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Hot Dry Rock Reservoir Characterization and Modeling

October 1, 1978—September 30, 1979

Final Report

University of California

LOS ALAMOS SCIENTIFIC LABORATORY

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Prepared for the
Los Alamos Scientific Laboratory

by

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HOT DRY ROCK RESERVOIR CHARACTERIZATION AND MODELING

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ABSTRACT

This report presents results of analytical and experimental studies on hydraulic fracturing and on the characterization and modeling of hot dry rock geothermal energy reservoirs. The first four Chapters are concerned with problems of thermal cracking and heat transfer, with fluid flow through large cracks, and with the stable and unstable growth of water-filled cracks under internal pressure and thermal loading. In Chapter 5 experiments are reported, which present visually observable hydraulic fractures in transparent materials to demonstrate the interaction between hydraulic fractures and the development of thermal cracks. In Chapter 6 seismic detection of hydraulic fractures is discussed, and a method to invert crack-scattering data is presented.

The six self-contained Chapters describe individual and joint efforts of investigators as indicated by the authorship of the chapters.

CHAPTER 1

THE STOPPING OF A RISING, LIQUID-FILLED CRACK IN THE EARTH'S CRUST BY A FREELY SLIPPING HORIZONTAL JOINT

by

J. Weertman*

1.1 INTRODUCTION

The extraction of heat from hot dry rock masses within the Earth's crust (Smith et al.¹) requires the creation of large liquid-filled cracks by hydraulic fracture. A serious potential problem to this scheme for the extraction of heat from hot dry rock [as proposed at the Los Alamos Scientific Laboratory (LASL)] is the possibility that a large water-filled crack will not remain where it is created. We pointed out (Weertman,²⁻⁵ and Weertman and Chang,⁶) that a crack which is filled with liquid that is less dense than the surrounding rock can rise to the Earth's surface by breaking open rock at the upper crack tip while simultaneously squeezing shut at its lower tip. Pollard and co-workers,⁷⁻⁹ through use of our analysis, calculated that any water-filled crack becomes unstable when the crack length has a relatively modest length. The equation for the stress intensity factor K at the upper tip of a two-dimensional, liquid-filled crack (see Eq. (13) of Weertman and Chang,⁶) is

$$K = (\pi L)^{1/2}[(\rho - \rho')gL],$$

where L is the half-length of a crack, ρ is the density of rock, ρ' is the density of the liquid, and g is the gravitational acceleration. (This expression for K is valid if the tectonic stress has a much smaller variation with depth than does the hydrostatic pressure in the rock or in the liquid.) Experimental

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measurements of the critical K_c value for Westerly Granite (Schmidt and Lutz,¹⁰) have determined that $K_c = 2.7 \text{ MPa m}^{1/2}$ for this rock. Setting $K = K_c$ and using $\rho - \rho' = 2000 \text{ kg m}^{-3}$ and $g = 9.8 \text{ m s}^{-2}$, it is found that any crack in granite longer than only $2L = 36 \text{ m}$ will move towards the Earth's surface. (Secor and Pollard⁹ found a somewhat larger length of $2L = 200 \text{ m}$ through use of a tectonic stress with an appreciable depth variation.)

The water-filled cracks envisioned for use in the extraction of heat from hot dry rock masses are of dimensions of the order of one or two kilometers. Will liquid-filled cracks of such large dimensions rise to the Earth's surface? They certainly should if the critical stress intensity factor K_c of the crustal rock is within an order of magnitude or so of $3 \text{ MPa m}^{1/2}$, and if the cracks are in competent rock without pre-existing horizontal or near horizontal joints. But it is reasonable to expect any large rock mass to contain joints. Thus the possibility exists that any rising crack might be stopped by a joint and that much larger stable liquid-filled cracks can be created in a hot dry rock mass. It is the purpose of this paper to examine this possibility for the idealized case of a freely slipping horizontal joint. The more realistic case of a joint with friction is also considered. (The analysis is also applicable to the problem of a rising magma-filled crack as well as to the problem of a descending water-filled crack in a glacier.) Although only vertical cracks and horizontal joints are considered in this paper, the results should be qualitatively correct for cracks that are only approximately vertical and joints that are only approximately horizontal. (A joint that deviates by a large angle from a horizontal orientation, of course, can itself be considered to be an approximately vertical crack.) The results also should be qualitatively correct for cracks that do not extend an infinite distance in the third direction.

1.2 THEORY

1.2.1 Review

Consider first a vertical, liquid-filled two dimensional crack such as is shown in Fig. 1.1. Let y represent distance measured in the vertical direction and let the positive direction be upwards. Let x represent distance measured in the horizontal direction. The crack plane exists at $x = 0$. A tensile tectonic stress T is considered to act normal to the crack plane. In addition to this stress hydrostatic pressure equal to $P_0 - \rho gy$ also acts across the crack plane where P_0 is the pressure at the point chosen for the origin ($y = 0$), g is the

Fig. 1-1. Vertical, liquid-filled crack.

gravitational acceleration, and ρ is the density of rock. For simplicity, it is assumed that both ρ and T are constants and are not functions of the depth y . The total normal stress across the crack plane, were the crack not present, is thus equal to $T - P_0 + \rho gy$.

The crack profile is given by ²⁻⁹

$$D(y) = D_0(1 + y/y_0)(1 - y^2/L^2)^{1/2}, \quad (1-1)$$

where $D(y)$ is the crack opening at position y , L is the half-width of the crack, and the terms D_0 and y_0 are given by

$$D_0 = 2V/\pi L, \quad (1-2)$$

and

$$y_0 = 2V\mu/\pi\alpha g(\rho - \rho')L^2, \quad (1-3)$$

where ρ' is the density of the liquid, μ is the shear modulus of the rock, $\alpha = 1 - \nu$ where ν is Poisson's ratio for the rock, and V is the volume of liquid in the crack per unit distance in the z direction. (It will be assumed that the density of the liquid is a constant and is independent of position y or of the hydrostatic pressure.) In Eq. (1-1) the origin is taken to be at the center of the crack. Thus the crack tips exist at $y = \pm L$. Since normally $D(y)$ must be a positive quantity, the largest permitted value of L (for a fixed volume V of fluid) is equal to

$$L_{\max} = \{2V\mu/\pi\alpha g(\rho - \rho')\}^{1/3}. \quad (1-4)$$

The hydrostatic pressure $P(y)$ within the liquid of the crack is equal to

$$P(y) = -T + P_0 + P' - \rho'gy, \quad (1-5)$$

where it can be shown that the term P' is equal to

$$P' = \mu V/\alpha \pi L^2. \quad (1-6)$$

The tensile stress that acts across the crack plane ahead of, but very close to the crack tip generally is equal to the term $K/(2\pi r)^{1/2}$ where r is the distance measured from the crack tip and the constant K is known as the stress intensity factor. It can be shown that K is equal to the expression

$$K = \{(\mu/\alpha)[\pi(L - |y|)/2]\}^{1/2} |dD/dy| \}_{|y| \rightarrow L}. \quad (1-7)$$

The values of stress intensity factor K at the two crack tips are given by

$$K = \{P' \pm (\rho - \rho')gL/2\}(\pi L)^{1/2}, \quad (1-8)$$

where the plus sign is used for the upper tip and the minus sign for the lower tip. It should be noted from Eq. (1-6) and Eq. (1-7) that the stress intensity factor of a crack filled with an incompressible fluid is independent of the T

and P_0 and depends only upon the crack length and the volume of fluid within the crack.

If K_c is the critical stress intensity factor of the rock that contains the liquid-filled crack, the values of K given by Eq. (1-8) cannot exceed K_c . Thus a crack with a stress intensity factor K at the upper crack tip that exceeds K_c must increase its length until its half-length satisfies the equation

$$K_c = \{(\mu V / \alpha \pi L^2) + (\rho - \rho') g L / 2\} (\pi L)^{1/2}. \quad (1-9)$$

Once the crack half-length reaches the value given by (1-9) no further change in the crack will occur, provided that the value of K at the bottom crack tip has a value larger than zero. A negative value of K at the bottom crack tip requires the existence of negative values of the crack opening displacement $D(y)$.

The largest value L can take without K taking on a negative value at the bottom tip is found by setting $K = 0$ in Eq. (1-8) for the bottom tip. Thus $L_{\max}$ is given by Eq. (1-4). The value of K at the upper crack tip when $L = L_{\max}$ is equal to

$$\begin{aligned} K &= (\rho - \rho') g L_{\max} (\pi L_{\max})^{1/2} \\ &= \{2\mu V (\rho - \rho') g / \alpha\}^{1/2}. \end{aligned} \quad (1-10)$$

Now if K given by Eq. (1-10) is larger than K_c , rock at the upper crack tip will break open. But the resultant increase in crack length would require K at the lower crack tip to have a negative value. This cannot happen at the lower crack tip because the crack faces will come together and shorten the crack length. Thus the crack will attempt to keep its length constant by moving upwards. As fast as the upper crack tip breaks rock and moves up, the crack faces at the lower crack tip come together and this tip also moves upward. Once a crack contains more than a critical volume of fluid it must propagate upwards towards the Earth's surface.²⁻⁹ This is the result discussed in the Introduction. If large water-filled cracks that are created by hydraulic fracture were to rise to the Earth's surface, they would be useless for the extraction of heat from hot dry rock masses.

1.2.2 Effect of a Freely Slipping Horizontal Joint

Suppose a horizontal joint were to exist in the path of an upward propagating crack. Could such a joint stop this crack?

Consider a fluid-filled crack shown in Fig. 1.2 that is either above or below a horizontal joint. Take the case in which the joint interface is freely slipping. That is, the interface cannot support a shear traction stress across it but it can support a normal compressive traction stress. In this situation approximate values of the stresses everywhere in the rock on the side of the joint that contains the crack can be found by removing the joint and placing an image crack (such as is shown in Fig. 1-2) on the opposite side of where the joint was and at the same distance that the real crack is from the joint. The stresses in the rock are the sum of the pre-existing stresses and stress fields of the real and the image cracks. (On the side of the joint that does not contain

Fig. 1-2. (a) Real liquid-filled crack below a freely slipping joint and image crack above the joint.

(b) Real crack above the joint and image crack below the joint.

the crack the stresses at the crack plane can be found by removing the joint and placing at the real crack a crack of the same half-length L , but with a displacement $D(y)$ that is twice that of the real crack. However, this artifice does not give the correct value of the stresses at positions away from the crack plane.)

A formal solution for the crack profile can be found using the image crack and the image dislocation concepts. However, if the center of the crack is at a distance d from the interface that is large compared with the half-length L an approximate solution can be found that is a slight modification of the equations in the previous subsection. The image crack produces a stress field at the real crack when $d \gg L$ that is the same as a dislocation dipole of a strength equal to that of the volume of the crack. (The stress field of the distribution of image infinitesimal dislocations on the joint that is also required in the problem falls more rapidly than the inverse square of distance and can be ignored.) Thus the image crack, when it is below the interface, produces a tensile stress σ_{di} across the $x = 0$ plane equal to

$$\begin{aligned}\sigma_{di} &= (\mu V / \pi \alpha) / (2d+y)^2 \\ &\approx (\mu V / \pi \alpha 4d^2) (1-y/d),\end{aligned}\tag{1-11}$$

where y is measured from the center of the real crack. If the image crack is above the interface the plus and minus signs in Eq. (1-11) are reversed.

