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OPTIMAL CONTROL STUDIES OF SOLAR HEATING SYSTEMS

Final Report

MASTER

By
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Work Performed Under Contract No. AS02-77CS34519

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Solar Energy

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FINAL REPORT
ON
CONTRACT NO. EG-77-S-02-4519
ON THE SUBJECT OF
OPTIMAL CONTROL STUDIES OF SOLAR HEATING SYSTEMS

PREPARED FOR
THE U.S. DEPARTMENT OF ENERGY

BY
DEPARTMENT OF MECHANICAL ENGINEERING
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ABSTRACT

OPTIMAL CONTROL FOR SOLAR HEATED BUILDINGS

In the past few years fuel prices have seen steady increases. Also, the supply of fuel has been on the decline. Because of these two problems there has been an increase in the number of solar heated buildings. Since conventional fuel prices are increasing and as a solar heating system represents a high capital cost it is desirable to obtain the maximum performance from a solar heating system. The control scheme that is used in a solar heated building has an effect on the performance of the solar system. The best control scheme possible would, of course, be desired.

This report deals with the control problems of a solar heated building. The first of these problems is to control the inside temperature of the building and to minimize the fuel consumption. This problem applies to both solar and conventionally heated buildings. The second problem considered in this paper is to control the collector fluid flow to maximize the difference between the useful energy collected and the energy required to pump the fluid. The third problem is to control the enclosure temperature of a building which has two sources of heat, one solar and the other conventional, in a manner that maximizes the solar contribution and minimizes the auxiliary contribution.

ACKNOWLEDGEMENTS

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1. EXECUTIVE SUMMARY

1.1 OBJECTIVES

The objectives of this study were to determine control strategies that lead to reduced fuel consumption in space heating and service hot water applications.

1.2 ACCOMPLISHMENTS

- Developed the optimal controller of the first kind: How to heat your house and use less fuel.
- Developed the optimal controller of the second kind: How to collect more solar energy and require less energy in the collection process.
- Developed the optimal controller of the third kind: How to collect and distribute solar energy optimally.
- Developed an optimal control strategy for use of off-peak energy with a solar system.
- Developed a Luenberger Observer for predicting ambient temperatures.
- Designed and built a μ -processor based controller for implementing these control strategies.
- Designed and built a μ -processor based Advanced Control and Instrumentation System (ACI) for analyzing the performance of optimal controllers.
- Interfaced with industrial organizations for the purpose of commercialization of these devices.
- Initiated patent applications.
- Supported an M.S. student (Ed Hoover) who is now working at Sandia Laboratories (on energy problems).
- Supported another student (Dwight Ellsworth Hull, III) who obtained his Ph.D. and is now working at Martin-Marietta Corp. (on energy problems).
- Supported another M.S. student (David Hays) who is now working at Solar Environmental Engineering Co. (SEEC), Inc. (on solar energy problems).

1.3 OPTIMAL CONTROLLERS OF THE FIRST KIND

- Determined an optimal control strategy for house heating.
Question: Which temperature history would you prefer to have in your house:

(A)

BANG-BANG CONTROL

(B)

OPTIMAL CONTROL

- Performed sensitivity analyses; determined the effect of house parameters (UA , C_E) on the performance of the optimal controller. Not sensitive to C_E , highly sensitive to UA .
- Developed an optimal estimation method for determining UA .
- Determined fuel savings that may be realized through the use of improved controls.

Oil Consumption for Heating

$$20\% \text{ of } 20 \text{ MBPD} = 4 \text{ MBPD}$$

$$10\% \text{ Savings via Improved Control} = 0.4 \text{ MBPD}$$

$$\text{At } \$12/\text{Bbl}, \text{ Savings} \approx \$5 \times 10^6/\text{day} \text{ (considerably larger at current prices)}$$

- Investigated methods for implementation and commercialization.
- Published Results

1. Hull, D.E. and C.B. Winn, "Optimal Temperature Control Using a Microprocessor," IEEE Decision and Control Conference, New Orleans, LA., December 1977.
2. Hoover, E.R. and C.B. Winn, "Design and Computer Simulation of Solar Heat Pump Systems," Proceedings of the 1978 Annual Meeting of AS/ISES, Publications Office, University of Delaware, Newark, Delaware, 1978.

1.4 OPTIMAL CONTROLLERS OF THE SECOND KIND

- Determined the control strategy that maximizes the difference between solar energy collected and parasitic losses incurred in collecting the energy.
- These optimal controllers eliminate cycling of the pumps and thereby reduce system noise and pump wear.
- Investigated methods for implementation and commercialization.
- Published Results
 1. Winn, C.B. and D.E. Hull, "Optimal Controllers of the Second Kind," Proceedings of the AS/ISES Annual Meeting, Denver, Colorado, August 1978.
 2. Winn, C.B. and D.E. Hull, "Optimal Control of Active Solar Systems," Workshop on the Control of Solar Energy Systems for Heating and Cooling, U.S. Department of Energy, Hyannis, Mass., May 1978.

Background

- Kovarik and Lesse, 1975
 - Obtained a numerical solution
 - Discussed drawbacks to solution
 - Optimal control not a function of any system state variable
 - Must know weather conditions in advance
 - Numerical solution not easily implemented
- Proportional controllers
 - May perform better than fixed $\Delta T_{ON} - \Delta T_{OFF}$ bang-bang controllers during periods of low solar insolation.
- Adjustable bang-bang controllers
 - Give optimal performance if parasitic losses are excluded

Present Solution

- Closed form analytical solution

- The optimal controller is a function of measureable states of the system.
- The optimal controller does not require prior information regarding the weather
- Easy to implement

Typical Results

A. TEMPERATURE AND RADIATION DATA

<u>DATE</u>	SOLAR <u>HORIZONTAL (kJ/m²)</u>	AVERAGE AMBIENT <u>TEMPERATURE</u>
2/8	14292	-7°C
2/1	14190	2°C
1/30	9751	-5°C
2/12	6482	-7°C

B. PERCENTAGE IMPROVEMENT RELATIVE TO BANG-BANG CONTROLLER (WATER SYSTEM)

<u>DATE</u>	INCREASE IN USEFUL <u>ENERGY</u>	INCREASE IN OBJECTIVE <u>FUNCTION</u>
2/8	5%	6%
2/1	4%	5%
1/30	22%	26%
2/12	74%	82%

Calculation Procedure

1.5 OPTIMAL CONTROLLERS OF THE THIRD KIND

- Determined the optimal control strategy to be used with a solar heating system so that the solar utilization is maximized, the use of the conventional heating system is minimized, and the temperature is maintained at the desired level.
- Investigated methods for implementation and commercialization.
- Published Results
 1. Hull, Dwight, E., III, "Optimal Control for Solar Heated Dwellings," Ph.D. Thesis, Department of Mechanical Engineering, Colorado State University.

1.6 OPTIMAL CONTROL OF OFF-PEAK DEVICES

- Developed an optimal control strategy to avoid peak load demand for solar heated buildings while maintaining temperature control and optimizing the use of off-peak electricity for charging the storage.
- Developed methods for short-term weather forecasting (temperature and insolation).
- Developed methods for estimating future heating requirements.
- Published Results:
 1. Winn and Duong, "An Optimal Control Strategy for Peak Load Reduction," Workshop on the Control of Solar Energy Systems for Heating and Cooling, U.S. Department of Energy, Hyannis, Mass., May 1978.

1.7 TEMPERATURE PREDICTING

- Developed a method for predicting outside air temperature; the method can be used to improve the performance of controllers (especially with off-peak heating and solar assisted heat pumps).
- Designed and built a μ -processor based device for implementing the method.

Luenberger Observer Theory Applied to Temperature Prediction

Logical Process

Cycles - 1/minute to 1/5 minutes

Note: Only 11 multiplies

- While a more complicated model would do a better job this works very well.
- It is easily implemented on a μ -processor.
- The correcting gains can be set to provide best near-term (6 hr) or long term (24 hr) performance.

1.8 FUTURE ACTIVITIES

- Complete the construction of the controller test facility and compare the performance figures of various controllers.
- Install the μ -processor based controller in CSU House I (and/or II) to test optimal control strategies.
- Investigate optimal control strategies for the passive house to be built at CSU.
- Refine our analysis based on the results obtained with the ACI.
- Continue to provide educational opportunities to students who are interested in energy related problems.

2. DETAILED TECHNICAL REPORT

CHAPTER I

INTRODUCTION

In President Carter's outline of his energy program, he indicated a hope that by 1985 there would be 2.5 million solar heated homes in the United States (Moss, 1977). This would mean that about 3 percent of the homes in the United States would be solar heated. With dwindling supplies of oil and natural gas, President Carter's plan would reduce the amount of money sent abroad for importing oil. Today the United States imports about half of its oil (Moss, 1977). So, of the present daily consumption of 18 million barrels of oil in the United States, approximately 9 million barrels are imported. At \$10.60 per barrel the United States is spending 4 million dollars per hour for imported oil (Kending, 1977). Since about 20 percent of the energy consumed in the United States is for space heating, and if by 1985 there are 2.5 million homes heated by solar energy, then the oil imported by the United States could be reduced by 65,000 barrels of oil per day. Going by the current oil prices of \$10.60 per barrel--which will undoubtedly be higher by 1985--this would mean a savings of 700,000 dollars to the United States each day. This is assuming that the solar homes are heated 60 percent of the time by solar energy.

If these solar systems could be controlled in a manner to improve their efficiency then an even greater savings could be realized. Say the efficiency of the solar homes is improved by

5 percent, then the United States would realize a 60,000 dollar further reduction in the cost of imported oil per day at \$10.60 per barrel.

This efficiency is possible by use of a new way of controlling the heating systems of both conventional and solar heated homes. This new control strategy is discussed in this paper.

The conventional heating system in homes today is handled by a rather simple controller--the thermostat. The conventional thermostat has seen numerous small refinements but no major changes. A major change was not needed because conventional homes have only one heat source to contend with. Also, with energy so inexpensive for so long, there seemed little economic need to develop new controllers. Now with energy costs constantly increasing, more efficiency in the use of our energy is of great economic importance. And with the advent of solar homes and their more complicated systems of obtaining heat, a change is needed in the home controller. The solar system is more complicated in two ways. First, solar homes have two sources of heat to be controlled. These sources are the solar system and the auxiliary system. Also, since solar energy can only be collected during the day and has to have a storage system during times when there is no available solar energy, the controller has another task to handle.

This paper deals with three basic control problems of both conventional and solar heated homes. These are discussed below.

Control Problem One

The first problem is that of temperature control of the enclosure. The problem here is to heat the enclosure and use a minimum amount of fuel. This problem is not restricted to a solar home but can also be considered for a conventionally heated home. The solution to this problem is given in analytical form in Chapter II. The major assumption made is that the ambient temperature is known as a function of time. Also discussed in Chapter II are three methods of updating the ambient temperature estimate so that actual weather data can be used with the control strategy. There are two optimal control strategies discussed in Chapter II. Control strategy one is formulated with the heat source in the objective function to the second power. The control function is a continuous function of time. This means that there are not abrupt changes in the amount of heat delivered to the enclosure. Any time the ambient temperature is below the enclosure temperature the heat source will supply some heat to the enclosure. Control strategy two is formulated with the heat source in the objective function to the first power. The optimal control strategy in this case changes into a problem of determining switching time. The problem is to determine when the control changes from an off position to a full or partial capacity position. This means that control strategy two can turn the heat source from the off position to the on position.

Control Problem Two

The second control problem applies only to solar heated homes. Here again the goal is to reduce the fuel consumption. Heat

is collected during the day to decrease the amount of fuel needed to heat the house, but, if the cost of pumping the collecting fluid through the collectors is greater than the savings acquired by using solar energy in the first place, it becomes uneconomical to use the solar energy. So, the problem is to reduce the cost of pumping the fluid through the collector while at the same time increasing or maximizing the amount of energy collected.

M. Kovarik and P.F. Lesse in their paper "Optimal Control of Flow in Low Temperature Solar Heat Collectors," (1976) addressed this problem. Their solution falls short in that it cannot be implemented. There are two basic reasons as to why their solution cannot be implemented. The first is that their control is not based on a measurable state of the system; the second is that one would have to be able to predict the exact solar and weather conditions a day in advance. The work done by Kovarik and Lesse is accomplished by using the rate of flow of the collector fluid as the control variable. This results in a nonlinear two point boundary value problem, which can only be solved numerically.

One can avoid having to solve a two point boundary value problem by recognizing that the useful energy collected is given by the product of the heat removal factor and the available heat term. The heat removal factor is the only term which has the flow rate as part of its equation. In Chapter III a transformation from the flow rate as the control variable to the heat removal factor as the control variable is made. By use of this transformation it is possible to obtain an analytical solution to the collector flow rate problem. The advantage of the analytical solution is that it can be

applied to real weather data. This is because the optimal control is a function of measurable states only and it is not necessary to predict the weather conditions in advance.

Within this second problem are two parts. The first part concerns uniform storage of the heat and the second part concerns stratified storage of the heat. This division is necessary because there are two basic solar systems in use today--air and water. The water system tends to consist of uniform storage and the air system tends to consist of stratified storage. In Chapter III an analytical solution is given for both types of storage.

Control Problem Three

The third problem applies only to solar homes. In a solar heated home there are two sources of heat. One source of heat is the solar storage system and the other source of heat is the auxiliary system. Since it does not cost as much to use the heat from storage as it does to use the heat from the auxiliary system it is more economical to use the heat from storage as much as possible while using the auxiliary energy source as sparingly as possible.

Also included in this third problem is the situation concerning the collector flow rate, which was discussed earlier under the second problem.

Problem three is now a combination of problem one, plus an addition of another heat source, and problem two. In solving problem three mathematically, problem two becomes independent of problem one and its additional heat source. Consequently, the solution to problem two remains the same.

Problem one, plus the additional heat source, represents a two heat source problem. One source of heat is the auxiliary system and the other source of heat is the solar storage system. The solar storage system can heat the entire enclosure if enough heat is available from the storage. If the storage cannot provide enough heat for the enclosure then the auxiliary system is capable of heating the entire enclosure. This is not a two zoned enclosure. The enclosure temperature is assumed to be the same throughout the entire enclosure.

