac{Q_0}{2\pi} \ln |2(X_s - X_{s0})| \quad (5.16)$$

$$-1 \leq X_s \leq 1, \quad Y_s = 0^-; \quad \Phi = -\frac{Q_0}{2\pi} (\theta_c + \theta_w)$$

where

$$\theta_c = \arctan \frac{\sqrt{1 - X_s^2}}{X_s}, \quad \theta_w = \operatorname{Im} \ln(Z_s - Z_{s0}) \quad (5.17)$$

The branch cuts for θ_c and θ_w are taken such that

$$0 \leq \theta_c \leq \pi, \quad -\pi \leq \theta_w \leq \pi \quad (5.18)$$

The jump in μ is produced by the jump in θ_w , which has the following values along the element:

$$+1 \leq X_s \leq X_{s0}, \quad Y_s = 0^+; \quad \theta_w = \pi \quad (5.19)$$

$$-1 \leq X_s \leq X_{s0}, \quad Y_s = 0^-; \quad \theta_w = -\pi \quad (5.20)$$

$$X_{s0} \leq X_s \leq 1, \quad Y_s = 0; \quad \theta_w = 0 \quad (5.21)$$

Since the derivative of θ_c with respect to X_s is continuous at X_{s0} and that of θ_w vanishes along the entire element, $\partial\mu_s/\partial s$ is continuous along the element. A plot of μ_s is given in Figure 20. The reader may verify that (5.13) has a behavior of the form (3.23) near the tips of the crack. Thus, Ω_s meets all conditions listed above.

The function Ω_t , used to model a similar behavior for crack t is of the form,

$$\Omega_t = -\frac{Q_0}{2\pi} \ln \left[\frac{Z_t + \sqrt{(Z_t - 1)(Z_t + 1)'}}{2(Z_t - Z_{t0})} \right] \quad (5.22)$$

where Z_t and Z_{t0} are defined for crack t in the same way as Z_s and Z_{s0} for crack s . The complete complex potential for the main element of the crossing, Ω_m , is the sum of Ω_s and Ω_t ,

$$\Omega_m = \frac{Q_0}{2\pi} \ln \left[\frac{Z_s + \sqrt{(Z_s - 1)(Z_s + 1)'}}{2(Z - Z_{s0})} \right] - \frac{Q_0}{2\pi} \ln \left[\frac{Z_t + \sqrt{(Z_t - 1)(Z_t + 1)'}}{2(Z - Z_{t0})} \right] \quad (5.23)$$

It may be noted that the logarithmic singularity at z_0 is a removable one; the

Figure 20. The behavior of μ_s along the element.

singular parts of the first and second terms in (5.23) cancel. However, the branch cuts for $\ln(Z - Z_{s0})$ and $\ln(Z - Z_{t0})$ are different, so that the imaginary parts do not cancel.

Even though the above function has all properties desired of the cross-function, it is not yet suitable for a numerical model. Each of the two complex potentials corresponds to a flow pattern that is symmetric with respect to the normal through the center of the element. If, for example, the water would flow from crack s into crack t at z_0 , then the distribution of the outflow from crack t would not be affected by the location of z_0 on the crack. The other elements would be capable of correcting for this effect, but at the cost of efficiency. A better element is obtained if the distribution of the inflow or outflow along the two elements is made dependent upon the location of z_0 on the two cracks. An element that provides this feature is presented below.

Auxiliary Element.

The auxiliary element is added to both Ω_s and Ω_t in order to obtain a good approximation of the behavior near z_0 , with a minimal number of elements. The exact behavior at z_0 is not known but can be estimated from the extreme case of cracks of infinite permeability, for which an exact solution can be found, and from results obtained from SYLENS. Both indicate that the specific discharge vector is rather small in the neighborhood of z_0 , as compared to other points of the cracks. Therefore, the element is chosen such that the complete cross-element does not contribute to the specific discharge vector at z_0 . It may be noted that this does not imply that the flow stagnates at z_0 ; other

elements will contribute to both the inflow into the cracks and the tangential component of flow at z_0 . In particular, those of the elements of the type discussed in Sec. 4 that contain z_0 will contribute to the flow into and out of the cracks at z_0 .

The auxiliary elements extend over the full length of the cracks. The function will be presented first for the s-crack, will be referred to as Ω_s^* , and is chosen such that it meets the following 4 conditions:

- 1) The flow described by $\Omega_s + \Omega_s^*$ stagnates at z_0
- 2) The potential is continuous across the element, and the stream function jumps.
- 3) The function Ω_s^* does not affect the free-field; Ω_s is of order $1/z$ near infinity
- 4) The complex potential is of the form (3.23) near the tips of the crack

The following complex potential,

$$\Omega_s^* = \frac{Q_0}{2\pi} \left[(Z_s - 2Z_{s0}) \sqrt{(Z_s - 1)(Z_s + 1)} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] \quad (5.24)$$

meets the conditions listed, as will be shown. Condition 1) is met if the derivative $W = -\{\Omega_s + \Omega_s^*\}'$, where the prime stands for differentiation with respect to Z_s , is of the following form near Z_{s0} :

$$\{\Omega_s + \Omega_s^*\}' = A(Z_s - Z_{s0}) \quad (Z_s \rightarrow Z_{s0}) \quad (5.25)$$

where A is a complex constant. Differentiation of (5.13) and (5.24) and addition gives, ignoring the term $-Q_0/(2\pi) \ln\{2(Z_s - Z_{s0})\}$ whose effect on the flow field is canceled by the corresponding term in Ω_m ,

$$\{\Omega_s + \Omega_s^*\}' = \frac{Q_0}{\pi} (Z_s - Z_{s0}) \left[Z_s (Z_s^2 - 1)^{-1/2} - 1 \right] \quad (5.26)$$

so that condition (5.25) is met. The expressions for Φ and Ψ along the element are obtained from (5.24) by the use of (5.14),

$$-1 \leq X_s \leq 1, \quad Y_s = 0^+ \quad \Phi = \frac{Q_0}{2\pi} \left[-(X_s - X_{s0})^2 + (X_{s0}^2 + \frac{1}{2}) \right] \quad (5.27)$$

$$-1 \leq X_s \leq 1, \quad Y_s = 0^+ \quad \Psi = \frac{Q_0}{2\pi} (X_s - 2X_{s0}) \sqrt{1 - X_s^2}$$

and

$$-1 \leq X_s \leq 1, \quad Y_s = 0^- \quad \Phi = \frac{Q_0}{2\pi} \left[-(X_s - X_{s0})^2 + (X_{s0}^2 + \frac{1}{2}) \right] \quad (5.28)$$

$$-1 \leq X_s \leq 1, \quad Y_s = 0^- \quad \Phi = -\frac{Q_0}{2\pi} (X_s - 2X_{s0}) \sqrt{1 - X_{s0}^2}$$

It is seen from (5.27) and (5.28) that the condition 2) above is fulfilled. That the third condition is met is seen as follows. The expansion of (5.24) about infinity is obtained after writing (5.24) as follows:

$$\Omega_s^* = \frac{Q_0}{2\pi} \left[Z_s (Z_s - 2Z_{s0}) \sqrt{1 - Z_s^{-2}} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] \quad (5.29)$$

Expansion of the term $(1 - Z_s^{-2})^{1/2}$ about $(Z_s^{-2}) = 0$ and subsequent substitution in (5.29) shows that indeed Ω_s^* is of the order Z_s^{-1} near infinity. That the fourth condition is met is seen from the occurrence of the term $(Z_s^2 - 1)^{1/2}$ in (5.24).

The complete complex potential for the auxiliary element, Ω_a , is obtained by adding a term for the t crack similar to (5.29),

$$\begin{aligned} \Omega_a = \frac{Q_0}{2\pi} & \left[(Z_s - 2Z_{s0}) \sqrt{(Z_s - 1)(Z_s + 1)} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] - \\ & - \frac{Q_0}{2\pi} \left[(Z_t - 2Z_{t0}) \sqrt{(Z_t - 1)(Z_t + 1)} - (Z_t - Z_{t0})^2 + (Z_{t0}^2 + \frac{1}{2}) \right] \quad (5.30) \end{aligned}$$

The complete cross function is the sum of (5.23) and (5.30),

$$\Omega_{cr} = \Omega_m + \Omega_a \quad (5.31)$$

The contour plots obtained by the use of the above cross function are agreement with those obtained from SYLENS, as is shown in Sec. 7.

6 NUMERICAL MODEL

This section is devoted to a brief description of the numerical model applied in BACRACK, along with a listing of the main functions and constants involved. The computer program is written on a Perkin and Elmer 3220 mini computer with a memory of $\frac{1}{2}$ megabyte. Some of the features of the program and the analysis are developed in order to resolve difficulties arising from the relatively low accuracy (6 significant digits) of a mini computer. A general sketch of the model will be given below before treating some of the details.

The cracks are divided into elements, and control points (points where a boundary condition is enforced) are distributed uniformly between tips and crossings. Next, the matrix of coefficients is constructed, representing the system of linear equations evolving from the conditions at the control points. The system of equations is subsequently solved. Once the equations are solved all of the parameters are known for each crack ($\mu_1^0, \mu_{n+1}^0, \mu_j^k, j=2, \dots, n, k=0,1,2$) and crossing (Q_0), and the program is ready to contour a grid of piezometric heads.

It is to be noted that all integrations and differentiations are performed analytically. It therefore is sufficient to calculate and store all numerical values for the parameters once.

The analytical procedure aimed at eliminating approximately half of the unknowns beforehand is very valuable as far as storage requirements are concerned; it reduces the memory space necessary to store the matrix by a factor of 4. However, a disadvantage of this procedure is that 6 times as many functions must be evaluated for each value of Ω as would be the case for a conventional formulation. Therefore, once the parameters are known, the 6 functions are combined into a single one, which reduces the calculation time approximately by a factor of 6.

If N represents the total number of cracks, n_r the number of elements in crack r and n_c the total number of crossings, the complete complex potential may be written as

$$\Omega = \sum_{r=1}^N \left[\sum_{j=1}^{n_r} \Omega_j^r(z) \right] + \sum_{r=1}^{n_c} \Omega_{cr}^r(z) + \Omega_g(z) \quad (6.1)$$

The superscript r refers to the crack in Ω_j (defined in Sec. 4) and to the crossing in Ω_{cr} (defined in Sec. 6). The function $\Omega_g(z)$ represents a given potential, at present being a field of uniform flow.

6.1 THE COMPLEX POTENTIAL FOR THE CRACKS

The complex potentials involved in each crack, derived in Sec. 4, will be listed below along with all constants. The reader is referred to Sec. 4 for a complete description.