The effect of this dipole stress field on the crack can be found by replacing in the equations of the previous subsection the term P_0 with the term

$$P_0 \rightarrow P_0 - (\mu V / 4\pi \alpha d^2)\tag{1-12}$$

and the term $\rho'g$ with the term

$$\rho'g \rightarrow \rho'g \pm (\mu V / 4\pi \alpha d^3).\tag{1-13}$$

In Eq. (1-13) the minus sign is used if the real crack is below the interface and the plus sign is used if it is above it.

The substitution of Eq. (1-12) has no effect at all on the crack parameters, but the substitution of Eq. (1-13) does. In particular, Eq. (1-10) becomes

$$L_{\max} = \{ [\pi\alpha(\rho-\rho')g/2\mu V] \pm (1/8d^3) \}^{-1/3}. \quad (1-14)$$

The stress intensity factor at the upper tip of a crack with the maximum half-length given by Eq. (1-14) is now given by

$$K = \{ (2\mu V/\alpha) [(\rho-\rho')g \pm (\mu V/4\pi\alpha d^3)] \}^{1/2} \quad (1-15)$$

instead of by Eq. (1-4). In both Eq. (1-14) and Eq. (1-15) the plus sign is used if the crack is below the interface and the minus sign if it is above it. Thus a joint attracts a crack to it.

Consider next a fluid-filled crack that is beneath a joint and whose upper tip just touches the joint interface (see Fig. 1-3).

$$K = (C\mu/\alpha)(\pi L/2)^{1/2}. \quad (1-16)$$

The largest value that C can have, therefore, is

$$C = K_C(\alpha/\mu)(2/\pi L)^{1/2}. \quad (1-17)$$

Fig. 1-3. Liquid-filled crack almost touching a freely slipping joint.

At the upper crack tip, where the crack meets the joint, the stress intensity factor is always equal to zero. Because of sliding on the joint the crack tip displacement $D(0) > 0$ and the crack tip is effectively blunted. No infinite stress singularity exists on the crack plane immediately ahead of the upper crack tip. Thus the crack is stopped.

The pressure within the fluid of the crack is still equal to the expression given by Eq. (1-5). The term P' of that equation is now given by

$$P' \approx (\mu/2\alpha)\{C(1+n_1)\} + \{4(\rho-\rho')gL/\pi\}(1+n_2), \quad (1-18)$$

instead of by Eq. (1-6). Here $n_1 = 0.04$ and $n_2 = 0.03$.

1.2.3 Leakage of Fluid Into a Horizontal Joint

Once a horizontal joint has stopped, an upward propagating crack the size of the crack can be changed if fluid from the crack enters the horizontal joint. The fluid pressure $P(0)$ at $y = 0$ (in Fig. 1-3) is given by

$$P(0) = P_0 - T + P', \quad (1-19)$$

where T is again the tectonic tensile stress, P_0 is the lithostatic pressure, and P' is given by Eq. (1-18).

A crack that has propagated upwards for a distance greater than $2L$ until it reaches a joint and then is stopped will have a stress intensity factor at its lower tip that is equal to $K = 0$. When $K = 0$ at the lower tip it is necessary that $C = 0$. Thus the crack length is given by

$$L \approx \{3V\mu/8\alpha(\rho-\rho')g\}^{1/3} \quad (1-20)$$

and the fluid pressure at $y = 0$ is equal to

$$\begin{aligned} P(0) &\approx P_0 - T + 4(\rho-\rho')gL/\pi \\ &= P_0 - T + \{4(\rho-\rho')g/\pi\}\{3V\mu/8\alpha(\rho-\rho')g\}^{1/3}. \end{aligned} \quad (1-21)$$

If the upper tip of an upward propagating crack moves a distance less than about $2L$ before it encounters a joint, the stress intensity factor K of the lower tip is not necessarily equal to zero because that tip has not moved through fractured rock. The value of K at the lower tip will equal zero only if the lower tip has had a chance also to propagate upwards through previously fractured rock. In this situation the constant C is not equal to zero and the fluid pressure $P(0)$ given by Eq. (1-21) is equal to

$$P(0) \approx P_0 - T + (V\mu/2\pi\alpha L^2) - [4(\rho - \rho')gL/3\pi]. \quad (1-22)$$

The stress intensity factor at the lower tip is given by

$$K \approx (\pi L/2)^{1/2} \{ (V\mu/\pi\alpha L^2) - [32(\rho - \rho')gL/3\pi] \}, \quad (1-23)$$

where K is restricted to values in the range of $0 \leq K \leq K_c$.

1.2.4 Leakage of Fluid from Crack to Joint

Leakage of fluid from the crack into the joint will occur whenever the fluid pressure P is larger than the normal compressive stress σ_N across the joint at the crack-joint intersection. (σ_N is considered to be a positive quantity when the normal stress is compressive.) The normal compressive stress σ_N at the crack-joint intersection is equal to the overburden hydrostatic pressure P_0 plus the stress produced by the presence of the crack. It can be shown that on the crack plane ($x = 0$) the traction tensile or compressive stress component and the non-traction tensile or compressive stress component (that is, the stress component across a plane that is perpendicular to the crack plane) produced by a dislocation are actually equal to each other. The nontraction stress component of the crack is actually a normal traction stress component for the joint. Thus the compressive stress σ_N across the joint at the crack-joint intersection is equal to

$$\sigma_N = P_0 + P', \quad (1-24)$$

where P' is given by Eqs. (1-20), (1-23) or (1-24). The fluid pressure at $y = 0$ is equal to

$$P(0) = P_0 + P' - T.$$

(1-25)

Whenever T is tensile in character $\sigma_N > P(0)$ and fluid will not flow from the crack into the joint. Only if the tectonic stress is compressive can there be fluid flow from the crack into the joint.

The normal compressive stress σ_N will be equal to P_0 across the joint at distances from the crack-joint intersection that are of the order of or larger than the distance L . It is conceivable that slippage may take place on a joint that is not a perfectly flat surface. Thus the joint may be propped open in the vicinity of the crack-joint interface out to distances comparable with the crack half-length L . Fluid from the crack could move into the joint whenever $P(0) = P_0$. In this situation, fluid will flow into the joint and the crack volume will be reduced if the fluid volume is fixed or the crack length will change if fluid is continuously pumped into the crack.

The largest possible crack half-length is found by setting $P(0) = P_0$ and $K = 0$ to give

$$L = \pi T / 4(\rho - \rho')g.$$

(1-26)

The crack length that exists when $P(0) = P_0$ when the stress intensity factor K at the bottom tip is equal to $K = K_c$ and fluid is pumped continuously into the crack is determined by the equation

$$K_c = (2\pi L)^{1/2} \{T - [4(\rho - \rho')gL/\pi]\}.$$

(1-27)

Equation (1-27) is found by combining Eq. (1-24) and Eq. (1-25). When T is compressive or is small in magnitude there is no value of L that satisfies (1-29). In this situation either $L = 0$ or L has a finite value equal to or smaller than that given by Eq. (1-26) and the value of the stress intensity factor K is smaller than K_c . If T is tensile in character and of sufficient magnitude there are, in general, two values of L that satisfy Eq. (1-27). Figure 1-4 presents a plot of L versus T calculated using Eq. (1-27). The smaller crack length is an unstable solution. A crack with this length will grow until its length is equal to the larger value when fluid is pumped continuously into the crack.

Fig. 1-4. Tectonic stress T versus crack half length L for $K = 0$ and for $K = K_c$ calculated from equation (1-30) using $K_c = 3 \text{ MPa m}^{1/2}$ and $\rho - \rho' = 2000 \text{ kg m}^{-3}$

Equation (1-26) is identical to that found previously⁴ for the depth of a liquid free crevasse (for which $\rho' = 0$). The value of the crack length $2L$ given by Eq. (1-27) for a tectonic stress T of 10 MPa (100 bars) with $\rho - \rho' = 2000 \text{ kg m}^{-3}$ is $2L = 0.8 \text{ km}$. This crack length is over an order of magnitude larger (see the Introduction) than the permitted crack length of a crack in unjointed rock that has a critical stress intensity factor of $K_c = 3 \text{ MPa m}^{1/2}$. Of course, when the joint is not propped and $T > 0$ there is no limit on the length of crack that a freely slipping joint can stop.

1.2.5 Effect of Joint Friction

It has been assumed up to now that a joint is freely slipping. A real joint in a rock mass, of course, can support a finite shear traction stress. We do not believe, however, that the qualitative conclusion that a joint can stop a fluid-filled crack must be changed if the joint can support a finite shear traction stress provided that the joint can be pried open whenever fluid in it has a pressure that exceeds the lithostatic overburden pressure.

The lithostatic overburden pressure at $y = L$ is equal to $P_0 - \rho g L$. To this pressure across the joint must be added the additional (negative) pressure produced by the presence of the crack. From the well-known solution of the stress field around a crack tip this latter pressure is equal to $-K_c/4|x|^{1/2}$ where x is the distance measured along the joint from the crack-joint intersection. Thus the total compressive stress σ_N across the joint is given by

$$\sigma_N = P_0 - \rho g L - K_c/4|x|^{1/2}. \quad (1-28)$$

The stress σ_N is smaller than the fluid pressure at the crack tip out to distances smaller than given by the equation

$$|x|^{1/2} \leq (K_c/4)/\{T - K_c/(\pi L)^{1/2} - (\rho - \rho')g/2\}. \quad (1-29)$$

When L is equal to the maximum length given by (1-11) for $K = K_c$ this last equation becomes

$$|x|^{1/2} \leq 1/\{(4T/K_c) - 6[(\rho - \rho')g/\pi K_c]^{1/3}\}. \quad (1-30)$$

If the tectonic stress T is smaller than $T \leq 0.6$ MPa (6 bars), the fluid pressure at the crack tip is larger than σ_N everywhere on the joint (if $K_C = 3 \text{ MPa m}^{1/2}$ and $\rho - \rho' = 2000 \text{ kg m}^{-3}$). For $T = 6$ MPa (60 bars) this pressure is larger than σ_N out to $|x| = 2$ cm and for $T = 60$ MPa (600 bars) it is larger out to $x = 2$ mm. But the maximum displacement $D(0)$ of a crack with a half-length given by Eq. (1-10) for $K_C = 3 \text{ MPa m}^{1/2}$ is only of the order of 2 mm (see Table I of Weertman³). Thus it appears that a joint can be wedged open by fluid pressure to a sufficient extent to permit frictionless slip out to distances that are sufficient to blunt the tip of the fluid-filled crack. Of course, once such slip takes place the values of both the fluid pressure within the crack and the traction compressive stress σ_N will be altered to values that approach those found in the previous sections.