Recall that in problem one that there were two optimal control strategies discussed. Optimal control strategy one is desirable for use with the storage system of a solar heated home. This is because the control strategy is less sensitive to errors in the heat capacity and the thermal resistance to heat flow in the model of the house. This will allow the control strategy to continue to work with some errors in the heat capacity and the thermal resistance to heat flow with little effect on the system response.

Strategy two is desirable for use with the auxiliary heater in the solar heated home. This is because the auxiliary heater should turn off whenever the storage can provide the total heat load for the house.

Implementation and Conclusions

Chapter V discusses the methods of implementation of the optimal control strategies. In order to implement the optimal control strategies discussed in this paper a microprocessor-based controller would probably be required. Though the optimal control

strategy developed for the collector fluid flow may not necessarily require a microprocessor-based controller.

My conclusions and recommendations are discussed in Chapter VI.

Derivations

The appendices contain all the mathematical details for the control strategies developed in this paper. Appendix I contains the mathematics for the two control strategies discussed in Chapter II. The first strategy is the quadratic optimal control strategy. The second control strategy is the singular optimal control strategy. Both strategies control the temperature of the enclosure. Appendix II contains the mathematics for the optimal fluid flow rate through a solar collector. This is problem two and is discussed in Chapter III. Found in Appendix II is the solution for problem two using two different plants. The first plant considered had a uniform storage system; the second plant considered had a stratified storage system. Appendix III contains the mathematics for control problem three. This includes both the singular optimal control strategy and the quadratic optimal control strategy.

CHAPTER II

ENCLOSURE TEMPERATURE CONTROL STRATEGIES

The objective for the first control problem was to minimize the amount of energy required to heat the enclosure and to minimize the amount of discomfort. There are many objective functions which could be chosen to meet the above desired objective. Two of the possible objective functions are discussed in this chapter.

Quadratic Optimal Control Problem

The first objective function chosen was the following:

$$J = \int_0^{tf} \{C_1 x_1^2 + C_2 u_2^2\} dt$$

In this objective function the discomfort is represented by x_1 which is the temperature difference between the actual enclosure temperature and the desired enclosure temperature. The energy delivered to the enclosure is given by u_2 . C_1 and C_2 are weighting coefficients of the objective or cost function.

The plant which is being controlled has only two components. The first is a building and the second is a source of heat. The differential equation used to represent the enclosure was the following:

$$\dot{x}_1 = \left\{ u_2 u_{2MAX} - UA \{x_1 + T_D - T_A\} \right\} / C_E$$

where

U_2 is the normalized energy term.

U_{1MAX} is the maximum output of the heat source.

UA is the area conductance product of the enclosure.

X_1 is the enclosure temperature minus the desired temperature.

T_A is the ambient temperature.

T_D is the heat capacity of the enclosure.

By applying optimal control theory to the above problem, the following results can be obtained:

$$U_2^* = U_{2MAX} / (2C_2C_E)$$

$$\lambda_1 = UA/C_E + 2C_1X_1$$

$$\dot{X}_1 = \lambda_1 U_{2MAX}^2 / (2C_2C_E^2) - UA(X_1 + T_D - T_A) / C_E$$

The optimal control is given by U_2^* . It varies with λ_1 only. λ_1 is a Lagrange multiplier which varies with time. λ_1 is the gradient of the objective function with respect to variations in the initial conditions, while holding the control constant and satisfying the constraints (Bryson and Ho, 1969).

Figure 1. Schematic of House Heating System.

The complete derivation of this problem can be found in Appendix I.

The assumption is made that the ambient temperature is a function of time, i.e. $T = B - A \sin(\omega t)$.

Since the X_1 and λ_1 differential equations are linear and since the ambient temperature is given as a function of time, then an analytical solution for both differential equations may be obtained. In order to complete the solution the boundary condition of the differential equations must be determined. The boundary condition may be given by two different methods. The first is to allow the temperature of the enclosure to be unspecified at the final time. This will yield the following conditions:

$$\begin{aligned} X_1(t_0) &= X_0 \quad \text{known} \\ (\lambda_1)_{t_f} &= 0. \end{aligned}$$

The solution for λ_1 is a function of the following three variables (as shown in Appendix I):

1. λ_1 at the initial time.
2. X_1 at the initial time.
3. time.

The solution for X_1 is a function of the same three variables. In order to determine X_1 and λ_1 for all time it is necessary to know λ_1 at the initial time. This can be determined by setting the λ_1 equation equal to the boundary condition at the final time, $(\lambda_1(t_f) = 0)$, and solving for λ_1 at the initial time.

The second way to specify the boundary conditions is to specify the enclosure temperature at the final time. This leads to the following boundary conditions:

$$X_1(t_0) = X_0 \text{ (known)}$$

$$X_1(t_f) = X_f \text{ (known)}$$

Again it is necessary to find λ_1 at the initial time in order to determine X_1 and λ_1 for all time. In this case the equation for X_1 is set to the final condition for X_1 and then solved for λ_1 at the initial time. The equations for λ_1 at the initial time can be found in Appendix I.

Predicting Ambient Temperature

The optimal control can be determined for all time in either case provided the ambient temperature follows the function of time it was assumed to have followed. Obviously, the ambient temperature does not follow the assumed function exactly. Therefore a method of changing the optimal control strategy to account for changes in the ambient temperature is needed. There are three methods of upgrading the optimal control strategy to allow for the changes in the ambient temperature. The first method uses a short time span. Since it is a short time period the predictions of ambient temperature are fairly accurate. The problem is that if the first boundary condition is used, where X_1 at the final time is unspecified, then the control goes to zero at the final time where the final time is represented by the end of the prediction interval. This means that the control is turned on and off frequently. For example, let the prediction interval be equal to 15 minutes. The equation for the optimal control (U_2^*) is as follows:

$$U_2^* = \lambda_1 U_{2MAX} / 2C_2C_E$$

The only time dependent variable in this equation is λ_1 . When X_1 is unspecified then λ_1 at the final time is equal to zero (a boundary condition). The final time is equal to the prediction interval in this case. This means that the heat source is going to be turned off every 15 minutes (at the end of a prediction interval). It will be turned on at the start of the next prediction interval which is the end of the preceding prediction interval. This means that the heat source will be turned on and off 96 times in 24 hours if a 15 minute prediction interval is used.

The second method is to use the neighboring optimal control policy (Bryson and Ho, 1969). The solution to the neighboring policy is identical to the solution to the original problem except for some changes in constants. The third method is to vary the initial time of the original solution. This method leaves the final time at the same value. This is normally assumed to be 24 hours.

Recall that λ_1 is a function of the following three variables:

1. λ_1 at the initial time.
2. X_1 at the initial time.
3. time.

The third method of updating the solution changes the initial time to the present time. Assume that the prediction interval is 15 minutes. This would mean that every 15 minutes the enclosure temperature is measured to determine X_1 at the new initial time and λ_1 is calculated at the new initial time. The ambient temperature is also measured every 15 minutes so as to obtain a new prediction on the ambient temperature. The new initial time is the old initial time plus 15 minutes or the prediction interval. The third method

is probably somewhat easier to implement than the second method. Although any one of the three methods could be used, the third method was used to obtain the results shown in this paper.

Results of Quadratic Optimal
Control Strategy

Shown in Table 1 is the data used to obtain the results discussed below.

TABLE 1
INPUT DATA FOR THE ENCLOSURE TEMPERATURE
CONTROL STRATEGIES

Constants of the Model	Data
Heat Capacity of Enclosure (C_E)	18890.0 kJ/°C
Maximum Heat Source Output (U_{2MAX})	105500.0 kJ/hr
Heat Loss Coefficient of Enclosure (UA)	949.5 kJ/hr/°C
Desired Temperature of Enclosure (T_D)	20.8 C

Shown in Figure 2 is a plot of the enclosure temperature minus the desired enclosure temperature versus time. The initial temperature error of the enclosure was zero. The enclosure temperature declines gradually as the ambient temperature declines. The enclosure temperature rises gradually as the ambient temperature rises. Around hour 21 there is a marked decline in the enclosure temperature. This is due to the control going to zero at the final time. There was a weighting coefficient on the temperature error

HOUSE TEMPERATURE FROM DESIRED TEMPERATURE

Figure 2. Enclosure Temperature Error for Quadratic Control Strategy.

of 1 and on the heat source of 100. The temperature error at the final time was 2°C below the desired temperature.

Shown in Figure 3 is a plot of the control variable versus time. The control variable represents a fraction of the maximum output of the heat source. This means that the energy output of the heat source at any time may be obtained by multiplying the maximum output of the heat source by the control variable. For example, at the initial time (time = 0), the control variable is 0.21. This means that the heat source was delivering 22155 kJ/hr to the enclosure at this time. The output of the heat source increases as the ambient temperature decreases. Also as the ambient temperature increases the output of the heat source decreases. At hour 21 the control variable starts to approach zero. This is approximately the same time that the enclosure temperature started to show a drop from the desired temperature of the enclosure.

Shown in Figure 4 is the ambient temperature. The results shown in Figure 2 and Figure 3 were obtained from this data. This is a sine wave with a period of 24 hours, an amplitude of 8°C and an intercept of -6°C .

Shown in Figure 5 is a plot of the enclosure temperature minus the desired enclosure temperature versus time. The initial temperature error in this case was zero. Also, real measured weather data was used. Similar responses in the enclosure temperature are noted in both Figure 2 and Figure 5. In Figure 5, the enclosure temperature error at the final time is approximately 3°C . This is 1°C lower than the enclosure temperature was when the sine wave was used as the ambient temperature. This is due to a lower ambient

Figure 3. Control Variable for Quadratic Control Strategy.

AMBIENT TEMPERATURE

Figure 4. Artificial Ambient Temperature.

HOUSE TEMPERATURE FROM DESIRED TEMPERATURE

Figure 5. Enclosure Temperature Error for Quadratic Control.

temperature at the final time with the real measured weather data. Shown in Figure 6 is a plot of the control variable. It can be seen from Figure 6 that the heat source output varies directly with the ambient temperature but in the opposite direction. This can be seen better with the real measured weather data because the measured data tends to rise and fall more sharply.

Shown in Figure 7 is a plot of the ambient temperature versus time. This is measured weather data from the Denver-Boulder, Colorado area.

Singular Optimal Control Strategy

The second objective function chosen was very similar to the first except that the control variable was not squared. The second objective function is as follows:

$$J = \int_0^{t_f} \left\{ C_1 x_1^2 + C_2 u_2 \right\} dt$$

The variables are defined in the same way as in the first objective function. The plant or system is identical to the plant of the first control problem and the differential equation describing the plant is the same. Again by applying optimal control theory to the new problem the following results may be obtained:

1. If $\lambda_1 u_{2MAX} - C_2 C_E > 0$ then $u_2^* = 1$.
2. If $\lambda_1 u_{2MAX} - C_2 C_E < 0$ then $u_2^* = 0$.
3. If $\lambda_1 u_{2MAX} - C_2 C_E = 0$ then $u_2^* = UA(x_1 + T_0 - T_A)/U_{1MAX}$.

Figure 6. Control Variable for Quadratic Control Strategy.

Figure 7. Real Measured Ambient Temperature.

The optimal control is given by U_2^* . It varies with λ_1 only. λ_1 is a Lagrange multiplier which varies with time. X_1 is the temperature error of the enclosure.

Since the energy term was not squared, the solution becomes a singular solution. The problem reduces to a problem of determining switching times (i.e., at what time does the control switch from case 1 to case 2).

Shown in Figure 8 is a phase plane plot of X_1 versus λ_1 . The horizontal line in the center of Figure 8 is the dividing line for the control variable. Above this line the control variable is set to 1 (the heat source is at full capacity). Below this line the control variable is set to 0 (the heat source is turned off). The enclosure temperature error, X_1 , is a stationary point at the intermediate solution and is given by the following equation:

$$X_1 = -C_2 U A / (2 C_1 U_{2MAX})$$

λ_1 is a constant at the intermediate solution and is given by the following equation:

$$\lambda_1 = C_2 C_E / U_{2MAX}$$

This means that in Figure 8 there is only one point on the horizontal line which is the intermediate solution. One curve in the upper family of curves in Figure 8 and one curve in the lower families of curves in Figure 8 will have this point in common. This means that the only way that the heat source can switch from full capacity to the off position is to pass through the intermediate solution or intermediate point. Boundary conditions on the differential equations for λ_1 and X_1 are needed to complete the solution.

PHASE PLANE

Figure 8. Phase Plane.

There are two methods of specifying these boundary conditions. First, specify X_1 at the initial time and allow X_1 to be free at the final time. This leads to following boundary conditions:

$$X_1(t_0) = X_0 \quad (\text{known})$$

$$\lambda_1(t_f) = 0.$$

If X_0 is less than the value of X_1 at the intermediate solution then the optimal control will operate in the following manner. First, the control strategy turns on the heat source to full capacity and then follows the one curve from the family of curves above the horizontal line which intersects the intermediate point--which lies on the horizontal line. Second, once at the intermediate point the control variable will remain there for a period of time. Third, the control will switch from the intermediate point in such a manner that it can follow one curve from the family of curves below the horizontal line so that λ_1 is equal to zero at the final time. This is to satisfy the boundary condition stated above for λ_1 at the final time.

If X_0 is greater than the value of X_1 at the intermediate point then the first step in the control sequence will change. The heat source will be in the off position and the control variable will follow the one curve from the family of curves below the horizontal line that intersects the intermediate point.

Second, specify X_1 at both the initial and the final time. This leads to the following boundary conditions:

$$X_1(t_0) = X_0 \quad (\text{known})$$

$$X_1(t_f) = X_f \quad (\text{known}).$$

The control policy will be the same as described for the first set of boundary conditions, except for the last step. The last step is discussed below.

If X_f is greater than the intermediate value of X_1 then the heat source will switch from the intermediate solution to full capacity at a time so as to arrive at X_f at the final time. If X_f is less than the intermediate value of X_1 then the heat source turns off at a time so that X_1 will equal X_f at the final time.