The complex potential Ω_j , $j = 2, 3, \dots, n$, for internal elements is given by (4.69),

$$\Omega_j = \frac{1}{2\pi} \sum_{k=0}^2 \left[\mu_j^k F^k(Z_j) + \mu_{j+1}^k G^k(Z_j) \right] L_j^k \quad (6.2)$$

where

$$\mu_j^k F^k(Z_j) = \mu_j^k(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] + P^k(Z_j) \quad (6.3)$$

and

$$\mu_{j+1}^k G^k(Z_j) = \mu_{j+1}^k(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] + Q^k(Z_j) \quad (6.4)$$

where

$$\mu_j^k f^k(Z_j) = \sum_{m=0}^s \mu_j^k a_m^k Z_j^m, \quad \mu_{j+1}^k g^k(Z_j) = \sum_{m=0}^s \mu_{j+1}^k a_m^k (-1)^{m+k} Z_j^m \quad (6.5)$$

and

$$\mu_j^k P^k(Z_j) = \sum_{m=0}^4 \mu_j^k a_m^k Z_j^m, \quad \mu_{j+1}^k Q^k(Z_j) = \sum_{m=0}^4 \mu_{j+1}^k a_m^k (-1)^{m+k+1} Z_j^m \quad (6.6)$$

The complex potential for the first element is introduced in (4.83),

$$\Omega_1 = \frac{1}{2\pi} \left[\mu_1^0 F_1(\chi_1) + \sum_{k=0}^2 \mu_1^k G_1^k(\chi_1) \{L_1\}^k \right] \quad (6.7)$$

where

$$\mu_1^k F_1^k(\chi_1) = \mu_1^k(\chi_1) \ln \left[\frac{\sqrt{\chi_1^k} - 1}{\sqrt{\chi_1^k} + 1} \right] + P_1^k(\chi_1) \quad (6.8)$$

and

$$\mu_1^k G_1^k(\chi_1) = \mu_1^k(\chi_1) \ln \left[\frac{\sqrt{\chi_1^k} - 1}{\sqrt{\chi_1^k} + 1} \right] + Q_1^k(\chi_1) \quad (6.9)$$

where

$$f_1(\chi_1) = \sum_{m=0}^3 b_m \chi_1^{m+1/2}, \quad g_1(\chi_1) = \sum_{m=1}^3 \frac{k}{b_m} \chi_1^{m+1/2} \quad (6.10)$$

and

$$P_1(\chi_1) = \sum_{m=0}^3 \beta_m \chi_1^m, \quad Q_1(\chi_1) = \sum_{m=0}^3 \frac{k}{\beta_m} \chi_1^m \quad (6.11)$$

The potential for the last element, finally, is given by (4.91),

$$\Omega_n = -\frac{1}{2\pi} \left[\mu_{n+1} F_1(\chi_n) + \sum_{k=0}^2 \frac{k}{\mu_n} G_1(\chi_n) \{-L_n\}^k \right] \quad (6.12)$$

The constants $\frac{k}{a_m}$ are presented in Table 2, $\frac{k}{b_m}$ in Table 3, $\frac{k}{\alpha_m}$ and $\frac{k}{\beta_m}$ in Table 4, and b_m and β_m in Table 5. The values for $\frac{k}{a_m}$, $\frac{k}{b_m}$, and b_m are given without proof or derivation because their determination is straightforward. The expressions for $\frac{k}{\alpha_m}$, $\frac{k}{\beta_m}$ and β_m are derived in Appendix C.

TABLE 2. Values of the constant $\frac{k}{a_m}$

m →		0	1	2	3	4	5
k	0	1/2	-15/16	0	5/8	0	-3/16
↓	1	5/32	-7/32	-3/16	5/16	1/32	-3/32
	2	1/64	-1/64	-1/32	1/32	1/64	-1/64

TABLE 3. Values of the constant $\frac{k}{b_m}$

m →		0	1	2	3
k	0	0	35/8	-21/4	15/8
↓	1	0	-5/2	4	-3/2
	2	0	1/2	-1	1/2

Numerical evaluation of the above functions at points away from the element leads to considerable inaccuracies. These inaccuracies are the result of errors

TABLE 4. Expressions for the constants α_m^k and β_m^k

m	0	1	2	3	4
α_m^k	$\frac{2}{5}a_5 + \frac{2}{3}a_3 + 2a_1$	$\frac{2}{3}a_4 + 2a_2$	$\frac{2}{3}a_5 + 2a_3$	$2a_4$	$2a_5$
β_m^k	$2(b_0 + \frac{1}{3}b_1 + \frac{1}{5}b_2 + \frac{1}{7}b_3)$	$2(b_1 + \frac{1}{3}b_2 + \frac{1}{5}b_3)$	$2(b_2 + \frac{1}{3}b_3)$	$2b_3$	0

TABLE 5. Expressions for the constants b_m and β_m

m	0	1	2	3
b_m	1	-3	+3	-1
β_m	32/35	-22/5	16/3	-2

induced by subtracting large numbers. The function $p^k(Z_j)$ in (6.3), for example, has been determined such that it cancels the first term in (6.3) for large values of Z . But $p^k(Z_j)$ by itself is a 4th order polynomial and becomes large for large values of Z_j . In case $|Z_j|$ is larger than a given value (3 in the present program), it is better to replace Ω_j , Ω_1 and Ω_n by their asymptotic expansions, where the singular terms are removed analytically. The inaccuracy in the transition from the function to its asymptotic expansion is less than the machine accuracy of 6 digits when 12 terms are taken in the expansions. The series expansions are derived in Appendix C and are given below. The expansions of $F^k(Z_j)$ and $G^k(Z_j)$, see (6.3) and (6.4), are

$$F_j^k(Z_j) = \sum_{m=1}^{\infty} \gamma_m^k Z_j^{-m}, \quad G_j^k(Z_j) = \sum_{m=1}^{\infty} \gamma_m^k (-1)^{k+m+1} Z_j^{-m} \quad (|Z_j| > 1) \quad (6.13)$$

where

$$\gamma_m^k = \left[-1 - (-1)^n \right] \left[\frac{\frac{k}{a_1}}{n+1} + \frac{\frac{k}{a_3}}{n+3} + \frac{\frac{k}{a_5}}{n+5} \right] + \left[-1 + (-1)^n \right] \left[\frac{\frac{k}{a_0}}{n} + \frac{\frac{k}{a_2}}{n+2} + \frac{\frac{k}{a_4}}{n+4} \right] \quad (6.14)$$

The expansions of $F_1(\chi)$ and $G_1(\chi)$ have similar forms:

$$F_1(\chi) = \sum_{m=1}^{\infty} \delta_m \chi^{-m}, \quad G_1(\chi) = \sum_{m=1}^{\infty} \delta_m^k \chi^{-m} \quad (6.15)$$

where

$$\delta_m = -2 \left[\frac{b_0}{2m+1} + \frac{b_1}{2m+3} + \frac{b_2}{2m+5} + \frac{b_3}{2m+7} \right] \quad (6.16)$$

and

$$\delta_m^k = -2 \left[\frac{\frac{k}{b_1}}{2m+3} + \frac{\frac{k}{b_2}}{2m+5} + \frac{\frac{k}{b_3}}{2m+7} \right] \quad (6.17)$$

It may be noted that the above functions are all fully known; the constants listed are calculated once, at the beginning of the program.

6.2 THE COMPLEX POTENTIAL FOR THE CROSSING

A special complex potential, Ω_{cr} , is added for each crossing. the various functions involved in this complex potential are listed below, where the indices s and t refer to the two cracks. Ω_{cr} is given by (5.31),

$$\Omega_{cr} = \Omega_m + \Omega_a \quad (6.18)$$

where Ω_m is the main function, see (5.23)

$$\Omega_m = \frac{Q_0}{2\pi} \ln \left[\frac{Z_s + \sqrt{(Z_s-1)(Z_s+1)'}}{2(Z-Z_{s0})} \right] - \frac{Q_0}{2\pi} \ln \left[\frac{Z_t + \sqrt{(Z_t-1)(Z_t+1)'}}{2(Z-Z_{t0})} \right] \quad (6.19)$$

and Z_s is given by (5.8)

$$Z_s(z) = X_s(z) + iY_s(z) = \frac{z - \frac{1}{2}(z_{se} - z_{si})}{\frac{1}{2}(z_{se} - z_{si})} \quad (6.20)$$

The endpoints of the crack are z_{s1} and z_{se} and the intersection is at z_{s0} ,

$$Z_{s0} = Z_s(z_0) = X_{s0} \quad (Y_{s0} = 0) \quad (6.21)$$

The complex variables Z_t and Z_{t0} are defined similarly for the t crack. The auxiliary function Ω_a is given by (5.30),

$$\Omega_a = \frac{Q_0}{2\pi} \left[Z_s (Z_s - 2Z_{s0}) \sqrt{1 - Z_s^{-2}} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] -$$

$$- \frac{Q_0}{2\pi} \left[Z_t (Z_t - 2Z_{t0}) \sqrt{1 - Z_t^{-2}} - (Z_t - Z_{t0})^2 + (Z_{t0}^2 + \frac{1}{2}) \right] \quad (6.22)$$

Asymptotic expansions for Ω_s and Ω_s^* (see the first terms in (6.19) and (6.22), compare (5.13) and (5.24)) are derived in Appendix C and are:

$$\Omega_s + \Omega_s^* = \frac{Q_0}{2\pi} \ln \left[\frac{Z_s + \sqrt{(Z_s - 1)(Z_s + 1)}}{2(Z_s - Z_{s0})} \right] +$$

$$\frac{Q_0}{2\pi} \left[Z_s (Z_s - 2Z_{s0}) \sqrt{1 - Z_s^{-2}} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] = \sum_{m=1}^{\infty} C_m Z_s^{-m} \quad (6.23)$$

where

$$C_m = A_m + \frac{Z_{s0}^m}{m} - \frac{1 * 3 * 5 * \dots * (m-1)}{2 * 4 * 6 * \dots * m^2} ; \quad m=2j, \quad j=1,2,\dots \quad (6.24)$$

$$C_m = -2Z_{s0} A_m + \frac{Z_{s0}^m}{m} ; \quad m=2j+1, \quad j=0,1,2,\dots \quad (6.25)$$

and

$$A_m = (-1)^m \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-m+1)}{m!} \quad (6.26)$$

6.3 THE SYSTEM OF EQUATIONS

The parameters involved in the various functions are determined by solving a system of linear equations. The first two terms in the expression for the complex potential, (6.1), contain all unknowns and will be combined to one function Ω_c , so that

$$\Omega = \Omega_c + \Omega_g \quad (6.27)$$

First, the case will be considered that the cracks do not intersect. Then, if there are N cracks and if crack r consists of n_r elements, the total number of unknowns, N_u , is (compare (4.93))

$$N_u = \sum_{r=1}^N (3n_r - 1) \quad (6.28)$$

The N_u unknowns may be arranged in a one dimensional array M_j ($j = 1, 2, \dots, N_u$) by placing all of the unknowns for crack r , μ_1^0, μ_j^k ($k = 0, 1, 2$; $j = 2, \dots, n_r$) and $\mu_{n_r+1}^0$ behind those for crack $r-1$ ($r = 2, 3, \dots, N$). There are n_r control points z_j^c on these cracks. The first and last control points are chosen at fixed distances from the crack tips and the remaining $n_r - 2$ are distributed uniformly between the first and the last, see Figure 21. The boundary condition

Figure 21. Distribution of control points on a single crack

(4.55) is applied at each control point, which gives with (6.27),

$$D \frac{\partial \Phi_c}{\partial s} - (\Phi_c^+ - \Phi_c^-) = -D \frac{\partial \Phi_g}{\partial s} ; \quad z = z_j^c \quad (j = 1, 2, \dots, N_u) \quad (6.29)$$

where the differentiation is performed at z_j^c and in the direction of the crack. The Equations (6.29) may be represented in matrix form using a coefficient matrix A_{ij} as follows:

$$\sum_{j=1}^{N_u} A_{ij} M_j = B_i \quad (i = 1, 2, \dots, N_u) \quad (6.30)$$

where B_i represents the right hand side of (6.29). The elements of B_i are known. The parameters M_j are obtained upon inversion of (6.30).