1.3 CONCLUSIONS

It is clear from this analysis that a freely slipping horizontal joint can stop a liquid-filled crack that propagates towards the Earth's surface. If the nonhydrostatic tectonic stress component across the crack plane is compressive all the fluid of the crack will enter the joint. However, if the tectonic stress component T is tensile rather than compressive, no fluid will enter the joint that stops the crack provided that the joint surfaces are close fitting. In this situation there is no limit to the size of the crack that can be stopped. On the other hand, if the freely slipping joint surfaces are propped open over distances from the crack of the order of the crack length, fluid can enter the joint and the crack-half length is restricted to lengths no larger than $\pi T / 4(\rho - \rho')g$ where ρ is the rock density, ρ' is the fluid density, and g is the gravitational acceleration. (This expression, for the case of $\rho' = 0$, is the same one obtained previously for the depth of liquid free crevasses at the Earth's surface.)

If the compressive stress in the rock is not equal to the value $P_0 - \rho g y$ that was used in the analysis but rather to the value $P_0 - c^* y$ where c^* is some constant, the term ρg in the various equations should be replaced by the constant c^* . If $c^* < \rho' g$ a fluid-filled crack will tend to propagate down rather than up.

Cracks can also be stopped by nonfreely slipping joints provided that the joint can be easily pried open whenever the pressure of fluid in contact with it exceeds the lithostatic overburden pressure.

It appears, therefore, that it is possible in jointed material to have vertical, stable liquid-filled cracks exist that have lengths that are much larger than those calculated for a liquid-filled crack in jointless rock.

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CHAPTER 2

CHARACTERIZATION OF A PENNY-SHAPED RESERVOIR IN HOT DRY ROCK

by

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In studying the feasibility of geothermal energy extraction, it is important to consider the stability of a reservoir. On this subject, the papers by Secor and Pollard,¹ Abe et al.,² and Hsu and Santosa³ can be cited. These studies, however, take no account of the thermoelastic effect on the stress intensity factor (which is the parameter controlling the instability behavior of a crack-like reservoir). In other words, the validity of these results may be confined to the early stage of the operation of a geothermal energy system. When the system operates effectively, the heat energy is extracted steadily from the surface of the crack-like reservoir. The extraction of heat gives rise to a sufficiently large thermal contraction of the rock and increases the width of the reservoir. Therefore, the thermoelastic effect should be taken into account in the study of the characterization of a reservoir.

On the basis of fracture mechanics concepts extended to include thermoelastic effects, the present research deals with the characterization of a penny-shaped reservoir in the effective stage of operation of a geothermal energy system. The surrounding rock (granite) is assumed to be homogeneous and isotropic, and impermeable. After formulating the stable condition of a reservoir that is not changing its radius by propagation or closing, the flow rate necessary for a stable crack (critical flow rate) is plotted against the radius of the reservoir for various pressures of fluid (water) at the outlet on the earth surface. The instability behavior of the reservoir is also studied when the inlet flow rate exceeds the critical value.

The stable condition of a reservoir, that is not changing its radius by propagating or closing, requires

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$$0 < k < k_c$$

(2-1)

where $k = k_T + k_E$ and k_c is the fracture toughness of the rock. The constants k_T and k_E are the stress intensity factors of the penny-shaped crack due to the stationary thermal stress and the internal pressure inside the crack, respectively. They are calculated from Kassir's analysis⁴ and Sneddon and Lowengrub⁵.

Then, (2-1) becomes

$$0 < \frac{E\alpha_c \rho_f m_0 (T_0 - T_i)}{8\pi(1-\nu)\lambda R^{1/2}} + \frac{2R^{1/2}}{\pi} \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh + \rho_f (1 + fh/2R_0)(m_0/\pi R_0^2)^2/2\} < k_c, \quad (2-2)$$

where E is Young's modulus, α = thermal expansion coefficient of rock, ρ_f = fluid density, m_0 = outlet flow rate, T_0 = outlet temperature, T_i = inlet temperature, ν = Poisson's ratio, λ = conductivity of the rock, R is the radius of the crack, p_0 = outlet pressure, β = volumetric expansion coefficient of the fluid, g is the gravitational acceleration, h the height of crack center from the earth surface, K_a the coefficient of active rock pressure, ρ_r = density of rock, f = friction factor, and R_0 is the radius of the outlet hole. From this inequality, the upper and lower critical outlet flow rates, m^* and m_* , can be obtained against the radius R for various pressures p_0 when the fluid temperatures T_i and T_0 and the geometrical and physical quantities are given.

The behavior of the reservoir will be studied when the inlet flow rate m_i is changed. According to the value of the outlet flow rate m_0 , the following three cases are considered:

Case 1. $m_0 = m^*$,

Case 2. $m^* > m_0 > m_*$,

Case 3. $m_0 = m_*$.

Since it is assumed that the rock is impermeable and the fluid is incompressible, we obtain

$$m_i = \dot{V} + m_0, \quad (2-3)$$

where V is the volume of the crack and a dot denotes differentiation with respect to time t . V can be calculated from the crack opening displacement. It is obtained as

$$V = 2(1 + \nu)\alpha c_p \rho_f m_0 (T_0 - T_i) R^2 / 5\lambda + \frac{16(1 - \nu^2)R^3}{3E} \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh\}. \quad (2-4)$$

Case 1. When the outlet flow rate m_0 reaches the upper critical flow rate m^* , the reservoir starts to propagate. Then, we have that $k = k_c$ in (2-2) and $dR/dt > 0$. The outlet flow rate m_0 and the volume V are given by

$$m_0 = \frac{8\pi(1-\nu)\lambda}{E\alpha c_p \rho_f (T_0 - T_i)} [k_c R^{1/2} - \frac{2R}{\pi} \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh\}],$$

$$V = \frac{16(1-\nu^2)}{15E} [3\pi k_c^{5/2} - R^3 \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh\}]. \quad (2-5)$$

Substituting (2-5) into (2-3), we have the relation between R and t . The initial condition is specified by

$$R = R_s \text{ at } t = t_s. \quad (2-6)$$

Case 2. If $m^* > m_0 > m_*$, the radius of the reservoir is unchanged but its volume is changed. Substitution of (2-4) into (2-3) leads to

$$\frac{2(1+\nu)\alpha c_p \rho_f (T_0 - T_i) R^2}{5\lambda} \frac{dm_0}{dt} + m_0 = m_i. \quad (2-7)$$

When the initial condition is specified by

$$m_0 = m_s \text{ at } t = t_s, \quad (2-8)$$

the relationship between m_0 and t is obtained.

Case 3. For the case of $m_0 = m_*$, it follows that $k = 0$ and $dR/dt \neq 0$. Then the flow rate m_0 and the volume V are given by

$$m_0 = - \frac{16(1-\nu)\lambda R}{E\alpha c_p \rho_f (T_0 - T_i)} \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh\},$$

$$\nu = - \frac{16(1-\nu^2)R^3}{15E} \{p_0 + \rho_f [1 - \beta(T_0 - T_i)]gh - K_a \rho_r gh\}. \quad (2-9)$$

When (2-9) is substituted into (2-3), the relation between R and t is obtained.

The aforementioned analysis will be applied to the study of the characterization of reservoir in the LASL program. In the numerical calculations, the following data are used: $T_i = 70^\circ \text{C}$, $T_0 = 170^\circ \text{C}$, $\rho_f = 9.43 \times 10^2 \text{ kg/m}^3$, $\eta = 2.47 \times 10^{-7} \text{ m}^2/\text{s}$, $c_p = 4.23 \times 10^3 \text{ J/kg}\cdot\text{K}$, $\beta = 9.1 \times 10^{-4}/\text{K}$, $E = 3.5 \times 10^{10} \text{ Pa}$, $\nu = 0.12$, $\alpha = 1.2 \times 10^{-5}/\text{K}$, $\lambda = 3.0 \text{ W/m}\cdot\text{K}$, $\rho_r = 2.65 \times 10^3 \text{ kg/m}^3$, $K_a = 0.75$, $k_c = 1.6 \times 10^6 \text{ N/m}^{3/2}$, $h = 3.0 \times 10^3 \text{ m}$, $R_0 = 1.22 \times 10^{-1} \text{ m}$, $e = 2.0 \times 10^{-3} \text{ m}$ and $g = 9.8 \text{ m/s}^2$. These values are collected from literature (Washburn,⁶ Binder,⁷ Abe et al.,² Tester and Smith,⁸ and Hsu and Santosa³).

From (2-2), the upper and lower critical flow rates, m^* and m_* , are plotted against the radius R of the reservoir for various fluid pressures p_0 at the outlet on the earth surface in Fig. 2-1. The values of the upper and lower critical flow rates are very close and are shown by a solid line in the figure. This means that there is almost no stable situation of the reservoir. It is seen from Fig. 2-1 that the critical flow rates increase linearly with the radius of the reservoir. This results from the fact that the last term in the braces on (2-2) is negligibly small. Then, it can be easily understood that the critical flow rates are linear with the pressure p_0 . Here we should note that the possible flow rate is fairly small compared with the economically acceptable flow rate ($\geq 50 \text{ kg/s}$) reported by Tester and Smith.⁸

When the inlet flow rate m_i exceeds the critical values m^* and m_* , the reservoir becomes unstable. The numerical results are shown in figures. Figures 2-2(a) and 2-2(b) show, respectively, the variations of the radius R of the reservoir and the outlet flow rate m_0 against the time $t - t_s$ at $p_0 = 10 \text{ MPa}$. The propagating behavior of the reservoir is shown in the cases A, B and C, and the closing behavior in the cases D, E, and F. The values of the constant inlet flow rate m_i and the initial values of the outlet flow rate and the radius are given in Table 2-I. The radius of the reservoir, which has become stationary, is denoted by $\bar{R}$ and their values are also shown in this table. Figures 2-2(a) and 2-2(b) reveal that the response of the reservoir is slower.

Fig. 2-1. Upper and lower critical flow rates, m^* and m_* , versus the radius R of the reservoir for various pressures p_0 .

TABLE 2- I

	A	B	C	D	E	F
$M_0(m^3/s)$	3.5297×10^{-3}	3.2949×10^{-3}	3.0601×10^{-3}	2.8114×10^{-3}	2.5771×10^{-3}	2.3428×10^{-3}
$\bar{R}(m)$	600	560	520	480	440	400
$m_s(m^3/s)$	2.9437×10^{-3}			2.9285×10^{-3}		
$R_s(m)$	500			500

Fig. 2-2(a). Radius R of the reservoir versus $t - t_s$.

The lower and upper bounds m_* and m^* of the outlet flow rate were obtained under an assumption of perfectly elastic behavior of fractured rock (both thermal and hydraulic loading agents were considered). Under such assumptions, the usual LEFM analysis can be applied, and the term "stable" is then synonymous with a stationary crack, which neither closes nor propagates. The upper and lower critical flow rates, m^* and m_* , could thus be evaluated in terms of the reservoir radius (R), the fluid temperatures at the inlet (T_i) and outlet (T_0), and other thermoelastic and physical quantities characteristic of the geothermal system. A numerical example pertinent in characterization of the reservoir considered in the LASL program, cf. Sekine and Mura,⁹ has demonstrated that the upper and lower critical flow rates are very close to each other, and therefore, there is almost no stable situation in the reservoir. It was concluded that "the domain of the stable situation of the reservoir is extremely narrow." Since the economically

Fig. 2-2(b). Outlet flow rate m_0 versus $t - t_s$.

feasible rate of flow (>50 kg/s) exceeded substantially both limits, m^* and m_* , found by Sekine and Mura, it was recommended that a multiple fracture system be introduced to gain better efficiency, on the one hand, and on the other, to operate the geothermal system within its confines of safety.

