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MASTER

COHERENT BACKSCATTERING IN THE SOFT X-RAY REGION

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1. INTRODUCTION

Several authors have proposed schemes to produce coherent radiation in the soft x-ray region of the spectrum by stimulated amplification of laser light in a magnetic undulator. In all these schemes, a relativistic electron beam is density modulated in an optical klystron stage.⁽¹⁾ In some of the proposals the magnetic modulator is realized with two interfering lasers.⁽²⁾

Following a separate line of research, Compton scattering of laser photons on an electron beam has been investigated as a means to produce high energy photons.⁽³⁾

The present paper shows how coherent polarized soft x-rays can be produced by a combination of the two techniques, i.e. by Compton scattering of a laser beam from a relativistic electron beam, whose current or charge density is periodically modulated.

The basic arrangement is shown in Fig. 1: an electron beam from, say, a Van de Graaff and a laser beam propagate through an undulator along the same line. Inside the undulator, the laser electromagnetic wave produces a modulation of the energy of the electrons. After some drift space, the modulation of the electron energy transforms into a modulation of the beam longitudinal charge density. The laser photons are reflected by a concave mirror against the electrons and are backscattered. In the process, their energy is greatly increased. If electron and laser photon energy are matched in a proper way, the modulated electron beam may act as a moving diffraction grating, and the backscattered X-rays show a high degree of coherence.

Fig. 1 General Layout - Two halves of a Tandem Van de Graaff electrostatic accelerator are shown together with the undulator and the laser. In the case shown bunching and scattering lasers coincide.

2. THE MECHANISM OF MODULATION

To produce a modulation of the electron energy in the modulator, the laser must be polarized, and a resonance condition must be fulfilled. That is, if the undulator is of the transverse type, the laser must have its electric vector in the plane of the electron trajectories, and the frequency of the laser must be equal to the fundamental or to one of the odd harmonics of the spontaneous radiation emitted by the undulator,⁽⁴⁾ given by

$$\omega_h = h \omega_0 \frac{2\gamma^2}{1+K^2} \quad h=1, 3, 5, \dots \quad (2-1)$$

where: γ is the electron energy in units of its rest mass energy mc^2 , $\omega_0 = 2\pi/\lambda_0$ with λ_0 undulator period, h the harmonic number, and K the undulator parameter

$$K = \frac{1}{\sqrt{2}} \frac{\Omega_0}{\omega_0} ; \Omega_0 = \frac{e}{mc} B_0 \quad (2-2)$$

with B_0 undulator peak magnetic field.

Let us discuss the case of a transverse undulator, and assume, without great loss of generality that the magnetic field of the undulator is purely sinusoidal with only one component along, say, the y -axis. The linearly polarized electromagnetic laser wave has an electric field $\vec{E}$ and a magnetic field $\vec{E}/c$. With reference to Fig. 2 and introducing the reduced electric and magnetic fields

$$\vec{\Omega} = \frac{e}{mc} \vec{B} \quad (2-3)$$

$$\vec{\tilde{E}} = \frac{e}{mc^2} \vec{E}$$

Fig. 2 Pattern of the static magnetic field of the transverse undulator and of the e.m. field of the laser wave.

one can write the components of the total electromagnetic field as

$$\Omega_x = \Omega_z = 0$$

$$\Omega_y = \Omega_0 \cos \omega_0 z - \epsilon_0 \cos \omega(\tau - z)$$

(2-4)

$$\epsilon_x = -\epsilon_0 \cos \omega(\tau - z)$$

$$\epsilon_y = \epsilon_z = 0$$

where $\tau = ct$, and $\phi_0 = \omega \tau_0$ is the initial phase ($\tau = \tau_0, z = 0$) for a given electron entering the undulator.

In the laboratory frame of reference, the relativistic equations of motion of the electrons can be written in the following way.⁽⁵⁾

$$\begin{aligned}\frac{\dot{\tau}}{\tau} &= \beta \\ \gamma \dot{\vec{\beta}} &= \vec{E} + \vec{\beta} \times \vec{\Omega} - \vec{\beta}(\vec{\beta} \cdot \vec{E}) \\ \dot{\gamma} &= \vec{\beta} \cdot \vec{E}\end{aligned}\tag{2-5}$$

where $\vec{\beta}$ is the velocity of the electron in units of the speed of light c , and the derivatives are taken with respect to τ .

Let us use the expression (2-4) for the field and limit ourselves to the "small signal regime",⁽⁶⁾ which corresponds to the case of the magnetic field of the wave much smaller than the static magnetic field of the undulator

$$\epsilon_0 \ll \Omega_0$$

Furthermore, assume that the motion, in the transverse undulator considered, is all confined in the plane x - z . We obtain in this case

$$\begin{aligned}\dot{\gamma} \beta_x &= -\beta_z \Omega_0 \cos \omega_0 z \\ \dot{\gamma} \beta_z &= \beta_x \Omega_0 \cos \omega_0 z \\ \dot{\gamma} &= \beta_x \epsilon_0 \cos \omega(\tau - z)\end{aligned}\tag{2-6}$$

Equations (2-6) separate: the first two equations describe the trajectory of the electrons as determined solely by the static undulator magnet field, and the third equation describes the energy variation of the electrons as determined only by the laser electric field. We can integrate the first two Eqs. (2-6) approximatively by observing that $z \approx \tau$ and that $\beta_x \ll \beta_z$, and obtain for the transverse displacement of the orbit x , the transverse velocity β_x and the longitudinal velocity β_z ⁽⁵⁾

$$x = \frac{\sqrt{2}}{\omega_0} \frac{\kappa}{\gamma} (\cos \omega_0 z - 1)$$

$$\beta_x = -\sqrt{2} \frac{\kappa}{\gamma} \sin \omega_0 z \quad (2-7)$$

$$\beta_z = \langle \beta_z \rangle_u + \frac{1}{2} \frac{\kappa^2}{\gamma^2} \cos 2\omega_0 z$$

The average longitudinal velocity throughout the undulator is

$$\langle \beta_z \rangle_u = 1 - \frac{1+\kappa^2}{2\gamma^2} \quad (2-8)$$

The last Eq. (2-6) can be written, by virtue of Eqs. (2-7)

$$\dot{\gamma} = -\frac{\sqrt{2}}{2} \frac{\kappa}{\gamma} \epsilon_0 [\sin \phi^+ - \sin \phi^-] \quad (2-9)$$

where expressions of the relative phase between the electron oscillation and the electromagnetic laser wave appear explicitly. Namely

$$\phi^\pm = \omega(\tau - z) \pm \omega_0 z \quad (2-10)$$

In order to obtain an overall energy change through the undulator, these phases should not change too rapidly. If they remain approximatively constant, the electrons retain the phases they had at the entrance in the undulator, and will gain or lose energy at a constant rate as far as they proceed along. In this case, as shown in Appendix A, they can be written as a function of one phase only and the resulting differential equation for the energy modulation becomes

$$\dot{\gamma} = \frac{\sqrt{2}}{2} \frac{K\epsilon_0}{\gamma} F_h(b) \sin \phi_0 \quad (2-11)$$

where γ is close to the "resonant energy" γ_r (see Eq. (2-1))

$$\gamma_r = \left[\frac{\omega}{h\omega_0} \frac{1+K^2}{2} \right]^{\frac{1}{2}} ; h = 1, 3, 5, \dots \quad (2-12)$$

and we have introduced the function

$$F_h(b) = J_{\frac{h-1}{2}}(b) - J_{\frac{h+1}{2}}(b) \quad (2-13)$$

of the parameter

$$b = h \frac{K^2}{2(1+K^2)}$$

In Eq. (2-13) h is the harmonic number of the undulator defined earlier, and J_h are Bessel functions of integer order.