When intersections occur, one unknown parameter, represented as Q_0 in (6.19) through (6.22), is added per crossing. There are n_c crossings and the parameter for crossing j , $j = 1, 2, \dots, n_c$ is represented as Q_{0j} . Two of the control points are removed for each crossing, one for each of the intersecting cracks, and are replaced by 3 integration paths as discussed in Sec. 5.1. Thus, the total number of equations becomes

$$N_u = \sum_{r=1}^N [3n_r - 1] - 2n_c + 3n_c = \sum_{r=1}^N [3n_r - 1] + n_c \quad (6.31)$$

The control points are redistributed over each crack such that the distances between control points are approximately equal, whereby the points of intersection are counted as control points. The total number of equations thus matches the total number of unknowns, and (6.30) becomes

$$\sum_{j=1}^{N_u - n_c} A_{ij} M_j + \sum_{j=N_u - n_c + 1}^{N_u} A_{ij} Q_j \quad (i = 1, 2, \dots, N_u) \quad (6.32)$$

The equations corresponding to $j = N_u - n_c + 1$ through $j = N_u$ are obtained by the application of the set of equations of the form (5.5) through (5.7) to each point of intersection, which yields,

$$\int_{\xi_1}^{z_0} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) ds + \int_{z_0}^{\xi_4} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) dt - \{\Phi_c(\xi_4) - \Phi_c(\xi_1)\} = \Phi_g(\xi_4) - \Phi_g(\xi_1) \quad (6.33)$$

$$\int_{\xi_1}^{z_0} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) ds + \int_{z_0}^{\xi_2} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) ds - \{\Phi_c(\xi_2) - \Phi_c(\xi_1)\} = \Phi_g(\xi_2) - \Phi_g(\xi_1) \quad (6.34)$$

$$\int_{\xi_1}^{z_0} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) ds + \int_{z_0}^{\xi_3} \frac{1}{D} (\mathcal{F}_c^+ - \mathcal{F}_c^-) dt - \{\Phi_c(\xi_3) - \Phi_c(\xi_1)\} = \Phi_g(\xi_3) - \Phi_g(\xi_1) \quad (6.35)$$

where ξ_1 , ξ_2 , ξ_3 and ξ_4 represent the endpoints of the integration path and z_0

is the point of intersection (see Figure 17 and Figure 18 in Sec. 5).

The thickness b of the cracks is presently taken to be constant along each crack between the first and last control points. From the tip to these points, however, the thickness varies as the $\frac{1}{2}$ th power of the distance from the tip for the linear law and as the $\frac{1}{6}$ th power for the cubic law (see (3.5) and (3.22)). Thus, each crack has its own single parameter b to define its thickness. It may be noted that the first and last control points are always chosen such that intersections do not occur between the tip of a crack and the first control point.

The integrations in (6.33) through (6.35) are performed analytically. For all elements other than the cross elements, these integrals may be reduced to integrals of polynomials that are readily obtained and will not be given here. For the cross-element these integrals are somewhat involved and are written below. Integration of $\mathcal{F}^+ - \mathcal{F}^-$ for either one of the complex potentials Ω_s and Ω_t in Ω_m , see (5.13), (5.22) and (5.23), gives along a section $X_A \leq X \leq X_B$,

$$\int_{X_A}^{X_B} (\mathcal{F}^+ - \mathcal{F}^-) ds = \frac{Q_0 L}{2\pi} \left[X(\theta_c - \theta_w^+) - \sqrt{1-X^2} \right] \Big|_{X_A}^{X_B} \quad (6.36)$$

where θ_c and θ_w are defined by (5.18) and θ_w^+ represents the value of θ_w along the + side of the element, which is either π or 0. The variable X may represent either X_s or X_t . Similar integration for the two functions, Ω_s^* and Ω_t^* involved in Ω_a gives

$$\int_{X_A}^{X_B} (\mathcal{F}^+ - \mathcal{F}^-) ds = \frac{Q_0 L}{2\pi} \left[-\frac{1}{3}(1-X)^{3/2} - X_0 \{ X \sqrt{1-X^2} + \arcsin X \} \right] \Big|_{X_A}^{X_B} \quad (6.37)$$

where X_0 may either be X_{s0} or X_{t0} .

All functions presented above are programmed as separate complex function subroutines in BACRACK, which are called when the coefficients of the matrix A_{ij} are determined. As is mentioned above, the functions are written in a different form after all matrix coefficients have been determined. For example, the function Ω_j in (6.2) is then written as

$$\Omega_j = h_j(Z_j) \ln \frac{Z_j - 1}{Z_j + 1} + p(Z_j) \quad (6.38)$$

where

$$h_j(Z_j) = \sum_{m=0}^5 A_m^j Z_j^m, \quad p(Z_j) = \sum_{m=0}^4 B_m^j Z_j^m \quad (6.39)$$

The six coefficients A_m^j and the five coefficients B_m^j are stored per element in order to reduce calculation times. It may be noted that the calculation of these coefficients is done after the system of linear equations has been solved so that the memory space used for A_{ij} has become available for storage of coefficients. The 6 asymptotic expansions of (6.38) are compressed into one in a similar fashion. The same procedure is applied to Ω_1 and Ω_n .

7 VERIFICATIONS

As was remarked earlier, the linear law for flow in the fissures is used primarily for the purpose of verification. When the linear law is used, the fissures may be viewed as zones of a high permeability. Flow in aquifers with discontinuities in the permeability can be modeled by the use of an analytic technique which is rather similar in principle to that presented here. Boundaries of inhomogeneities in the permeability are modeled by the use of line doublets, which account for the jumps in the potential Φ , caused by the discontinuity in the permeability across the boundaries of the cracks. The use of doublets has the advantage that the condition of continuity of flow across the boundary is met exactly. The complex potential for a line doublet is obtained by that for a line dipole by multiplication by a factor $-i$, and a function that incorporates a parabolic strength distribution for the line doublet is applied in SYLENS.

Comparison of the exact solution for a single crack with SYLENS has been discussed in Sec. 3. This comparison served to verify the assumption that the crack may be modeled as a slot of infinitesimal width. The first verification of the numerical model is to compare results for a single crack with the exact solution. The piezometric contours obtained from the model and from the exact solution appear to be indistinguishable and are not presented.

The second verification of the numerical model consists of a comparison with 8 runs done with SYLENS for 2 cases of intersecting cracks. For the first case, case A, the cracks are orthogonal and for the second case, case B, the cracks enclose an angle of 50° . The lengths of all cracks are 6 units and the width of each crack is 0.1 units so that

$$\frac{b_{\max}}{L} = 1/60 \quad (7.1)$$

The cracks are shown in Figure 22(a) for case A and in Figure 22(b) for case B. There is uniform flow of discharge q_0 [L/T] whose direction is defined by the angle γ as indicated in the Figure. The angle between the cracks is α . Four runs are made for each of the two cases and the data are presented in Table 6

TABLE 6. Data for case A

Case	α	$b_{\max}/L$	k^*/k	γ	q_0/k
A.1	90°	1/60	200	$+45^\circ$	1
A.2	90°	1/60	2000	$+45^\circ$	1
A.3	90°	1/60	200	-45°	1
A.4	90°	1/60	2000	-45°	1

Figure 22. The verification problems; cases A and B

TABLE 7. Data for case B

Case	α	$b_{\max}/L$	k^*/k	γ	q_0/k
B.1	50°	1/60	200	$+45^\circ$	1
B.2	50°	1/60	2000	$+45^\circ$	1
B.3	50°	1/60	200	0	1
B.4	50°	1/60	2000	0	1

for case A and in Table 7 for case B.

The total number of elements used in SYLENS is 80 for both cases. The elements are uniformly spaced along the boundaries of the cracks, except near the tips and near the intersections, where shorter elements are used than elsewhere; the lengths of the elements vary between 7 units and 25 units.

The results are produced in the form of plots of piezometric contours in Figures 23 through 30. The solid contours are obtained from SYLENS and the dashed ones from BACRACK. The runs of BACRACK are all done with 6 elements, 3 on each crack. It is seen from Figure 30 that an erroneous closed contour

occurs near one of the tips of the crack labeled as crack t in Figure 22(b). This contour disappears when more elements are used, as is shown in Figure 31, which is obtained from BACRACK with 8 elements, 4 on each crack. It is seen that the contours obtained from BACRACK compare well with those produced by SYLENS.

In addition to the contour plots, graphs are produced representing the piezometric levels along the two cracks for one case, case B.3. In order to illustrate the effect of the distribution of control points and the number of elements, runs are made for various distributions of control points in BACRACK, using first 3 and second 4 elements per crack. The distribution of control points is shown for the case of three elements per crack in Figure 32. The larger squares in the Figure represent the nodes; the four smaller squares in the neighborhood of the intersection are the endpoints of the integration paths. The control points are marked by dashes. The difference between Figure 32(a) and Figure 32(b) is the location of the first control point, which is much closer to the tip for (b) than for (a). Similar distributions of control points are applied for the case of 4 elements per crack.

The piezometric levels along the cracks are represented in Figure 33 and Figure 34. The cracks are the straight lines, with the coordinates s and t defined in Figure 22(b). Figure 33 corresponds to the crack labeled crack s in Figure 22(b), and Figure 34 corresponds to crack t. The plots labeled (a) are obtained with 3 elements per crack and those labeled (b) with 4. The solid curves represent piezometric levels obtained from SYLENS; the others are produced by BACRACK. The piezometric levels in SYLENS are obtained for points along the centerline of the crack, and the level is zero at the point of intersection. The dashed curves correspond to the cases that the first and last control points are chosen further away from the crack tips, see Figure 32(a); these control points are closer to the crack tips for the dashed curves, see Figure 32(b). It is apparent from the graphs that the results improve when the first and last control points are near the crack tips, and when the number of elements per crack is increased from 3 to 4.

As an example, contourlines are produced for a case of 9 cracks and 11 intersections in Figure 35; a detail is shown in Figure 36. The factor k^*b/k is 50 for cracks and q_0/k is 1. The cracks are modeled by using 5 elements, and the number of equations solved for this problem is $9 * (3n-1) + 11 = 137$ (see (6.32)).

Inaccuracies occur, particularly for high transmissivities of the cracks. The inaccuracies are due to the large variations in the magnitudes of the gradients throughout the flow domain. These variations are apparent particularly in Figure 35; the gradients inside the domains enclosed by cracks are much smaller than those outside these domains. Small errors in the magnitudes of the coefficients in the functions therefore will tend to cause

Figure 23. Comparison for case A.1; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 24. Comparison for case A.2; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 25. Comparison for case A.3; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 26. Comparison for case A.4; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 27. Comparison for case B.1; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 28. Comparison for case B.2; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 29. Comparison for case B.3; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 30. Comparison for case B.4; the solid lines are obtained from SYLENS and the dashed ones from BACRACK.

Figure 31. Piezometric contours obtained from BACRACK for case B.3 using 4 elements per crack.

(a)

(b)

Figure 32. Distribution of control points for case B.3 and three elements per crack.

Figure 33. Plot of ϕ versus s for 3 elements (a) and 4 elements (b); the solid lines are obtained from SYLENS.

Figure 34. Plot of ϕ versus s for 3 elements (a) and 4 elements (b); the solid lines are obtained from SYLENS.

large relative errors in piezometric levels. The projected implementation of finite boundaries is expected to eliminate most of these errors. The specification of piezometric heads at points where the cracks intersect boundaries will enforce a gradient along the cracks, whereas at present the gradients along the cracks are left free; the boundary condition is at infinity. The implementation of finite boundaries is therefore a desirable feature.

Figure 35. Piezometric contours for flow with 9 cracks and 11 intersections.

Figure 36. Piezometric contours for flow with 9 cracks and 11 intersections; detail.

8 ALTERNATIVE APPROACH

Prior to the development of BACRACK, another computer program, CRAPOS, was written by C. Detournay. CRAPOS is capable of modeling non-intersecting cracks in an infinite porous medium and is based upon the use of elements that are considerably simpler in terms of their complex potential than those used in BACRACK. This alternative approach has been abandoned primarily because it requires the use of series expansions of a much higher order than the fifth order polynomials used in BACRACK, which caused undesirable inaccuracies in CRAPOS. On a large computer, however, these inaccuracies may become less important, and the use of simpler elements may then be preferable. This alternative approach therefore is sketched briefly below.