Results of Singular Optimal Control Strategy

Shown in Figure 9 is a plot of the enclosure temperature minus the desired enclosure temperature versus time. The initial temperature error was zero. Once the enclosure temperature reached the intermediate point it remained there until the heat source was turned off. The heat source was turned off at approximately hour 20. This meets the boundary condition that λ_1 at the final time equals zero. The weighting coefficient for the temperature error was 1 and was 100 for the heat source. The temperature error at the final time for the singular optimal control strategy is greater than the temperature error for the quadratic optimal control strategy. If the weighting coefficients of the singular optimal control strategy are changed to 25 for the heat source, then the temperature error at the final time is almost identical to the temperature error for the quadratic optimal control strategy shown in Figure 2.

Shown in Figure 10 is a plot of the control variable versus time. The assumption made in this problem was that the ambient

HOUSE TEMPERATURE DIFFERENCE

Figure 9. Enclosure Temperature Error for Singular Control Strategy.

CONTROL VS TIME

Figure 10. Control Variable for Singular Control Strategy.

temperature is constant. Because of this assumption the control variable has a tendency to vary between 0 and 1 near the initial time. The control variable follows the ambient temperature until hour 20 when it goes to zero. This is to meet the boundary condition that λ_1 has to be zero at the final time. The ambient temperature used to obtain the results shown in Figure 9 and 10 was the sine wave shown in Figure 4.

Shown in Figure 11 is a plot of the enclosure temperature minus the desired enclosure temperature versus time, using the real measured weather data shown in Figure 7. The temperature goes to the intermediate point and there remains until approximately hour 20 when the heat source is turned off. The temperature error at the final time is approximately 5.5°C below the desired enclosure temperature. It is larger than when the sine wave ambient temperature is used because the real weather temperature falls at a faster rate and is colder at the final time.

Shown in Figure 12 is a plot of the control variable versus time. The control variable increases when the ambient temperature decreases and decreases when the ambient temperature increases. This is again similar to other control responses shown earlier in this chapter.

"Bang-Bang" Control Strategy

The standard temperature control strategy used in buildings today is called a "Bang-Bang" control strategy and works in the following manner:

Figure 11. Enclosure Temperature Error for Singular Control Strategy.

CONTROL VS TIME

Figure 12. Control Variable for Singular Control Strategy.

1. If the temperature of the enclosure is one degree celsius below the desired temperature then the heat source is turned on to its total capacity.
2. If the temperature of the enclosure is one degree celsius above the desired temperature then the heat source is turned off.

Comparisons of Control Strategies

Shown in Table 2 is a comparison between the optimal control strategies and the "Bang-Bang" control strategy. The second optimal control strategy is referred to as the singular optimal control strategy and the results appear in column 4 of Table 2. The first optimal control strategy and its results appear in column 3 of Table 2. In column 2 of Table 2 are shown the results of the "Bang-Bang" control strategy. The optimal control strategies consumed less fuel to heat the enclosure than the "Bang-Bang" control strategy consumed. This was true in all cases. The fuel savings change with changes in the weighting coefficients. The fuel savings for the optimal control strategies ranged from a low of 4 percent to a high of 30 percent. The temperature drop at the final time ranged from a low of 1.1°C below the desired enclosure temperature to a high of 7.5°C below the desired enclosure temperature.

Even though there is a change in the percent of energy savings using the same weighting coefficient, but different weather data, there is no change in the percent of improvement made by the objective function. The improvement in the quadratic objective function for the quadratic optimal control strategy is approximately 80 percent in all cases. The improvement in the singular objective

TABLE 2
COMPARISON OF CONTROL STRATEGIES

Quantity Being Measured	Bang-Bang Control Strategy	Quadratic Optimal Control Strategy	Singular Optimal Control Strategy
$C_1 = 1 \quad C_2 = 25$			
Energy Consumed	590800.	568700.	541600.
Quadratic Objective Function	151.	33.	36.
Singular Objective Function	151.	135.	133.
$C_1 = 1 \quad C_2 = 100 \quad 2$			
Energy Consumed	590800.	544500.	492300.
Quadratic Objective Function	571.	128.	176.
Singular Objective Function	571.	521.	497.
$C_1 = 1 \quad C_2 = 400$			
Energy Consumed	590800.	491400.	442200.
Quadratic Objective Function	2251.	460.	602.
Singular Objective Function	2251.	1907.	1791.
$C_1 = 1 \quad C_2 = 25 \quad 1$			
Energy Consumed	622400.	597100.	572400.
Quadratic Objective Function	158.	36.	38.
Singular Objective Function	158.	143.	140.

Table 2. (continued)

Quantity Being Measured	Bang-Bang Control Strategy	Quadratic Optimal Control Strategy	Singular Optimal Control Strategy
$C_1 = 1 \quad C_2 = 100 \quad 1 \quad 2$			
Energy Consumed	622400.	562000.	504900.
Quadratic Objective Function	601.	133.	172.
Singular Objective Function	601.	543.	521.
$C_1 = 1 \quad C_2 = 400 \quad 1$			
Energy Consumed	622400.	488200.	433100.
Quadratic Objective Function	2370.	457.	535.
Singular Objective Function	2370.	1929.	1808.

1 Real Weather Data

2 Coefficients used for the plots shown

function for the singular optimal control strategy ranges from 12 to 24 percent. The very large improvement in the quadratic objective function is caused by the control variable (U_2) being squared in the objective function. This control variable is always less than 1 and when squared becomes even smaller. With the "Bang-Bang" control strategy the control variable is either equal to 1 or 0. The energy savings and the enclosure temperature drop at the final time is the same for both optimal control strategies when the correct weighting coefficients are used. In this case, if the quadratic optimal control strategy weighting coefficient is four times the singular optimal control strategy weighting coefficient then the results are very similar.

This ratio of 4:1 on the weighting coefficient would change if different input data were used. The reason for it being about 4:1 in this case is because the control variable is approximately 0.5 most of the time. When 0.5 is squared the result is a 4:1 ratio. If the temperature drop of the enclosure is compared for the two optimal control strategies, using the same weighting coefficient, then it is noticed that a greater drop occurs with the singular optimal control strategy. This can be seen by comparing Figures 2 and 9. Though a weighting coefficient of 25 was used instead of 100 for the singular optimal control strategy, a maximum temperature error of 2°C is seen. This is the same as seen in Figure 2 for the quadratic optimal control strategy.

The quadratic optimal control strategy has been simulated with as much as 50 percent error in the thermal resistance to heat flow of the enclosure. The result was a change in the enclosure

temperature of about 0.5°C . Therefore it could be said that the quadratic optimal control strategy is not very sensitive to errors in the model. On the other hand, by looking at the solution for the singular optimal control strategy, it becomes obvious that in order to keep the enclosure temperature stationary (the intermediate solution) there can be no errors in the coefficient for thermal resistance to heat flow of the building. If there are any errors in the coefficient then the singular optimal control strategy would be constantly turning on and off.

Recall that with the singular optimal control strategy the enclosure temperature is constant at the intermediate solution. This means that the problem of errors in the coefficient might be solved by using feedback from the enclosure temperature to maintain a constant enclosure temperature (state feedback). When the control strategy is at the intermediate point and starts to deviate from this stationary point the intermediate control is changed in a manner so as to drive the enclosure temperature to the stationary point.

In our opinion, if the problem of errors in the coefficients of the enclosure model can be solved easily by use of state feedback then the singular optimal control strategy would be the easiest to implement.

CHAPTER III.

COLLECTOR FLOW RATE CONTROL STRATEGIES

The second control problem, the control of the fluid flow rate through the collector, has two plants associated with it. The first plant is a uniform storage type system such as usually occurs in a liquid solar system. A schematic of this plant can be seen in Figure 13. The second plant is a stratified storage system such as is usually found in a solar air system. A schematic of this plant can be seen in Figure 14. The objective function is the same for both plants and is as follows: maximize the useful energy collected and minimize the pumping cost.

Numerical Optimal Control Strategy

The pumping cost of the collector fluid is assumed to be proportional to the cube of the flow rate. Therefore the objective function would be written as follows:

$$J = \int_0^{t_f} \left\{ Q_u - C_f \dot{m}^3 \right\} dt$$

This derivation was done by Kovarik and Lesse in their paper "Optimal Control of Flow in Low Temperature Solar Heat Collectors."

The storage tank temperature (T_s) and the useful energy (Q_u) are given by the following equations:

Figure 13. Schematic of Collector with Uniform Storage.

Figure 14. Schematic of Collector with Stratified Storage.

$$\dot{T}_S = Q_U / C_S$$

$$Q_U = F_r f$$

$$f = A \left\{ H_T (\tau\alpha)_e - U_L (T_S - T_A) \right\}$$

$$F_r = \dot{m} C_p / A U_L \left\{ 1 - e^{(-F' A U_L / \dot{m} C_p)} \right\}$$

where

T_S is the storage temperature.

Q_U is the useful energy collected.

C_S is the heat capacity of the storage tank.

F_r is the heat removal factor.

$\dot{m}$ is the fluid flow rate through the collector.

f is the available energy.

C_p is the specific heat of the fluid in the collector.

A is the collector area.

U_L is the heat loss coefficient of the collector.

F' is the collector efficiency factor.

H_T is the solar insolation on the tilted surface.

$(\tau\alpha)_e$ is the effective transmittance absorptance product.

When the optimal control theory is applied to this problem two coupled nonlinear differential equations result. The initial condition is known on one differential equation and the final condition is known on the other equation. Thus, a two point boundary value problem which can normally only be solved numerically results. The numerical results for one case are given in reference 4.

Analytical Optimal Control Strategy

If instead of using the collector fluid flow rate as the control variable, the heat removal factor is used as the control variable, then an analytical solution to this problem may be obtained. In order to do this the pumping cost of the collector fluid must be related to the heat removal factor. The heat removal factor is a function of the collector fluid flow rate and so a relationship between the heat removal factor and the pumping cost was developed.

The following approximation was made in developing this relationship:

$$e^{-F'AU_L/\dot{m}C_p} = 1 - F'AU_L/\dot{m}C_p + \left[F'AU_L/\dot{m}C_p\right]^2 \frac{1}{2}$$

This approximation is a truncation of a Taylor series expansion. This approximation is good in that the exponent of the exponential function is less than 0.5 (Appendix II). By substituting the approximation into the heat removal factor equations and by rearranging the terms, the following equation is obtained:

$$\dot{m} = \left\{ F'^2 AU_L / 2C_p \right\} \left(F' - F_r \right)^{-1}$$

Recall the pumping cost was assumed to be proportional to the cube of the flow rate. With this assumption and the above equation it can be shown that the pumping cost is inversely proportional to the cube of the difference between the collector efficiency and the heat removal factor. The details of this relationship are shown in Appendix II. By substituting the new pumping cost into the

original objective function the following objective function is obtained:

$$J = \int_0^{tf} \left\{ C_4 Q_U - C_5 \left(\bar{T}' - F_r \right)^{-3} \right\} dt$$

By assuming that the available energy term (f) is constant for short periods of time, the following optimal control strategy may be derived (shown in Appendix II):

$$U_1^* = F_r^* = F_r - \sqrt[4]{\frac{3C_5}{C_4 f}}$$

This assumption is adequate for short time periods because the available energy varies with the solar insolation and the ambient temperature and neither of these change at a rapid rate.

The second plant to be considered was one with stratified storage and is modeled by the following differential equations:

$$\dot{x}_1 = n P_1 Q_U / C_5 \quad I = 1, 2, 3, \dots, n$$

where

$$P_1 = \dot{m} C_p (T_{out} - x_1) / Q_U$$

$$I = 2, 3, 4, \dots, n$$

$$P_I = \dot{m} C_p (x_{I-1} - x_1) / Q_U$$

n = the number of segments in storage.

Assumptions similar to those made in the case of uniform storage were again made. That is, P_T and f are constant for short time intervals. The same optimal control strategy was obtained as with the uniform storage. In order to obtain the original control variable (collector fluid flow) from the control variable used

for the analytical optimal control strategy simply solve the equation for $\dot{m}$ as a function of F_p (page 48).

Optimal Flow with No Pumping Cost

In Kovarik and Lesse's paper there was a discussion of the optimal flow rate through the collector if the pumping cost is not considered. It was stated that for uniform storage the optimal control strategy used was a "Bang-Bang" control strategy. This may be shown by looking at the useful energy collected equation, which is the product of the heat removal factor and the available energy term. The heat removal factor is zero when the collector flow equals zero and is equal to one when the collector flow rate is infinite. The collector flow rate which gives the maximum useful energy collected would be an infinite flow rate when the available energy term is positive. Otherwise the collector flow rate will be zero.

It would seem that the above discussion would be true for both uniform storage and for the stratified storage. This is not the case. When the flow rate is set to infinity then the temperature of the fluid out of the collector is almost equal to the inlet temperature of the fluid. With stratified storage the lower segment would be the only segment absorbing energy. More useful energy could be gained if at least half of the storage were gaining energy. It would be better to reduce the flow rate so that the temperature out of the collector is greater than the average temperature of the storage.

Comparison of the Optimal
Control Strategies

The analytical optimal control strategy was compared with both the "Bang-Bang" control strategy and to the numerical solution shown in Reference 4. The "Bang-Bang" control used in most solar installations works in the following manner. The collector fluid flow is turned on when the outlet temperature of the collector fluid is a specified amount above the storage temperature (11°C). The collector fluid flow is shut off when the outlet temperature of the collector fluid is a specified amount above the storage temperature (3°C).

Table 3 and Table 4 were developed using artificial weather data. The solar insolation for this day was a sine wave which peaked at noon with 800 watts per square meter. The sine wave had a half period of 12 hours. The ambient temperature was a constant of 10°C . The artificial weather data was used so that the results could be compared with the results from Reference 4.