In the case of electrons with energy close to resonance ($\gamma = \gamma_r$), Eq. (2-11) can be integrated immediately to give

$$\gamma = \gamma_0 + \frac{\sqrt{2}}{2} \frac{K\epsilon_0}{\gamma_r} F_h(b) z \sin \phi_0 \quad (2-14)$$

At the end of the undulator, where $z = N_0 \lambda_0$, with N_0 the number of periods, the electron relative energy variation is

$$\frac{\Delta Y}{Y} = D \sin \phi_0 \quad (2-15)$$

with the modulation amplitude

$$D = \frac{\sqrt{2}}{2} \frac{K \epsilon_0}{Y_r^2} F_h(b) N_0 \lambda_0 \quad (2-16)$$

The energy modulation described by Eq. (2-14) naturally gives rise to a beam current modulation, since the faster electrons tend to catch up with the slower electrons, started earlier, thus producing a periodic enhancement of the current, or of the electron density downstream.

The behavior can be represented by the classical klystron Applegate diagram,⁽⁷⁾ which shows how the electrons, after reaching the first maximum, disband, giving rise to two-peaked profiles which subsequently decay. Afterwards new peaks are formed periodically.

To describe the charge density modulation, let us consider an electron starting at the beginning of the undulator ($z = 0$), with a phase

$$\phi_0 = \omega \tau_0 = \frac{2\pi}{\lambda_b} \tau_0 \quad (2-17)$$

where λ_b is the wavelength of the modulating laser. This electron reaches a position z , past the undulator, at a time given by

$$\tau = \tau_0 + \left[1 - u(z - L_u) \right] \int_0^z \frac{dz}{\beta_{z,u}} + u(z - L_u) \int_{L_u}^z \frac{dz}{\beta_{z,d}} \quad (2-18)$$

where $u(z - L)$ is a unitary step function defined by

$$u(z-L_u) \begin{cases} = 0 & \text{for } z \leq L_u \\ = 1 & \text{for } z > L_u \end{cases} \quad (2-19)$$

$L_u = N_o \lambda_o$ is the undulator length, and $\beta_{z,u}$ and $\beta_{z,d}$ are the longitudinal velocities inside the undulator and the drift space, respectively. $\beta_{z,u}$ is given by the third of Eqs. (2-7) and

$$\beta_{z,d} = 1 - \frac{1}{2\gamma^2}$$

The energy γ varies inside the undulator, according to Eq. (2-16), but in the drift space remains constant and equal to

$$\gamma = \gamma_o + \Delta\gamma$$

with $\Delta\gamma$ given by Eq. (2-15). After integration of Eq. (2-18), the arrival time at the position z is found to be, in the approximation $K \ll 1$

$$\tau = \tau_o + \frac{z}{\langle \beta \rangle} - \frac{D}{\gamma^2} \left\{ \frac{z^2}{2L_u} - u(z-L_u) \left[\frac{z^2}{2L_u} - z + \frac{L_u}{2} \right] \right\} \sin \phi_o \quad (2-20)$$

It is immediate to verify that it is

$$\tau = \tau_o \quad \text{for } z = 0$$

$$\tau = \tau_o + \frac{L_u}{\langle \beta \rangle} - \frac{DL_u}{2\gamma_o^2} \sin \phi_o \quad \text{for } z = L_u$$

$$\tau = \tau_o + \frac{z}{\langle \beta \rangle} - \frac{D}{\gamma_o^2} \left(z - \frac{L_u}{2} \right) \sin \phi_o \quad \text{for } z > L_u$$

If $\rho(z, \tau)$ is the electron beam charge density at a position z and at a time τ , conservation of electric charge requires that

$$\rho(z, \tau) |d\tau| = \rho(0, \tau_0) |d\tau_0| \quad (2-21)$$

This equation can be conveniently rewritten, using the derivative $|d\tau/d\tau_0|$ obtained from Eq. (2-20), as

$$\rho(z, \tau) = \frac{\rho(0, \tau_0)}{|1 - \eta \cos \phi_0|} \quad (2-22)$$

where we have introduced the "bunching parameter"

$$\eta = \frac{\omega D}{\gamma_0^2} \left[\frac{z^2}{2L_u} - u(z - L_u) \left(\frac{z^2}{2L_u} - z + \frac{L_u}{2} \right) \right] \quad (2-23)$$

Equation (2-22) shows that the charge density diverges periodically in time where $\eta = 1$ and exhibits double divergence peaks for $\eta > 1$.

The condition $\eta = 1$ takes place at a distance from the undulator entrance, according to Eq. (2-23) given by

$$z = z_b \begin{cases} = \gamma_0 \sqrt{\frac{2L_u}{\omega D}} & \text{for } z_b < L_u \\ = \frac{\gamma_0^2}{\omega D} + \frac{L_u}{2} & \text{for } z_b > L_u \end{cases} \quad (2-24)$$

We will restrict ourselves to the second case, i.e., to the case where the maximum bunching is reached past the undulator. It is, in the latter case

$$\eta = \frac{\omega D}{\gamma_0^2} \left(z - \frac{L_u}{2} \right) = \frac{z - \frac{L_u}{2}}{z_b - \frac{L_u}{2}} \quad (2-25)$$

The charge density $\rho(z, \tau)$ in Eq. (2-22) is a function of time τ and z through Eqs. (2-20) and (2-23), respectively.

Following the classical treatment of the density bunching in a klystron,⁽⁸⁾ let us expand $\rho(z, \tau)$ in a Fourier series of the angle

$$\phi = \omega \left(\tau - \frac{z}{\langle \beta \rangle} \right) = \phi_0 - \eta \sin \phi_0 \quad (2-26)$$

A cosine series can be written as follows

$$\rho(z, \tau) = \rho(0, \tau_0) \left[1 + \sum_1^{\infty} a_n \cos n\phi \right] \quad (2-27)$$

with the coefficients

$$\rho(0, \tau_0) a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \rho(\phi) \cos n\phi \, d\phi \quad (2-28)$$

Let us change integration variable from ϕ to ϕ_0 , and obtain because of Eq. (2-22)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n(\phi_0 - \eta \sin \phi_0) \, d\phi_0 \quad (2-29)$$

where each integral defines the n -th Bessel function.

Finally, we obtain for a monoenergetic electron beam the following expression for the modulated charge density

$$\rho(z, \tau) = \rho(0, \tau_0) \left\{ 1 + 2 \sum_{n=1}^{\infty} J_n(n\eta) \cos n\omega \left(\tau - \frac{z}{\langle \beta \rangle} \right) \right\} \quad (2-30)$$

This equation, written in the laboratory frame, shows that the electron beam charge density propagates as a superposition of waves, all moving at the same velocity $\langle \beta \rangle$, and with relative amplitudes expressed by the Bessel function coefficients. In turn, these coefficients change as a function of the distance z from the undulator, reaching a maximum value for $\eta \approx 1$, where the condition for maximum bunching is obtained.

3. EFFECTS OF ELECTRON BEAM ENERGY SPREAD, FINITE ELECTRON BEAM EMITTANCE AND UNDULATOR IMPERFECTIONS

Electron beam energy spread, finite dimensions and angle spread and finally undulator imperfections produce a decrease in the value of the modulation depth for a given undulator structure and laser beam power.

3.1 Effect of the Energy Spread

Assume that the electrons in the beam have a gaussian distribution of energies at the modulator entrance, of width σ_e .

For an electron which started with an energy $\gamma_e + \delta\gamma$ slightly varied respect to the resonant energy, Eqs. (2-15) and (2-26) become

$$\frac{\Delta\gamma}{\gamma} = D \sin \phi_0 + \frac{\delta\gamma}{\gamma} \quad (3-1)$$

and

$$\phi = \phi_0 - \eta \sin \phi_0 + \frac{\eta}{D} \frac{\delta\gamma}{\gamma} = \phi + \frac{\eta}{D} \frac{\delta\gamma}{\gamma} \quad (3-2)$$

respectively.