A single main element is used in CRAPOS to replace the multiple element cracks presented in Sec. 4. This element is derived from the exact solution presented in Sec. 3, see (3.10), and has the form

$$\Omega = f(Z_j) \sqrt{(Z_j - 1)(Z_j + 1)} + p(Z_j) \quad (8.1)$$

where Z_j is defined by (4.2) and

$$f(Z_j) = \sum_{m=0}^n A_j Z_j^m, \quad p(Z_j) = \sum_{m=0}^{n-1} B_j Z_j^m \quad (8.2)$$

where the coefficients A_j and B_j are real. The function (8.1) has a behavior of the form (3.23) near both crack tips ($Z_j = 1$ and $Z_j = -1$) and the behavior along the element is given by (compare (3.20) and (3.21))

$$\begin{aligned} -1 \leq X_j \leq 1, \quad Y_j = 0^+; \quad \Phi = p(X_j) \\ -1 \leq X_j \leq 1, \quad Y_j = 0^+; \quad \Phi = f(X_j) \sqrt{1 - X_j^2} \end{aligned} \quad (8.3)$$

and

$$\begin{aligned} -1 \leq X_j \leq 1, \quad Y_j = 0^-; \quad \Phi = p(X_j) \\ -1 \leq X_j \leq 1, \quad Y_j = 0^-; \quad \Phi = -f(X_j) \sqrt{1 - X_j^2} \end{aligned} \quad (8.4)$$

The coefficients B_j may be expressed in terms of A_j after expanding (8.1) about $Z_j^{-1} = 0$ and by requiring that $\Omega(Z_j)$ is of order Z_j^{-1} at infinity. The internal boundary condition (4.55) is applied at N points of the crack for an order $N-1$ of the polynomial $f(Z_j)$. Asymptotic expansions are used at some distance away from the crack.

The disadvantage of this approach is that the order of the polynomial is proportional to the number of control points. For 14 constraint points, for

example, as occur for cracks modeled with 5 elements in BACRACK, the order of $f(Z_j)$ becomes 13. Thus, the number of control points is limited by the maximum order of the polynomial that can be handled by the computer system, while preserving a smooth transition between the functions and their asymptotic expansions.

REFERENCES

- Abramowitz, M. and I.A. Stegun. 1965. Handbook of Mathematical Functions. Dover Publications, New York.
- Aifantis, E.C. 1977. Introducing a multiporous medium, Developments in Mechanics 8:209 - 211.
- Aifantis, E.C. 1979. On the Response of Fissured Rock. Proceedings of the 16th Midwestern Mechanics Conference 10.
- Aifantis, E.C. 1980. On the Problem of Diffusion in Solids. Acta Mechanica 37:265 - 296.
- Barenblatt, G.I., Iu.P. Zheltov and I.N. Kochina. 1960. Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks. PPM 24 (5):852 - 864.
- Barenblatt, G.I., and Iu.P. Zheltov. 1960. On the Fundamental Equations of Seepage of Homogeneous Fluids in Jointed Rocks. Dokl. Akad. Nauk. SSR (3):545 - 548.
- Bear, J. and C. Braaster. 1972. On the Flow of Two Immiscible Fluids in Fractured Porous Media. Developments in Soil Science 2, Fundamentals of Transport Phenomena in Porous Media, 177 - 202, Elsevier, New York.
- De Josselin de Jong, G. and S.C. Way. 1972. Dispersion in Fissured Rock. Geoscience Department New Mexico Institute of Mining and Technology, Socorro, New Mexico.
- Duguid, Jr., J.O. 1973. Flow in Fractured Rock. Dissertation, Princeton University, New Jersey.
- Mushkhelishvili, N.I. 1958. Singular Integral Equations. Wolters - Noordhoff, Groningen, The Netherlands.
- Lamb, H. 1932. Hydrodynamics, Dover Publications, New York.
- Louis, C. 1974. Introduction de l'Hydraulique des Roches. Publication Bureau de Recherches Geologiques et Minieres, Orleans, France.
- Sneddon, I.N. 1951. Fourier Transforms. Mc Graw - Hill, New York.

- Strack, O.D.L., and H.M. Haitjema. 1981a. Modeling Double Aquifer Flow Using a Comprehensive Potential and Distributed Singularities, I, Solution for Homogeneous Permeability. Water Resour. Res. 17(5): 1529 - 1534.
- Strack, O.D.L., and H.M. Haitjema. 1981b. Modeling Double Aquifer Flow Using a Comprehensive Potential and Distributed Singularities, II, Solution for Inhomogeneous Permeability. Water Resour. Res. 17(5): 1535 - 1549.
- Verruijt, A. 1970. Theory of Groundwater Flow. Macmillan, London, England.
- Wilson, C.R. and P.A. Witherspoon. 1970. An Investigation of Laminar Flow in Porous Rocks, Soil Mech. Bituminous Mat. Res. Lab. Rept. 70-6, Dept. of Civil Eng. University of California, Berkeley, California.
- Witherspoon, P.A. et al. 1980. Validity of Cubic Law for Fluid Flow in a Deformable Rock Fracture. Water Resour. Res. 16(6): 1016 - 1024.
- Wittke, W. 1970. Three Dimensional Percolation of Regularly Fissured Rock. Proceedings of the Symposium Planning Open Pit Mines, Johannesburg, South Africa.

APPENDIX A EQUATION OF MOTION FOR FLOW IN THE FISSURES

The approximate equations for flow through the fissures will be derived below. It is assumed that the fissures are thin and saturated with water. The walls of the fissures are symmetric with respect to planes through the centers of the fissures, see Figure A.1. Cartesian coordinates x_1 , x_2 and x_3 are

Figure A.1. A crack.

introduced such that x_3 is vertical. Local coordinates x_i^* ($i=1,2,3$) are defined such that x_1^* and x_2^* are in the plane of the fissure.

The flow in the fissures is taken to be slow and viscous, so that the inertia terms in the Navier-Stokes equation may be neglected. Furthermore, the fluid is taken as incompressible, and the remaining terms in the Navier-Stokes equations are

$$\rho g_i^* - \partial_i^* p + \mu \partial_j^* \partial_j^* v_i^* = 0 \quad (\text{A.1})$$

where the Einstein-summation convention is adopted. In the above equation, ρ stands for the unit mass of water, g_i^* represents the components of the acceleration of gravity in the x_i^* system, p is the pressure, μ the dynamic viscosity and v_i^* denotes the components of the velocity vector in the x_i^* system. The symbol ∂_i^* is defined as

$$\partial_i^* = \frac{\partial}{\partial x_i^*} \quad (\text{A.2})$$

The term ρg_i^* may be written as

A.2

$$\rho g_i^* = -\rho g \partial_i^* x_3 \quad (\text{A.3})$$

and introduction of the piezometric head ϕ^* as

$$\phi^* = \frac{p}{\rho g} + x_3 \quad (\text{A.4})$$

in (A.1) yields

$$\rho g \partial_i^* \phi^* = \mu \partial_j^* \partial_j^* v_i \quad (\text{A.5})$$

The boundary surfaces of the fissure may be represented by

$$x_3^* = \pm \frac{1}{2}b (x_1^*, x_2^*) \quad (\text{A.6})$$

where b is small as compared to the extent of the fissures. With the assumptions listed above, the problem of viscous flow in the fissures is rather similar to the flow in a Hele-Shaw apparatus, where a viscous fluid flows in the narrow interspace between two parallel plates. According to Lamb [1932], the velocity component v_3^* may be neglected with respect to v_1^* and v_2^* , as well as the variation of ϕ^* in the x_3^* direction with respect to those in the x_1^* and x_2^* directions. Furthermore, the partial derivatives of v_a^* ($a=1,2$) with respect to x_1^* and x_2^* are negligible with respect to the partial derivatives in the x_3^* direction. The validity of these approximations for the Hele-Shaw analogue has been proven by many experiments; the results appear to be in rather good agreement with the approximate equations, even for large gradients in the plane of flow, such as occur near singular points in the model experiments.

With the above approximations, the Navier-Stokes equation (A.5) may be written as

$$\rho g \partial_a^* \phi^* = \mu \frac{\partial^2 v_a^*}{\partial (x_3^*)^2} \quad (a=1,2) \quad (\text{A.7})$$

Integration over the thickness of the fissure yields

$$\rho g \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \partial_a^* \phi^* dx_3^* = \mu \left[\left[\frac{\partial v_a^*}{\partial x_3^*} \right] (x_3^* = \frac{1}{2}b) - \left[\frac{\partial v_a^*}{\partial x_3^*} \right] (x_3^* = -\frac{1}{2}b) \right] \quad (a=1,2) \quad (\text{A.8})$$

The terms at the right of (A.8) represent the shear stresses at the boundaries

A.3

of the fissure. A parabolic velocity distribution is assumed, see Lamb [1932], pp 581 - 582,

$$v_a = \frac{6Q_a}{b^3} \left[\left(\frac{1}{2}b \right)^2 - (x_3^*)^2 \right] \quad (a = 1, 2) \quad (\text{A.9})$$

where Q_a represents the discharge vector defined as

$$Q_a = \int_{-\frac{1}{2}b}^{\frac{1}{2}b} v_a^* dx_3^* \quad (\text{A.10})$$

With (A.9) and (A.10), (A.8) becomes

$$\rho g \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \partial_a^* \phi^* dx_3^* = - \frac{12\mu}{b^2} Q_a \quad (\text{A.11})$$

The integral in (A.11) may be written as

$$\begin{aligned} & \rho g \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \partial_a^* \phi^* dx_3^* = \\ & \rho g \left[\partial_a^* \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \phi^* dx_3^* - \frac{1}{2} \left[(\phi^*)_{(x_3^* = \frac{1}{2}b)} + (\phi^*)_{(x_3^* = -\frac{1}{2}b)} \right] \partial_a^* b \right] \quad (a = 1, 2) \end{aligned} \quad (\text{A.12})$$

where use is made of Leibnitz's rule. With

$$\bar{\phi} = \frac{1}{b} \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \phi^* dx_3^* \quad (\text{A.13})$$

representing the average piezometric head, and making the approximation that

$$\bar{\phi} \sim \frac{1}{2} \left[(\phi^*)_{(x_3^* = \frac{1}{2}b)} + (\phi^*)_{(x_3^* = -\frac{1}{2}b)} \right] \quad (\text{A.14})$$

equation (A.12) becomes

$$\rho g \int_{-\frac{1}{2}b}^{\frac{1}{2}b} \partial_{\alpha}^* \phi^* dx_3^* = \rho g \left[\partial_{\alpha}^* (b \tilde{\phi}) - \tilde{\phi} \partial_{\alpha}^* b \right] = \rho g b \partial_{\alpha}^* \tilde{\phi} \quad (\alpha = 1, 2) \quad (A.15)$$

Thus, the discharge Q_{α} , see (A.11), becomes, approximately,

$$Q_{\alpha} = - \frac{\rho g b^3}{12\mu} \partial_{\alpha}^* \tilde{\phi} \quad (\alpha = 1, 2) \quad (A.16)$$

This equation is identical to that used by Hele-Shaw, and derived by Lamb [1932] in a slightly different manner, for the case that b is a constant. Introduction of a coefficient β^* , defined as

$$\beta^* = \rho g / (12\mu) \quad (A.17)$$

in (A.16) yields

$$Q_{\alpha} = - \beta^* b^3 \partial_{\alpha}^* \tilde{\phi} \quad (A.18)$$

The effect of roughness of rock face may be incorporated in an approximate fashion by modifying the expressions for v_{α}^* , see (A.9), by multiplying them by a factor less than 1. This will lead to a modified value of the coefficient β^* in (A.18).