Here, we intend to show that the inclusion of ductile material behavior, which is likely to occur under high-tectonic pressures present in the hot dry rock, cf. Bridgman¹⁰ and a more recent study by Pugh and Chandler,¹¹ will widen the domain of stable operating range of the reservoir. While the lower limit m_* , computed by Sekine and Mura⁹, remains unchanged in the analysis which follows, the upper flow rate limit, m^* , corresponding to a spontaneous enlargement of the reservoir can indeed be allowed to increase substantially. It will be shown that an enhancement of the upper flow rate nearing 40% is possible, for a certain range of geometrical and physical constants characterizing the geothermal system and depending on the way in which hydraulic fracturing is conducted.

Analysis given here indicates that an extensive stable crack growth occurs even under rising load, i.e. under an increasing inlet flow rate. Therefore, a significant margin of safety may exist even when the onset of crack growth is imminent. Attempts at accurate calculation of this margin of safety by extending the LEFM or the J-integral type analyses into the domain of stable crack growth are impeded by the mathematical complexity of the problem; compare a recent review paper by Rice.¹²

The "final stretch" model of crack tip behavior, proposed by Wnuk,^{13,14} is employed here to study the quasi-static and stable extension of a geothermal reservoir. This model assumes existence of a structural "end zone" which precedes the crack front and exerts a profound influence on the crack propagation. Screening of the energy flowing through this end zone toward the crack tip is the major factor controlling the stable phase of crack extension.

Numerical examples show that the theoretical estimate of the upper critical flow rate of the fluids pumped through the reservoir can be considerably increased (as compared with the values obtained within the LEFM framework) when the non-linear inelastic behavior of rocks containing the reservoir is accounted for. Available data indicate that the rate of crack extension during the early stage of crack growth is reduced by several orders of magnitude. Such a result implies widening of the safety margin for hot dry geothermal system.

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CHAPTER 3

EFFECTS OF TEMPERATURE PROFILE AND ANISOTROPY ON INSTABILITY OF PARALLEL THERMAL CRACKS IN ROCK*

by

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3.1 ABSTRACT

When a system of parallel equidistant cooling cracks propagates into a half-space, it reaches at a certain depth a critical state at which the growth of every other crack is arrested. Later these cracks reach a second critical state at which they close. The intermediate cracks, at doubled spacing, open about twice as wide and advance further as cooling penetrates deeper. Determination of the critical states requires calculation of the derivatives of the stress intensity factors with regard to crack lengths, which is here accomplished by finite element method. It is found that the crack depth-to-spacing ratio at which the critical state is reached is extremely sensitive to the temperature profile. This ratio greatly increases as the cooling front becomes steeper. The effect of transverse isotropy of the material upon the location of critical states is found to be relatively mild. The results are of interest for one recently proposed geothermal energy scheme for hot dry rock, as well as for shrinkage cracks in concrete.

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3.2 STATEMENT OF PROBLEM

Extraction of geothermal heat from hot dry rock is of great interest because geothermal basins which are dry are far more numerous than those which have a natural water circulation. In one recently proposed scheme,^{16,12} a huge vertical crack of about 1 km in size is to be produced in the hot rock mass at the depth of several kilometers by the hydraulic fracturing process, in which water is forced under high pressure into a vertical bore. The crack must then be intersected by another bore, and by injecting cool water into one bore, hot water may be collected from the other bore to drive a power plant.

The withdrawal of heat by the circulating water gradually cools the rock near the crack walls, and it appears that after a few months of operation the rate of heat transport to the crack would become insufficient.^{12,5} Nevertheless, various mechanisms can be expected to increase the heat output. One of them is the thermal cracking, which would no doubt be induced in the rock. We may expect that, due to cooling, a system of secondary vertical cracks normal to the original (main) vertical crack would propagate into the hot rock mass, and part of the flux of circulating water would get diverted to these secondary cracks where the rock is still hot. The diverted flux is roughly proportional to the cube of secondary crack width, w , and to the inverse of crack spacing, b , while thermal strain calculation shows that w is proportional to b . The combined effect^{12,15} is, therefore, that the diverted flux of water is roughly proportional to b^2 . So, the heat extraction from the secondary cracks can become significant only if their spacing is sufficiently large.^{12,15} It is the purpose of this report to investigate when this happens.

It may be of interest to note that the question of spacing is also important for drying shrinkage cracks in reinforced concrete. There, one wants the opposite to happen. To ensure sufficient transfer of shear stresses across the rough crack surfaces, to prevent corrosion of the embedded reinforcement, and to achieve good fatigue resistance and ductility, the cracks must remain hairline thin, and so one needs the crack spacing to remain sufficiently small. Furthermore, crack spacing is of interest to geologists, for example, to interpret the cooling cracks observed in solidified lava beds in the ocean,¹¹ or the cracks in dried-up mud flats,¹⁰ or permafrost soils.

The situation may be idealized as a system of parallel Mode I cracks of lengths a_i ($i = 1, 2, \dots, n$), propagating into a brittle elastic halfspace $x \geq 0$

along planes $y = \text{const.}$ (x, y, z are cartesian coordinates); see Fig. 3-1. The loading consists of a prescribed thermal strain $\alpha\Delta T(x)$ (or shrinkage strain) as a function of depth x (α = thermal dilatation coefficient, and ΔT = temperature drop). We will consider fixed temperature profiles with the same value of the surface temperature drop $\Delta T(0)$, and the loading will then be fully characterized by the cooling penetration depth, D . Thus, temperature depends only on x/D and is assumed to be independent of y and z , which is certainly a simplification when water circulates through the secondary thermal cracks. We shall assume that the cracks are planar. This is of course a simplification; the actual crack pattern must be expected to be three-dimensional, and cracks could also be curved.

Solving this problem of fracture mechanics, we notice an interesting feature--the solution is not unique. We can find solutions of any chosen crack spacing s , as well as solutions with the same crack length or alternating crack lengths. What is causing the lack of uniqueness?

We must of course suspect instability. This was first investigated in Ref. 1 (and summarized in Ref. 2), in which the second variation of the work needed to create the cracks as well as the conditions of adjacent equilibrium were analyzed and the stability conditions of a system of interacting cracks were determined. (In general terms these results were first given in previous unpublished work in 1976; see Acknowledgment at the end of this chapter.) The later work of some other authors on this problem was commented upon in Ref. 1.

In this work, after summarizing the stability conditions from Ref. 1, we will determine the critical states of crack arrest for various temperature or shrinkage stress profiles in a halfspace. Although for linear homogeneous materials and simple geometries these problems can be solved more accurately by a Green's function approach leading to singular integral equations, the finite element approach has been adopted because it has required the least human effort and cost (since a suitable finite element program was already available), and because it is simple, general, and can be easily extended to various surface geometries, arbitrary crack paths, as well as nonhomogeneous, nonlinear and anisotropic materials. Investigation of the effect of anisotropy will be also included in this study because anisotropy is an important characteristic of rock.

3.3 GRIFFITH CRACKS

Consider the work, W , that needs to be supplied to create cracks of length a_i :

Fig. 3-1. Critical states of crack arrest and bifurcation of equilibrium path, and parallel crack system.

$$W = U(a_1, a_2, \dots, a_n; D) + \sum_{i=1}^N \int_0^{a_i} 2\gamma_i da_i \quad (3-1)$$

Here U represents the strain energy of the elastic body, $2\gamma_i$ is the specific energy of crack extension of the i^{th} crack, considered to be a material property, and D is a loading parameter (in our problem, the penetration or depth of cooling or drying). The equilibrium state of the cracks is determined by vanishing of the first variation of work:

$$\delta W = \sum_{i=1}^m \left(\frac{\partial U}{\partial a_i} + 2\gamma_i \right) \delta a_i + \sum_{j=m+1}^k \frac{\partial U}{\partial a_j} \delta a_j = 0, \quad (3-2)$$

where $i = 1, \dots, m$ are the cracks that extend ($\delta a_i > 0$); $j = m+1, \dots, k$, are those which close ($\delta a_j < 0$); and $i = k+1, \dots, N$ are further cracks remain stationary ($\delta a_i = 0$). Because Eq. (3-2) must be satisfied for any admissible δa_i , the equilibrium crack extensions are characterized by (3-1):

$$\text{for } \delta a_i > 0: -\frac{\partial U}{\partial a_i} = 2\gamma_i; \text{ for } \delta a_i < 0: \frac{\partial U}{\partial a_i} = 0. \quad (3-3)$$

The first of these conditions is the well-known Griffith fracture criterion, and $-\partial U/\partial a_i = G_i$ is the energy release rate of the i^{th} crack.⁹ Equation (3-2) can be equivalently stated in terms of the Mode I stress intensity factor, which is here defined as $K_i = \lim \sigma_y [2\pi(x-a_i)]^{1/2}$ where σ_y = normal stress on the crack extension line,⁸ and not as $K_i = \lim \sigma_y (x-a_i)^{1/2}$. For plane strain, we have $\partial U/\partial a_i = -K_i^2/E'$ where $E' = E/(1-\nu^2)$, E = Young's modulus, ν = Poisson ratio.⁹ We may, therefore, rewrite Eq. (3-3) in the form:¹

$$\text{for } \delta a_i > 0: K_i = K_{c_i}; \text{ for } \delta a_i < 0: K_i = 0 \quad (3-4)$$

in which $K_{c_i} = (2\gamma_i E')^{1/2}$ = critical value of the stress intensity factor for the given material.

