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A Collinear n-Body Problem of Classical Electrodynamics

R. D. Driver[†]

University of Rhode Island, Kingston, RI 02881

M. J. Norris[‡]Sandia National Laboratories, Albuquerque, NM 87185[#]**MASTER**Abstract

One model for the motion of n charged particles on the x -axis leads to a system of delay differential equations with delays which depend on the unknown trajectories. If appropriate past histories of the trajectories are given, say on $[-r, 0]$, then for sufficiently small $t \geq 0$ one has a system of n^2 ordinary differential equations of the form

$$y' = f(t, y) \quad \text{with} \quad y(0) = y_0 \quad \text{given.} \quad (*)$$

The function f , which involves the known past histories of the trajectories, is continuous; so existence of solutions is assured. However, f does not satisfy the Lipschitz condition usually used for proving uniqueness.

The main new result is that the solution of $(*)$ is unique provided, for some integer $m \leq n^2$,

$$f_j(t, \xi) < 1 \quad \text{for} \quad j = 1, \dots, m,$$

and

$$\|f(t, \xi) - f(t, \eta)\| \leq K \sum_{j=1}^m |g_j(t - \xi_j) - g_j(t - \eta_j)| + K \sum_{j=m+1}^{n^2} |\xi_j - \eta_j|,$$

where $K > 0$ is constant and each g_j is a continuous function of bounded variation.

This generalized Lipschitz-type condition is indeed satisfied by the function f in $(*)$. The m components of y which play the special role in the above uniqueness criterion are the $n(n-1)$ delays of the original n -body problem. Eventually one finds that solutions of the original equations of motion exist and are unique as long as no two particles collide.

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R. D. Driver

University of Rhode Island, Kingston, RI 02881

M. J. Norris

Sandia National Laboratories, Albuquerque, NM 87185

Consider n charged particles on the x -axis; and assume that each is influenced only by the retarded fields of the others, and no two particles collide during the time considered.

If $x_i(t)$ is the position of particle i at time t , and c is the speed of light, then at time t particle j is influenced by the electromagnetic fields which were produced by particle i ($i \neq j$) at some earlier instant $t - r_{ij}(t)$. The delay $r_{ij}(t)$, representing the time required for fields to propagate from particle i to particle j must satisfy the functional equation.

$$cr_{ij}(t) = |x_j(t) - x_i(t - r_{ij}(t))| \quad (1)$$

To shorten the notation, we will sometimes represent $x_j(t)$ and $r_{ij}(t)$ by x_j and r_{ij} , respectively. Then Eq. (1) becomes

$$cr_{ij} = |x_j - x_i(t - r_{ij})|$$

We shall also use the symbol

$$v_j = v_j(t) \equiv x'_j(t)/c$$

for the velocity of particle j in units of c .

The following theorem, proved in [1], lists some basic properties of Eq. (1).

Theorem 1. Let $\alpha < 0 < \beta$ and let x_i and x_j be given differentiable functions with $|x'_i(t)| < c$, $|x'_j(t)| < c$ on $[\alpha, \beta)$ and with $x_i(t) \neq x_j(t)$ on $[\alpha, \beta)$. Assume Eq. (1) has a solution when $t = 0$. (A sufficient condition

for this would be

$$|x'_i(t)| \leq c - |x_j(0) - x_i(0)|/|\alpha| \quad \text{for } \alpha \leq t \leq 0 .)$$

Then Eq. (1) has a unique solution for all $t \in [0, \beta)$ and (1) is equivalent to

$$cr_{ij} = \sigma_{ij}[x_j - x_i(t - r_{ij})] , \quad (1')$$

where

$$\sigma_{ij} \equiv \text{sgn} [x_j(0) - x_i(0)] .$$

Moreover

$$r_{ij} \geq |x_j - x_i|/2c .$$

If, in addition, $v_i = x'_i/c$ and $v_j = x'_j/c$ are continuous, then

$$r'_{ij} = \frac{\sigma_{ij}[v_j - v_i(t - r_{ij})]}{1 - \sigma_{ij}v_i(t - r_{ij})} \quad \text{for } 0 \leq t < \beta . \quad (2)$$

Finally, if $r_{ij}(0)$ satisfies Eq. (1') at $t = 0$ and if r_{ij} satisfies (2) for $0 \leq t < \beta$, then $r_{ij} = r_{ij}(t)$ satisfies Eq. (1') for $0 \leq t < \beta$.