APPENDIX B CONDITIONS OF CONTINUITY AT NODAL POINTS

Consider the behavior of the complex potential $\Omega(z)$ at a nodal point z_j of a crack of n elements, where $2 < j < n$. The complex potential may be written as (compare (4.10))

$$\begin{aligned} \Omega = \Omega^*(z) + \frac{1}{2\pi} \mu_{j-1} (Z_{j-1}) \ln \left[\frac{Z_{j-1} - 1}{Z_{j-1} + 1} \right] + \\ + \frac{1}{2\pi} \mu_j (Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] \end{aligned} \quad (B.1)$$

where $\Omega^*(z)$ is analytic at z_j and Z_j is defined by (4.2),

$$Z_j = \frac{z - \frac{1}{2}(z_j + z_{j+1})}{\frac{1}{2}(z_{j+1} - z_j)} \quad (B.2)$$

Functions $\tilde{\mu}_{j-1}(z)$ and $\tilde{\mu}_j(z)$ are introduced as follows:

$$\tilde{\mu}_{j-1}(z) = \mu_{j-1} [Z_{j-1}(z)] ; \quad \tilde{\mu}_j(z) = \mu_j [Z_j(z)] \quad (B.3)$$

Both functions $\tilde{\mu}_{j-1}(z)$ and $\tilde{\mu}_j(z)$ are analytic at z_j and thus may be expanded in the form of a Taylor series about z_j , and (B.1) may be written in the form:

$$\begin{aligned} \Omega = \Omega^*(z) + \frac{1}{2\pi} \left[\tilde{\mu}_{j-1}(z_j) + \tilde{\mu}'_{j-1}(z_j)(z - z_j) + \frac{1}{2!} \tilde{\mu}''_{j-1}(z_j)(z - z_j)^2 + \right. \\ \left. + \frac{1}{3!} \tilde{\mu}'''_{j-1}(z_j)(z - z_j)^3 + \dots \right] \ln \left[\frac{z - z_j}{z - z_{j-1}} \right] + \frac{1}{2\pi} \left[\tilde{\mu}_j(z_j) + \right. \\ \left. + \frac{1}{2!} \tilde{\mu}''_j(z_j)(z - z_j)^2 + \frac{1}{3!} \tilde{\mu}'''_j(z_j)(z - z_j)^3 + \dots \right] \ln \left[\frac{z - z_{j+1}}{z - z_j} \right] \end{aligned} \quad (B.4)$$

where the primes stand for differentiation with respect to z and where the arguments of the logarithms have been written in terms of z by the use of (B.2). Combining the singular logarithmic terms and defining $\tilde{\Omega}(z)$ as $\Omega^*(z)$ plus all other terms in (B.4) that are analytic at z_j , (B.4) becomes

B.2

$$\begin{aligned} \Omega = \tilde{\Omega}(z) + \frac{1}{2\pi} & \left[\{ \tilde{\mu}_{j-1}(z_j) - \tilde{\mu}_j(z_j) \} + \{ \tilde{\mu}'_{j-1}(z_j) - \tilde{\mu}'_j(z_j) \} (z - z_j) + \right. \\ & \left. \frac{1}{2} \{ \tilde{\mu}''_{j-1}(z_j) - \tilde{\mu}''_j(z_j) \} (z - z_j)^2 + \frac{1}{3!} \{ \tilde{\mu}'''_{j-1}(z_j) - \tilde{\mu}'''_j(z_j) \} (z - z_j)^3 + \dots \right] \\ & * \ln(z - z_j) \end{aligned} \quad (\text{B.5})$$

Hence, if the functions $\tilde{\mu}_{j-1}$ and $\tilde{\mu}_j$ are chosen such that

$$\tilde{\mu}_{j-1}(z_j) = \tilde{\mu}_j(z_j) \quad (\text{B.6})$$

$$\tilde{\mu}'_{j-1}(z_j) = \tilde{\mu}'_j(z_j) \quad (\text{B.7})$$

$$\tilde{\mu}''_{j-1}(z_j) = \tilde{\mu}''_j(z_j) \quad (\text{B.8})$$

then $\Omega = \Omega(z)$ becomes

$$\Omega = \tilde{\Omega}(z) + \frac{1}{2\pi} \left[\frac{1}{3!} \{ \tilde{\mu}'''_{j-1}(z_j) - \tilde{\mu}'''_j(z_j) \} (z - z_j)^3 + \dots \right] \ln(z - z_j) \quad (\text{B.9})$$

This function, along with its first and second order derivatives is finite and continuous at z_j . Using (B.3), (B.6) through (B.8) may be written in terms of the functions μ_{j-1} and μ_j of the local variables Z_{j-1} and Z_j as follows, noticing that $Z_{j-1}(z_j) = 1$ and $Z_j(z_j) = -1$,

$$\mu_{j-1}(1) = \mu_j(-1) \quad (\text{B.10})$$

$$\mu'_{j-1}(1) \frac{dZ_{j-1}}{dz} = \mu'_j(-1) \frac{dZ_j}{dz} \quad (\text{B.11})$$

$$\mu''_{j-1}(1) \left[\frac{dZ_{j-1}}{dz} \right]^2 = \mu''_j(-1) \left[\frac{dZ_j}{dz} \right]^2 \quad (\text{B.12})$$

where the prime stands for differentiation with respect to the local variables, and where it is used that d^2Z_{j-1}/dz^2 and d^2Z_j/dz^2 vanish. With (B.2), (B.10) through (B.12) become, using that $z_{j+1} - z_j = L_j e^{i\alpha}$, see (4.4),

$$\mu_{j-1}(1) = \mu_j(-1) \quad (\text{B.13})$$

$$\frac{2}{L_{j-1}} \mu'_{j-1}(1) = \frac{2}{L_j} \mu'_j(-1) \quad (\text{B.14})$$

$$\frac{4}{L_{j-1}^2} \mu''_{j-1}(1) = \frac{4}{L_j^2} \mu''_j(-1) \quad (\text{B.15})$$

These equations are in agreement with (4.61) through (4.63).

The continuity conditions at node z_2 are obtained in a similar fashion. The complex potential is now written in the form (compare (4.33))

$$\Omega = \Omega^*(z) + \frac{1}{2\pi}\mu_1(\chi_1) \ln \left[\frac{\sqrt{\chi_1'} - 1}{\sqrt{\chi_1'} + 1} \right] + \frac{1}{2\pi}\mu_2(Z_2) \ln \left[\frac{Z_2 - 1}{Z_2 + 1} \right] \quad (\text{B.16})$$

where χ_1 is defined by (4.27),

$$\chi_1 = \frac{z - z_1}{z_2 - z_1} \quad (\text{B.17})$$

Equation (B.16) may be written as:

$$\begin{aligned} \Omega = \Omega^*(z) + \frac{1}{2\pi}\mu_1(\chi_1) \ln \left[\frac{\sqrt{z - z_1'} - \sqrt{z_2 - z_1'}}{\sqrt{z - z_1'} + \sqrt{z_2 - z_1'}} \right] + \\ + \frac{1}{2\pi}\mu_2(Z_2) \ln \left[\frac{z - z_3}{z - z_2} \right] \end{aligned} \quad (\text{B.18})$$

Writing $\ln(z - z_2)$ as

$$\begin{aligned} \ln(z - z_2) &= \ln[z - z_1 - (z_2 - z_1)] = \\ &= \ln \left[\left[\sqrt{z - z_1'} - \sqrt{z_2 - z_1'} \right] \left[\sqrt{z - z_1'} + \sqrt{z_2 - z_1'} \right] \right] \end{aligned} \quad (\text{B.19})$$

all terms of (B.18) that are analytic at z_2 may be combined in a function $\tilde{\Omega}(z)$ so that (B.18) takes the form

$$\Omega = \tilde{\Omega}(z) + \frac{1}{2\pi} [\mu_1(\chi_1) - \mu_2(Z_2)] \ln \left[\sqrt{z - z_1'} - \sqrt{z_2 - z_1'} \right] \quad (\text{B.20})$$

By a reasoning similar to the one followed for the interior elements, it is seen, noticing that μ_1 is analytic at z_2 , that Ω along with its first and second derivatives is finite and continuous at z_2 if the following conditions are met:

$$\tilde{\mu}_1(z_2) = \tilde{\mu}_2(z_2) \quad (\text{B.21})$$

$$\tilde{\mu}_1'(z_2) = \tilde{\mu}_2'(z_2) \quad (\text{B.22})$$

$$\tilde{\mu}_1''(z_2) = \tilde{\mu}_2''(z_2) \quad (\text{B.23})$$

where the primes stand for differentiation with respect to z , $\tilde{\mu}_2(z)$ is

defined by (B.3), and

$$\tilde{\mu}_1(z) = \mu_1[\chi_1(z)] \quad (\text{B.24})$$

With (B.3) and (B.17), (B.21) through (B.23) may be written in terms of the local variables χ_1 and Z_2 ,

$$\mu_1(1) = \mu_2(-1) \quad (\text{B.25})$$

$$\mu_1'(1) \frac{d\chi_1}{dz} = \mu_2'(-1) \frac{dZ_2}{dz} \quad (\text{B.26})$$

$$\mu_1''(1) \left[\frac{d\chi_1}{dz} \right]^2 = \mu_2''(-1) \left[\frac{dZ_2}{dz} \right]^2 \quad (\text{B.27})$$

where it is used that $d^2\chi_1/dz^2 = 0$ and $d^2Z_2/dz^2 = 0$. These equations can be written in the form (4.74) through (4.76) by the use of (B.2) and (B.17).

The conditions of continuity at node z_n are obtained in a similar fashion, and are (compare Figure 16)

$$\mu_n(1) = \mu_{n-1}(1) \quad (\text{B.28})$$

$$\mu_n'(1) \frac{d\chi_n}{dz} = \mu_{n-1}'(1) \frac{dZ_{n-1}}{dz} \quad (\text{B.29})$$

$$\mu_n''(1) \left[\frac{d\chi_n}{dz} \right]^2 = \mu_{n-1}''(1) \left[\frac{dZ_{n-1}}{dz} \right]^2 \quad (\text{B.30})$$

where (compare (4.52))

$$\chi_n = - (z - z_{n+1}) \frac{e^{-ia}}{L_n} \quad (\text{B.31})$$

so that

$$\frac{d\chi_n}{dz} = - \frac{e^{-ia}}{L_n} \quad (\text{B.32})$$

With (B.2) and (B.32), Equations (B.28) through (B.30) may be written in the form (4.88) through (4.90).

APPENDIX C ASYMPTOTIC EXPANSIONS

The coefficients of the polynomials $\overset{k}{P}(Z_j)$, $\overset{k}{Q}(Z_j)$, $P_1(\chi_1)$ and $\overset{k}{Q}_1(\chi_1)$, listed in Sec. 6, will be derived in this Appendix, as well as the asymptotic expansions of the functions defined in Secs. 4 and 5.