Shown in Table 3 is a comparison of the objective function for all three control strategies. The results of the numerical solution from Reference 4 are shown in column 2 of Table 3. Shown in columns 3 and 4 of Table 3 are the results from the analytical optimal control strategy developed in this paper. The results from the "Bang-Bang" control strategy are shown in columns 5 and 6 of Table 3. By comparing the entries in column 2 with the entries in column 3 of Table 3 the amount of agreement between the numerical optimal solution from Reference 4 and the analytical optimal solution can be determined. The maximum percent difference is 0.1 percent. The

TABLE 3
OBJECTIVE FUNCTION COMPARISON

Time of Day	Numerical Optimal Control Strategy (Reference 4)	Analytical Optimal Control Strategy		Bang-Bang Control Strategy	
	J2	J1	J2	J1	J2
0700	1764	1768	1768	842	842
0800	7045	7056	7060	3704	3704
0900	15034	15052	15055	11416	11416
1000	24779	24797	24800	21550	21550
1100	35258	35269	35276	32360	32360
1200	45432	45432	45468	42840	42840
1300	54360	54360	54360	52020	52056
1400	61200	61164	61164	47168	57168
1500	65232	65196	65196	59472	59472
1600	66168	66096	66132	60120	60120

$$J_1 = \int [Q_U - C_5(F' - F_r)^{-3}] dt$$

$$J_2 = \int (Q_U - C_3 \dot{m}^3) dt$$

$$C_5 = 5.58 \times 10^{-5}$$

$$C_3 = 1000.$$

TABLE 4
USEFUL ENERGY COLLECTED.

Time of Day	Numerical Optimal Control Strategy (Reference 4)	Analytical Optimal Control Strategy	Bang-Bang Control Strategy
0700	1768	1782	846
0800	7063	7103	3712
0900	15073	15142	12571
1000	24847	24941	21517
1100	35359	35467	32389
1200	45576	45684	42876
1300	55332	54684	52092
1400	61380	61488	57204
1500	65448	65556	59058
TOTAL	66384	66492	60156

The units used were kJ.

difference could be accounted for in just the numerical handling of the problem.

The optimal control strategy does provide for a 9 percent improvement over the "Bang-Bang" control strategy in the objective function.

The input data for these results are shown in Table 5.

Shown in Table 4 is a comparison of the useful energy collected by all three control strategies. The results of the numerical solution from Reference 4 are shown in column 2 of Table 4. Shown in column 3 of Table 4 are the results from the analytical optimal control strategy developed in this paper. The results from

TABLE 5
INPUT DATA COLLECTOR CONTROL PROBLEM

Constants	Data
Collector Area (A)	8 m ²
Specific heat of collecting fluid (C _p)	4.187 kJ/kg/°C
Storage thermal capacitance (C _s)	837 kJ/°C
Thermal conductance between collecting fluid and surroundings (U _L)	14.4 kJ/hr/°C/m ²
Effective Transmittance absorptance products (τα) _e	0.7

the "Bang-Bang" control strategy are shown in column 4 of Table 4. By comparing the entries in column 2 with the entries in column 3 of Table 4 another measure of the agreement between the two optimal control strategies can be seen. The largest difference seen in these two columns is less than 1 percent. This indicates that the two optimal control strategies are in excellent agreement. The optimal control strategy does provide for a 10 percent improvement over the "Bang-Bang" control strategy in the useful energy collected.

Algorithm for Implementation of Optimal Control Strategy

The numerical optimal control strategy cannot be implemented for two reasons. The first reason is that it is not based on a measurable state of the system; the second reason is that one would have to be able to predict the exact solar weather conditions a day in advance.

The second optimal control strategy can be implemented as a state feedback system. There are three variables which must be measured. They are as follows: storage tank temperature, the ambient temperature, and finally either the collector outlet temperature or the solar insolation on the tilted surface. The available energy term (f) is the term which needs the feedback. Recall that the available energy term was given by the following equation:

$$f = A \{ H_T(\tau\alpha)_e - U_L(T_{in} - T_A) \}$$

If the solar insolation on the tilted surface, the ambient temperature and the storage tank temperature are all measured, then the available energy term (f) is easily calculated because all variables on the right side of the equation are known.

If the solar insolation on the tilted surface is not measured directly then it must be estimated. Since the value that is desired to be known is the available energy term, the solar insolation will not be estimated but the available energy term will be estimated. This can be done by looking at the collector outlet temperature as follows:

Case 1 - the collector flow is on.

$$f = \dot{m} C_p (T_{out} - T_{in}) / F_r$$

Case 2 - the collector flow is off.

$$f = A U_L (T_{out} - T_{in})$$

It is more practical to measure the collector outlet temperature because of the high cost associated with measuring the solar insolation on the tilted surface.

Comparison of the Control Strategies

Shown in Table 6 are comparisons of the useful energy and cost function obtained when using the "Bang-Bang" control strategy and the optimal control strategy using both measurements of solar insolation and collector fluid outlet temperature. The solar insolation and the ambient temperature were from observed weather data and can be seen in Figures 15 and 16.

TABLE 6
COMPARISON OF COLLECTOR FLUID FLOW
CONTROL STRATEGIES

Control Strategy	Useful Energy Collected	Objective Function
Bang-Bang	370100	360100
Optimal 1	387700	382100
Optimal 2	387300	381900

1 - measuring solar insolation on the tilted surface;

2 - measuring collector fluid outlet temperature.

The input data used to generate Table 6 can be seen in Table 7 under the water system heading.

By measuring the solar insolation and by using the optimal control strategy, there is an improvement of 5 percent over the "Bang-Bang" control strategy. There is also a 5 percent improvement when the collector outlet temperature is measured.

Figure 15. Measured Solar Insolation.

Figure 16. Measured Ambient Temperature

TABLE 7
INPUT DATA

Model Parameters	Symbol	Units	Water System	Air System
Collector Area	A	m ²	55.76	55.76
Collector Thermal Loss Coefficient	U _L	kJ/HR-°C-m ²	14.4	14.4
Heat Capacity of Storage	C _S	kJ/°C	19000.	19000.
Specific Heat of the Fluid	C _p	kJ/kg-°C	4.187	1.0
Collector Efficiency Factor	F'	---	.98	.98
Effective Transmittance Absorptance Product	($\tau\alpha$) _e	---	.7	.7
Pumping Cost		W-sec ³ /kg ³	1000.	1000.
Collector Fluid Flow Rate ¹	m	kg/HR	2722.	2636.

¹Used for the Bang-Bang control strategy only.

There is no real advantage in measuring solar insolation because results just as good can be obtained by measuring the collector outlet temperature, which is much more easily accomplished.

Shown in Table 7 are the data that were used for an air and water solar system. These data are representative of both systems. In this case the collectors are identical. The storage size and the collector fluid flow rate (for "Bang-Bang") were sized using the following rules of thumb for both systems.

Water system:

1. 2 gallons of storage for each square foot of collector.
2. The collector flow rate is about .02 gallons per minute per square foot of collector.

Air system:

1. About 50 pounds of rock per square foot of collector.
2. The collector flow rate is two cubic feet per minute per square foot of collector.

The results shown in Table 8 for the water system are identical to the results shown in Table 6 for the optimal control strategy which measures the collector outlet temperature. The optimal control strategy shows an improvement of 6 percent in the cost function and a 5 percent improvement in the useful energy collected, when compared to the "Bang-Bang" control strategy. The cost of pumping in this case for the "Bang-Bang" control strategy is 3 percent of the useful energy collected. On the other hand, the cost of pumping the collector fluid for the optimal control strategy is only 1 percent of the useful energy collected.

The cost of pumping the collector fluid for the optimal control strategy was approximately 50 percent less than the cost of pumping the collector fluid for the "Bang-Bang" control strategy.

TABLE 8
COLLECTOR CONTROL COMPARISON

System Type	Control Strategy	Useful Energy Collected (kJ)	Objective Function
Water	Bang-Bang	370100	360100
	Optimal	387300	381900
Air	Bang-Bang	390800	362500
	Optimal	394600	382300

Shown in Figure 17 is a plot of the collector fluid flow rate for the optimal control strategy for the water system. The collector fluid flow rate increases as the solar insolation increases and decreases as the solar insolation decreases. At noon (Time = 12) the flow rate remains fairly constant for an hour.

The optimal control strategy for the air system collected 1 percent more useful energy than did the "Bang-Bang" control strategy, and gave a 5.5 percent improvement in the cost function. The low percent improvement in useful energy collected could be due to the collector fluid inlet temperature being nearly constant. The percent improvement in useful energy collected is greater when energy is being removed from the storage as seen in Chapter IV. These results are shown in Table 8. The cost of pumping the collector fluid for the "Bang-Bang" control strategy is 7 percent of the useful energy col-

Figure 17. Collector flow rate.

control strategy is 3 percent of the useful energy collected. The objective function represents the difference between the useful energy collected and the cost of pumping the collector fluid. This means that an improvement in the objective function represents the total energy savings. This means that both the water and air optimal control strategies showed an improvement of slightly more than 5 percent in the total energy savings when compared to the "Bang-Bang" control strategy for this particular case.

In comparing the air system to the water system, it is interesting to note that the cost functions are almost identical. The air system was able to collect more useful energy than the water system by 2 percent.

The air system had more than double the pumping cost of the water system. There is a 0.1 percent difference between the objective function of the air and water systems. This means that both the air and water systems performed identically in total energy savings.

Comparison of Control Strategies with Different Weather Data

The measured weather data used for all the results discussed to this point was collected on February 8. Three other days were also chosen in order to test the effect of different weather conditions on the optimal control strategy. These three days are February 1, January 30, and February 12. Shown in Table 9 is a brief summary of the weather data for all four days.

February 1 was chosen because on that date the solar insolation on the horizontal surface was approximately the same as found

on February 8 while the average ambient temperature was higher. This was done in order to see the effect that the ambient temperature would have on the percent improvement of the optimal control strategy over the "Bang-Bang" control strategy.

January 30 was chosen because on that date the solar insolation on the horizontal surface was less than on February 8 while the average ambient temperature was approximately the same.

February 12 was chosen because on that date the solar insolation on the horizontal surface was less than half the solar insolation as recorded on February 8. The average ambient temperature is approximately the same as recorded on February 8.

January 30 and February 12 were chosen so as to see the effect the amount of solar insolation would have on the optimal control strategy.

In Table 10 is shown the amount of improvement that the optimal control strategy would have over the "Bang-Bang" control strategy.

There are three points to be noticed in Table 10. First, the percent improvement in the objective function and in the useful energy for the water system increases with a decrease in the solar insolation. Second, the air system shows practically no improvement in the amount of useful energy collected. Third, the objective function for the air system had an increase in the percent improvement with a decrease in the solar insolation.

There is an energy savings to be made by the use of the optimal control strategy for the collector fluid flow rate. There is some cost involved, for in order to implement the optimal control

TABLE 9
WEATHER DATA

Date	Solar Horizontal	Average Ambient Temperature
2/8	14292	-7°C
2/1	14190	2°C
1/30	9751	-5°C
2/12	6482	-7°C

TABLE 10
COMPARISON OF CONTROL STRATEGIES WITH
DIFFERENT WEATHER DATA

Date	Useful Energy	Objective Function
<u>Water</u>		
2/8	5%	6%
2/1	4%	5%
1/30	22%	26%
2/12	74%	82%
<u>Air</u>		
2/8	1%	5%
2/1	1%	4%
1/30	-1%	9%
2/12	1%	23%

strategy, a microprocessor type of controller would probably have to be used. This would mean a slightly higher initial cost of the controller which could be offset by the energy savings.

An example of how the additional initial cost could be offset by the fuel savings is seen in the following analysis. It has been shown that there is at least a 5 percent savings to be realized by use of the optimal control strategy in a home which has a thermal loss coefficient of $950 \text{ kJ/hr/}^{\circ}\text{C}$ and the home which uses electricity as the auxiliary heat system at $80.04/\text{KW} \cdot \text{HR}$. A larger thermal loss coefficient will give a larger energy savings and a smaller thermal loss coefficient will give a smaller energy savings. Assume that the home is located in an area where there are 7000 heating degree days. The savings using the optimal control strategy would be $\$50.00/\text{year}$. It can be estimated that the new controller might cost an additional $\$250.00$. If the additional initial cost of $\$250.00$ were borrowed at 9 percent interest then the loan plus the interest would be paid off by the energy savings in 7 years.

CHAPTER IV.

SOLAR HOUSE CONTROL PROBLEM

The third control problem is a two heat source problem. The system considered in this paper is a solar heating system, air or water, with an auxiliary heater as the second source. A schematic of the solar heating system considered can be seen in Figure 18. In the control of the house temperature, the control problem is similar to the control problem covered in Chapter 2, except that there are now two sources of heat. The standard control strategy used in most solar heated buildings is described as follows. The heat from the storage to the building is turned on when the enclosure temperature is 1°C below the desired enclosure temperature. It is turned off when the enclosure temperature is 1°C above the desired enclosure temperature. In some systems the heat from the storage to the building is also turned off any time the auxiliary heat source is turned on. The auxiliary heat source is turned on when the enclosure temperature is 2°C below the desired enclosure temperature. The auxiliary heat source is turned off when the enclosure temperature is 1°C below the desired enclosure temperature. In some solar systems, if the storage temperature is below a minimum temperature no attempt is made to heat from storage.

Figure 18. Schematic of Solar Space Heating System.

Quadratic Optimal Control Strategy

The objective for the optimal control strategy would be as follows:

1. Minimize the amount of discomfort.
2. Minimize the amount of auxiliary energy.
3. Minimize the amount of energy from storage.
4. Maximize the amount of useful energy collected.
5. Minimize the pumping cost for the collector fluid flow.

The objective function is as follows:

$$J = \int_0^{t_f} \left\{ c_1 x_1^2 + c_2 u_2^2 + c_3 u_3^2 - c_4 Q_U + c_5 \left(F' - u_1 \right)^{-3} \right\} dt$$

where

C_1, C_2, C_3, C_4, C_5 are weighting coefficients.

X_1 is the enclosure temperature error.

U_2 is the auxiliary energy.

U_3 is the energy from storage to the enclosure.

U_1 is F_r which is the heat removal factor.

Q_U is the useful energy collected.

F' is the collector efficiency factor.

There are two plants associated with this solar system. The first plant is the enclosure for which the temperature error (X_1) is given by the first differential equation below. The second plant is the storage system; the second differential equation below describes the storage tank temperature (X_2).

$$\begin{aligned}\dot{X}_1 &= \left\{ U_2 U_{2MAX} + U_3 U_{3MAX} - \left\{ X_1 + T_D - T_A \right\} \right\} / C_E \\ \dot{X}_2 &= \left\{ Q_U - U_3 U_{3MAX} - UA_{LOSS} \left\{ X_2 - X_1 - T_D \right\} \right\} / C_S\end{aligned}$$

The storage tank described here is for uniform storage. The control strategy is the same if the storage is stratified. The development for stratified storage is shown in Appendix III.