After these preliminaries we can write the equation of motion for particle j as follows. When Eq. (1) has a unique solution, then for t in $[\alpha, \beta)$ one has

$$\frac{v'_j}{(1 - v_j^2)^{3/2}} = \sum_{i \neq j} \frac{K_{ij}}{r_{ij}^2} \frac{1 + \sigma_{ij}v_i(t - r_{ij})}{1 - \sigma_{ij}v_i(t - r_{ij})} , \quad (3)$$

where K_{ij} is a constant. Actually K_{ij} and K_{ji} will have opposite signs.

The last assertion of Theorem 1 will essentially let us consider Eqs. (2) and (3) in place of (1) and (3).

The "natural" initial data problem for this system of functional differential equations is as follows.

Problem P. Let continuously differentiable functions φ_i be given on an interval $[\alpha, 0]$ for $i = 1, \dots, n$ with the properties that each $|\varphi_i'(t)| < c$ on $[\alpha, 0]$ and $\varphi_i(0) \neq \varphi_j(0)$ when $i \neq j$. Let β be positive. A solution of Problem P on $[\alpha, \beta)$ will be a set of continuously differentiable functions x_i ($i = 1, \dots, n$) on $[\alpha, \beta)$ such that there exists a collection of functions $\{r_{ij}(t) | i, j = 1, \dots, n; i \neq j\}$ on $[0, \beta)$ with the x_i 's and r_{ij} 's satisfying

- (a) $x_i(t) = \varphi_i(t)$ on $[\alpha, 0]$
- (b) $x_i(t) \neq x_j(t)$ when $i \neq j$ and $0 \leq t < \beta$
- (c) $|v_i(t)| < 1$ for $0 \leq t < \beta$ and each i (where $v_i \equiv x_i'/c$)
- (d) Eqs. (1) and (3).

Note that we made no assumption that the "initial trajectories," φ_i , satisfy any particular equations.

In the context of the functional differential equations presented here, it will be quite natural to assume that enough appropriate initial data has been given so that Eq. (1) does have a solution at $t = 0$. (This assumption was eliminated in the two-body problem by Travis [6].)

In formulating a problem, such as P above, which presumably is to relate to a real-world situation one must be concerned that the problem is "well-posed." The requirement that the φ_i 's be continuously differentiable is significant in this regard in that the condition is used in assuming a solution exists in certain cases. However, one does not have a guarantee of uniqueness of solution on the basis of requirements mentioned so far. One must then consider what restrictions on the φ_i 's are relevant to such matters.

When this problem was originally treated for the case of two particles [1], it was assumed that each φ_i' was Lipschitz continuous. But studies of the two-body problem in three dimensions - involving equations of neutral type - led to the conclusion that this was an unreasonable hypothesis [2].

It seemed more realistic to require each φ_i' to be just absolutely continuous. This indeed turned out to be adequate for the collinear motion of two particles [4], and was later found to be sufficient for local existence and uniqueness in the case of two particles moving in three dimensions [3].

However the proofs used in [4] and [3] do not extend to the case $n \geq 3$.

To resolve this difficulty the following uniqueness theorem for ordinary differential equations has recently been proved [5].

Theorem 2. The system of m equations

$$y' = f(t, y) \quad \text{with} \quad y(0) = y_0$$

has a unique solution (locally) if in some neighborhood of $(0, y_0)$ in R^{1+m} f is continuous and for some integer $p \in [0, m]$

$$f_j(t, \xi) < 1 \quad \text{for} \quad j = 1, \dots, p$$

and f satisfies the generalized Lipschitz-type condition

$$\|f(t, \xi) - f(t, \eta)\| \leq K \sum_{j=1}^p |g_j(t - \xi_j) - g_j(t - \eta_j)| + K \sum_{j=p+1}^m |\xi_j - \eta_j|,$$

where $\|\cdot\|$ is any norm in R^m , $K > 0$ is a constant, and each g_j is a continuous function of bounded variation.

The theorem proved in [5] is actually a little more general, but the above will suffice for our present purpose.

The following (main) theorem says that the collinear n -body problem does have a unique solution (prior to collisions) even assuming only absolutely continuous initial velocities - or slightly less.

Theorem 3. For $i=1, \dots, n$ let φ_i be a given function on $[\alpha, 0]$ such that φ_i' is continuous and of bounded variation there. Assume that

- (i) $\varphi_j(0) \neq \varphi_i(0)$ for each i, j with $i \neq j$
- (ii) $|\varphi_i'(t)| < c$ on $[\alpha, 0]$ for each i

~~$r_{ij}(0) = 0$~~

(iii) Eq. (1) has a solution r_{0ij} at $t = 0$ for each i, j with $i \neq j$.