THE COEFFICIENTS OF THE POLYNOMIALS $\overset{k}{P}$ AND $\overset{k}{Q}$

The functions $\overset{k}{F}$ involved in the internal elements of a crack will be considered first. The function (6.3),

$$\overset{k}{F}(Z_j) = \overset{k}{f}(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] + \overset{k}{P}(Z_j) \quad (C.1)$$

is written as

$$\overset{k}{F}(Z_j) = \overset{k}{\rho}(Z_j) + \overset{k}{P}(Z_j) \quad (C.2)$$

where

$$\overset{k}{\rho}(Z_j) = \overset{k}{f}(Z_j) \ln \left[\frac{Z_j - 1}{Z_j + 1} \right] \quad (C.3)$$

and

$$\overset{k}{f}(Z_j) = \sum_{m=0}^s \overset{k}{a}_m Z_j^m \quad (C.4)$$

see (6.5). Equation (C.3) may be written as

$$\overset{k}{\rho}(Z_j) = Z_j^s \sum_{m=0}^s \overset{k}{a}_{s-m} Z_j^{-m} \ln \left[\frac{1 - Z_j^{-1}}{1 + Z_j^{-1}} \right] \quad (C.5)$$

Introduction of functions $\overset{k}{u}(Z_j^{-1})$ and $\overset{k}{v}(Z_j^{-1})$ as

$$\overset{k}{u}(Z_j^{-1}) = \sum_{m=0}^s \overset{k}{a}_{s-m} (Z_j^{-1})^m, \quad \overset{k}{v}(Z_j^{-1}) = \ln \left[\frac{1 - Z_j^{-1}}{1 + Z_j^{-1}} \right] \quad (C.6)$$

in (C.5) yields

$$\rho^k(Z_j^{-1}) = Z_j^s h^k(Z_j^{-1}) \quad (C.7)$$

where

$$h^k(Z_j^{-1}) = u^k(Z_j^{-1}) v(Z_j^{-1}) \quad (C.8)$$

The function $h^k(Z_j^{-1})$ is analytic at $Z_j^{-1} = 0$ and therefore may be expanded in a Taylor series about $Z_j^{-1} = 0$ so that (C.7) may be written as

$$\rho^k(Z_j) = Z_j^s \left[h^k(0) + h'^k(0) Z_j^{-1} + \frac{h''^k(0)}{2} Z_j^{-2} + \dots + \frac{h^{(m)k}(0)}{m!} Z_j^{-m} + \dots \right] \quad (C.9)$$

The objective is to express the general coefficient $h^{(n)k}/n!$ in this series in terms of the known coefficients a_m^k . It follows from the definition (C.8) that

$$\frac{h^{(m)k}}{m!} = \frac{(uv)^{(m)k}}{m!} \quad (C.10)$$

Application of Leibnitz's rule for higher derivatives of products gives

$$\frac{h^{(m)k}}{m!} = \frac{1}{m!} \left[\frac{k^{(m)}}{u} v + \binom{m}{1} \frac{k^{(m-1)}}{u} v' + \dots + \binom{m}{n} \frac{k^{(m-n)}}{u} v^{(n)} + \dots + \frac{k}{u} v^{(m)} \right] \quad (C.11)$$

With

$$\binom{m}{n} = \frac{m!}{n! (m-n)!} \quad (C.12)$$

(C.11) becomes

$$\frac{h^{(m)k}}{m!} = \frac{k^{(m)}}{u} \frac{1}{m!} v + \frac{k^{(m-1)}}{u} \frac{1}{(m-1)!} v' + \dots + \frac{k^{(m-n)}}{u} \frac{1}{(m-n)!} \frac{v^{(n)}}{n!} + \dots + \frac{k}{u} \frac{v^{(m)}}{m!} \quad (C.13)$$

It follows from (C.6) that

$$\frac{k^{(m)}(0)}{m!} = a_{s-m}^k; \quad 0 \leq m \leq 5 \quad (C.14)$$

$$\frac{k^{(m)}(0)}{m!} = 0; \quad m > 5 \quad (C.15)$$

and from the well-known expansion of $\ln[(1-x)/(1+x)]$ about $x=0$ that

$$\frac{v^{(m)}(0)}{m!} = \frac{-2}{m} ; \quad m = 2j+1, \quad j=0,1,2,3,\dots \quad (C.16)$$

$$\frac{v^{(m)}(0)}{m!} = 0 ; \quad m=2j, \quad j=0,1,2,3,\dots \quad (C.17)$$

It thus follows that, for $0 \leq m \leq 5$,

$$\frac{k^{(m)}(0)}{m!} = -2 \left[\frac{k}{a_6 - m} + \frac{1}{3} \frac{k}{a_8 - m} + \frac{1}{5} \frac{k}{a_{10} - m} + \dots \right] \quad 0 < m \leq 5 \quad (a_j = 0, j > 5) \quad (C.18)$$

where use is made of (C.13) through (C.17). Hence,

$$\begin{aligned} \frac{k^{(0)}}{h^{(0)}} &= 0, \quad \frac{k'}{h'} = -2a_s, \quad \frac{k''}{h''} = -2a_4, \quad \frac{k^{(3)}}{h^{(3)}} = -2(a_s + \frac{1}{3}a_s), \\ \frac{k^{(4)}}{h^{(4)}} &= -2(a_2 + \frac{1}{3}a_4), \quad \frac{k^{(5)}}{h^{(5)}} = -2(a_1 + \frac{1}{3}a_3 + \frac{1}{5}a_s); \quad Z_j^{-1} = 0 \end{aligned} \quad (C.19)$$

The function $\rho^k(Z_j)$, defined by (C.5) and represented as (C.7) thus has the following expansion about infinity

$$\rho^k(Z_j) = +Z_j^s \sum_{m=0}^s \frac{k^{(m)}(0)}{m!} Z_j^{-m} + O(Z_j^{-1}) \quad (C.20)$$

or

$$\rho^k(Z_j) = - \sum_{m=0}^4 \frac{k}{a_m} Z_j^m + O(Z_j^{-1}) \quad (C.21)$$

where 0 stands for 'order of' and where

$$\begin{aligned} \frac{k}{a_s} &= -\frac{k^{(0)}}{h^{(0)}} = 0, \quad \frac{k}{a_4} = -\frac{k'}{h'} = 2a_s, \quad \frac{k}{a_3} = -\frac{k''}{h''} = 2a_4, \\ \frac{k}{a_2} &= -\frac{k^{(3)}}{h^{(3)}} = 2(a_s + \frac{1}{3}a_s), \quad \frac{k}{a_1} = -\frac{k^{(4)}}{h^{(4)}} = 2(a_2 + \frac{1}{3}a_4), \\ \frac{k}{a_0} &= -\frac{k^{(5)}}{h^{(5)}} = 2(a_1 + \frac{1}{3}a_3 + \frac{1}{5}a_s) \end{aligned} \quad (C.22)$$

These expressions are listed in Table 4, Sec. 6.

The function $\overset{k}{P}(Z_j)$ is defined such that it combines with $\overset{k}{\rho}(Z_j)$ to render a function $\overset{k}{F}(Z_j)$ that is of order Z_j^{-1} near infinity, see (C.2). Hence, $\overset{k}{P}(Z_j)$ must be of the form

$$\overset{k}{P}(Z_j) = \sum_{m=0}^4 \overset{k}{a}_m Z_j^m \quad (C.23)$$

where it is used that $\overset{k}{a}_5 = 0$, see (C.22).

The coefficients of $\overset{k}{Q}(Z_j)$, involved in $\overset{k}{G}(Z_j)$ defined by (6.4), are determined as follows. The coefficients in the polynomial $\overset{k}{g}(Z_j)$, see (6.5), are obtained from the $\overset{k}{a}_m$ in $\overset{k}{f}(Z_j)$ by multiplication by a factor $(-1)^{m+k}$. It appears from (C.22) upon application of the above factor that

$$\overset{k}{Q}(Z_j) = \sum_{m=0}^4 \overset{k}{a}_m (-1)^{m+k+1} \quad (C.24)$$

ASYMPTOTIC EXPANSIONS OF $\overset{k}{F}$ AND $\overset{k}{G}$

All terms of an order larger than Z_j^{-1} cancel in the expansion of $\overset{k}{F}(Z_j)$ about $Z_j = \infty$. Hence, the remaining terms in this asymptotic expansion are:

$$\overset{k}{F}(Z_j) = \overset{k}{\rho}(Z_j) + \overset{k}{P}(Z_j) = \sum_{m=6}^{\infty} \frac{\overset{k}{h}^{(m)}(0)}{m!} Z_j^{s-m} \quad (C.25)$$

Using (C.13), (C.15), (C.16) and (C.17), the following expressions are obtained

$$\frac{\overset{k}{h}^{(m)}(0)}{m!} = -2 \left[\frac{\overset{k}{a}_1}{m-4} + \frac{\overset{k}{a}_3}{m-2} + \frac{\overset{k}{a}_5}{m} \right] ; \quad m=2j+1, \quad j=3,4,\dots \quad (C.26)$$

and

$$\frac{\overset{k}{h}^{(m)}(0)}{m!} = -2 \left[\frac{\overset{k}{a}_0}{m-5} + \frac{\overset{k}{a}_2}{m-3} + \frac{\overset{k}{a}_4}{m-1} \right] ; \quad m=2j, \quad j=3,4,5,\dots \quad (C.27)$$

C.5

Introduction of new coefficients γ_n^k with

$$\gamma_m^k = \frac{h^{(m+s)}(0)}{(m+5)!} \quad (C.28)$$

or

$$\gamma_m^k = -2 \left[\frac{a_1^k}{m+1} + \frac{a_3^k}{m+3} + \frac{a_5^k}{m+5} \right] ; \quad m=2j, \quad j=1,2,\dots \quad (C.29)$$

and

$$\gamma_m^k = -2 \left[\frac{a_0^k}{m} + \frac{a_2^k}{m+2} + \frac{a_4^k}{m+4} \right] ; \quad m=2j+1, \quad j=0,1,2,\dots \quad (C.30)$$

in (C.25) yields

$$F(Z_j) = \sum_{m=1}^{\infty} \gamma_m^k Z_j^{-m} \quad (C.31)$$

The expansion for $G(Z_j)$ is obtained in a similar fashion and becomes (compare (C.24))

$$G(Z_j) = \sum_{m=1}^{\infty} \gamma_m^k (-1)^{k+m+1} Z_j^{-m} \quad (C.32)$$

THE COEFFICIENTS OF THE POLYNOMIALS $P_1(\lambda_1)$ AND $Q_1(\lambda_1)$

The functions involved in the first element of a crack are given by (6.7) through (6.11). The function (6.8),

$$F_1(\lambda_1) = + f_1(\lambda_1) \ln \left[\frac{\sqrt{\lambda_1'} - 1}{\sqrt{\lambda_1'} + 1} \right] + P_1(\lambda_1) \quad (C.33)$$

will be written as

$$F_1(\lambda_1) = \rho_1(\lambda_1) + P_1(\lambda_1) \quad (C.34)$$

where

$$\rho_1(\lambda_1) = f_1(\lambda_1) \ln \left[\frac{\sqrt{\lambda_1} - 1}{\sqrt{\lambda_1} + 1} \right] \quad (C.35)$$

and

$$f_1(\lambda_1) = \sum_{m=0}^3 b_m \lambda_1^{m + \frac{1}{2}} \quad (C.36)$$

Equation (C.36) may be written as

$$f_1(\lambda_1) = \lambda_1^{1/2} \sum_{m=0}^3 b_m \lambda_1^m \quad (C.37)$$

or, with

$$\zeta = \lambda_1^{1/2} \quad (C.38)$$

$f_1(\lambda_1)$ may be written as

$$f_1(\lambda_1) = \zeta \sum_{m=0}^3 b_m \zeta^{2m} = \zeta' \sum_{m=0}^3 b_m \zeta^{2m-6} = \zeta' \sum_{m=0}^3 b_{3-m} \zeta^{-2m} \quad (C.39)$$

so that (C.35) becomes

$$\rho_1(\lambda_1(\zeta)) = \rho_1^*(\zeta) = \zeta' \sum_{m=0}^3 b_{3-m} \zeta^{-2m} \ln \left[\frac{1 - \zeta^{-1}}{1 + \zeta^{-1}} \right] \quad (C.40)$$