Let us fold now the integrals (2-29) for the Fourier coefficients of the charge density, with the energy distribution, as follows

$$\rho(0, \tau_0) a_n = \frac{\rho_0}{\pi/2\pi\sigma} \int_{-\pi}^{\pi} d\phi_0 \quad (3-3)$$

$$- \frac{(\delta\gamma/\gamma)^2}{2\sigma^2} \cos\left(n\phi + \frac{n}{D} \frac{\delta\gamma}{\gamma}\right)$$

$$\int_{-\infty}^{\infty} d\left(\frac{\delta\gamma}{\gamma}\right) e$$

The inner integration is immediate and the new coefficients for the Fourier series are the following

$$a_n = e^{-\frac{1}{2} \frac{n^2}{n_{ce}^2}} \frac{1}{\pi} \int_{-\pi}^{\pi} d\phi_0 \cos n\phi \quad (3-4)$$

with an energy distribution dependent cut-off harmonic number given, at maximum bunching, by

$$n_{ce} = \frac{D}{\sigma} \quad (3-5)$$

Equations (3-4) and (3-5) show that high order modulation is obtained only with very good electron energy spread and/or deep energy modulation.

3.b Effects of the Emittance

A finite emittance of the electron beam plays a similar role and can be treated as an equivalent energy spread.

There are two effects due to a spread in the transverse velocity β_{x0} of the electrons entering the undulator. The first is clearly understood considering that two electrons of the same total energy but different β_{x0} have different longitudinal velocity or, conversely, if they have the same β_{x0} , they have different energy. The energy variation equivalent to an initial β_{x0} is given by

$$\frac{\delta \gamma}{\gamma} \Big|_{\theta} = \frac{1}{2} \gamma^2 \beta_{x0}^2 \quad (3-6)$$

Assume that the electron beam has a gaussian distributed transverse velocity at the undulator entrance

$$d\rho(o, \tau_o) = \frac{\rho_o}{\sqrt{2\pi}\sigma_\theta} e^{-\frac{\beta_{x0}^2}{2\sigma_\theta^2}} d(\beta_{x0}) \quad (3-7)$$

If the width of this distribution is small compared with the average transverse velocity of the "central" electron in the undulator, all considerations that have led to Eq. (2-14), in particular the assumption that the phase does not change much, remain valid.

By repeating the argument of the previous section, an angle dependent cut-off harmonic number is found at maximum bunching, namely

$$n'_{c\theta} = \frac{D}{\gamma^2 \sigma_\theta^2} \quad (3-8)$$

A second angle effect arises from the integration of the third of Eqs.

(2-6). By inserting a more general expression for the transverse velocity

$$\beta_x = \beta_{x0} - \sqrt{2} \frac{K}{\gamma} \sin \omega_o z \quad (3-9)$$

the differential equation for the energy becomes

$$\dot{\gamma} = \beta_{x0} \epsilon_o \cos \omega(\tau - z) - \frac{\sqrt{2}}{2} \frac{K \epsilon_s}{\gamma} F_h(b) \sin \phi \quad (3-10)$$

Integration of Eq. (3-10) shows that in addition to the usual modulating term, a second term appears, proportional to the transverse initial velocity of the electron β_{x0} . Since there is no correlation between β_{x0} and the phase of each individual electron, Eq. (2-15) becomes

$$\frac{\delta Y}{Y} = D \sin \phi_0 + \frac{\delta Y}{Y} \Big|_{\theta''} \quad (3-11)$$

with the "equivalent" energy spread

$$\begin{aligned} \frac{\delta Y}{Y} \Big|_{\theta''} = & -\frac{\sqrt{2}}{2\pi} \frac{\lambda_0}{L_u} \frac{DY}{hKF_h(b)} \\ & \cdot \beta_{x0} \left[(\cos h\omega_0 z - 1) \sin \phi_0 - \sin h\omega_0 z \cos \phi_0 \right] \end{aligned} \quad (3-12)$$

We can now repeat the same calculation which has led us to Eq. (2-14), and fold over the angle distribution in analogy to Eq. (3-3)

$$\begin{aligned} \rho(o, \tau_0) = & \frac{\rho_0}{\pi\sqrt{2\pi}\sigma_\theta} \int_{-\pi}^{\pi} d\phi_0 \\ & \cdot \frac{\beta_{x0}^2}{2\sigma_\theta^2} \cdot \\ & \int d(\beta_{x0}) e^{\cos n(\phi + \xi \beta_{x0} \sin \phi_0)} \end{aligned} \quad (3-13)$$

with

$$\xi = \sqrt{2} \frac{DY}{KF_h(b)}$$

and obtain a critical harmonic number $n_{c\theta}$ for the horizontal angle spread, given by

$$n_{c\theta}' = \frac{\sqrt{2}}{4} \frac{KF_h(b)}{DY\sigma_\theta} \quad (3-14)$$

3.c Effect of Undulator Imperfections

The amplitude of high order harmonics is severely affected by imperfections in the undulator. Let us assume that these imperfections could be represented by a noise in the value of the undulator parameter K defined by Eq. (2-2), in what the peak values of the magnetic field or the undulator period do not reproduce exactly from period to period. K is then represented by a nominal $\bar{K}$ value plus a random δK with a gaussian distribution of zero average and standard deviation σ_K (undulator errors)

$$K = \bar{K} + \delta K \quad (3-15)$$

Let us introduce this expression for K in the differential equation (2-11) for γ , where we have made the assumption that the errors in K are small enough not to invalidate the approximate constance of the phase ϕ through the undulator. Equation (2-14) can be rewritten, with $z = L_u$

$$\gamma = \bar{\gamma} + \frac{\sqrt{2}}{2} \frac{\epsilon_0}{\gamma_r} \sin \phi_0 \int_0^{L_u} \delta \left[K F_h(b) \right] dz \quad (3-16)$$

where $\bar{\gamma}$ is the unperturbed solution of Eq. (2-14) for $K = \bar{K}$.

The integral in Eq. (3-16) can be expressed as a sum of integrals calculated in each undulator period. The contribution to the energy modulation due to the error of K in the i .th period can be written

$$\left. \frac{\delta \gamma}{\gamma} \right|_{k,i} = \frac{\sqrt{2}}{2} \frac{\epsilon_0}{\gamma_r^2} \lambda_0 \delta \left[K_i F_h(b) \right] \sin \phi_0 \quad (3-17)$$

The variation of the function of K appearing in this equation can be evaluated by expanding the Bessel functions contained in F_h to first order in their (small) argument. It is obtained

$$\delta [KF] \approx \frac{h+1}{2} F_h(\bar{b}) \delta K \quad (3-18)$$

Since the electrons randomly walk along the undulator, the errors due to the imperfections in every single period add up quadratically and generate a final error in the energy modulation with a gaussian distribution of zero average and dispersion given by

$$\left| \frac{\delta \gamma}{\gamma} \right| = \sqrt{\sum \left(\frac{\delta \gamma}{\gamma} \right)_i^2} \quad (3-19)$$

With the assumption that the errors are all equal (e.g. they are determined by the accuracy of the magnetic field measuring system), we obtain an overall error

$$\left| \frac{\delta \gamma}{\gamma} \right|_K = \frac{\sqrt{2}}{8} (h+1) F_h(b) \frac{\epsilon_0 \lambda_0 \sqrt{N_0}}{\gamma_r^2} \delta K \sin \phi_0 \quad (3-20)$$

which results proportional to the square root of the number of undulator periods N_0 .⁽⁹⁾

The time of arrival of the electrons at a position z is affected by the equivalent energy spread given by Eq. (3-20). With a similar argument used in the previous section, the phase to be used in evaluating the Fourier coefficients results

$$\phi = \phi_0 - \omega \frac{\sqrt{2}}{8} (h+1) z \frac{\epsilon_0 \lambda_0 \sqrt{N_0}}{\gamma_r^4} F_h(b) \delta K \sin \phi_0 \quad (3-21)$$

Since the errors in the undulator modify the trajectory of an electron in an unpredictable way, depending on the initial coordinates, angle and energy of the electron, producing an additional energy spread, we can assume that the elec-

trons in the incoming beam effectively see a statistically distributed K . Therefore, as done before, folding over this distribution and averaging over the initial phase of the electrons, another critical harmonic number is obtained, namely

$$n_{cK} \approx \frac{\sqrt{2}}{\pi} \frac{1}{h(h+1)} \frac{\gamma^2}{\epsilon_0 z_b F_h(b) \sqrt{N_0}} \frac{1}{\sigma_K} \quad (3-22)$$

It will be recognized that in many practical cases this is the largest effect contributing to the suppression of high order harmonics in the electron beam charge modulation.