The case $K_i > K_{c_i}$ or $-\partial U/\partial a_i > 2\gamma_i$ is impossible because Eq. (3-2) yields $\delta W < 0$ for $\delta a_i > 0$, which is an unstable condition (energy is released rather

than consumed by crack extension). The case $K_i < 0$ or $-\partial U/\partial a_i < 0$ yields also $\delta W < 0$ for $\delta a_i < 0$, which is again unstable. Therefore, it is necessary that $0 \leq \partial U/\partial a_i \leq 2\gamma_i$ or $0 \leq K_i \leq K_{Ci}$ at any time. Consequently, only the following variations of crack lengths are admissible:

$$\begin{aligned} &\text{for } K_i = K_{Ci} : \delta a_i \geq 0 ; \\ &\text{for } 0 < K_i < K_{Ci} : \delta a_i = 0 ; \\ &\text{for } K_i = 0 : \delta a_i \leq 0 . \end{aligned} \quad (3-5)$$

3.4 INSTABILITY OF A SYSTEM OF GRIFFITH CRACKS

Let us now explore the possibility of a critical state (bifurcation point) in which there exists an adjacent equilibrium state characterized by different crack lengths $a_i + \delta a_i$ but the same loading parameter D . The equilibrium state is characterized by the condition $W_i = \partial W/\partial a_i = 0$, and this condition would have to hold both for a_j and $a_j + \delta a_j$ at the same D . Thus, there must be no change in W_i between these two states, i.e. $\delta W_i = \sum_j (\partial W_i/\partial a_j) \delta a_j = 0$ at constant D , and because $\partial W_i/\partial a_j = \partial^2 W_i/\partial a_i \partial a_j$, we thus obtain the critical state condition:

$$\sum_{j=1}^n W_{ij} \delta a_j = 0, \quad (3-6)$$

in which

$$W_{ij} = \frac{\partial^2 W}{\partial a_i \partial a_j} = \frac{\partial^2 U}{\partial a_i \partial a_j} + 2 \frac{\partial \gamma_i}{\partial a_i} \delta a_i H(\delta a_i), \quad (3-7)$$

according to Eq. (3-1); $j = 1, \dots, n$ runs through the given set of crack lengths which are assumed to vary independently; H is the Heaviside step function; and for a homogeneous material, in which $\gamma_i = \gamma = \text{constant}$, we have $W_{ij} = \partial^2 U/\partial a_i \partial a_j = \partial G_i/\partial a_j$. Noting that $\partial U/\partial a_i = -K_i^2/E'$, Eq. (3-6) yields for a homogeneous material ($\partial \gamma_i/\partial a_i = 0$) the expression $W_{ij} = K_i \partial K_i/\partial a_j$, and because $W_{ij} = W_{ji}$ we have the following useful relation:

$$K_i \frac{\partial K_i}{\partial a_j} = K_j \frac{\partial K_j}{\partial a_i} \quad (3-8)$$

Alternatively, we may obtain Eqs. (3-6) to (3-8) directly from the condition that a crack system that is in equilibrium ($\delta W = 0$) is stable if the second variation $\delta^2 W$ is always positive, and unstable if it can be negative,¹ i.e.,

$$\delta^2 W = 1/2 \sum_{i=1}^n \sum_{j=1}^n W_{ij} \delta a_i \delta a_j \quad \begin{cases} > 0 \text{ stable (all admissible } \delta a_i) \\ = 0 \text{ critical (some admissible } \delta a_i) \\ < 0 \text{ unstable (some admissible } \delta a_i) \end{cases} \quad (3-9)$$

Obviously $2\delta^2 W = \sum \sum \delta a_i (W_{ij} \delta a_j)$ and this equals zero if Eq. (3-6) holds; so, Eq. (3-6) provides the critical state according to Eq. (3-9). If matrix W_{ij} is positive definite, stability is assured. Note, however, that if it is not positive definite, the crack system may still be stable if those δa_i which make $\delta^2 W$ negative are not admissible according to Eq. (3-5). This last restriction marks an important difference from buckling problems. There indeed exist cases where W_{ij} is not positive definite yet the crack system is stable (3-1).

In our problem of cracked halfspace (Fig. 3-1), we have infinitely many cracks, but because we assume a periodic pattern of crack lengths we may restrict ourselves to only a few independent crack lengths, whose number we denote as N ($N = 2$ in Fig. 3-1). We may then consider the body to be limited by the symmetry lines (with sliding supports, as pictured in Fig. 3-2).

If Eq. (3-6) is satisfied for some non-zero δa_i that are admissible by Eq. (3-5), we have a critical state in which the crack lengths vary at no change of loading. Since Eq. (3-6) is a system of homogeneous linear equations, the critical state can be obtained only if some principal minor, $\det_n(W_{ij})$, of $(N \times N)$ matrix W_{ij} vanishes ($n \leq N$). Positive definiteness of the quadratic form in Eq. (3-9) is assured if each principal minor $\det_n(W_{ij})$ is positive. Thus

$$\det_n(W_{ij}) = \begin{cases} > 0 \text{ stable (all } n, \text{ any admissible } \delta a_i) \\ = 0 \text{ critical (some } n, \text{ admissible } \delta a_i) \\ < 0 \text{ unstable (some } n, \text{ admissible } \delta a_i). \end{cases} \quad (3-10)$$

An important point is that we must check the principal minors of all sizes $n \leq N$.

Fig. 3-2. Finite element grid used (each rectangle consists of four triangles).

The eigenvector which corresponds to the vanishing of $\det_n(W_{ij})$ may be determined from Eq. (3-6) and must satisfy Eq. (3-5) to obtain instability.

When we consider, for example, two independent crack lengths a_1 and a_2 (Fig. 3-1), one possibility is that only a_2 extends while a_1 remains constant. Then, we need to check $\det_n(W_{ij}) = W_{22} = \partial^2 W / \partial a_2^2$, and Eq. (3-9) yields $\delta^2 W = (1/2)W_{22} \times (\delta a_2)^2$. Noting that for a homogeneous material $W_{22} = \partial^2 U / \partial a_2^2 = -\partial G_2 / \partial a_2$ where $G_2 = K_2^2 / E'$, and assuming that $K_2 \neq 0$, we thus have the condition:¹

$$\frac{\partial K_2}{\partial a_2} \begin{cases} < 0 \text{ stable} \\ = 0 \text{ critical} \\ > 0 \text{ unstable.} \end{cases} \quad (3-11)$$

at $D = \text{const.}$, $\delta a_2 > 0$ ($K_2 = K_c$) and $\delta a_1 = 0$. This condition is indeed found to govern in our problem.¹

Furthermore, we must check the (2×2) determinant condition. The critical state is now characterized by $\det_2(W_{ij}) = W_{11}W_{22} - W_{12}W_{21} = 0$, and the eigenvector $(\delta a_1, \delta a_2)$ is given by $W_{11} \delta a_1 + W_{12} \delta a_2 = 0$ and $W_{21} \delta a_1 + W_{22} \delta a_2 = 0$. Thus, if $K_1 > 0$ and $K_2 > 0$, we have $\delta a_2 / \delta a_1 = -W_{11} / W_{21} = -W_{12} / W_{22}$, and because $W_{12} = W_{21}$ (Eq. 3-8) we get $W_{12} = W_{21} = \pm \sqrt{W_{11}W_{22}}$. For the system of parallel cracks in a halfspace, numerical calculations indicate that $\partial K_1 / \partial K_2$ is always negative, i.e., $W_{12} > 0$. So, $W_{12} = W_{21} = \sqrt{W_{11}W_{22}}$, and we have for the critical state

$$W_{11}W_{22} - W_{12}^2 = 0 \text{ at } \frac{\delta a_2}{\delta a_1} = \sqrt{\frac{W_{11}}{W_{22}}}, \quad (3-12)$$

provided that $\delta a_1 \neq 0$ and $\delta a_2 \neq 0$ are admissible increments according to Eq. (3-5). Obviously, if $K_1 = K_2 = K_c$, the signs of W_{11} and W_{22} are both positive (Eq. 3-11), and so $\delta a_2 / \delta a_1$ is obtained from Eq. (3-12) as negative.¹ This, however, violates the admissibility conditions in Eq. (3-5) because $K_1 = K_2 = K_c$. Therefore, violation of the (2×2) determinant condition does not indicate instability of our system of parallel cooling cracks when the cracks are equally long (see Ref. 1; this fact and Eqs. (3-6) to (3-12) were obtained in the unpublished work mentioned in Acknowledgement at the end of this chapter).

The foregoing conclusion is contingent upon the negativeness of $\partial K_1 / \partial a_2$. There exist other problems where $\partial K_1 / \partial a_2$ can be positive,¹ yielding $\delta a_1 / \delta a_2 > 0$, and then the (2×2) determinant condition may indicate unstable crack extension.

In case that $K_1 = 0$ and $K_2 = K_c$, it is interesting to examine the (2×2) determinant condition again. Because $\partial U / \partial a_1 = -K_1^2 / E'$, we get $\det_2(W_{ij}) = 4K_1K_2 \times (K_{1,1}K_{2,2} - K_{1,2}K_{2,1}) / E'^2$ where $K_{i,j} = \partial K_i / \partial a_j$. Thus, at $K_1 = 0$ we must always have $\det_2(W_{ij}) = 0$. To determine the corresponding eigenvector, we first note that, according to Eq. 3-8, $K_2K_{2,1} = K_1K_{1,2} = 0$ if $K_1 = 0$. Thus, we see that:³

$$\det_2(W_{ij}) = 0; \quad \partial K_2 / \partial a_1 = 0 \quad (\text{at } K_1 = 0). \quad (3-13)$$

The eigenvector corresponding to the (2×2) determinant condition satisfies Eq. (3-6), i.e. $W_{22}\delta a_2 + W_{21}\delta a_1 = 0$. Therefore, $K_2K_{2,2}\delta a_2 = -K_2K_{2,1}\delta a_1$, and so, assuming that $\partial K_2 / \partial a_2 \neq 0$ and substituting Eq. (3-13), we conclude that the eigenvector $(\delta a_1, \delta a_2)$ which corresponds to Eq. (3-13) is³:

$$\delta a_1 < 0, \quad \delta a_2 = 0 \quad (\text{at } K_1 = 0 \text{ and constant } D), \quad (3-14)$$

which is admissible according to Eq. (3-5). So, here we have found a critical state of parallel cracks in a halfspace which is characterized by vanishing of the (2×2) determinant.

We now arrive at the following picture of the system of parallel cooling cracks in a halfspace.¹ The cracks, which may be assumed to start at equal spacing $b(b = 2h)$ that must not be smaller than the minimum possible spacing (see Appendix I), grow at first at equal length as the penetration depth of cooling D increases. To examine possible transition to a case of alternating crack lengths, two crack lengths a_1, a_2 and the halfspace strip within the symmetry lines (Figs. 3-1 and 3-2) may be considered. The (2×2) determinant vanishes first but this does not signify instability¹ as explained before. Subsequently, $\partial K_2 / \partial a_2$ (at constant a_1 and D) vanishes, and this represents a critical state of crack arrest (bifurcation). Further crack extension at $a_1 = a_2$ represents an unstable, impossible path because $\partial K_2 / \partial a_2 > 0$. As shown qualitatively in Ref. 1 and quantitatively in Ref. 3, cracks a_2 at critical state jump ahead at constant a_1 and constant temperature (constant D); see Fig. 3-3. Then cracks a_2 gradually extend in a stable manner with increasing D while cracks a_1 stop growing and K_1 gradually diminishes at increasing a_1 and D until K_1 becomes zero; this represents the second critical state of crack closing (Fig. 3-3). In this state, which corresponds to vanishing (2×2) determinant, cracks a_1 suddenly close over a finite length at constant a_2 and constant temperature (constant D) (Fig. 3-3), and

Fig. 3-3. Bifurcations, post critical paths and critical states of crack arrest.

subsequently cracks a_2 grow in a stable manner at increasing D . Because cracks a_1 are closed, this is equivalent to the initial situation of a single crack length, but with doubled spacing. The entire process is then repeated at doubled spacing, then at quadrupled spacing, etc.