This equation has a form now that is similar to the one discussed for the interior elements, and can be treated in the same fashion. Defining $u_1(\zeta^{-1})$ and $v(\zeta^{-1})$ as

$$u_1(\zeta^{-1}) = \sum_{m=0}^3 b_{3-m} \zeta^{-2m}, \quad v(\zeta^{-1}) = \ln \left[\frac{1 - \zeta^{-1}}{1 + \zeta^{-1}} \right] \quad (C.41)$$

it follows that the function $h_1(\zeta^{-1})$, with

$$h_1(\zeta^{-1}) = u_1(\zeta^{-1}) v(\zeta^{-1}) \quad (C.42)$$

may be expanded about $\zeta^{-1} = 0$ as

$$h_1(\zeta^{-1}) = h_1(0) + h_1'(0) \zeta^{-1} + \frac{h_1''(0)}{2!} \zeta^{-2} + \dots + \frac{h_1^{(n)}(0)}{n!} \zeta^{-n} + \dots \quad (C.43)$$

The first seven coefficients $h(0), \dots, h^{(7)}(0)$ are obtained by the same approach as that followed previously; compare (C.11) through (C.18), where it is noted that all derivatives with respect to ζ^{-1} of u of order higher than 7 vanish at $\zeta^{-1} = 0$. These derivatives of $u_1(\zeta^{-1})$ are, at $\zeta^{-1} = 0$,

$$\begin{aligned} u_1 = b_3, \quad u_1' = 0, \quad \frac{u_1''}{2} = b_2, \quad \frac{u_1^{(3)}}{3!} = 0, \quad \frac{u_1^{(4)}}{4!} = b_1, \\ \frac{u_1^{(5)}}{5!} = 0, \quad \frac{u_1^{(6)}}{6!} = b_0; \quad \zeta^{-1} = 0 \end{aligned} \quad (C.44)$$

and the derivatives of $v(\zeta^{-1})$ are given by (C.16) and (C.17). Application of (C.13) gives, replacing h by h_1 and u by u_1 , and using (C.44),

$$\begin{aligned} h_1 = 0, \quad h_1' = -2b_3, \quad \frac{h_1''}{2} = 0, \quad \frac{h_1^{(3)}}{3!} = -2(b_2 + \frac{1}{3}b_3), \\ \frac{h_1^{(4)}}{4!} = 0, \quad \frac{h_1^{(5)}}{5!} = -2(b_1 + \frac{1}{3}b_2 + \frac{1}{5}b_3), \quad \frac{h_1^{(6)}}{6!} = 0 \\ \frac{h_1^{(7)}}{7!} = -2(b_0 + \frac{1}{3}b_1 + \frac{1}{5}b_2 + \frac{1}{7}b_3); \quad \zeta^{-1} = 0 \end{aligned} \quad (C.45)$$

The function $\rho_1(\chi_1)$, see (C.40), thus has the following free-field expansion

$$\rho_1 = \zeta^7 \sum_{m=0}^7 \frac{h_1^{(m)}(0)}{m!} \zeta^{-m} + O(\zeta^{-2}) \quad (C.46)$$

or, including only the non-zero terms,

$$\rho_1 = - \sum_{m=0}^3 \beta_m \zeta^{+2m} + O(\zeta^{-2}) \quad (C.47)$$

where

$$\begin{aligned} \beta_3 = 2b_3, \quad \beta_2 = 2(b_2 + \frac{1}{3}b_3), \quad \beta_1 = 2(b_1 + \frac{1}{3}b_2 + \frac{1}{5}b_3), \\ \beta_0 = 2(b_0 + \frac{1}{3}b_1 + \frac{1}{5}b_2 + \frac{1}{7}b_3) \end{aligned} \quad (C.48)$$

The polynomial $P_1(\chi_1)$ is defined such that it cancels all terms of $\rho_1(\chi_1)$ of order higher than χ_1^{-1} . Therefore, $P_1(\chi_1)$ is of the form

$$P_1(\lambda_1) = \sum_{m=0}^3 \beta_m \zeta^{2m} = \sum_{m=0}^3 \beta_m \lambda_1^m \quad (C.49)$$

where use is made of (C.38).

The function $\overset{k}{G}_1(\lambda_1)$ has the same form as $F_1(\lambda_1)$, and the polynomial $Q_1(\lambda_1)$ is

$$Q_1(\lambda_1) = \sum_{m=0}^3 \overset{k}{\beta}_m \lambda_1^m \quad (C.50)$$

where

$$\begin{aligned} \overset{k}{\beta}_3 &= 2b_3, \quad \overset{k}{\beta}_2 = 2(b_2 + \tfrac{1}{3}b_3), \quad \overset{k}{\beta}_1 = 2(b_1 + \tfrac{1}{3}b_2 + \tfrac{1}{5}b_3), \\ \overset{k}{\beta}_0 &= 2(b_0 + \tfrac{1}{3}b_1 + \tfrac{1}{5}b_2 + \tfrac{1}{7}b_3) \end{aligned} \quad (C.51)$$

which is analogous to (C.48).

THE ASYMPTOTIC EXPANSIONS OF $F_1(\lambda_1)$ AND $\overset{k}{G}_1(\lambda_1)$

All terms of order higher than λ_1^{-1} cancel in the expansions of $F_1(\lambda_1)$ and $\overset{k}{G}_1(\lambda_1)$ about $\lambda_1 = \infty$. The remaining terms in the expansion of $F_1(\lambda_1)$ are those of $\rho_1(\lambda_1)$, see (C.46), with $m > 7$,

$$F_1(\lambda_1(\zeta)) = \rho_1 = \zeta^7 \sum_{m=8}^{\infty} \frac{h_1^{(m)}(0)}{m!} \zeta^{-m} \quad (C.52)$$

All derivatives of u_1 of order higher than 7 vanish for $\zeta^{-1} = 0$. Using this, (C.44), (C.16), (C.17), and (C.13) with h_1 replacing h and u_1 replacing u , the following expression is obtained for $h_1^{(m)}(0)/m!$,

$$\frac{h_1^{(m)}(0)}{m!} = -2 \left[\frac{b_0}{m-6} + \frac{b_1}{m-4} + \frac{b_2}{m-2} + \frac{b_3}{m} \right]; \quad m=2j+1, \quad j=3,4,5,\dots \quad (C.53)$$

$$h_1^{(m)}(0) = 0; \quad m=2j, \quad j=4,5,\dots \quad (C.54)$$

Thus, (C.52) may be written as

$$F_1(\chi_1(\zeta)) = \sum_{m=1}^{\infty} \frac{h_1^{(m)}(0)}{m!} \zeta^{-m} \quad (C.55)$$

In view of (C.53) and (C.54) this reduces to

$$F_1(\chi_1) = \sum_{m=1}^{\infty} \frac{h_1^{(2m+1)}(0)}{(2m+1)!} \zeta^{-2m} = \sum_{m=1}^{\infty} \delta_m \chi_1^{-m} \quad (C.56)$$

where

$$\delta_m = \frac{h_1^{(2m+1)}(0)}{(2m+1)!} = -2 \left[\frac{b_0}{2m+1} + \frac{b_1}{2m+3} + \frac{b_2}{2m+5} + \frac{b_3}{2m+7} \right] \quad (C.57)$$

The asymptotic expansion of $G_1^k(\chi_1)$ is analogous,

$$G_1^k(\chi_1) = \sum_{m=1}^{\infty} \delta_m^k \chi_1^{-m} \quad (C.58)$$

where δ_m^k is obtained from (C.57) by replacing b_i by b_i^k ($i=0,1,2,3$) and by noticing that $b_0^k=0$, see Table 3, Sec. 6,

$$\delta_m^k = -2 \left[\frac{b_1^k}{2m+3} + \frac{b_2^k}{2m+5} + \frac{b_3^k}{2m+7} \right] \quad (C.59)$$

THE CROSS FUNCTION

Asymptotic expansions for both the main element, Ω_s , and the auxiliary element, Ω_s^* will be presented below. The function Ω_s , see (5.13), has the form

$$\Omega_s = \frac{Q_0}{2\pi} \ln \left[\frac{Z_s + \sqrt{(Z_s-1)(Z_s+1)'}}{2(Z_s - Z_{s0})} \right] \quad (C.60)$$

This function may be written as follows:

$$\Omega_s = \frac{Q_0}{2\pi} \ln \left[Z_s + \sqrt{Z_s^2 - 1} \right] - \frac{Q_0}{2\pi} \ln [2(Z_s - Z_{s0})] \quad (C.61)$$

or

$$\Omega_s = \frac{Q_0}{2\pi} \operatorname{arccosh}(Z_s) - \frac{Q_0}{2\pi} \ln[2(Z_s - Z_{s0})] \quad (C.62)$$

The expansion of $\operatorname{arccosh}(Z)$ about $Z = \infty$ is well-known (see for example Abramowitz and Stegun [1965]) and has the form

$$\operatorname{arccosh}(Z_s) = \ln(2Z_s) - \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2m)^2} Z_s^{-2m} \quad (|Z_s| > 1) \quad (C.63)$$

Equation (C.61) may be written as

$$\Omega_s = \frac{Q_0}{2\pi} \left[-\ln \left[1 - \frac{Z_{s0}}{Z_s} \right] + [\operatorname{arccosh}(Z_s) - \ln(2Z_s)] \right] \quad (C.64)$$

or, expanding the first term about $Z_{s0}/Z_s = 0$ and applying (C.63) to the second,

$$\Omega_s = \frac{Q_0}{2\pi} \sum_{m=1}^{\infty} \left[\frac{1}{m} \left[\frac{Z_{s0}}{Z_s} \right]^m - \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2m)^2} Z_s^{-2m} \right] \quad (|Z_s| > 1) \quad (C.65)$$

The function Ω_s^* , see (5.24), has the form

$$\Omega_s^* = \frac{Q_0}{2\pi} \left[(Z_s - 2Z_{s0}) \sqrt{(Z_s - 1)(Z_s + 1)} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] \quad (C.66)$$

which may be written as follows:

$$\Omega_s^* = \frac{Q_0}{2\pi} \left[Z_s^2 \sqrt{1 - Z_s^{-2}} - 2Z_{s0}Z_s \sqrt{1 - Z_s^{-2}} - (Z_s - Z_{s0})^2 + (Z_{s0}^2 + \frac{1}{2}) \right] \quad (C.67)$$

The expansion of $(1 - Z_s^{-2})^{1/2}$ about $Z_s^{-2} = 0$ is, see Abramowitz and Stegun [1965],

$$\sqrt{1 - Z_s^{-2}} = 1 + \sum_{m=1}^{\infty} A_m Z_s^{-2m} \quad (|Z_s^{-2}| < 1) \quad (C.68)$$

where

$$A_m = (-1)^m \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-m+1)}{m!} \quad (C.69)$$

Substitution of expression (C.68) for $(1 - Z_s^{-2})^{1/2}$ in (C.67) gives, after some rearrangement,

$$\Omega_s^* = \frac{Q_0}{2\pi} \sum_{m=1}^{\infty} B_m Z_s^{-m} \quad (|Z_s| > 1) \quad (C.70)$$

where

$$B_m = A_m ; \quad m=2j, \quad j=1,2,\dots \quad (C.71)$$

$$B_m = -2Z_{s0} A_m ; \quad m=2j+1, \quad j=0,1,2,\dots \quad (C.72)$$

The expansions (C.65) and (C.70) may be combined as follows:

$$\Omega_s + \Omega_s^* = \sum_{m=1}^{\infty} C_m Z_s^{-m} \quad (C.73)$$

where

$$C_m = A_m + \frac{Z_{s0}^m}{m} - \frac{1*3*5*\dots*(m-1)}{2*4*6*\dots*m^2} ; \quad m=2j, \quad j=1,2,\dots \quad (C.74)$$

$$C_m = -2Z_{s0} A_m + \frac{Z_{s0}^m}{m} ; \quad m=2j+1, \quad j=0,1,2,\dots \quad (C.75)$$