From Eqs. (3-5), (3-8), (3-14) and (3-22) an overall harmonic attenuation coefficient α and a critical number n_c for the highest attainable harmonic in the modulation are obtained. It is

$$\alpha = e^{-\frac{1}{2} \frac{n^2}{n_c^2}} \quad (3-22)$$

and

$$\frac{1}{n_c^2} = \frac{1}{n_{c\epsilon}^2} + \frac{1}{n_{c\theta}^2} + \frac{1}{n_{c\theta'}^2} + \frac{1}{n_{cK}^2} \quad (3-23)$$

4. SCATTERING OF A LIGHT WAVE BY A BUNCHED ELECTRON BEAM

As it has been shown in the preceding section, after passage through an undulator together with an electromagnetic wave of the proper wavelength and polarization, the charge of an electron beam is modulated in space and time according to Eq. (2-30). This equation is an harmonic expansion with coefficients given by the integrals of Eq. (2-29), in the Laboratory frame (LAB) and in the assumption of an ideal electron beam with zero energy spread and zero emittance

and of a perfect magnetic undulator. Section 3 showed how the coefficients are modified and actually reduced for a real beam and a real undulator.

Assume now that an electromagnetic wave is made to scatter on the electron beam. It can be the same wave we used for modulation, reflected on a mirror, or a second wave of suitable wavelength. We will show in this section that the scattered wave is partially coherent, since the density modulated electron beam acts as a moving grating, and we will calculate the amount of coherence.

In the electron rest frame (ERF), the scattered wave retains the frequency of the incoming wave. In the LAB system, the frequency of the Thomson scattered wave is boosted up by the double Lorentz transformation.⁽¹⁰⁾

4.a Unidimensional Case

Let us consider a process of elastic scattering of a plane electromagnetic wave with a spatially extended distribution of electrons, in the electron rest frame (ERF). Assume first an unidimensional array of N electrons, uniformly distributed at a spacing a along the z -direction, as shown in Fig. 3.

Define:

$$\vec{k}(\vec{k}') = \frac{2\pi}{\lambda(\lambda')} \quad \begin{array}{l} \text{vector momentum of the incoming} \\ \text{(outgoing) photon} \end{array}$$

θ, θ' incident and scattering angles

a electron lattice spacing

If $\vec{E}$ is the electric vector amplitudes of the radiation scattered from the first electron, the amplitude $\vec{E}_T$ scattered by the ensemble of N electrons is given by

$$\vec{E}_T = \sum_{n=1}^N \vec{E}_n = \vec{E} \sum_{n=1}^N e^{i(n-1)a} \left[|\vec{k}| \cos\theta - |\vec{k}'| \cos\theta' \right] \quad (4-1)$$

as it stems from simple geometrical considerations (Fig. 3).

The expression appearing in Eq. (4-1)

$$q_z = |\vec{k}| \cos\theta - |\vec{k}'| \cos\theta', \quad \vec{q} = \vec{k} - \vec{k}' \quad (4-2)$$

is simply the momentum transfer component along the z-direction, and thus the photon intensity measured at the angle θ' can be written as

$$|\vec{E}_T|^2 = |\vec{E}|^2 \left| \sum_{n=0}^{N-1} e^{inaq_z} \right|^2 \quad (4-3)$$

This shows that the intensity obtained with the unidimensional array of N electrons of Fig. 3 is enhanced with respect to the single electron intensity

$|\vec{E}|^2$ by the factor

$$|F(q_z)|^2 = \left| \sum_{n=0}^{N-1} e^{inaq_z} \right|^2 \quad (4-4)$$

$$= \left| \frac{1 - e^{iNaq_z}}{1 - e^{iaq_z}} \right|^2 = \frac{\sin^2 \frac{Naq_z}{2}}{\sin^2 \frac{aq_z}{2}}$$

which is the square of the charge form factor of the electron distribution.

When the Bragg condition

$$q_z = m \frac{2\pi}{a} ; m = 0, 1, 2, \dots \quad (4-5)$$

Fig. 3 Uni-dimensional array of scattering electrons.

is satisfied, the phase shifts among the different components of the total electromagnetic field of Eq. (4-3) are integer multiples of 2π and therefore they add constructively giving rise to a new coherent electromagnetic field.

Moreover, under the same Bragg condition of Eq. (4-5), Eqs. (4-3) and (4-4) show that the intensity of this radiation is concentrated in very narrow intensity peaks centered around the directions defined by Eq. (4-5), whose heights are N^2 times larger than those obtained with a single electron and whose total widths are $2\pi/Na$, as shown in Fig. 4. Since we will only consider wavelengths of the incoming wave in the ERF much larger than the Compton wavelength $\lambda_c (= h/mc = 2.43 \cdot 10^{-2} \text{ \AA})$, the scattering is a pure Thomson process with no energy shift. Consequently, we have

Fig. 4 Form factor of the one-dimensional array of six scattering electrons.

$$|\vec{k}| = |\vec{k}'| = k = \frac{2\pi}{\lambda} \quad (4-6)$$

and for $\theta = 0$, in the backscattering case ($\theta' = \pi$)

$$q_z = 2K \quad \text{and} \quad a = m \frac{\lambda}{2} \quad m = 0, 1, 2 \dots \quad (4-7)$$

This shows that in order to have production of backscattered coherent radiation, the spacing of the electron lattice a must be an integral multiple of one-half wavelength of the incoming (outgoing) radiation in the ERF.

4b. Three-dimensional Case

The considerations of the previous sub-section can be extended to the case where one has an ensemble of N scattering electrons confined in a volume V with a spatial distribution $\rho(x,y,z)$. In this case, the enhancement form factor of Eq. (4-4) is naturally generalized to the

$$F(\vec{q}) = \sum_j^N e^{i\vec{q} \cdot \vec{r}_j} \quad (4-8)$$

where $\vec{r}_j$ is the vector position of the j -th particle. In this case and independently upon what the electron spatial distribution is, Eq. (4-4) can always be rewritten as

$$\left| \sum_j^N e^{i\vec{q} \cdot \vec{r}_j} \right|^2 = \sum_j^N \sum_{h=1}^N e^{i\vec{q} \cdot (\vec{r}_j - \vec{r}_h)} = N + \sum_{j \neq h} e^{i\vec{q} \cdot (\vec{r}_j - \vec{r}_h)} \quad (4-9)$$

and since N can be very large, the second term may be approximated with the average value of the function $e^{i\vec{q} \cdot \vec{r}}$ over the electron distribution $\rho(\vec{r})$

$$\sum_h \sum_{i \neq h} e^{i\vec{q} \cdot (\vec{r}_j - \vec{r}_h)} \approx N(N-1) \left| \int_V d\vec{r} e^{i\vec{q} \cdot \vec{r}} \rho(\vec{r}) \right|^2 \quad (4-10)$$