3.5 NUMERICAL ANALYSIS

The finite element method, with grids exemplified in Fig. 3-2 was used to solve the problem. All rectangular finite elements consist of four constant strain triangles and the displacements of the central node are eliminated in advance by static condensation. The stress intensity factors and their derivatives are calculated from the first and second derivatives of the total strain energy of the system^{6,8,13,14} which was evaluated as

$$\begin{aligned}
 U &= \int_V \left(\frac{1}{2} \right) C_{ijkl} \epsilon_{ij} \epsilon_{kl} - \sigma_{ij}^0 \epsilon_{ij} \, dV \\
 &= \int_V \left[G \epsilon_{ij} \epsilon_{ij} + \frac{G\nu}{1-2\nu} (\epsilon_{kk})^2 - \frac{E\alpha}{1-2\nu} (T - T_0) \epsilon_{kk} \right] dV,
 \end{aligned}
 \tag{3-15}$$

where the first expression applies for anisotropic materials and the second one for isotropic materials; G = elastic shear modulus, E = Young's modulus, ν = Poisson's ratio, T_0 = initial uniform temperature, T = current temperature, C_{ijklm} = anisotropic elastic moduli, σ_{ij}^0 = thermal stresses produced by $T - T_0$ at $\epsilon_{ij} = 0$; indices i, j, k, m refer to Cartesian axes $x_1 = x, x_2 = y, x_3 = z$ and summation over $i = 1, 2, 3$ is implied by repeated indices. The derivatives $\partial U / \partial a_1$ and $\partial^2 U / \partial a_1 \partial a_2$ are calculated from their central finite difference approximations:

$$\begin{aligned} \left(\frac{\partial U}{\partial a_2} \right)_{r,s} &= \frac{U_{r,s+1} - U_{r,s-1}}{2\Delta x}, \quad \left(\frac{\partial^2 U}{\partial a_2^2} \right)_{r,s} \approx \frac{U_{r,s+1} - 2U_{r,s} + U_{r,s-1}}{\Delta x^2} \\ \left(\frac{\partial^2 U}{\partial a_1 \partial a_2} \right)_{r,s} &\approx \frac{U_{r+1,s+1} - U_{r-1,s+1} - U_{r+1,s-1} + U_{r-1,s-1}}{4\Delta x^2}, \end{aligned} \quad (3-16)$$

where the subscripts refer to cracks terminating at adjacent grid nodes $r-1, r, r+1$, and $s-1, s, s+1$ shown in Fig. 3-1; Δx = spacing of these nodes.

To get an idea of accuracy, several cases of known K -values have been solved by finite elements. E.g., for the case of equally long cracks (Fig. 3-1) with $a_1 = a_2 = 2h = 10\text{m}$ and $\Delta x = 2\text{m}$, grid depth of 40 m and a uniform stress $\sigma_y = 1$ at infinity, the solution is known to be¹⁷ $K_I/\sqrt{h} = 0.999$ while the finite element calculation provided 1.01. For the case of a strip of length $2h$ and width $2h$, which is subject to a uniform uniaxial tension $\sigma_y = 1$ at ends and contains in the mid-length cross section two symmetrical cracks penetrating from the boundary to the depth of $a_1 = h/2$, the finite element method gave $K_I\sqrt{2}/h = 1.311$ while more accurate methods previously¹⁷ provided 1.311 and 1.330.

In calculations for the hot dry rock geothermal energy scheme, the material properties typical of granite have been used: linear thermal dilatation coefficient $\alpha = 8 \times 10^{-6}$ per $^\circ\text{C}$, Young's modulus $E = 37600 \text{ MN/m}^2$, Poisson ratio $\nu = 0.305$, and surface energy density $\gamma = 104 \text{ J/m}^2$, which yields $K_c = 2.94 \text{ MNm}^{-3/2}$. The halfspace surface was assumed to be cooled by 100°C , i.e. $T_s - T_0 = 100^\circ\text{C}$, and the initial crack spacing was $b = 2h = 1\text{m}$. The results can be, however, transformed to other spacings and K_c values because the expression

$$K_I = \frac{K_I(1 - \nu)}{E\alpha(T_0 - T_s)\sqrt{2h}} \quad (3-17)$$

is independent of E , ν , $2h$, α and $T_0 - T_s$ and may be regarded as the nondimensional stress intensity factor; for derivation see Ref. 3. (Note that the use of $1 - 2\nu$ instead of $1 - \nu$, made by some, would be incorrect and would result in a 78% error in κ_1 .)

The diagrams of crack length a_1 versus cooling penetration depth D (equilibrium path of the crack system) are calculated as follows. We fix the crack lengths a_2 and a_1 , choose various values of D and run the finite element program to obtain K_2 . Then we use Newton iteration to find the D -value which yields $K_2 = K_c$ (given critical value). We repeat this for various lengths a_2 (plotting the points in Figs. 3-1, 3-3, 3-4, 3-5, and for each a_2 we also evaluate $\partial K_2 / \partial a_2$ using Eq. (3-16) (by running the program also for $a_2 + \Delta x$ and $a_2 - \Delta x$). Then we use Newton iteration with respect to a_2 so as to finally locate the a_2 -value which gives $\partial K_2 / \partial a_2 = 0$. For calculation of the post-critical path and the second critical state, see Ref. 4.

3.6 EFFECT OF TEMPERATURE PROFILE (LOADING)

The previous conclusion on the doubling of crack spacing appears to be relatively favorable for heat withdrawal from the secondary thermal cracks in the hot dry rock mass. The spacing of the opened cracks indeed increases, roughly in proportion to the penetration depth D . For the error function profile of temperature, it has been calculated³ that the opening width w of the leading cracks fluctuates roughly around the values $w = (a - x/D)^2$, $\alpha(T_s - T_0)D/2$ (where α = thermal dilatation coefficient, T_0 = initial uniform temperature, T_s = temperature to which the halfspace surface is cooled) and the total vertical flux of water along the secondary thermal cracks is roughly between 9% and 12% of the vertical flux in the main crack⁴ and does not depend on D . However, these results have been obtained assuming the error function profile of temperature, which is correct only if water circulation in the secondary cracks has a negligible effect on temperatures and all heat is transported by conduction in rock. This assumption of course cannot be quite true because the vertical water flux in the secondary cracks is not negligible.⁴

The water circulating in the thermal cracks provides additional cooling of the rock below the halfspace surface. We neglect here the fact that the temperature distribution becomes two-dimensional and consider that the same (averaged) temperature profile, depending only on x/D , may be considered to exist along all normals $y = \text{const}$. This profile must be at small depth x more uniform than the

Fig. 3-4. Results from Fig. 3-1 replotted in relation to cooling penetration depth D .

Fig. 3-5. Post-critical behavior and second critical state for linear temperature profile.

error function profile whereas near the cooling front it must have a steeper gradient. Therefore, we have arbitrarily selected several temperature profiles shown in Fig. 3-4. Profile 1 is $\Delta T(x) = \text{erfc}(x/3D)$, where erfc is the complementary error function, the case already mentioned. Profile 3 is linear, and profiles 2, 4, and 5 have a temperature drop in the form of a halfwave of cosine function extending over lengths D , $D/2$ and $D/4$ from the cooling front, respectively.

The equilibrium crack lengths and the critical states of crack arrest (bifurcation points) obtained for various temperature profiles by finite element calculations are plotted in Figs. 3-1 and 3-4. It is seen that generally the steeper the cooling front, the longer are the cracks for the same cooling penetration depth, and the deeper is the location of the critical point at which every other crack would stop growing. For profile 5, which has a very steep cooling front, no critical state has been found, i.e., the cracks continuously extend at equal length, equal spacing and equal width.

To organize the results, can we introduce some simple approximate characteristic of the steepness of the temperature profile? The parameter x_c/D where x_c is the distance of the centroid of the profile (i.e., of the cross-hatched area in Fig. 3-1) from the halfspace surface is suitable for this purpose. This parameter can vary between 0 (for a temperature drop that is concentrated at the surface) and 0.5 (for a temperature drop that is concentrated at cooling penetration depth). The results are tabulated as a function of x_c/D in Table 3-I. It is seen that the critical state values of $a_2/2h$ and a_2/D vary monotonically as a

TABLE 3-I
CRITICAL STATES OF CRACK ARREST FOR VARIOUS TEMPERATURE PROFILES

Profile	Profile No.	x_c/D	a_2/D	$a_2/2h$	Eq. (3-19)	Eq. (3-18)
1	0.250	0.538	1.53	0.539	1.59
2	0.297	0.787	3.07	0.783	2.51
3	0.333	0.891	3.20	0.868	3.89
4	0.383	0.893	6.97	0.899	8.94
5	0.439	--	?(>30)	0.900	50

function of x_c/D , which confirms the suitability of using x_c/D as the profile characteristic. The following formulas approximate the results tabulated in Table 3-1:

$$\left(\frac{a_2}{2h}\right)_{\text{crit}} = \left(\frac{0.29}{0.48 - x_c/D}\right)^2 \text{ if } \frac{x_c}{D} < 0.48; \text{ else } \infty \quad (3-18)$$

$$\left(\frac{a_2}{D}\right)_{\text{crit}} = 0.9 - 107(0.4 - \frac{x_c}{D})^3 \text{ if } \frac{x_c}{D} \leq 0.40; \text{ else } 0.9 \quad (3-19)$$

More accurate formulas could no doubt be obtained by using two variables: x_c/D , and the second moment of the temperature profile divided by its area and by D^2 . By including profile moments of high-enough order, one could of course obtain formulas as accurate as desired.

The critical states of crack closing (second critical states) were calculated in Ref. 3 for the error function profile. How are these critical states influenced by the temperature profile? This is apparent from Figs. 3-5 and 3-6 where some finite element results are plotted for the linear profile of temperature and for the error function profile. For the former profile, the critical state of crack closing is seen to occur at $a_2/2h = 4.90$, whereas for the latter profile³ it is found to occur at $a_2/2h = 2.56$. So, we see that the effects of temperature profile on the critical states of crack closing and of crack arrest are similar. The deeper the centroid of the profile, the deeper both critical states are located.

The profiles of crack width at critical states are plotted in Fig. 3-7. We see that, approximately, the crack becomes closed over its entire length when $K_1 = 0$ is attained.

Now that we have seen the numerical results we consider their practical consequences. We see that the results are not particularly favorable for the heat transfer effectiveness of the secondary thermal cracks in the hot dry rock geothermal scheme.

We have already noted that, because of cooling by water circulating in the secondary cracks, the temperature profile must have a steeper front than the error function profile. However, we see that for a steeper front things get worse; i.e., the point where the crack spacing is doubled (which means roughly a doubling in the opening width of the leading cracks), is pushed deeper into the half-space. This tends to decrease the water flux and the heat withdrawal rate in the

Fig. 3-6. Post-critical behavior and second critical state for error function profile of temperature.

Fig. 3-7. Profiles of crack width at the critical states.

secondary cracks. We must keep in mind, though, that these conclusions are reached under certain simplifying assumptions, stated before.

The temperature profile that is established is obviously somewhere between profile 1 which implies no heat transport through the secondary cracks yet gives the highest water flux in these cracks, and profile 5, which implies the largest heat transfer from rock to the secondary cracks yet gives the smallest crack width and the smallest water flux in the secondary cracks. The present results can be used in the analysis of the water flow and heat convection to determine the actual temperature profile, but this is beyond the scope of this work.