APPENDIX D CONTINUITY OF $\partial u / \partial s$

Consider an infinitesimal neighborhood of the point of intersection of the cracks, see Figure 19, that is sufficiently small that variations of the components of the specific discharge vector within zones 1, 2, 3 and 4 may be neglected. Cartesian coordinates x, y and x^*, y^* are introduced as shown in Figure D.1. Crack t makes an angle α with crack s and the components of the

Figure D.1. Local coordinates at the crossing

specific discharge vector in the x, y system are related to those in the x^*, y^* system, and vice versa, as

$$q_x^* = q_x \cos \alpha + q_y \sin \alpha \quad (D.1)$$

$$q_y^* = -q_x \sin \alpha + q_y \cos \alpha \quad (D.2)$$

$$q_x = q_x^* \cos \alpha - q_y^* \sin \alpha \quad (D.3)$$

$$q_y = q_x^* \sin \alpha + q_y^* \cos \alpha \quad (D.4)$$

The components of the specific discharge vectors in the four zones are labeled by the superscript i , $i=1,2,3,4$. The head is continuous across the cracks and therefore the components of the specific discharge vector tangential to the cracks must be continuous across the cracks. This leads to the following 4 equations:

D.2

$$q_x^1 = q_x^4 \quad (D.5)$$

$$q_x^2 = q_x^3 \quad (D.6)$$

$$q_x^{1*} = q_x^{2*} \quad (D.7)$$

$$q_x^{4*} = q_x^{3*} \quad (D.8)$$

Using (D.1) and (D.2), (D.7) and (D.8) may be written as

$$q_x^1 \cos \alpha + q_y^1 \sin \alpha = q_x^2 \cos \alpha + q_y^2 \sin \alpha \quad (D.9)$$

$$q_x^4 \cos \alpha + q_y^4 \sin \alpha = q_x^3 \cos \alpha + q_y^3 \sin \alpha \quad (D.10)$$

Using (D.5) and (D.6) in (D.10) and subtraction from (D.9) yields after division by $\sin \alpha$

$$q_y^1 - q_y^4 = q_y^2 - q_y^3 \quad (D.11)$$

from which it follows that the inflow along crack s is continuous at the point of intersection.

Combination of (D.5) and (D.6) with (D.3) and (D.4) gives

$$q_x^{1*} \cos \alpha - q_y^{1*} \sin \alpha = q_x^{4*} \cos \alpha - q_y^{4*} \sin \alpha \quad (D.12)$$

$$q_x^{2*} \cos \alpha - q_y^{2*} \sin \alpha = q_x^{3*} \cos \alpha - q_y^{3*} \sin \alpha \quad (D.13)$$

and application of (D.7) and (D.8) to (D.13) followed by subtraction from (D.12) gives, after division by $\sin \alpha$,

$$q_y^{2*} - q_y^{1*} = q_y^{3*} - q_y^{4*} \quad (D.14)$$

Hence, the inflow along crack t is continuous at the point of intersection as well.

DISTRIBUTION

No. of
Copies

No. of
Copies

OFFSITEUNITED STATES

	John Bird Geology Department Cornell University Ithaca, NY 14853
A. A. Churm DOE Chicago Patent Group 9800 South Cass Avenue Argonne, IL 60439	Albin Brandstetter Office of Nuclear Waste Isolation Battelle Memorial Institute 505 King Avenue Columbus, OH 43201
27 DOE Technical Information Center	
2 Argonne National Laboratory Technical Information Services Report Section Building 2 of 3, CE 125 9700 South Cass Avenue Argonne, IL 60439	2 Brookhaven National Laboratory Reference Section Information Division Upton, Long Island, NY 11973
10 Battelle Memorial Institute Office of Nuclear Waste Isolation Attn: Beverly Rawles 505 King Avenue Columbus, OH 43201	Harry C. Burkholder Office of Nuclear Waste Isolation Battelle Memorial Institute 505 King Avenue Columbus, OH 43201
Michael J. Bell, Chief High-Level Waste Technical Development Branch Waste Management Division U.S. Nuclear Regulatory Commission Washington, D.C. 20555	Neville G.W. Cook Dept. of Materials Science and Mineral Engineering Hearst Mining Building University of California Berkeley, CA 94720
	Carl R. Cooley DOE Office of Nuclear Waste Management Washington, DC 20545

Distr.2

No. of
Copies

No. of
Copies

Mike Cullingford
Probablistic Analysis Staff
Nuclear Regulatory Commission
Mail Stop 3103
Washington, DC 20555

Stanley N. Davis, Hydrologist
The University of Arizona
College of Earth Sciences
6540 W. Box Canyon Drive
Tuscon, AZ 85705

James O. Duguid
Battelle Memorial Institute
Office of Nuclear Waste Isolation
505 King Avenue
Columbus, OH 43201

Dan Egan
Environmental Protection Agency
Office of Radiation Programs
Technology Assessment Division
M/S AW559
Washington, DC 20460

Warren Eister
Department of Energy
Division of Waste Isolation
Washington, DC 20545

Bruce R. Erdal
Los Alamos Scientific Laboratory
CNC - 11, MS - 514
Los Alamos, NM 87545

Dr. Charles Fairhurst
Head of the Department
of Civil and Mineral
Engineering
112 MinMet Building
221 Church Street S.E.
Minneapolis, MN 55455

Larry R. Fitch
Office of Licensing
Systems Integration
Basalt Waste Isolation Project
Rockwell Hanford Operations
P.O. Box 800
Richland, WA 99352

Robert Garrells
Dept. of Geological Sciences
Northwestern University
Evanston, IL 60201

Henk M. Haitjema
955 Dawn Ave
St Paul, MN 55112

Colin Heath
NUS Corporation
4 Research Place
Rockwell, MD 20805

William M. Hewitt
Office of Nuclear Waste Isolation
Battelle Memorial Institute
505 King Avenue
Columbus, OH 43201

Distr.3

No. of
Copies

No. of
Copies

Peter L. Hoffmann
Office of Nuclear Waste Isolation
Battelle Memorial Institute
505 King Avenue
Columbus, OH 43201

Muzaffer Kehnemuyi
Office of Nuclear Waste Isolation
Battelle Memorial Institute
505 King Avenue
Columbus, OH 43201

John F. Kircher
Office of Nuclear Waste Isolation
Battelle Memorial Institute
505 King Avenue
Columbus, OH 43201

C. Kreitler
Texas Bureau of Economic Geology
University Station
P.O. Box X
Austin, TX 78712

Terry R. Lash
Director of Science and Public
Policy
Keystone Center
Box 38
Keystone, CO 80435

2 Lawrence Berkeley Laboratory
Reference Library
University of California
Berkeley, CA 94720

2 Lawrence Livermore Laboratory
Reference Library
P.O. Box 808
Livermore, CA 94550

Stan E. Logan
Los Alamos Technical Assoc.
P.O. Box 410
Los Alamos, NM 87544

Tom Longo
Department of Energy
Division of Waste Isolation
Washington, DC 87544

2 Los Alamos Scientific Laboratory
Reference Library
P.O. Box 1663
Los Alamos, NM 87544

J. B. Martin
Asst. Director for Radioactive
Waste Management Branch
NRC Division of Materials and
Fuel Cycle Facility Licensing
Washington, DC 20555

John T. McGinnis
Office of Nuclear Waste Isolation
Battelle Memorial Institute
505 King Avenue
Columbus, OH 43201

M. Merritt
WIPP, Division 4512
Sandia Laboratories
Albuquerque, NM 87115

Martin A. Molecke
Nuclear Waste Experimental
Programs

Division 4512
Sandia Laboratories
Albuquerque, NM 87185

Distr.4

No. of
Copies

No. of
Copies

2 Barry N. Naft NUS Corporation 4 Research Place Rockville, MD 20805	Robert O. Pohl Laboratory of Atomic and Solid State Physics Cornell University Ithaca, NY 14853
2 Jeff O. Neff Department of Energy Columbus Program Office 505 King Avenue Columbus, OH 43201	H. L. Ragsdale Department of Biology Emory University Atlanta, GA 30322
2 Neil A. Norman Environmental Sciences Department Bechtel National Inc. P.O. Box 3965 San Francisco, CA 94105	Gilbert E. Raines Office of Nuclear Waste Isolation Battelle Memorial Institute 505 King Avenue Columbus, OH 43201
2 Oak Ridge National Laboratory Central Research Library Document Reference Section Oak Ridge, TN 37830 Frank L. Parker Dept. of Environmental Engineering Vanderbilt University Nashville, TN 37235 T. H. Pigford University of California Berkeley, CA 94720 George F. Pinder Dept. of Civil Engineering Princeton University Princeton, NY 08540	Herbert R. Shaw U.S. Department of the Interior Geological Survey Branch of Experimental Geochem. and Mineralogy 345 Middlefield Road Menlow Park, CA 94025 D. B. Shipler Battelle Memorial Institute Office of Nuclear Waste Isolation 505 King Avenue Columbus, OH 43201 2 Savannah River Laboratory Reference Library Aiken, SC 29801 Arthur J. Soinski Energy Resources Conservation and Development Commission 1111 Howe Avenue Sacramento, CA 95825

Distr.5

No of
Copies

No of
Copies

Howard P. Stephens
Sandia Laboratories
P.O. Box 5800
Albuquerque, NM 87115

Paul A. Witherspoon
Earth Science Division
Lawrence Berkeley Laboratory
Building 90
Berkeley, CA 94720

David B. Stewart
U.S. Geological Survey
National Center 959
Reston, VA 22092

ONSITE

5 DOE Richland Operations Office

25 Otto D. L. Strack
23 Black Oak Road
St Paul, MN 55110

O. J. Elgert
R. B. Goranson
H. E. Ransom
J. J. Schreiber
F. R. Standerfer

Martin S. Tierney
Environmental Assessment
Division 4514
Sandia Laboratories
Albuquerque, NM 87115

12 Rockwell Hanford Operations

Newell J. Trask, Geologist
U.S. Department of the Interior
Geological Survey
Reston, VA 22092

R. C. Arnett
R. L. Bielefeld
D. J. Carrell
R. A. Deju
F. A. Deluca
L. R. Fitch
R. E. Gephart
Rockwell Document Control
Rockwell Library (2)

Bob E. Watt
1447 45th Street
Los Alamos, NM 87544

R. E. Wilems
INTERA Environmental
Consultants, Inc.
11999 Katy Freeway, Suite 610
Houston, TX 77079

Robert Williams
Electric Power Research Inst.
3412 Hillview Avenue
P.O. Box 10412
Palo Alto, CA 94304

No. of
Copies

91 Pacific Northwest Laboratory

E. M. Arnold
G. L. Benson
D. J. Bradley
T. D. Chikalla
M. O. Cloninger
C. R. Cole
W. J. Deutch
P. G. Doctor
F. H. Dove
M. G. Foley (20)
T. E. Gates
F. N. Hodges
C. T. Kincaid
M. R. Kreiter
J. W. Lindberg
B. K. Marshall
G. L. McVay
R. W. Nelson (30)
R. D. Peters
G. M. Petrie
A. M. Platt
J. V. Robinson
R. J. Serne
J. W. Shade
J. K. Soldat
J. T. Zellmer
Technical Information Library (5)
Publishing Coordination (2)
Water and Land Resources Dept.
Library (10)