The optimal control strategy for all three control variables is as follows:

$$1. \quad U_1^* = F' = \sqrt[4]{\frac{3C_5}{C_4 F}} \quad (\text{collector flow rate})$$

$$2. \quad U_2^* = \lambda_1 U_{2MAX} / \left\{ 2C_2 C_E \right\} \quad (\text{auxiliary energy})$$

$$3. \quad U_3^* = \lambda_1 U_{3MAX} / \left\{ 2C_3 C_E \right\} \quad (\text{energy from storage})$$

If the optimal control variables (U_1^* , U_2^* , U_3^*) are substituted into the differential equation describing the two plants shown earlier then the two state differential equations, $\dot{X}_1$ and $\dot{X}_2$, are as shown below. The only Lagrange multiplier which appears in the optimal control variable equation is λ_1 . It is described by the differential equation, $\dot{\lambda}_1$, shown below. The problem now is to solve the following three differential equations:

$$\dot{X}_1 = \frac{\lambda_1 U_{2MAX}^2}{2C_2 C_E^2} + \frac{\lambda_1 U_{3MAX}^2}{2C_3 C_E^2} - \frac{UA(X_1 + T_D + T_A)}{C_E}$$

$$\dot{X}_2 = \left\{ \left[F - 4 \sqrt{\frac{3C_5}{C_4 F}} \right] F - \frac{\lambda_1 U_{3MAX}^2}{2C_3 C_E} - UA_{LOSS}(X_2 - X_1 - T_D) \right\} / C_5$$

$$\dot{\lambda}_1 = \lambda_1 UA/C_E + 2C_1 X_1$$

Only two of these three differential equations are coupled. In order to determine the optimal control strategy only λ_1 as a function of time needs to be known. This is because λ_1 is the only time dependent variable in the equations of the optimal control variable. Therefore, the only two differential equations which must be solved are $\dot{X}_1$ and $\dot{\lambda}_1$. The problem is now to solve two coupled linear differential equations. The solution is shown in Appendix III.

The solution is identical to the quadratic control strategy discussed in Chapter II.

Singular Optimal Control Strategy

It can be observed that the auxiliary energy and the energy from storage must be on and off at the same time. During periods when the storage could heat the building without any auxiliary energy the optimal control strategy requires that the auxiliary be on to some degree. Therefore, it was decided that it would be better if the auxiliary heat source were allowed to be turned on and off. Recall from Chapter II that the singular optimal control strategy allowed the heat source to be turned off at certain times. The objective function was changed to the following so that the control of the auxiliary energy source would have a singular solution representing a more realistic situation.

$$J = \int_0^{t_f} \left\{ C_1 \dot{x}_1^2 + C_2 U_2 + C_3 U_3^2 - C_4 Q_U + C_5 (F' - U_1)^{-3} \right\} dt$$

The optimal control strategy for the collector fluid flow rate and the heat removed from storage remain unchanged. The optimal control strategy for the auxiliary energy changes to the following:

1. If $\lambda_1 U_{2MAX} - C_2 C_E > 0$ then $U_2^* = 1$
2. If $\lambda_1 U_{2MAX} - C_2 C_E < 0$ then $U_2^* = 0$
3. If $\lambda_1 U_{2MAX} - C_2 C_E = 0$ then $U_2^* = \text{singular solution.}$

Again the singular solution gives the temperature error as a stationary point and the following control strategy results:

$$U_2^* = \frac{C_2 (UA)^2}{2C_1 (U_{2MAX})^2} - \frac{C_2 (U_{3MAX})^2}{2C_3 (U_{2MAX})^2} + \frac{UA}{U_{2MAX}} (T_D - T_A)$$

"Bang-Bang" Control Strategy

In all the results given, there are two "Bang-Bang" control strategies. The first "Bang-Bang" control strategy is the system where the storage heat system is turned on as needed until the storage temperature reaches the enclosure temperature.

The first "Bang-Bang" control strategy works in the following manner:

1. The transfer of heat from storage is turned on when the enclosure temperature is 1°C below the desired temperature.
2. The transfer of heat from storage is turned off when the enclosure temperature is 1°C above the desired enclosure temperature.
3. The transfer of heat from storage is zero when the storage temperature is equal to or less than the enclosure temperature.
4. The auxiliary heater is turned on when the enclosure temperature is 2°C below the desired temperature.
5. The auxiliary heater is turned off when the enclosure temperature is 1°C below the desired temperature.

The collector control strategy is the same as described in Chapter III for the "Bang-Bang" control strategy. The second "Bang-Bang" control strategy is the same as the first, except that the heat from the storage is turned off when the storage temperature is below 30°C . One assumption which was made in the simulation was that the effectiveness of the heat exchanger between the storage tank and the enclosure was constant. The effectiveness of the

heat exchanger was set to 0.7 for the water system and it was set to 1.0 for the air system.

Results for the Water Solar System

The input data for all the results are shown in Table 11. The data shown in Table 11 are the same as shown in Table 7 of Chapter III, except that the data for the house are now included in Table 11. The weather data used for these results are the same real weather data shown in Chapter III, Figures 15 and 16.

Recall that in Chapter II the two optimal control strategies allowed the temperature error to be free at the final time. In the results shown in this chapter, the temperature error was required to be zero at the final time thereby eliminating the temperature drop at the final time as seen in Chapter II.

Shown in Table 12 are the results obtained for the water system with uniform storage. There are results for three cases shown in Table 12 with only the starting temperature of the storage being changed. The starting temperature of storage in Chapter II was 10°C. This is different from the three starting storage tank temperatures shown in Table 12.

The first starting tank temperature of 30°C shows that both optimal control strategies used less auxiliary energy than the second "Bang-Bang" control strategy. The first "Bang-Bang" control strategy used less auxiliary energy than the optimal control strategies. This does not mean that the first "Bang-Bang" control strategy provided for an overall cost or energy savings. The quantities which must be compared are the objective functions because they show the overall performance improvement. In comparing the "Bang-Bang" control strategies

TABLE 11
INPUT DATA

Model Parameters	Symbol	Units	Water System	Air System
Collector Area	A	m ²	55.76	55.76
Collector Thermal Loss Coefficient	U _L	kJ/HR-°C-m ²	14.4	14.4
Heat Capacity of Storage	C _S	kJ/°C	19000.	19000.
Specific Heat of Collector Fluid	C _p	kJ/kg-°C	4.187	1.0
Collector Efficiency Factor	F'	---	.98	.98
Effective Transmittance Absorptance Product	($\tau\alpha$) _e	---	.7	.7
Storage Thermal Loss Coefficient	UA _{LOSS}	kJ/HR-°C	20.07	53.14
Collector Fluid Flow Rate ¹	m	kJ/HR	2722.	2636.
Heat Capacity of Enclosure	C _E	kJ/°C	18090.	18890.
Enclosure Thermal Loss Coefficient	UA	kJ/HR-°C	949.5	949.5
Maximum output of Auxiliary Heater	U ₂ MAX	kJ/HR	105500.	105500.
Desired Enclosure Temperature	T _D	°C	20.	20.
Weighting Coefficients on:				
1. Temperature error	C ₁	---	10.	10.
2. Auxiliary energy ²	C ₂	---	4000.	4000.
3. Storage energy	C ₃	---	500.	500.
4. Useful energy	C ₄	---	.01	.01
5. Pumping costs	C ₅	---	.0006	.042

¹Used for Bang-Bang control strategy only.

²Four time the value shown was used for the Quadratic Optimal Control Strategy.

TABLE 12
CONTROL STRATEGY COMPARISON FOR A WATER SOLAR SYSTEM

Quantity Measured	Initial Storage Temperature	"Bang-Bang" Control Strategy		Quadratic Optimal Control Strategy	Singular Optimal Control Strategy
		1	2		
30°C					
Useful Energy Collected		323000.	300000.	330000.	323000.
Auxiliary Energy Consumed		190000.	317000.	298000.	294000.
Storage Energy Consumed		389000.	295000.	323000.	326000.
Quadratic Objective Function		38075.	51482.	8499.	---
Singular Objective Function		16475.	15482.	---	13600.
50°C					
Useful Energy Collected		281000.	281000.	280000.	286000.
Auxiliary Energy Consumed		0.	0.	123000.	30200.
Storage Energy Consumed		605000.	605000.	513000.	597000.
Quadratic Objective Function		6969	6969	2200.	---
Singular Objective Function		6969	6969	---	4884
70°C					
Useful Energy Collected		215000.	215000.	220000.	220000.
Auxiliary Energy Consumed		0.	0.	51300.	0.
Storage Energy Consumed		655000.	655000.	591000.	641000.
Quadratic Objective Function		3524.	3524.	- 91.	---
Singular Objective Function		3524.	3524.	---	282.

¹The storage was allowed to heat the enclosure until the storage temperature was less than or equal to the enclosure temperature.

²No heat was drawn from storage after storage temperature reached 30°C .

it can be seen that the value of the singular objective function is less for the second "Bang-Bang" control strategy and made an overall cost savings or performance improvement when compared with the first "Bang-Bang" control strategy.

The singular optimal control strategy shows a 12 percent improvement in the objective function when compared to the second "Bang-Bang" control strategy. This would mean that the singular optimal control strategy showed an overall performance improvement when compared with either "Bang-Bang" control strategy. The quadratic optimal control strategy shows a 78 percent improvement over the first "Bang-Bang" control strategy and an 83 percent improvement over the second "Bang-Bang" control strategy in the cost function. There is a larger improvement in the quadratic optimal control strategy cost function than in the singular optimal control strategy cost function because the auxiliary control function, which is a fraction, is squared in the quadratic optimal control strategy's cost function and so reduces at a faster rate.

In looking at the case where the initial storage tank temperature was 50°C, it can be seen that the "Bang-Bang" control strategies use no auxiliary energy. Both optimal control strategies used some auxiliary energy. The quadratic optimal control strategy used some auxiliary energy all day, but most of the auxiliary energy was used at the final time when it was driving the enclosure temperature to the desired enclosure temperature. The same is true for the singular optimal control strategy except it uses no auxiliary energy until the final time when it drives the enclosure temperature to the desired enclosure temperature. Again, there is an improvement made by the optimal control strategies in the objective functions which are the real measure of the overall savings.

When the storage tank temperature was initialized at 70°C , the singular optimal control strategy and the two "Bang-Bang" control strategies use no auxiliary energy. There was some auxiliary energy used for the quadratic control strategy. This is because in that formulation the auxiliary heater must be on whenever heat is being removed from storage. The total energy delivered to the enclosure through use of the optimal control strategies is less than the total energy delivered through use of the "Bang-Bang" control strategies. The optimal control strategies also collected more useful energy than the "Bang-Bang" control strategies collected.

Results of the Air Solar System

Shown in Table 13 are the results of a solar heated home using air as the collector fluid. The air solar storage system was modeled as a stratified storage system. The air system was started at the same initial storage temperature as was the water system.

It can be seen in the case where the top segment of storage was initialized at 30°C that both optimal control strategies used 37 percent less auxiliary energy than the second "Bang-Bang" control strategy used. The singular optimal control strategy used an almost identical amount of auxiliary energy as the first "Bang-Bang" control strategy used. Again the real measure of the overall improvement can be seen by comparing the objective function values of the different strategies. There was approximately a 90 percent improvement in the cost function made by the optimal control strategies.

The second initial storage temperature was 50°C . It can be seen that the "Bang-Bang" control strategies used no auxiliary energy,

TABLE 13
CONTROL STRATEGY COMPARISON FOR AN AIR SOLAR SYSTEM

Quantity Measured	Initial Storage Temperature Top Segment	"Bang-Bang" Control Strategy		Quadratic Optimal Control Strategy	Singular Optimal Control Strategy
		1	2		
30°C					
Useful Energy Collected		354000.	339000.	359000.	357000.
Auxiliary Energy Consumed		162000.	262000.	195000.	165000.
Storage Energy Consumed		449000.	341000.	433000.	464000.
Quadratic Objective Function		31768.	40343	3798.	---
Singular Objective Function		37918.	50293.	---	7712.
50°C					
Useful Energy Collected		333000.	333000.	356000.	356000.
Auxiliary Energy Consumed		0.	0.	83500.	23400.
Storage Energy Consumed		652000.	652000.	556000.	616000.
Quadratic Objective Function		2938.	2938.	-321.	---
Singular Objective Function		2938.	2938.	---	409.
70°C					
Useful Energy Collected		312000.	312000.	356000.	355000.
Auxiliary Energy Consumed		0.	0.	635000.	21200.
Storage Energy Consumed		655000.	655000.	578000.	621000.
Quadratic Objective Function		1170.	1170.	-1152.	---
Singular Objective Function		1170.	1170.	---	-513.

¹ The storage was allowed to heat the enclosure until the storage temperature was less than or equal to the enclosure temperature.

² No heat was drawn from storage after storage temperature reached 30°C.

while both optimal control strategies used some auxiliary energy. However, the total energy delivered to the enclosure by the optimal control strategies was less than the amount delivered by the "Bang-Bang" control strategies. Both optimal control strategies collected more useful energy by 6 percent than the "Bang-Bang" control strategies collected. Of course the cost functions were reduced by the optimal control strategies and again this was by 90 percent.

The third starting storage temperature was initialized at 70°C for the top segment of the storage. The "Bang-Bang" control strategies used no auxiliary energy and the optimal control strategies used some auxiliary energy. The singular optimal control strategies used auxiliary energy in the morning when the collector fluid flow was turned on and also in the evening just before the collector fluid flow was shut off. The air system was modeled to heat the enclosure directly from the collector when the enclosure was calling for heat and the collector fluid was flowing. When the auxiliary energy was used the collector outlet temperature was low causing the need for extra heat. The optimal control strategies did use less total energy to heat the enclosure than either "Bang-Bang" control strategy. Twelve percent more useful energy was collected by the optimal control strategies than by the "Bang-Bang" control strategies in this case. Again the optimal control strategies showed an improvement in the objective function when compared to the "Bang-Bang" control strategies.