Similar phenomena are observed for shrinkage cracks on concrete. Here, the thermal strain $\alpha\Delta T$ must be replaced by linear shrinkage strain ϵ_{sh} , and the problem is perfectly analogous, except that the error function distribution of ϵ_{sh} does not satisfy the diffusion equation precisely because the dependence of ϵ_{sh} on pore relative humidity is nonlinear and the diffusivity is dependent on pore humidity.⁴ By contrast to the cases of interest for hot dry rock geothermal energy, the profile is usually steeper near the surface than the error function profile and then rather flat at greater depth below the surface. Extrapolating the results from Figs. 3-7 and 3-4 and Table 3-I, we may, therefore, expect that the critical state of shrinkage crack arrest is reached at a smaller crack depth than it is for the error function profile. Thus, the spacing of drying shrinkage cracks in plane concrete would increase more rapidly than the spacing of our cooling cracks in rock.

3.7 EFFECT OF MATERIAL ANISOTROPY

The preceding calculations are based on isotropic material properties. However, the granite rocks usually exhibit significant orthotropy. Therefore, the critical states have been calculated by finite elements also for several cases of transverse isotropy whose axis is normal to half-space surface. The material compliance matrix was the same as before except that the uniaxial compliances $1/E_{yy}$ and $1/E_{zz}$ in the y and z directions were altered. The uniaxial compliance $1/E_{xx}$ was kept equal $1/E$, all three shear compliances were kept equal $1/G$, and the off-diagonal terms were kept equal $-\nu/E$. Cases in which the transverse stiffness (E_{yy} , E_{zz}) are either higher or smaller than E_{xx} have been considered. The crack extension criterion was defined in terms of 2γ , the specific energy of crack extension, which was kept the same as before.

Some results for transversely isotropic rock are plotted in Figures 3-8 and 3-9. Although the transverse stiffnesses differ by a factor of 2, the effect of anisotropy is seen to be relatively small. The likely practical situation is

Fig. 3-8. Critical states of crack arrest and bifurcation of equilibrium path for error function profile of temperature and various orthotropic materials.

Fig. 3-9. Results from Fig. 3-6 replotted in relation to cooling penetration depth D .

that of a halfspace whose weaker direction is along the normal x (because the main hydraulic fracture, whose wall we treat as the halfspace surface, is likely to form in a plane normal to the weak direction). In this case the difference from the solution for isotropic properties is particularly small (Figs. 3-6, 3-9)

3.8 MINIMUM CRACK SPACING

Cracks may be expected to initiate with the minimum possible spacing, which may be determined as the spacing for which the surface energy of the cracks equals the drop in strain, U , caused by a complete relief of the stress on all planes $z = \text{const.}$ ⁴ (as proposed in the work mentioned in Acknowledgment). Approximating the initial temperature profile as parabolic, the strain energy in absence of cracks is $U_0 = \int_0^D 1/2 \alpha \Delta T(x) (\sigma_y^0 + \sigma_z^0) dx$ where $\Delta T(x) = (1 - x/D)^2 (T_s - T_0)$, $\sigma_y^0 = \sigma_z^0 = \alpha \Delta T(x) E / (1 - \nu)$ = thermal stress which is created in absence of cracks.⁴ By integration, $U_0 = 0.2 D \sigma^2 (T_s - T_0)^2 E / (1 - \nu)$ per unit area of halfspace surface. The strain energy U_1 that is left after the cracks are introduced must now be subtracted. Determining it we must not overlook that a significant part of U_1 is due to normal stress σ_z' in direction z parallel to halfspace surface as well as crack planes. If all stress in the (x, y) planes were relieved by cracking, the remaining energy would be due solely to uniaxial thermal stress $\sigma_z' = \alpha \Delta T(x) E$ and would equal $U_1 = \int_0^D (1/2) \alpha \Delta T(x) \sigma_z' dx = 0.1 D \sigma^2 (T_s - T_0)^2 E$. Consequently, the strain energy relief cannot be larger than $\Delta U = U_0 - U_1 = 0.1 D \alpha^2 (T_s - T_0)^2 E (1 + \nu) / (1 - \nu)$. Then, if $a_1 = a_2$, we must have $2\gamma a_1 \leq b \Delta U$, which yields:⁴

$$b = 2h \geq \frac{20(1-\nu)\gamma}{(1+\nu)\alpha^2(T_s-T_0)^2E} \frac{a_1}{D} = 10 \left(\frac{(1-\nu)K_c}{\alpha(T_s-T_0)E} \right)^2 \frac{a_1}{D}. \quad (3-20)$$

According to Fig. 3-1 and Ref 3, we may estimate that initially (for $D \ll 2h$) the ratio a_1/D is not less than about 1.5. For this value and properties of granite, Eq. 3-20 yields minimum spacing of about 7 cm. Note that this result is essentially independent of D . Note that, similar to Eq. 3-17, the use of $1 - 2\nu$ in Eq. 3-20 instead of $1 - \nu$, made by some, would be incorrect and would result in a 78% error in b .

3.9 SUMMARY AND CONCLUSIONS

When a system of parallel equidistant cooling cracks propagates into a half-space, it reaches at a certain depth a critical state at which the growth of every other crack is arrested. Later these cracks reach a second critical state at which they close. The intermediate cracks, at doubled spacing, open about twice as wide and advance further as cooling penetrates deeper. Determination of the critical states requires calculation of the derivatives of the stress intensity factors with regard to crack lengths, which is here accomplished by finite element method. It is found that the crack depth-to-spacing ratio at which the critical state is reached is extremely sensitive to the temperature profile. This ratio greatly increases as the cooling front becomes steeper. The effect of transverse isotropy of the material upon the location of critical states is found to be relatively mild. The results are of interest for one recently proposed geothermal energy scheme for hot dry rock, as well as for shrinkage cracks in concrete.

3.10 ACKNOWLEDGMENT

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Note.--Further work has been done on finite element analysis of the three-dimensional problem of a hexagonal system of cooling cracks in a halfspace. Part of the programs have already been written.

3.12 APPENDIX--NUMERICAL EXAMPLE

Example: Consider, as typical properties of Westerly granite, that $\nu = 0.305$, $E = 37600 \text{ MN/m}^2$, $\alpha = 8 \times 10^{-6} \text{ per } ^\circ\text{C}$, $\gamma = 104 \text{ J/m}^2$, $K_C = 2.94 \text{ MN m}^{-3/2}$, and assume that $T_s - T_0 = 100^\circ\text{C}$, and that $a_1/D = 0.9/\text{m}$ the initial crack spacing (see Fig. 3-4). From Eq. 3-20 we estimate

$$b = 2h = 10 \left(\frac{(1-0.305) 2.94}{8 \times 10^{-6} \times 100 \times 37600} \right)^2 0.9 = 0.0415 \text{ m} \quad (3-21)$$

as the initial spacing of cracks.

Furthermore, consider that after some time of operation the effect of water flow in secondary cracks produces profile No. 3 of Table 3-1. Then, from Table 3-1, $x_e/D = 0.333$. The approximate formulas (Eqs. 3-19 and 3-18) give $a_2/D = 0.868$ and $a_2/2h = 3.89$ for the first initial state of crack arrest, while the more accurate finite element results give $a_2/D = 0.891$ and $a_2/2h = 3.20$ (Table 3-1). Taking the latter values, and considering that the spacing of the opened cracks has already become, say, 1 m, then the critical crack length would be $a_2 = 3.20 \text{ m}$. For the error function profile, this would be only 1.53 m, i.e., much less.

CHAPTER 4

THERMALLY AND HYDRAULICALLY INDUCED TENSION CRACKS

by

S. Nemat-Nasser and L. M. Keer

FOREWORD

This chapter is divided into two parts. In Part I we summarize our research on thermally induced secondary cracks, and in Part II we examine the configuration of a system of parallel penny-shaped cracks.

PART I

THERMAL CRACKS

4.1 INTRODUCTION

The extraction of heat from hot dry rock masses involves several basic processes: (1) generation of a large fracture or a system of large fractures by hydraulic fracturing; (2) the circulation of water under pressure through the fractured zone and up to ground surface; (3) the cooling of the rock, the consequent secondary cracking, and their effects on flow and heat extraction processes; and (4) the crack configuration and geometry, and the manner by which they may be altered by fluid flow and by thermally induced stresses.

Estimates show that for a 1 kilometer diameter crack the initial crack opening is very small, of the order of a few millimeters. As the heat is extracted from the rock, the crack opening increases because of thermal contraction, and within several years the primary crack opening is of the order of centimeters. When the overburden pressure is considerably larger than the two horizontal principal tectonic stresses, the primary crack will most probably be vertical, and the secondary thermally induced cracks which may form perpendicular to the face of the primary crack (as the rock cools), take on the shape of vertically elongated hexagons. Assuming that secondary thermal cracks actually do form, it is

important to study the following items:

- (a) Initial crack spacing.
- (b) Crack growth regime: crack interaction; effect of heterogeneity; crack closure and snap through.
- (c) Thermally induced rock fragmentation: crack branching; crack curving; and spallation.

In the sequel we shall summarize our studies of items (a), (b), and (c) and give some specific illustrative examples. For a more detailed discussion the references listed at the end of this chapter should be consulted.

It should be noted that the actual problem of secondary cracks and their effects on the overall heat extraction process is a complicated coupled fluid-solid problem involving the following items:

- (A) Effect of secondary cracks on heat exchange and effect of secondary flow on thermally induced secondary cracks (coupled problem).
- (B) Effect of secondary cracks on plant life which, in turn, will involve estimates of: (B-1) effect of cooling on primary crack, and (B-2) effect of secondary cracks on primary flow and heat exchange.

Studies of items (A) and (B) are of considerable importance and are suggested for possible future effort.

4.2 INITIAL CRACK SPACING

There are several ways by which the initial crack spacing can be estimated. Three basic approaches are the energy approach, the stability approach, and the strength approach. We shall examine these below and present estimates for initial crack spacing.

In Fig. 4-1 we have schematically presented a primary crack and the state of stress. We have assumed that, initially, there are numerous microcracks that develop perpendicular to the face of the primary crack. Upon cooling, some of these microcracks extend perpendicular to the face of the primary crack, forming secondary cracks. We wish to know what is the minimum spacing of these cracks, when the thermal penetration depth δ is of the order of a few millimeters, i.e. soon after cooling takes place.

In the sequel we shall use the following two temperature profiles in the rock; see Fig. 4-2:

Fig. 4-1. Primary crack and the stress state.

Fig. 4-2. (a) Temperature profile (4-2);
(b) Temperature profile (4-1) with $n = 0.5$;
(c) A system of equally spaced edge cracks.

$$T = 0 \quad \text{for } 0 \leq y \leq \delta/(n+1),$$

$$T = \frac{T_0}{2} \left[1 - \cos \pi \frac{y(n+1) - \delta}{n\delta} \right] \quad \text{for } \delta/(n+1) \leq y \leq \delta,$$

$$T = T_0 \quad \text{for } \delta \leq y, \quad T_0 = T_r - T_w, \quad (4-1)$$

where $n = 0.5$ is used (see Fig. 4-2b); and

$$T = T_0 \operatorname{erf} \left[\frac{y\sqrt{3}}{\delta} \right], y \geq 0, \quad (4-2)$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ (see Fig. 4-2a).