with the normalization condition

$$\int_V \rho(\vec{r}) d\vec{r} = 1 \quad (4-11)$$

This way, we can replace the sum of Eq. (4-9) over the finite number N of electrons with an integral over the continuous distribution $\rho(\vec{r})$

$$|F(\vec{q})|^2 = \left| \sum_{j=1}^N e^{i\vec{q} \cdot \vec{r}_j} \right|^2 = N + N(N-1) |G(\vec{q})|^2 \quad (4-12)$$

with

$$|G(\vec{q})|^2 = \left| \int_V d\vec{r} e^{i\vec{q} \cdot \vec{r}} \rho(\vec{r}) \right|^2 \quad (4-13)$$

and the photon intensity measured with N electrons is

$$I(N, \vec{q}) = I_0(\vec{q}) [N + N(N-1) |G(\vec{q})|^2] \quad (4-14)$$

where the incoherent and coherent intensities are given, respectively, by

$$I_{inc}(N, \vec{q}) = N I_0(\vec{q}) \quad (4-15)$$

$$I_{coh}(N, \vec{q}) = N(N-1) |G(\vec{q})|^2 I_0(\vec{q})$$

and I_0 is the intensity measured with a single electron.

This result shows that, in addition to the usual incoherent term proportional to N , the scattering from a burst of N electrons exhibits always a

second coherent contribution which depends very strongly upon the electron spatial density through the presence of its charge form factor. When the momentum transfer is low enough that $\vec{q} \cdot \vec{r} \ll 1$ over the integration volume, the wavelength of the scattering radiation is much greater than the linear dimensions of the electron bunch. In this case it is $G(\vec{q}) \approx 1$ and the cooperative effects of the N electrons generate a scattered intensity N^2 times larger than the intensity from a single electron.

In the case when λ is much smaller than the linear dimensions of the electron bunch, the arguments in the exponent of Eq. (4-13) can be large and widely different in value, with the consequence that $G(\vec{q})$ falls rapidly to zero and only the incoherent part survives in the scattering. This is what happens in the usual backscattering experiments where the wavelengths involved are always much shorter than the electron bunch dimensions. ⁽³⁾

The use of a charge density modulated electron beam is a way to avoid this cancellation. If the electron density $\rho(\vec{r})$ is deformed in a way to create regular inhomogeneities inside the bunch on a length scale of the order of $|\vec{q}^{-1}|$, the form factor $G(\vec{q})$ can be different from zero even for very small wavelengths λ , and the scattering can receive contributions from the coherent part proportional to $N^2 G$.

4c. Spatial Grating

Let us now examine in detail the effect of the periodic density modulation of a gaussian electron bunch, obtained in the way discussed in Sec. 2.

We showed there that an energy modulation

$$\frac{\Delta E}{E} = D \sin \phi_0$$

around the energy E of the electron beam can produce a local charge density modulation, Eq. (2-30), which we will write in a more complete form as follows

$$\rho(x, y, z^L) = \frac{1}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_z^L} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} e^{-\frac{(z^L - \beta\tau)^2}{2\sigma_z^{L2}}} \quad (4-16)$$

$$\times \left[1 + 2 \sum_{n=1}^{\infty} J_n(n\eta) \cos \frac{2n\pi}{\beta\lambda_b} (z^L - \beta\tau^L) \right]$$

where $\sigma_x, \sigma_y, \sigma_z^L$ represent the electron bunch dimension in the x and y (transverse) and z (logitudinal) directions, and with η given by Eq. (2-25). The superscript L in Eq. (4-16) stands for LAB frame of reference. No energy or angular spread will be considered for the moment.

The coefficients in the harmonic expansion contain the Bessel functions $J_n(x)$, defined by their generating integrals

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\phi_0 - x \sin\phi_0) d\phi_0 \quad (4-17)$$

where ϕ_0 is the phase of the modulating wave electric field.

If we now apply the Lorentz transformations from the LAB to the ERF (variables with no subscripts) to the charge density

$$z = \gamma(z^L - \beta\tau^L) ; \tau = \gamma(\tau^L - \beta z^L) \quad (4-18)$$

and the approximation is made that, in the vicinity of maximum bunching,

$z^L = z_b$, it is

$$J_n \left(n \frac{z^L - \frac{1}{2} L_u}{z_b - \frac{1}{2} L_u} \right) = J_n \left(n \frac{z^L}{z_b} \right) \quad (4-19)$$

Equation (4-16) becomes

$$\rho(x, y, z) = \rho_0(x, y, z) \left[1 + 2 \sum_{1}^{\infty} J_n \left(n \frac{\gamma}{z_b} (z + \beta \tau) \right) \cos nk_b z \right] \quad (4-20)$$

where

$$\rho_0(x, y, z) = \frac{1}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_z} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2} \right)} \quad (4-21)$$

$$\sigma_z = \gamma \sigma_z^L$$

$$k_b = \frac{1}{\gamma} \frac{2\pi}{\lambda_b}$$

$$\beta = 1$$

This expression shows an electron pulse which is stationary in the ERF except for the time-dependent Bessel function describing the maximum bunching

$$z_b = \gamma(z + \beta \tau)$$

being swept across the pulse at a speed βc .

4d. Scattering

If a laser beam of wavelength λ_s in the LAB is directed head-on against the electrons, the momentum transfer components in the ERF expressed as a function of the angles θ and ϕ of Fig. 5 are given by the usual relations

Fig. 5 Frame of reference for the scattering kinematics.

$$k = |\vec{k}| = 2\gamma \frac{2\pi}{\lambda_s}$$

$$k' = |\vec{k}'|$$

$$\vec{q} = \vec{k} - \vec{k}'$$

$$q = k - k'$$

(4-22)

$$q_x = -k' \sin\theta \cos\phi = -k'l$$

$$q_y = -k' \sin\theta \sin\phi = -k'm$$

$$q_z = -(k - k' \cos\theta)$$

where the Thomson approximation $k = k'$ holds for a stationary electron in the ERF.

By inserting Eqs. (4-20) and (4-22) in Eq. (4-13), one has

$$G(\vec{q}) = \sum_n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz \rho_0(x, y, z) a_n \cos(nk_b z) e^{i(q_x x + q_y y + q_z z)} \quad (4-23)$$

where according to Eq. (2-29)

$$a_n = 2J_n \left[n \frac{\gamma}{z_b} (z + \beta \tau) \right] = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n \left[\phi_0 - \frac{\gamma}{z_b} (z + \beta \tau) \sin \phi_0 \right] d\phi_0 \quad (4-24)$$

and

$$a_0 = 1$$

To perform the integrations in x , y and in z separately, let us rewrite Eq. (4-23) in the following way

$$G(\vec{q}) = U(l, m) \sum_n I_n \quad (4-25)$$

with

$$I_n = \frac{1}{\sqrt{2\pi\sigma_z}} \int_{-\infty}^{\infty} dz a_n e^{-\frac{z^2}{2\sigma_z^2}} \cos(nk_b z) e^{iq_z z} \quad (4-26)$$

and

$$U(l, m) = \frac{1}{2\pi\sigma_x \sigma_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \rho_0(x, y) e^{-ik'(lx + my)} = e^{-\frac{1}{2} k'^2 (\sigma_x^2 l^2 + \sigma_y^2 m^2)} \quad (4-27)$$

is the Fourier transform of the transverse charge density distribution. Since $k'_x \sigma_x, \sigma_y \gg 1$, this function vanishes rapidly everywhere except at $\theta = 0, \pi$ where it is also $\ell, m = 0$. This result means that owing to the finite transverse dimension of the electron beam, any possible coherent contribution to the scattering can be confined only to two extremely narrow cones centered around the forward and backward directions, with semi-aperture angles given by

$$\frac{1}{k'_x \sigma_{x,y}} = \frac{1}{4\pi\gamma} \frac{\lambda_s}{\sigma_{x,y}}$$

On the other hand, in these two small angular regions of θ close to 0, π one has

$$\begin{cases} q_z = -(k-k' \cos\theta)_{\theta \approx 0} \approx q_z(0) - \frac{1}{2} k' \theta^2 \\ q_z = -(k-k' \cos\theta)_{\theta \approx \pi} \approx q_z(\pi) + \frac{1}{2} k' (\theta - \pi)^2 \end{cases} \quad (4-28)$$

and therefore, if the condition for the Fresnel number

$$r = \frac{1}{4\pi} \frac{\lambda_s \sigma_z^L}{\sigma_{x,y}^2} \ll 1 \quad (4-29)$$

is satisfied, it is $q_z = q_z(0, \pi)$ in Eq. (4-26) with no effect on Eq. (4-27).