Comparison of the Water and Air

Solar Systems

Shown in Table 14 is a comparison between the air and the water solar systems. The negative sign in the percent difference

TABLE 14
COMPARISON OF THE WATER TO THE AIR
SOLAR SYSTEMS

Quantity Measured	Initial Storage Temperature	Percent Difference ³
	30°C	
Useful Energy Collected		-11
Auxiliary Energy Consumed		44
Storage Energy Consumed		-42
Energy Delivered to Enclosure		- 1
	50°C	
Useful Energy Collected		-24
Auxiliary Energy Consumed		23
Storage Energy Consumed		- 3
Energy Delivered to Enclosure		- 2
	70°C	
Useful Energy Collected		-61
Auxiliary Energy Consumed		----- ¹
Storage Energy Consumed		- 3
Energy Delivered to Enclosure		----- ²

¹The water solar system used no auxiliary energy.

²Less than 1 percent.

The calculation was water system minus the air system quantity divided by the water system.

The singular optimal control strategy was used for this comparison.

column indicates that the air system number was larger than the water system number. The air system was able to collect more useful energy than the water system in all cases. The air system was able to collect the same amount of useful energy regardless of the initial temperature of the storage. In comparison, the water system reduced the amount of useful energy it was able to collect with increases in the initial storage temperature. This is the reason that the percent difference of the useful energy collected increases with an increasing initial storage temperature.

The total energy used to heat the enclosure is the same for both systems. The amount of energy which came from each source was different and varied with changes in the initial storage temperatures. When the initial temperature was 30°C then the water system used 44 percent more auxiliary energy than the air system. The air system used 23 percent less auxiliary energy than the water system when the initial storage temperature was 50°C. The water system used no auxiliary energy when the initial storage temperature was 70°C; the air system used some auxiliary energy.

Comparison of Control Strategies with Different Weather Data

There were three additional dates which were discussed in Chapter III. These three dates represented different weather conditions.

A good solar insolation and a cold average ambient temperature were recorded on February 8. The other three dates are compared to the February 8 weather data. Good solar insolation and a 9°C higher

average ambient temperature was recorded on February 1. About the same average ambient temperature and about half the solar insolation was recorded on January 30. Poor solar insolation and the same average ambient temperature was recorded on February 12. The data for these 4 days is shown in Table 9 (Chapter III). Measured weather data recorded on the first ten days of February was used to obtain the results for a ten day continuous simulation.

In Table 15 is shown the percent improvement that the optimal control strategy would make over the "Bang-Bang" control strategy. The singular optimal control strategy was used for the results shown in Table 15. The "Bang-Bang" control strategy in Table 15, used a cut-off temperature of 30°C ("Bang-Bang" control strategy 2).

The negative sign indicates that the numerical values of the optimal control strategy were less than the numerical values of the "Bang-Bang" control strategy. The positive sign indicated that the numerical values of the optimal control strategy were greater than the numerical values of the "Bang-Bang" control strategy. A positive sign is desired for the useful energy collected, and this indicates that more useful energy was collected by the optimal control strategy. A negative sign is desired for the auxiliary energy consumed and for the objective function. This indicates that there was less auxiliary energy consumed and that the objective function was minimized.

There are three main points to notice in Table 15. First, the optimal control strategy collected more useful energy in all cases than did the "Bang-Bang" control strategy. Second, the optimal control strategy reduced the auxiliary energy consumption in practically

TABLE 15
COMPARISON OF CONTROL STRATEGIES FOR DIFFERENT WEATHER DATA

Date	Quantity Measured	Percent Improvement	
		Water System	Air System
2/8	Useful energy collected	2	7
	Auxiliary energy consumed	1	1
	Storage energy consumed	-1	-6
	Objective function	-30	-86
2/1	Useful energy collected	8	12
	Auxiliary energy consumed	0	1
	Storage energy consumed	-5	-2
	Objective function	-101	-243
1/30	Useful energy collected	65	7
	Auxiliary energy consumed	-49	-55
	Storage energy consumed	14	17
	Objective function	-72	-36
2/12	Useful energy collected	119	37
	Auxiliary energy consumed	-27	-57
	Storage energy consumed	18	57
	Objective function	-69	-28
10 day	Useful energy collected	9	10
	Auxiliary energy consumed	-27	-54
	Storage energy consumed	9	13
	Objective function	-77	-59

all cases. Third, the ten day continuous simulation gives better indication of what the percent improvement would be for longer periods of time.

Conclusions

The results presented in this paper indicate there is a savings to be realized by using the optimal control strategy. One reason these savings could be higher than indicated in this paper is that the energy savings are accumulative. That is, a savings made one day will be reflected in the next day's energy usage. With a solar system, the energy is stored until needed and so the energy can be carried over to the next day. Two other major items which will have an effect on the energy savings are the input data and the weather conditions.

CHAPTER V.

IMPLEMENTATION

As shown earlier there is a fuel savings to be realized by using the optimal control strategy for temperature control of a building. The optimal control for the useful energy collected gives approximately a 5 percent increase in energy collected. The combined system showed an improvement of about 6 to 10 percent over the control strategy presently being used. The new optimal control strategy does three things. First, it maintains a temperature closer to the desired temperature of the enclosure. Second, it uses less energy to heat the enclosure. Third, it collects more useful energy.

Error Analysis

Since there is a savings to be realized by using the optimal control strategy it would seem reasonable to implement the optimal control strategy in an actual building. But, before implementing the optimal control strategy it should be tested for sensitivity to errors in the model constants. Some error analyses were conducted for the quadratic optimal control strategy; the controller was found to be fairly insensitive to errors in the heat capacity and the heat loss coefficient of the enclosure. The singular control strategy for temperature error is, by inspection, very sensitive to errors in the heat loss coefficient of the enclosure. This is because when the temperature of the enclosure is away from the singular solution

the heat is either on or off. Thus, with errors in the heat loss coefficient, the heat source will be acting like a "Bang-Bang" controller with a zero dead band.

Enclosure Temperature Controller

After the error analyses have been completed, the algorithms could be implemented on a microprocessor. In this case, since a lot of calculations are required by the optimal control strategies, it might be easier if a calculator chip and a microprocessor were used as the basis for the hardware of the controller. The only problem with the calculator chip is that it has a slow cycle time. But, since the equipment that the controller is connected to has a slow response time the slow cycle time should be no problem for this control situation.

Shown in Table 16 is a summary of the approximate number of arithmetic operations which need to be done when implementing the optimal enclosure temperature control strategies. This number of arithmetic operations is representative of the number of arithmetic operations that would have to be done each time the control variable and the ambient temperature were updated. This is in accordance with the third method of updating the control strategy as described in Chapter II. Implementing the control strategy also requires two temperature measurements. These temperature measurements are the enclosure temperature and the ambient temperature.

TABLE 16
NUMBER OF ARITHMETIC OPERATIONS FOR THE OPTIMAL
TEMPERATURE CONTROL STRATEGY

Type of Operation	Number of Times
Additions and Subtractions	22
Multiplications	29
Divisions	12
Exponential Function	2

Collector Fluid Flow Controller

The following three temperature measurements would be required to implement the collector fluid flow optimal control strategy:

1. Ambient temperature.
2. Collector fluid outlet temperature.
3. Storage temperature.

There would be even more temperature measurements required if the storage were considered to be stratified. The number of arithmetic calculations is fairly minimal for this type of problem. There are 3 multiplication, 3 subtraction and 2 division operations. There is also a fourth root which must be calculated.

This number of arithmetic operations was obtained by examining the following three equations:

$$1. \quad \dot{m} = \dot{m}_p (T_{OUT} - T_S) / F_r$$

$$2. \quad U_1^* = F' - \sqrt[4]{k_1 / f}$$

$$3. \quad \dot{m} = k_2 / (F' - U_1^*)$$

where

k_1 and K_2 are lumped constants.

Equation 1 is the calculation for f when the collector outlet temperature is measured. Equation 2 is the optimal control variable. Equation 3 is the calculation for obtaining the collector fluid flow rate from the control variable. All three of these equations can be found in Chapter III.

Because of the low number of arithmetic operations this control strategy could be easily implemented.

CHAPTER VI.

CONCLUSIONS AND RECOMMENDATIONS

In this paper there were three basic control problems which were discussed. An optimal control strategy was developed for each of these control problems. The optimal control strategy developed for the second control problem, which was the collector fluid flow rate problem, is the easiest, by far, to implement. This was shown in Chapter V. Since there was an energy savings to be realized by the use of the optimal control strategy for the collector fluid flow rate and the implementation is fairly easily accomplished, then it is recommended that this control strategy be implemented in solar systems.

The optimal control strategies for the first and third control problems show that an energy savings could be realized. These control strategies would probably involve the use of a microprocessor in order to be implemented. One problem with implementing the optimal control strategies for problems one and three is that the control strategy requires a variable output furnace. A variable output gas furnace is not available on the market today. If the house is heated by electric resistance then the variable output furnace is no problem.

There are three problems related to the work in this paper which need further investigation. The first problem is to develop a variable output natural gas furnace. This furnace needs to have

a variable capacity output with little change in the efficiency at any of the output capacities.

The second problem which should be added to the solar control problem is that of an optimal control for a domestic hot water system. This is very similar to the work done in Chapter III of this paper except the plant is a different type of system. The objective for this problem would be to maximize the useful energy collected minus the pumping cost--which is identical to the objective of problem 2. Because of this similarity, the solution would probably be the same as the solution to problem 2 which is shown in Chapter III.

The third problem is for a combined solar space heating and domestic hot water system. In this case it is desirable to get the maximum amount of energy to both the storage and the domestic hot water. One problem with the combined system is the inability to predict the load on each subsystem. Without knowledge of the loads it is difficult to determine the amount of useful energy to be delivered to each subsystem.

These are three problems which are related to the work done in this paper which need more investigation. Further investigation is needed into these areas.

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APPENDIX I.

The following is the derivation for the quadratic temperature optimal control strategy with a list of terms.

T_E = Temperature of enclosure.

T_D = Temperature desired in the enclosure.

T_A = Ambient temperature.

UA = Heat loss coefficient of the enclosure.

C_E = Heat capacity of the enclosure

U_{2MAX} = Maximum output of the heat source.

Let

$$x_1 = T_E - T_D$$

$$u_2 = \text{energy delivered to enclosure}/U_{2MAX}$$

objective function

$$J = \int_0^{t_f} \left\{ C_1 x_1^2 + C_2 u_2^2 \right\} dt$$

$$\dot{x}_1 = \left\{ U_2(U_{2MAX}) - UA(x_1 + T_D - T_A) \right\} / C_E$$

Let

$$\dot{x}_2 = - \left\{ C_1 x_1^2 + C_2 u_2^2 \right\}$$

$$H = \lambda_1 \left\{ U_2(U_{2MAX}) - UA(x_1 + T_D - T_A) \right\} / C_E \\ + \lambda_2 \left\{ -C_1 x_1^2 - C_2 u_2^2 \right\}$$

$$\frac{\delta H}{\delta u_2} = \frac{\lambda_1 U_{2MAX}}{C_E} - 2\lambda_2 C_2 u_2$$

$$\dot{\lambda}_1 = -\frac{\delta H}{\delta x_1} = \frac{\lambda_1 UA}{C_E} + 2\lambda_2 x_1$$

$$\dot{\lambda}_2 = \frac{\delta H}{\delta x_1} = 0$$

$$\therefore \dot{\lambda}_2 = \text{a constant.}$$

The final conditions on the $\dot{\lambda}$ equations are as follows:

1. If $x_1(t_f) = \text{specified}$ then $\lambda_1(t_f) = v$ and $\lambda_2(t_f) = 1$
2. If $x_1(t_f) = \text{free}$ then $\lambda_1(t_f) = 0$ and $\lambda_2(t_f) = 1$

$$\therefore \lambda_2 \equiv 1 \text{ for all time.}$$

By setting $\frac{\delta H}{\delta u_2} = 0$ we can solve for the optimal control u_2^* .

$$u_2^* = \frac{\lambda_1 u_{2\text{MAX}}}{2C_2 C_E}$$

$$\therefore \dot{x}_1 = \frac{\lambda_1 (u_{2\text{MAX}})^2}{2C_2 C_E^2} - \frac{UA}{C_E} (x_1 + T_D - T_A)$$

$$\dot{\lambda}_1 = \lambda_1 \frac{UA}{C_E} + 2C_1 x_1$$

Since $\dot{x}_1$ and $\dot{\lambda}_1$ are linear ordinary differential equations the solution can be determined.