Then there exists a positive β such that

(a) If $0 < \beta_1 \leq \beta$ then Problem P has a unique solution on $[\alpha, \beta_1]$

(b) If $\beta < \beta_1$, then Problem P does not have a solution on $[\alpha, \beta_1]$;

specifically, either $\beta = \infty$ or else there exist distinct i, j with

$$\lim_{t \rightarrow \beta^-} x_j(t) = \lim_{t \rightarrow \beta^-} x_i(t) \text{ -- a collision.}$$

Proof. Consider points $(t, v_1, \dots, v_n, r_{12}, \dots, r_{1n}, \dots, r_{n1}, \dots, r_{n,n-1})$ in a small neighborhood $U \subset \mathbb{R}^{1+n^2}$ of the point

$$(0, g_1(0), \dots, g_n(0), r_{012}, \dots, r_{0,n,n-1})$$

If U is sufficiently small then, in U ,

$$t - r_{ij} < 0 \quad \text{for each } i, j \ (i \neq j)$$

So, in the neighborhood U , Eqs. (2) and (3) are equivalent to the ordinary differential equations

$$r'_{ij} = \frac{\sigma_{ij} [v_j - g_i(t - r_{ij})]}{1 - \sigma_{ij} g_i(t - r_{ij})} \quad (4)$$

and

$$\frac{v'_j}{(1 - v_j^2)^{3/2}} = \sum_{i \neq j} \frac{K_{ij}}{r_{ij}^2} \frac{1 + \sigma_{ij} g_i(t - r_{ij})}{1 - \sigma_{ij} g_i(t - r_{ij})}, \quad (5)$$

and the initial conditions are $v_j(0) = g_j(0)$, $r_{ij}(0) = r_{0ij}$.

Local existence and uniqueness of a solution, say for $0 \leq t < h$, follow from Theorem 2. We have thus generated the unique solution of Eqs. (2) and (3) on $[0, h)$. Now define

$$x_i(t) = \begin{cases} \varphi_i(t) & \text{on } [u, 0] \\ \varphi_i(0) + \int_0^t c v_i(s) ds & \text{on } [0, h) \end{cases}$$

Then the resulting functions x_i ($i=1, \dots, n$) form a (unique) solution of Problem P on $[\alpha, h)$.

If, when solving Eqs. (4) and (5) in U, we made h as large as possible and found $h < \infty$, then either

$$\lim_{t \rightarrow h-} (t - r_{ij}(t)) = 0 \quad (6)$$

for some i, j , or else

$$\liminf_{t \rightarrow h-} r_{ij}(t) = 0 \quad \text{or} \quad \limsup_{t \rightarrow h-} |v_j(t)| = 1 \quad (7)$$

for some j or for some i, j , respectively.

If $h < \infty$ and (7) does not hold, then (6) must be true. In this case one can extend the solution to $[0, h]$ and then continue it further by repeating the above arguments with $t = h$ playing the role formerly played by $t = 0$ (the "method of steps").

One concludes that the solution can be continued to some maximal interval $[\alpha, \beta)$ where either $\beta = \infty$ or else (7) holds. We shall prove that if $\beta < \infty$, then

$$\liminf_{t \rightarrow h-} |x_i(t) - x_j(t)| = 0 \quad \text{for some } i, j (i \neq j) \quad (8)$$

Suppose (for contradiction) that $\beta < \infty$ and there exists $\delta > 0$ such that

$$|x_i(t) - x_j(t)| \geq \delta \quad \text{for each } i, j (i \neq j) \quad .$$

It then follows from Theorem 1 that each $r_{ij}(t) \geq \delta/2c$, which eliminates the first equation of (7). Thus it follows that

$$|v_i(t - r_{ij})| \leq a < 1 \quad \text{on } [0, \beta) \quad \text{for each } i, j (i \neq j) \quad .$$

Using the above information, we find from Eq. (3) that

$$\frac{d}{dt} \frac{v_j}{(1-v_j^2)^{1/2}} = \frac{v_j'}{(1-v_j^2)^{3/2}}$$

is bounded on $[0, \beta)$ for each j .

Since $\beta < \infty$ this means that $|v_j(t)|$ is bounded away from 1, and that eliminates the second equation of (7).

So if $\beta < \infty$ then (8) holds. The assertion of the theorem now follows from the fact that x_i and x_j have bounded derivatives. ■

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