It can now be easily shown that a small energy modulation in the Lab frame transforms to the ERF as follows

$$\left. \frac{\Delta E}{E} \right|_{\text{ERF}} = \frac{1}{2\beta^2} \left[\frac{\Delta E}{E} \right]_{\text{LAB}}^2 \quad (4-30)$$

and that consequently the third component of the momentum transfer at $\theta = \pi$ in the ERF becomes

$$q_z(\pi) = -2k \left[1 + D \sin\phi_0 \right] \quad (4-31)$$

where ϕ_0 is the phase appearing in the definition of the Bessel functions of Eq. (4-24), invariant under Lorentz transformation.

Since $D \ll 1$, with the aid of Eq. (4-31) and after some algebra, the integral of Eq. (4-26) can be reduced to the form

$$I_n \approx J_n\left(n \frac{\tau}{\tau_b}\right) e^{-\frac{(\gamma - \gamma_n)^2}{2\Delta\gamma_n^2}} \quad (4-32)$$

where

$$\tau_b = \frac{z_b}{n\gamma} = \frac{1}{2\pi} \frac{\gamma\lambda_b}{\beta D} + \frac{1}{2} \frac{L_u}{\beta\gamma} \quad (4-33)$$

$$\gamma_n^2 = \frac{n}{4} \frac{\lambda_s}{\lambda_b} \quad (4-34)$$

$$\frac{\Delta\gamma_n}{\gamma_n} = \frac{1}{4\pi} \frac{\lambda}{\sigma_z \frac{L}{z}} = \Gamma \quad (4-35)$$

In the backward direction, Eq. (4-25) becomes in turn

$$G\left[\vec{q}(\pi)\right] = U(l, m) \sum_n J_n\left(n \frac{\tau}{\tau_b}\right) e^{-\frac{(\gamma - \gamma_n)^2}{2(\Delta\gamma_n)^2}} \quad (4-36)$$

Equation (4-36) exhibits an infinite series of peaks exactly centered at the Bragg positions shown by Eq. (4-34), where λ is the wavelength of the scattered radiation in the LAB

$$\lambda = \frac{\lambda_b}{n} = \frac{\lambda_s}{4\gamma_n^2} \quad (4-37)$$

The peak-to-peak separation

$$\delta\gamma_n = \frac{1}{8\gamma_n} \frac{\lambda_s}{\lambda_b} ; \quad \frac{\delta\gamma_n}{\gamma_n} = \frac{1}{2n} \quad (4-38)$$

is found in general to be much larger than the width given by Eq. (4-35).

Consequently, Eq. (4-36) can receive only contributions from one term of the sum at a time, and one has

$$G[\vec{q}(\pi)] = U(\lambda, m) J_n \left(n \frac{\tau}{\tau_b} \right) e^{-\frac{(\gamma - \gamma_n)^2}{2\Delta\gamma_n^2}} \quad (4-39)$$

where γ closely satisfies Bragg condition of Eq. (4-34), equivalent to Eq. (4-7) found for the undimensional case.

Finally, since

$$U(0,0) = 1$$

and all

$$I_n(q_z \rightarrow 0) \rightarrow 0$$

except

$$I_0(q_z = 0) = 1$$

we obtain immediately from Eq. (4-25)

$$G(\vec{q} = 0) = 1$$

which satisfies the normalization condition (4-11).

5. PROPERTIES OF THE SCATTERED RADIATION

From Eq. (4-15), the total coherent intensity turns out to be

$$I^{\text{coh}}(N) = N^2 \int |G(\vec{q})|^2 I_0(\vec{q}) d\Omega \quad (5-1)$$

and since Eq. (4-39) is extremely peaked at $\theta = \pi$, it reduces to

$$I^{\text{coh}}(N) = N^2 I_0(\theta=\pi) e^{-\frac{(\gamma-\gamma_n)^2}{\Delta\gamma^2}} J_n^2\left(n \frac{\tau}{\tau_b}\right) \int_{4\pi} |U(l, m)|^2 d\Omega \quad (5-2)$$

Now, in the forward and backward direction it is

$$d\Omega = \sin\theta \, d\theta \, d\phi \approx d\ell \, dm$$

hence, by using Parseval's theorem one has ($k' = k$)

$$\left(\frac{k}{2\pi}\right)^2 \int_{4\pi} |U(\rho, m)|^2 d\ell dm = \iint_{-\infty}^{\infty} |\rho_0(x, y)|^2 dx dy = \frac{1}{4\pi\sigma_x\sigma_y} \quad (5-3)$$

and thus

$$I^{\text{coh}}(N) = \frac{1}{2} I_0(\theta=\pi) \left[NJ_n\left(n \frac{\tau}{\tau_b}\right) \right]^2 \frac{(\lambda_s/2\gamma)^2}{2\pi\sigma_x\sigma_y} e^{-\frac{(\gamma-\gamma_n)^2}{\Delta\gamma_n^2}} \quad (5-4)$$

This expression has this simple interpretation: in the plane wave description where the electron beam is uniformly illuminated by the laser, the $1/2(NJ_n)^2$ electrons longitudinally distributed in the n -th spatial harmonic are all backscattering in phase, and hence the emitted radiation adds up as the square of their number.

The obtained intensity is determined by the Thomson cross section and is confined in the diffraction limited solid angle

$$\Delta\Omega = \frac{(\text{Produced wavelength})^2}{\text{Beam area}} = \frac{(\lambda_s/2\gamma)^2}{2\pi\sigma_x\sigma_y} \quad (5-5)$$

giving rise to a beam with laser-like angular divergence and frequency bandwidth given by Eqs. (5-5) and (4-39) respectively.

5a. Energy and Angular Spread

The effects of the energy and angle spread of the electron beam together with that due to the undulator imperfections modify Eq. (5-4) with the introduction of the factor $\exp(-1/2 \frac{n^2}{n_c^2})$ in the expression of the number of electrons contributing to the scattering. Besides, Eq. (5-4) has to be integrated over the energy spread, and since $\Gamma \ll \sigma_e$, this causes a further decrease of the coherent production.

To account for both effects, Eq. (5-4) becomes

$$I^{\text{coh}}(N) = \frac{\sqrt{\pi}}{2} I_0(\theta=\pi) \left[N e^{-\frac{1}{2} \frac{n^2}{n_c^2}} J_n \right]^2 \Delta \Omega \frac{\Gamma}{\sigma_e} \quad (5-6)$$

As for the electron angular spread, since the coherent backscattering is concentrated at $\theta = \pi$, the transverse component of the momentum remains essentially unaffected in the process and, from the kinematical point of view, the only effect of the angular dispersion is to simulate an extra contribution in the energy spread. If $\Delta \theta_e$ is the angular dispersion of the beam in the LAB, the corresponding longitudinal momentum or effective energy spread results (see also Eq. (3-6))

$$\left. \frac{\Delta \gamma}{\gamma} \right|_{\text{ANGLE}} \approx \frac{1}{2} (\gamma \Delta \theta_e)^2 \quad (5-7)$$

As long as the following holds

$$\Delta \theta_e \ll \frac{1}{\gamma} \sqrt{2\sigma_e} \quad (5-8)$$

the effect of the angular spread of Eq. (5-7) can be neglected with respect to Eq. (5-6).