Assumption: $T_A = B - A \sin(\omega t + \rho)$

Let

$$k_1 = UA/C_E$$

$$k_2 = (u_{2\text{MAX}})^2 / 2C_2 C_E^2$$

$$k_3 = 2C_1$$

$$k_4 = UA(T_D - B)/C_E$$

$$k_5 = A(UA) \cos \rho / C_E$$

$$k_6 = A(UA) \sin \rho / C_E$$

then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} -k_1 & k_2 \\ k_3 & k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix} + \begin{bmatrix} k_4 + k_5 \sin(\omega t) + k_6 \cos(\omega t) \\ 0 \end{bmatrix}$$

$$\text{Let } \beta = \sqrt{k_1^2 + k_2 k_3}$$

$$\begin{aligned} x(t) = & \left\{ \frac{\beta - k_1}{2\beta} e^{\beta t} + \frac{\beta + k_1}{2\beta} e^{-\beta t} \right\} x_1(t_0) \\ & + \left\{ \frac{k_2}{2\beta} (e^{\beta t} - e^{-\beta t}) \right\} \lambda_1(t_0) \\ & + \left\{ \frac{k_4}{2\beta^2} \left[(\beta - k_1)e^{\beta t} - (\beta + k_1)e^{-\beta t} - 2k_1 \right] \right. \\ & + \frac{k_5 \omega}{\beta^2 + \omega^2} \left[\frac{k_1 \sin(\omega t)}{\omega} - \cos(\omega t) + \frac{\beta + k_1}{2\beta} e^{\beta t} + \frac{\beta - k_1}{2\beta} e^{-\beta t} \right] \\ & \left. + \frac{k_6}{2(\beta^2 + \omega^2)} \left[2k_1 \cos(\omega t) + 2\omega \sin(\omega t) + (\beta - k_1)e^{\beta t} - (k_1 + \beta)e^{-\beta t} \right] \right\} \end{aligned}$$

$$\begin{aligned} \lambda(t) = & \left\{ \frac{k_3}{2\beta} (e^{\beta t} - e^{-\beta t}) \right\} x_1(t_0) \\ & + \left\{ \frac{\beta + k_1}{2\beta} e^{\beta t} - \frac{(k_1 - \beta)}{2\beta} e^{-\beta t} \right\} \lambda_1(t_0) \\ & + \frac{k_3 k_4}{2\beta^2} (e^{\beta t} + e^{-\beta t} - 2) \\ & + \frac{k_3 k_5 \omega}{(\omega^2 + \beta^2)} \left[\frac{1}{2\beta} (e^{\beta t} - e^{-\beta t}) - \frac{\sin(\omega t)}{\omega} \right] \\ & + \frac{k_6 k_3}{2(\beta^2 + \omega^2)} \left[e^{\beta t} + e^{-\beta t} - 2 \cos(\omega t) \right] \end{aligned}$$

initial conditions

$$x_1(t_0) = \text{given as } x_0.$$

If $X(t_f) = \text{free}$ then $\lambda_1(t_f) = 0$

$$\begin{aligned} \lambda_1(t_0) = & \left\{ \frac{X_0 k_3}{2\beta} (e^{\beta t_f} - e^{-\beta t_f}) + \frac{k_3 k_4}{2\beta^2} (e^{\beta t_f} + e^{-\beta t_f} - 2) \right. \\ & + \frac{k_3 k_5}{(\omega^2 + \beta^2)} \left[\frac{\omega}{2\beta} (e^{\beta t_f} - e^{-\beta t_f}) - \sin(\omega t_f) \right] \\ & + \frac{k_6 k_3}{2(\omega^2 + \beta^2)} \left[e^{\beta t_f} + e^{-\beta t_f} - 2\cos(\omega t_f) \right] \Big\} \\ & + \left\{ \frac{(k_1 - \beta_1)}{2\beta} e^{-\beta t} - \frac{(k_1 + \beta_1)}{2\beta} e^{\beta t} \right\} \end{aligned}$$

If $X(t_f) = \text{specified as } X_f$

then

$$\begin{aligned} \lambda_1(t_0) = & \left\{ X_0 \left[\frac{(\beta - k_1)}{2\beta} e^{\beta t_f} + \frac{(\beta + k_1)}{2\beta} e^{-\beta t_f} \right] - X_{t_f} \right. \\ & + \frac{k_4}{2\beta^2} \left[(\beta - k_1)e^{\beta t_f} - (\beta + k_1)e^{-\beta t_f} - 2k_1 \right] \\ & + \frac{k_5 \omega}{(\beta^2 + \omega^2)} \left[\frac{(\beta - k_1)}{2\beta} e^{\beta t_f} + \frac{(\beta + k_1)}{2\beta} e^{-\beta t_f} + \frac{k_1 \sin(\omega t_f)}{\omega} - \cos(\omega t_f) \right] \\ & + \frac{k_5}{2(\beta^2 + \omega^2)} \left[2k_1 \cos(\omega t_f) + 2\omega \sin(\omega t_f) + (\beta - k_1)e^{\beta t_f} - (\beta + k_1)e^{-\beta t_f} \right] \Big\} \\ & + \left\{ \frac{k_2}{2\beta} (e^{-\beta t_f} - e^{\beta t_f}) \right\} . \end{aligned}$$

The following is the derivation for the singular temperature optimal control strategy.

Objective function

$$\begin{aligned} J &= \int_0^{t_f} \left\{ C_1 \dot{x}_1^2 + C_2 U_2 \right\} dt \\ \dot{x}_1 &= \left\{ U_2 (U_{2, \text{MAX}}) - UA(X_1 + T_D - T_A) \right\} / C_E \end{aligned}$$

$$\begin{aligned}
 \text{Let } \dot{x}_2 &= - \left\{ C_1 x_1^2 + C_2 U_2 \right\} \\
 H &= \lambda_1 \left\{ U_2 (U_{2\text{MAX}}) - UA(x_1 + T_0 - T_A) \right\} / C_E \\
 &\quad + \lambda_2 \left\{ -C_1 x_1^2 - C_2 U_2 \right\} \\
 \frac{\delta H}{\delta U_2} &= \frac{\lambda_1 U_{2\text{MAX}}}{C_E} - \lambda_2 C_2 \\
 \dot{\lambda}_1 &= - \frac{\delta H}{\delta x_1} = \frac{\lambda_1 UA}{C_E} + 2\lambda_2 C_1 x_1 \\
 \dot{\lambda}_2 &= - \frac{\delta H}{\delta x_2} = 0 \\
 \lambda_2(t_f) &= 1
 \end{aligned}$$

$\therefore \lambda_2 = 1$ for all time.

In this case U_2 does not appear in the partial of the Hamiltonian with respect to U_2 ; therefore the solution is as follows:

$$\begin{aligned}
 \text{If } \lambda_1 U_{2\text{MAX}} - C_2 C_E > 0 & \text{ then } U_2^* = 1 \\
 \text{If } \lambda_1 U_{2\text{MAX}} - C_2 C_E < 0 & \text{ then } U_2^* = 0 \\
 \text{If } \lambda_1 U_{2\text{MAX}} - C_2 C_E = 0 & \text{ then } U_2^* = ? \text{ (intermediate solution)}
 \end{aligned}$$

The intermediate solution:

$$\begin{aligned}
 \lambda_1 &= C_2 C_E / U_{2\text{MAX}} \\
 \frac{d(\delta H / \delta U_2)}{dt} &= \frac{U_{2\text{MAX}}}{C_E} \dot{\lambda}_1 \text{ (set to zero)} \\
 \frac{U_{2\text{MAX}}}{C_E} \left\{ \lambda_1 \frac{UA}{C_E} + 2C_1 x_1 \right\} &= 0
 \end{aligned}$$

$$\therefore \quad \dot{x}_1 = -\lambda_1 UA / (2C_1 C_E)$$

$$\frac{d^2(\delta H / \delta U_2)}{dt} = \frac{U_{2MAX}}{C_E} \left\{ \lambda_1 \frac{UA}{C_E} + 2C_1 \dot{x}_1 \right\} \text{ (set to zero)}$$

$$\frac{U_{2MAX}}{C_E} \left\{ \frac{UA}{C_E} \left[\lambda_1 \frac{UA}{C_E} + 2C_1 \dot{x}_1 \right] + 2C_1 \left[\frac{U_2 U_{2MAX}}{C_E} - \frac{UA}{C_E} (x_1 + T_D - T_A) \right] \right\} = 0$$

$$\therefore \quad U_2 = \frac{U_A^2}{U_{2MAX}} (T_D - T_A) + C_2 \left(\frac{UA}{U_{2MAX}} \right)^2$$

Now the optimal control problem becomes a problem of when does the control switch from one to another type of control. The description of the switching function is found in Chapter II.

APPENDIX II.

The development of the relationship between heat removal factor of the collector and the collector flow rate.

$$F_r = \frac{\dot{m}C_p}{AU_L} \left\{ 1 - e^{-\frac{F'AU_L}{\dot{m}C_p}} \right\}$$

If the quantity $\left\{ \frac{F'AU_L}{\dot{m}C_p} \right\} < 1$ then the following approximation can be made with fair accuracy. The smaller the quantity the better the accuracy.

$$e^{-\frac{F'AU_L}{\dot{m}C_p}} = 1 - \left\{ \frac{F'AU_L}{\dot{m}C_p} \right\} + \frac{1}{2} \left\{ \frac{F'AU_L}{\dot{m}C_p} \right\}^2$$

The following values were picked from the input data given in Chapter II.

$$F' = .98$$

$$A = 55.74 \text{ m}^2$$

$$\dot{m} = 2722 \text{ kg/HR}$$

$$U_L = 14.4 \text{ kJ} \cdot (\text{HR} \cdot ^\circ\text{C} \cdot \text{HR})^{-1}$$

$$C_p = 4.187 \text{ kJ} \cdot (\text{kg} \cdot ^\circ\text{C})^{-1}$$

$$\frac{F'AU_L}{\dot{m}C_p} = 0.07$$

Since the number was less than one the approximation can be made.

$$F_r = \frac{\dot{m} C_p}{AU_L} \left\{ 1 - \left[1 - \frac{F' AU_L}{\dot{m} C_p} + \frac{1}{2} \left(\frac{F' AU_L}{\dot{m} C_p} \right)^2 \right] \right\}$$

$$F_r = F' - \left\{ \frac{F'^2 AU_L}{2C_p} \right\} \frac{1}{\dot{m}}$$

$$\therefore \dot{m} = \left\{ \frac{F'^2 AU_L}{2C_p} \right\} (F' - F_r)^{-1}$$

Pumping cost is given by the following equation:

$$P = k \dot{m}^3$$

$$\dot{m}^3 = \left\{ \frac{F'^2 AU_L}{2C_p} \right\}^3 (F' - F_r)^{-3}$$

$$P = k \left\{ \frac{F'^2 AU_L}{2C_p} \right\}^3 (F' - F_r)^{-3}$$

List of terms:

H_T = Solar on the tilted surface.

F_r = Heat removal factor.

$\dot{m}$ = Collector flow rate.

C_p = Specific heat of the fluid.

A = Collector area.

U_L = Heat loss coefficient.

F' = Collector efficiency.

P = Pumping cost.

k = Pumping cost constant.

U_1 = Control variable.

Q_U = Useful energy collected.

f = Maximum energy available.

T_S = Temperature storage.

$(\tau\alpha)_e$ = The effective transmittance absorptance product.

The following is the derivation of the optimal collector flow rate considering pumping cost with uniform storage.

$$\text{objective function } J = \int_0^t f \left\{ C_4 Q_U - C_5 (F' - F_r)^{-3} \right\} dt$$

$$T_S = \frac{Q_U}{C_S} \quad \begin{cases} \text{no storage losses} \\ \text{no heat delivered to enclosure} \end{cases}$$

$$Q_U = F_r f$$

$$f = A [H_T (\tau\alpha)_e - U_L (T_S - T_A)]$$

$$F_r = \frac{mC_p}{AU_L} \left(1 - e^{-\frac{F' AU_L}{mC_p}} \right)$$

Let $U_1 = F_r$

$$X_1 = T_S$$

$$X_2 = C_4 U_1 f - C_5 (F' - U_1)^{-3}$$

$$H = \lambda_1 \left\{ \frac{U_1 f}{C_S} \right\} + \lambda_2 \left\{ C_4 U_1 f - C_5 (F' - U_1)^{-3} \right\}$$

$$\frac{\delta H}{\delta U_1} = \frac{\lambda_1 f}{C_S} + \lambda_2 C_4 f - 3 \frac{\lambda_2 C_5}{2} (F' - U_1)^{-4}$$

assume that f is constant

$$\dot{\lambda}_1 = - \frac{\delta H}{\delta X_1} = 0$$

$$\dot{\lambda}_2 = - \frac{\delta H}{\delta X_2} = 0$$

final conditions on λ_1 and λ_2 are:

$$\lambda_1(t_f) = 0 \quad (T_S \text{ is free at final time})$$

$$\lambda_2(t_f) = 1$$

$$\therefore \lambda_1 = 0 \quad \text{for all time.}$$

$$\lambda_2 = 1 \quad \text{for all time.}$$

Now set $\frac{\delta H}{\delta U_1}$ equal to zero and solve for U_1

$$U_1 = F' - \sqrt[4]{\frac{3C_5}{C_4 F}}$$

To get back to the collector flow rate do the following:

$$U_1 = F_r = \frac{\dot{m} C_p}{AU_L} \left\{ 1 - e^{-\frac{F' AU_L}{\dot{m} C_p}} \right\}$$

approximate by using $e^{-X} = 1 - X + X^2/2$

$$\dot{m} = \frac{F'^2 AU_L}{2C_p (F' - U_1)}$$

or by guessing $\dot{m}$ and trying it then in the original equation until it is as close as desired.

The following is the derivation of the optimal collector flow rate considering pumping cost with stratified storage.

Objective function

$$J = \int_0^{t_f} \left\{ C_4 Q_U - C_5 (F' - F_r)^{-3} \right\} dt$$

$$\dot{T}_{S_I} = P_I Q_U n / C_S \quad I = 1, 2, 3, \dots, n \text{ (number of segments)}$$

$$U_1 = F_r$$

$$\text{Let } \dot{X}_I = \dot{T}_{S_I} \quad I = 1, 2, 3, \dots, n$$

$$\dot{x}_{n+1} = C_4 U_1 f - C_5 (F' - U_1)^{-3}$$

$$P_1 = \frac{\dot{m}_1 C_p (T_{OUT} - x_1)}{Q_U}$$

$$P_I = \frac{\dot{m} C_p (x_{I-1} - x_I)}{Q_U} \quad I = 2, 3, \dots, n.$$

$$H = \sum_{I=0}^{n+1} \lambda_I \dot{x}_I$$

$$\frac{\delta H}{\delta U_1} + \sum_{I=1}^n \left(\frac{n P_I f}{C_S} \right) + \lambda_{n+1} C_4 f - 3 \lambda_{n+1} C_5 (F' - U_1)^{-4}$$

assume that P_I, f are constant. $I = 1, 2, 3, \dots, n.$

$$\dot{\lambda}_I = - \frac{\delta H}{\delta x_I} = 0 \quad I = 1, 2, 3, \dots, n.$$

$$\dot{\lambda}_{n+1} = \frac{\delta H}{\delta x_{n+1}} = 0$$

final conditions on λ_I with $I = 1, 2, 3, \dots, n+1$ are

$$\lambda_I(t_f) = 0 \quad I = 1, 2, 3, \dots, n.$$

$$\lambda_{n+1}(t_f) = 1$$

$\therefore \lambda_I = 0$ with $I = 1, 2, 3, \dots, n$ for all time.

$$\lambda_{n+1} = 1 \quad \text{for all time.}$$

Now set $\frac{\delta H}{\delta U_1}$ equal to zero and solve for U_1 .

$$U_1 = F' - \sqrt[4]{\frac{3C_5}{C_4 f}}$$

To get back to the collector flow rate do the same as was described earlier in this Appendix.