5a. Counting Rate and Time Structure

Owing to the presence of the Bessel functions in Eq. (5-6), the coherent production starts when $\tau/\tau_b \approx 1$, which corresponds to the peak of the electron pulse crossing the maximum bunching position $z^L = z_b$ in the LAB frame. Then, as the bunch proceeds further on, the intensity oscillates very rapidly and its maximum value drops down continuously.

If we assume to have a bunch travelling inside the laser field from the position $z^L = z_b$ to the position $z^L = z_b + L$ in the laboratory, it appears to be illuminated in the ERF during a time $L/\gamma\beta c$ and, according to Eq. (5-6), the total number of coherent photons produced inside the solid angle (5-5) will be given by

$$N_x^{\text{coh}} = \frac{\sqrt{\pi}}{2} I_0(\pi) N^2 e^{-n^2/n_c^2} \frac{\Delta\Omega \Gamma}{\sigma_\epsilon} \frac{z_b}{\gamma\beta c} \int_1^{1+L/z_b} J_n^2(nx) dx \quad (5-9)$$

Since the corresponding number of incoherent photons is

$$N_x^{\text{inc}} = N I_0(\pi) \Delta\Omega \frac{L}{\gamma\beta c} \quad (5-10)$$

the coherent/incoherent ratio inside $\Delta\Omega$ for the bunch of electrons travelling inside a laser field from $z^L = z_b$ to $z^L = z_b + L$ results

$$R = \frac{\text{Coherent}}{\text{Incoherent}} = \frac{\sqrt{\pi}}{2} N e^{-n^2/n_c^2} \frac{\Gamma}{\sigma_\epsilon} \bar{B} \quad (5-11)$$

where

$$\bar{B} = \frac{z_b}{L} \int_1^{1+L/z_b} J_n^2(nx) dx \quad (5-12)$$

On the other hand, if laser and electron beam are assumed cylindrical with the same cross section, $2\pi\sigma_{x,y}^2$, Eq. (5-10) specializes into

$$N_x^{\text{inc}} = N \frac{P_s}{2\pi\sigma_x^2 \epsilon_s} \frac{L}{\beta c} r_o^2 \Delta\Omega \quad (5-13)$$

where r_o is the classical electron radius (recall that the Thomson cross section at $\theta = \pi$ is r_o^2) P_s and ϵ_s are the laser power (watts) and laser photon energy (joules), respectively. Consequently, Eq. (5-9) becomes

$$N_x^{\text{coh}} = \frac{\sqrt{\pi}}{2} N^2 e^{-n^2/n_c^2} \frac{P_s}{2\pi\sigma_x^2 \epsilon_s} \frac{L}{\beta c} \frac{\tau}{\sigma_\epsilon} r_o^2 \Delta\Omega \bar{B} \quad (5-14)$$

The time distribution in the LAB of both incoherent (5-13) and coherent (5-14) radiation is, as usual, entirely determined by the length of the electron pulse σ_z^L . Accordingly, from Eqs. (2-15) and (4-31) the temporal coherence length in the LAB results

$$l_c = \lambda \left(\frac{\lambda}{d\lambda} \right) = \frac{\lambda}{2D} \quad (5.15)$$

with the meaning that after l_c the maximum phase-lag between any pair of photons in the bunch is 2π . Of course this is strictly valid only under the assumption that the laser frequency stability could be held as such that

$$\frac{\Delta\lambda_s}{\lambda_s} \ll 2D$$

Finally one has to remember that the coherent yield expressed in the ERF by Eq. (5.14) is concentrated in the LAB in the solid angle $\Delta\Omega/\gamma^2$ as a result of the Lorentz transformation from the (ERF) to the (LAB)-system.

6. EXAMPLES

It is quite clear from Eq. (4-40) that coherent backscattering in the soft X-rays region of commercially available lasers definitely requires very high order components in the Fourier expansion of the electric charge density (4-16) and consequently that all the limiting parameters like energy spread, angular emittance and undulator imperfections must be really pushed to state-of-the-art technology. Nevertheless, they appear within the present limits of feasibility.

We will have two lasers: laser (b) to modulate the electron beam, and laser(s) to be scattered by the electrons. The two lasers may coincide.

Let λ be the wavelength of the hard photon beam we want to produce. According to Eq. (4-40) the required electron energy is

$$\gamma = \frac{1}{2} \sqrt{\frac{\lambda_s}{\lambda}} \quad (6-1)$$

where λ_s is the wavelength of the s-laser. The parameter K of the transverse undulator is related to the electron energy γ and to the undulator period λ_0 by

$$K = \sqrt{2h\gamma^2 \frac{\lambda_b}{\lambda_0} - 1} \quad (6-2)$$

obtained from Eqs. (2-1) and (2-2), with h undulator harmonic number. Equation (6-2) shows that the period of the undulator must be smaller than a critical period given by

$$\lambda_c = 2h\gamma^2 \lambda_b \quad h = 1, 3, 5 \dots \quad (6-3)$$

However, according to Eq. (2-16), the undulator magnetic field

$$B_0^{(T)} = \frac{\sqrt{2K}}{(e/mc) \frac{\lambda_0}{2\pi}} = \frac{K}{66.029 \lambda_0 \text{ (m)}} \quad (6-4)$$

has to be high enough to obtain a sufficient modulation D with realistic laser powers. Such an undulator can be built with a permanent magnet structure of the "hybrid" type.⁽¹¹⁾ In a hybrid, magnetic field, period and gap g are connected by the empirical relationship

$$B_o = 3.3 B_r \exp \left[- \frac{g}{\lambda_o} \left(5.47 - 1.8 \frac{g}{\lambda_o} \right) \right] \quad (6-5)$$

where B_r , the remanent field of the permanent magnets used (Rare-Earth-Cobalt or Neodimium-Iron), is of the order of 1 Tesla. Equation (6-5) will be used to determine gap g.

Peak bunching of the electron beam takes place at a distance z_b from the beginning of the undulator given by Eq. (2-24). We require this position to be located past the undulator, in order to perform the backscattering in free space. This condition leads to

$$\frac{\gamma^2 \lambda_b}{\pi D} > L_u \quad (6-6)$$

or, using Eq. (2-1)

$$N_o < \frac{1+K^2}{2\pi h D} \quad (6-7)$$

Chosen the parameter K, the harmonic number h and a suitable number N_o of magnetic periods, Eq. (2-16) allows us to calculate the necessary b-laser electric field to obtain D. We have

$$E_o = \sqrt{2} \frac{mc^2}{e} \frac{\gamma^2}{F_h(b) L_u K} D \quad (6-8)$$

The maximum energy exchange between laser wave and electron beam is approximatively obtained when the b-laser is focused into the undulator with its waist w_0 at the undulator center and Rayleigh range equal to half the undulator length. (12) The cross section of the laser beam is therefore, at the waist location

$$2\pi w_0^2 = \frac{\lambda_b L_u}{2} \quad (6-9)$$

The pulsed power of the b-laser corresponding to this E_0 field is given by

$$P_b = \frac{\pi w_0^2}{Z_0} E_0^2 \quad (6-10)$$

where Z_0 is the impedance of free space ($= 377$ ohm).

In order to estimate the incoherent flux (5-13) let us assume an electron beam with average current $\langle I \rangle$ and a pulsed structure with repetition rate f . The number of electrons/bunch appearing in Eq. (5-13) is

$$N = \frac{\langle I \rangle}{ef}$$

with e the electron charge. Similarly assume a s-laser with average power $\langle P_s \rangle$ and the same cross section of the electron beam. If the laser/electron interaction takes place over a length L the laser pulse duration has to be at least $\tau \approx 2L/BC$ and the laser peak power of (5-13) becomes:

$$P = \frac{\langle P_s \rangle}{\tau f}$$

We have examined in detail two examples (Table I), to produce radiation in the soft x-ray region, at $120 \text{ \AA}$ and $40 \text{ \AA}$. This latter wavelength is in particular interest for x-ray microscopy and holographic work. The same laser has been considered both for bunching and scattering: a Carbon-Dioxide, TEA, pulsed, frequency doubled laser with wavelength $\lambda_p = \lambda_s = 5.03 \text{ \mu m}$. Lasers of this kind are commercially available with powers of several kw.

For the electron beam, the essential importance of a very small energy spread to produce high harmonic numbers, suggests the use of a recirculated Van de Graaf electrostatic accelerator, i.e. a machine with charge recovery. Accelerators of this type, actually an electron accelerator followed by an electron de-celerator of the same voltage, have been operated successfully⁽¹³⁾ with impressive results, providing beams of exceptionally high energy resolution and good emittance.

The undulator considered here is of the permanent magnet hybrid type, with a period of a few millimeters. Admittedly, this short period is difficult to achieve, however experiments are in program in this direction and promising results are available.⁽¹⁴⁾

For the two examples in Table I, the integral of the Bessel function squared over the region of backscattering has been calculated. Figure 6 shows the result. It is apparent that increasing the interaction region will increase the number of coherent x-ray photons, but increase also the incoherent background. The yield quoted for the two cases is referred to the interaction length which produces the coherent/incoherent ratio R shown in the Table.

It is interesting to compare our results with the coherent power obtainable for synchrotron radiation storage rings. Figure 7 shows this comparison.

Table I

Soft x-rays wavelength	λ_x	40	120	Å
ACCELERATOR				
Recirculated Van de Graaf				
Electron energy	E	09.30	5.37	MeV
Average current	$\langle I \rangle$	8		μA
Electron bunch length	σ_z	150		mm
Repetition rate		50		pps
Energy spread	σ_E	$0.7 \cdot 10^{-6}$		
Beam emittance	ϵ_e	$1.6 \cdot 10^{-8}$		m-rad
LASER				
CO ₂ , Doubled, TEA, pulsed				
Wavelength	$\lambda_D = \lambda_s$	5.3		μm
Average power		500		W
Peak power (scatering)		10		GW
UNDULATOR				
Permanent magnet, transverse				
Period	λ_0	3.5		mm
No. of periods	N_0	24		
Magnetic parameter (r.m.s.)	K	0.095		
Harmonic No.	h	1	3	
Accuracy of K	σ_K	$5 \cdot 10^{-4}$		
BUNCHING AND SCATTERING				
Modulation Depth	D	10^{-3}	$0.3 \cdot 10^{-3}$	
Bunching harmonic	n	1325	500	
Overall Harmonic attenuation coefficient	α	0.368	0.334	
Location of max. bunching from undulator end	z_b	238		mm
YIELD				
Coherent flux	n_x	$1.2 \cdot 10^8$	$3.5 \cdot 10^9$	photons/pulse
Coherent power		$3.4 \cdot 10^{-5}$	$3 \cdot 10^{-2}$	watt/0.1% bandwidth
Coherent length		2	18	μm
Coherent/Incoherent ratio	R	$6 \cdot 10^6$	$4 \cdot 10^6$	
Emission solid angle i.e. the coils	$\Delta\Omega$	$4.1 \cdot 10^{-11}$	$3.7 \cdot 10^{-10}$	sterad

Fig. 6 Coherent production and coherent/incoherent ratio vs. interaction length for the two cases of Table I.

Fig. 7 Comparison of the coherent flux obtained by backscattering with synchrotron radiation storage rings: arc sources and undulators.

APPENDIX A

The phases appearing in Eq. (2-10) of the text can be conveniently rewritten in terms of the longitudinal coordinate z only, using the relation

$$\tau = \tau_0 + \int_0^z \frac{dz}{\beta_z} \quad (\text{A-1})$$

with β_z given by Eq. (2-7). After some algebra, it is obtained

$$\phi^\pm = a^\pm - b \sin 2\omega_0 z \quad (\text{A-2})$$

with

$$\begin{cases} a^\pm = \phi_0 - \omega_0 z \left[\mp 1 - \frac{\omega}{\omega_0} \frac{1+k^2}{2\gamma^2} \right] \\ b = \frac{\omega}{4\omega_0} \frac{k^2}{\gamma^2} \end{cases} \quad (\text{A-3})$$

The sine of the phase (A-2) can be expressed as an infinite series of Bessel functions (*) as follows

$$\sin(a^\pm - b \sin 2\omega_0 z) = \sum_{n=-\infty}^{\infty} J_n(b) \sin \phi_n^\pm \quad (\text{A-4})$$

with

$$\phi_n^\pm = a^\pm - 2n\omega_0 z \quad (\text{A-5})$$

Now, Eq. (2-9) describes a monotonous energy exchange between laser and electrons only when the phase in Eq. (A-5) remains approximatively constant throughout the undulator, or when

$$\frac{d\phi_n^\pm}{dz} = 0$$

We can see that this happens when the energy of the electrons is chosen very close to a "resonant energy" γ_r defined, according to Eq. (2-1) as

$$\gamma_r = \left[\frac{\omega}{\hbar\omega_0} \frac{1+k^2}{2} \right]^{1/2} \quad (\text{A-6})$$

then, the derivative of the phase can be explicitly written as

$$\frac{d\phi_n^\pm}{dz} = -\omega_0 \left[\mp 1 + 2n - \hbar \frac{\gamma_r^2}{\gamma^2} \right] = -\omega_0 \left[\mp 1 + 2n - \hbar \right] \quad (\text{A-7})$$

This expression vanishes for two values of the index n given by

$$\begin{cases} n = \frac{\hbar+1}{2} \\ n = \frac{\hbar-1}{2} \end{cases} \quad (\text{A-8})$$

respectively. Hence, in the Bessel function series (A-4) we will retain only two terms. Moreover, since when the derivative vanishes, both phases are the same $\phi_n^\pm = \phi$, we will finally write

$$\sin\phi^\pm = J_{\frac{\hbar+1}{2}}(b) \sin\phi \quad (\text{A-9})$$

With these positions, the differential equation for energy and phase becomes

$$\begin{cases} \dot{\gamma} = \frac{\sqrt{2} \cdot k\epsilon_0}{2} \frac{\omega_0}{\gamma} \left[J_{\frac{\hbar-1}{2}}(b) - J_{\frac{\hbar+1}{2}}(b) \right] \sin\phi \\ \dot{\phi} = -\hbar\omega_0 \left[1 - \frac{\gamma_r^2}{\gamma^2} \right] \end{cases} \quad (\text{A-10})$$

They describe, in phase space, the motion of a parametric pendulum, and as such can be solved, e.g. by iterative expansion in power series of the (small) parameter $k\varepsilon_0/\gamma$. However, for the sake of the present considerations, it is completely adequate to disregard the equation for the phase and put

$$\phi \equiv \phi_0$$

in the first of Eqs. (A-10).

CONCLUSION

The production of high flux of coherent x-rays by Compton backscattering appears feasible with the use of a commercially available high power laser, a modified electrostatic accelerator and a very short period magnetic undulator.

The proposed system will compare favorably with synchrotron radiation sources for the attainable flux and for the considerable lesser complexity, especially where electrostatic accelerators are already available.

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