The following are ways of estimating f by measuring the collector outlet temperature.

$$f = A \left\{ H_T(\tau\alpha)_e - U_L (T_{IN} - T_A) \right\}$$

Case I: There is a flow through the collector.

$$Q_U = fF_r = \dot{m}C_p(T_{OUT} - T_{IN})$$

known quantities are $\dot{m}$, T_{IN} , T_A , and T_{OUT} .

$$F_r = \frac{\dot{m}C_p}{AU_L} \left\{ 1 - e^{-\frac{F'AU_L}{\dot{m}C_p}} \right\}$$

$\therefore F_r$ is known.

$$\therefore f = \dot{m}C_p(T_{OUT} - T_{IN})/F_r$$

Case II: There is no flow through the collector.

$$Q_U = 0$$

Assume that there is no capacitance in the collector.

$$(0) \dot{T}_p = A \left\{ H_T(\tau\alpha)_e - U_L (T_p - T_A) \right\}$$

$$\therefore 0 = A \left\{ H_T(\tau\alpha)_e - U_L (T_p - T_A) \right\}$$

$$\therefore T_p = T_{OUT} = T_A + H_T(\tau\alpha)_e/U_L$$

$$\therefore H_T(\tau\alpha)_e = U_L (T_{OUT} - T_A)$$

$$f = A \left\{ H_T(\tau\alpha)_e - U_L (T_{IN} - T_A) \right\}$$

$$f = A \left\{ U_L (T_{OUT} - T_A) - U_L (T_{IN} - T_A) \right\}$$

$$f = AU_L (T_{OUT} - T_{IN})$$

APPENDIX III.

The derivation of the combined collector flow rate and temperature error optimal controller for uniform storage.

Control variables:

U_1 - F_r is heat removal factor.

U_2 - part of the auxiliary energy maximum output.

U_3 - part of the storage energy maximum output.

Objective function

$$J = \int_0^{t_f} \left\{ C_1 \dot{x}_2^2 + C_2 U_2^2 + C_3 U_3^2 - C_4 Q_U + C_5 (F' - U_1)^{-3} \right\} dt$$

$$\dot{T}_S = \dot{x}_1 = \left\{ Q_U - U_3 U_{3MAX} - UA_{LOSS}(x_1 - x_2 - T_D) \right\} / C_S$$

$$x_2 = T_E - T_D$$

$$\dot{x}_2 = \left\{ U_2 U_{2MAX} + U_3 U_{3MAX} - UA(x_2 + T_D - T_A) \right\} / C_E$$

Let $\dot{x}_3 = -C_1 \dot{x}_2^2 - C_2 U_2^2 - C_3 U_3^2 + C_4 Q_U - C_5 (F' - U_1)^{-3}$

$$H = \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3$$

$$U_{3MAX} = \dot{m}_{MAX}(\text{storage}) C_p (x_1 - x_2 - T_D)$$

$$U_{2MAX} = \text{constant (maximum output of auxiliary heater)}$$

$$f = A \left\{ H_T(\tau\alpha)_e - U_L(x_1 - T_A) \right\}$$

Assume that U_{3MAX} and f are constant.

$$\frac{\delta H}{\delta U_1} = \frac{\lambda_1 f}{C_S} + 3C_4 f - 3C_5(F' - U_1)^{-4}$$

$$\frac{\delta H}{\delta U_2} = \frac{\lambda_2 U_{2MAX}}{C_E} - 2\lambda_3 C_2 U_2$$

$$\frac{\delta H}{\delta U_3} = -\frac{\lambda_1 U_{3MAX}}{C_S} + \frac{\lambda_2 U_{3MAX}}{C_E} - 2\lambda_3 C_3 U_3$$

$$\dot{\lambda}_1 = -\frac{\delta H}{\delta X_1} = \frac{\lambda_1 U_{LOSS}}{C_S}$$

$$\dot{\lambda}_2 = -\frac{\delta H}{\delta X_2} = \frac{\lambda_1 U_{LOSS}}{C_S} + \lambda_2 \frac{UA}{C_E} + 2\lambda_3 C_1 X_2$$

$$\dot{\lambda}_3 = -\frac{\delta H}{\delta X_3} = 0$$

The storage tank temperature is unspecified at the final time.

$$\therefore \lambda_1(t_f) = 0$$

$$\text{solution to } \dot{\lambda}_1 \text{ is } \lambda_1(t) = \lambda_0 e^{\left(\frac{U_{LOSS}}{C_S}\right) t}$$

$$\lambda_1(t_f) = 0 = \lambda_0 e^{\left(\frac{U_{LOSS}}{C_S}\right) t_f}$$

$$\text{since } e^{\left(\frac{U_{LOSS}}{C_S}\right) t_f} \text{ is not equal to zero}$$

$$\therefore \lambda_0 = 0$$

$$\therefore \lambda_1 = 0 \text{ for all time.}$$

$$\lambda_3(t_f) = 1$$

$$\therefore \lambda_3 = 1 \text{ for all time}$$

$$\therefore U_1^* = F' - \sqrt[4]{\frac{3C_5}{C_4 f}}$$

$$u_2^* = \frac{\lambda_2 u_{3\text{MAX}}}{2C_2 C_E}$$

$$u_3^* = \frac{\lambda_2 u_{3\text{MAX}}}{2C_3 C_E}$$

$\dot{x}_2$ and $\dot{\lambda}_2$ are coupled linear differential equations.

$$\dot{x}_2 = \dot{\lambda}_2 \left\{ \frac{u_{2\text{MAX}}^2}{2C_2 C_E^2} + \frac{u_{3\text{MAX}}^2}{2C_3 C_E^2} \right\} - \frac{UA}{C_E} (x_2 + T_D - T_A)$$

$$\dot{\lambda}_2 = \lambda_2 \frac{UA}{C_E} + 2C_1 x_2$$

Let $k_1 = UA/C_E$

$$k_2 = \frac{C_3 u_{2\text{MAX}}^2}{2C_2 C_3 C_E^2} = \frac{C_2 u_{3\text{MAX}}^2}{2C_2 C_3 C_E^2}$$

$$k_3 = 2C_1$$

$$k_4 = \frac{UA}{C_E} (T_A - T_D)$$

$$\beta = \sqrt{k_1^2 + k_2 k_3}$$

assuming T_A constant

$$\begin{aligned} x(t) = & \left\{ \frac{\beta + k_1}{2\beta} e^{-\beta t} + \frac{\beta - k_1}{2\beta} e^{\beta t} \right\} x_0 \\ & + \left\{ \frac{k_2}{2\beta} (e^{\beta t} - e^{-\beta t}) \right\} \lambda_0 \\ & + \frac{k_4 k_1}{\beta^2} - \frac{k_4 (\beta + k_1)}{2\beta^2} e^{-\beta t} + \frac{k_4 (\beta - k_1)}{2\beta^2} e^{\beta t} \\ x(t) = & \left\{ \frac{k_3}{2\beta} (e^{\beta t} - e^{-\beta t}) \right\} x_0 \end{aligned}$$

$$+ \left\{ \frac{\beta - k_1}{2\beta} e^{-\beta t} + \frac{\beta + k_1}{2\beta} e^{\beta t} \right\} \lambda_0$$

$$+ \frac{k_3 k_4}{2\beta^2} \left\{ e^{\beta t} + e^{-\beta t} - 2 \right\}$$

Since the collector flow rate and the enclosure temperature error separated within the combined derivation there is no need to show the derivation for stratified storage because it follows directly from the previous work shown.

If x_2 is specified at t_f to be x_f , then

$$\lambda_0 = \left\{ \left[\frac{\beta + k_1}{2\beta} e^{-\beta t_f} + \frac{\beta - k_1}{2\beta} e^{\beta t_f} \right] x_0 - x_f \right.$$

$$+ \frac{k_4 k_1}{2\beta} + \frac{k_4(\beta - k_1)}{2\beta^2} e^{\beta t_f} - \frac{k_4(\beta + k_1)}{2\beta^2} e^{-\beta t_f} \left. \right\}$$

$$+ \left\{ \frac{k_2}{2\beta} e^{-\beta t_f} - e^{\beta t_f} \right\}$$

If x_2 is free at t_f then $\lambda_2(t_f) = 0$

$$\lambda_0 = \left\{ \frac{k_3}{2\beta} \left[e^{-\beta t_f} - e^{\beta t_f} \right] x_0 - \frac{k_3 k_4}{2\beta^2} \left[e^{\beta t_f} + e^{-\beta t_f} - 2 \right] \right\}$$

$$+ \left\{ \frac{\beta - k_1}{2\beta} \left[e^{-\beta t_f} + \frac{\beta + k_1}{2\beta} e^{\beta t_f} \right] \right\}$$

The following is the derivation of the combined collector flow rate and the temperature error optimal controller with the auxiliary as a singular term for uniform storage.

Control variables

$U_1 = F_r$ - Heat Removal factor

U_2 - part of the auxiliary energy maximum output.

U_3 - part of the storage energy maximum output.

Objective function

$$J = \int_0^{t_f} \left\{ C_1 \dot{x}_2^2 + C_2 U_2 + C_3 U_3^2 - C_4 Q_U + C_5 (F' - U_1)^{-3} \right\} dt$$

$$\dot{T}_S = \dot{x}_1 = \left\{ Q_U - U_3 U_{3MAX} - UA_{LOSS} (x_1 - x_2 - T_D) \right\} / C_S$$

$$x_2 = T_E - T_D$$

$$\dot{x}_2 = \left\{ U_2 U_{2MAX} + U_3 U_{3MAX} - UA (x_2 + T_D - T_A) \right\} / C_E$$

Let $\dot{x}_3 = -C_1 \dot{x}_2^2 - C_2 U_2 - C_3 U_3^2 + C_4 Q_U - C_5 (F' - U_1)^{-3}$

$$H = \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 + \lambda_3 \dot{x}_3$$

$$\frac{\delta H}{\delta U_1} = \frac{\lambda_1 f}{C_S} + \lambda_3 C_4 f - 3\lambda_3 C_5 (F' - U_1)^{-4}$$

$$\frac{\delta H}{\delta U_2} = \frac{\lambda_2 U_{2MAX}}{C_E} - \lambda_3 C_2$$

$$\frac{\delta H}{\delta U_3} = \frac{\lambda_2 U_{3MAX}}{C_E} - \frac{\lambda_2 U_{3MAX}}{C_S} - 2\lambda_3 C_3 U_3$$

Assume f and U_{3MAX} are constant.

$$\dot{\lambda}_1 = \frac{\delta H}{\delta x_1} = \lambda_1 \frac{UA_{LOSS}}{C_S}$$

$$\dot{\lambda}_2 = -\frac{\delta H}{\delta x_2} = \lambda_2 \frac{UA}{C_E} - \lambda_1 \frac{UA_{LOSS}}{C_S} + 2\lambda_3 C_1 x_1$$

$$\dot{\lambda}_3 = \frac{\delta H}{\delta x_3} = 0$$

Solution for $\dot{\lambda}_1$ is

$$\lambda_1(t) = \lambda_0 e^{\left(\frac{UA_{LOSS}}{C_5}\right)t}$$

and $\lambda_1(t_f) = 0$ because T_S is unspecified at the final time.

$$\therefore \lambda_0 = 0$$

$$\therefore \lambda_1 = 0 \text{ for all time}$$

$$\lambda_3(t_f) = 1$$

$$\therefore \lambda_3 = 1 \text{ for all time.}$$

By setting $\frac{\delta H}{\delta U_1}$, $\frac{\delta H}{\delta U_2}$ and $\frac{\delta H}{\delta U_3}$ to zero

then

$$U_1^* = F' - \sqrt[4]{\frac{3C_5}{C_4 f}}$$

$$U_3^* = \frac{\lambda_2 U_{2MAX}}{2C_3 C_E}$$

$$\text{If } \lambda_2 U_{2MAX} - C_2 C_E > 0 \text{ then } U_2^* = 1$$

$$\text{If } \lambda_2 U_{2MAX} - C_2 C_E < 0 \text{ then } U_2^* = 0$$

$$\text{If } \lambda_2 U_{2MAX} - C_2 C_E = 0 \text{ then } U_2^* = ? \text{ (indeterminate solution)}$$

$$\text{Case 1: } U_2^* = 1$$

$$\dot{x}_2 = \frac{\lambda_2 (U_{2MAX})^2}{2C_3 (C_E)^2} + \frac{U_{2MAX}}{C_E} - \frac{UA}{C_E} (x_1 - T_D - T_A)$$

$$\dot{\lambda}_2 = \lambda_2 \frac{UA}{C_E} + 2C_1 x_2$$

$$\lambda_2 U_{2MAX} - C_2 C_E > 0$$

Let

$$k_1 = UA/C_E$$

$$k_2 = \frac{U_{3MAX}^2}{(2C_3C_E^2)}$$

$$k_3 = 2C_1$$

$$k_4 = \frac{UA}{C_E} (T_A - T_D) + U_{2MAX}/C_E$$

The solution is now identical to the solution shown for the quadratic auxiliary optimal controller.

Case II: $U_2^* = 0$

$$\dot{x}_2 = \frac{\lambda_2 U_{3MAX}^2}{2C_3C_E^2} - \frac{UA}{C_E} (x_1 + T_D - T_A)$$

$$\dot{\lambda}_2 = 2 \frac{UA}{C_E} + 2C_1 x_2$$

$$\lambda_2 U_{2MAX} - C_2 C_E < 0$$

Let

$$k_1 = UA/C_E$$

$$k_2 = \frac{U_{3MAX}^2}{(2C_3C_E^2)}$$

$$k_3 = 2C_1$$

$$k_4 = \frac{UA}{C_E} (T_A - T_D)$$

Again, the solution is identical to the one previously shown in Appendix III.

Case III:

$$\lambda_1 = \frac{C_2 C_E}{U_2 \text{ MAX}}$$

$$x_1 = \frac{C_2 U_A}{2C_1 U_2 \text{ MAX}}$$

$$U_2^* = \frac{U_A}{U_2 \text{ MAX}} x_1 + \frac{U_A}{U_2 \text{ MAX}} (T_D - T_A) - \frac{C_2 (U_3 \text{ MAX})^2}{2C_3 (U_2 \text{ MAX})^2}$$

$$U_3^* = \frac{\lambda_2 U_3 \text{ MAX}}{2C_3 C_E}$$