The temperature profile (4-1) assumes that there is convective heat transfer, and it represents a case where the heat of the rock to the depth $\delta/n+1$ is completely removed. On the other hand, the temperature profile (4-2) is the error function, and assumes no convective heat transfer. The two profiles, therefore, may be regarded as limiting cases, and therefore useful for obtaining good estimates.

4.2.1 Energy Approach

We express the temperature profile by $f(y/\delta)$, and calculate the total elastic energy in a strip of unit thickness and width b , as follows:

$$W = \frac{1}{2} \alpha^2 T_0^2 b \frac{E}{1-\nu} \int_0^\infty [1 - f(y/\delta)]^2 dy = K_0 \frac{\alpha^2 T_0^2 E b \delta}{1-\nu}, \quad (4-3)$$

where K_0 is the numerical factor depending on the temperature profile, and where the quantity $1 - \nu$ must be replaced by $1 - 2\nu$ when there is no free surface. We now equate to the surface energy the fraction θ of this total elastic energy, which has been used to generate new surfaces, and obtain

$$\theta W = 2h\gamma, \quad (4-4)$$

where γ is the specific surface energy, and h is the crack depth. Hence, we have

$$b = 2K_0^{-1} \left[\frac{(1-\nu)\gamma}{\alpha^2 E} \right] \left(\frac{h}{\theta\delta} \right) / T_0^2, \quad (4-5)$$

with $n = 0.5$, $K_0 = 19/48$ for temperature profile (4-1), and for temperature profile (4-2), $K_0 = 1/10$. The quantity $h/\theta\delta$ may be taken approximately equal to 1, and (4-5) rewritten as

$$b = K_0^{-1} A / T_0^2, \quad A = 2(1-\nu)\gamma / \alpha^2 E. \quad (4-6)$$

It is difficult to establish accurately the values of the material constants which enter (4-6)₂. The following values are often used:

$$\begin{aligned} \alpha &= 8 \times 10^{-6} / ^\circ\text{C}, \quad E = 4 \times 10^{10} \text{ N/m}^2, \\ \nu &= 0.22, \quad \gamma = 100\text{--}1,000 \text{ J/m}^2. \end{aligned} \quad (4-7)$$

With these values Table 4-I gives the estimates of the initial minimum crack spacing, for indicated values of T_0 and γ .

TABLE 4-I
MINIMUM CRACK SPACING EQ. (4-6)

$T_0 \text{ } ^\circ\text{C}$	$\gamma = 100 \text{ J/m}^2$	$\gamma = 1000 \text{ J/m}^2$	Temperature Profile
100	2 cm	16 cm	(4-1)
130	1 cm	10 cm	
100	7 cm	65 cm	(4-2)
130	4 cm	38 cm	

4.2.2 Stability Approach

If we introduce the dimensionless parameter

$$N = K_c / [\sqrt{2\pi} \hat{\beta} T_0 \sqrt{b}], \quad \hat{\beta} = \alpha E / (1-\nu), \quad (4-8)$$

where $K_c = \sqrt{2\gamma E / (1-\nu^2)}$ is the critical value of the stress intensity factor, the stability of the growth regime of a system of edge cracks in plane strain can be discussed in terms of the dimensionless parameter N . The result is only dependent on the temperature profile. Using the stability theory developed by the present writers, we have obtained the universal results shown in Figs. 4-3 and 4-4, for temperature profiles (4-1) and (4-2). The curves in these figures are explained later on. However, from a stability consideration, N cannot exceed 0.282 for temperature profile (4-1), and 0.19 for temperature profile (4-2). Hence, the minimum crack spacing is bounded as follows:

$$b \geq \frac{1}{(1+\nu)N_A^2} A/T_0^2 = \begin{cases} 3.3A/T_0^2 & \text{for (4-1)} \\ 7.4 A/T_0^2 & \text{for (4-2)} \end{cases} \quad (4-9)$$

Fig. 4-3. Stability chart for temperature profile (4-1); $n = 0.5$, $\Delta = \delta/b$, $a_1 = h_1/b$.

Fig. 4-4. Stability chart for temperature profile (4-2); $\Delta = \delta/b$, $a_1 = h_1/b$.

Note that these results confirm the energy estimate given by Eq. (4-6).

4.2.3 Effect of Material Heterogeneity

As will be discussed in Section 4-3, the growth regime of thermally induced tension cracks is highly imperfection-sensitive, in the sense that the introduction of a few percent inhomogeneity in the material parameters may change considerably the stage at which a given growth pattern becomes unstable. This will also affect the initial crack spacing, because the maximum value of the dimensionless stress intensity factor N_A in (4-9) will be changed. For example, if we introduce 5% inhomogeneity in the value of K_C , then N_A for temperature profile (4-1) reduces to 0.257, and if we introduce 10% inhomogeneity in K_C , then N_A reduces to 0.244. The corresponding minimum crack spacings then become $b = 12$ cm and 13 cm, respectively.

4.2.4 Strength Approach

We have also estimated the minimum crack spacing, using strength approaches. There are two possibilities. In one case, one calculates the maximum value that the dimensionless stress intensity factor N can possibly attain, see Figs. 4-3 and 4-4, and then estimates the minimum crack spacing from Eq. (4-8)₁. This yields much smaller crack spacings which, according to the energy consideration, cannot be produced. The second approach is to calculate the crack spacing such

that, while the stress intensity factor at the tips of the cracks does not exceed the critical value, the tension at the surface, halfway between the two cracks, does not exceed the tensile strength of the rock (about 80 bars). When δ is very small, calculations show that, again, the energy consideration yields the proper estimate for the minimum initial crack spacing. It therefore appears that Eq. (4-6) provides a reliable estimate for the initial crack spacing.

4.3 CRACK GROWTH REGIME

As the thickness of the thermal layer, δ , increases, secondary cracks grow into the rock. To establish the growth regime pattern of cracks of this kind, we require an assessment of the stability of their configuration.

The theory of the stability of the growth pattern of tension cracks in brittle solids has been developed recently by the present writers; see the references listed at the end of this chapter. The theory involves: α) Crack interaction and growth. β) Effect of material heterogeneity and imperfection. γ) Effect of anisotropy or initial stress state ($\sigma_1 \neq \sigma_2 \neq \sigma_3$).

On the other hand, the actual analysis turns out to be extremely difficult, and demands careful analytical and numerical efforts. For this, the present writers, together with their coworkers, have developed and used the following methods: δ) Analytical methods for simple geometries. ϵ) Combined analytical and finite-element approach. This approach has proved exceedingly effective with potential for application to difficult geometries.

In the sequel we briefly outline each of the above items.

4.3.1 Crack Interaction

Initially, the cracks are all equal in length, and grow in a stable manner, as the thickness of the thermal layer, δ , is increased; δ is used as a measure of the load parameter. As long as the crack spacing b is large compared with the common crack length h , there is very weak interaction between adjacent cracks, and such a crack growth pattern is inherently stable. In this case, each crack may be regarded as an isolated one, and since for a fixed δ , there is a fixed amount of elastic energy available, an extension of a crack at constant δ would release a certain amount of elastic energy, which results in a reduction of the corresponding stress intensity factor at the crack tip, and therefore crack growth will be arrested. If K is the stress intensity factor, then we have, in this case, $\partial K / \partial h < 0$; see Fig. 3 of Ref. 9.

As the common crack length increases with increasing δ , the interaction between adjacent cracks becomes more important. In Fig. 4-5 two interacting cracks are shown in a unit cell, with h_1 and h_2 as their corresponding lengths, and K_1 and K_2 the stress intensity factors; the critical value of the stress intensity factor is K_c . In Ref. 9 the following results have been obtained and numerically illustrated.

The equilibrium regime

$$h_1 = h_2 \quad \text{and} \quad K_1 = K_2 = K_c \quad (4-10)$$

is stable as long as

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} < 0, \quad (4-11)$$

and it is unstable for

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} > 0, \quad (4-12)$$

the critical state corresponding to

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} = 0. \quad (4-13)$$

When the critical state (4-13) is reached, one crack (say, crack 2) stops growing,

as the other crack extends spontaneously (i.e. without change in δ) by an infinitesimal amount. As the load parameter is increased, crack 2 remains stationary while crack 1 grows at a faster rate.

During this process, K_2 continues to decrease with increasing δ , and for temperature profile (4-1), it actually becomes zero at a new critical state,

Fig. 4-5. Two interacting edge cracks.

after which crack 2 snaps closed, while crack 1 extends by an additional finite amount.

In a recent work, Sumi et al.¹⁸ have shown that, before the critical state corresponding to (4-13) is reached, there are infinitely many stable critical states, at each of which two distinct crack growth patterns become possible: (1) equal crack growth regime defined by (4-10); and (2) a crack growth regime corresponding to

$$dh_1 > 0, dh_2 = 0, K_1 = K_c, K_2 \leq K_c, \quad (4-14)$$

i.e. one crack remaining stationary, as the other crack grows with increasing δ . These and related results are discussed in the following Subsections.

4.3.2 Fundamental Equilibrium Path

Consider the h_1, h_2, δ -space; Fig. 4-6. The fundamental equilibrium path in this space is the locus of points for which (4-10) holds. In Fig. 4-6 this is denoted by curve AB_uB . Points on this curve define equilibrium states, not all of which are stable.

4.3.3 Stable Bifurcation Points

When the common crack length, $h = h_1 = h_2$, is small relative to the crack spacing, the two cracks in the unit cell of Fig. 4-5 have a weak interaction. In this case, if at constant h_2 and δ , h_1 is increased by an infinitesimal amount, the stress field will be released at both crack tips, and therefore, $\partial K_1 / \partial h_1 < 0$ and $\partial K_2 / \partial h_1 < 0$. The weak interaction then implies that the stress is released more at crack 1 than at crack 2, because of the extension of crack 1, while keeping h_2 and δ fixed. Hence, we must have $|\partial K_2 / \partial h_1| < |\partial K_1 / \partial h_1|$. For this regime we have

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} < \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} < 0, h_1 = h_2. \quad (4-15)$$

The only possible crack extension regime then is defined by (4-10), namely the fundamental equilibrium path.

To prove this assertion, first observe that since $K_i = K_i(h_1, h_2, \delta)$,

Fig. 4-6. Various equilibrium states for two interacting cracks: B^* is a stable, and B_u an unstable bifurcation point; B_s^* and B_s are snap-through critical points.

$$dK_1 = \frac{\partial K_1}{\partial h_1} dh_1 + \frac{\partial K_1}{\partial h_2} dh_2 + \frac{\partial K_1}{\partial \delta} d\delta,$$

$$dK_2 = \frac{\partial K_2}{\partial h_1} dh_1 + \frac{\partial K_2}{\partial h_2} dh_2 + \frac{\partial K_2}{\partial \delta} d\delta.$$

(4-16)

Now, for $d\delta > 0$ and $dh_1 > 0$, i.e. $dK_1 = 0$, obtain

$$d\delta = - \left(\frac{\partial K_1}{\partial h_1} dh_1 + \frac{\partial K_1}{\partial h_2} dh_2 \right) / \frac{\partial K_1}{\partial \delta}, \quad (4-17)$$

and, upon substitution into (4-16)₂, arrive at

$$dK_2 = \left(\frac{\partial K_1}{\partial h_2} - \frac{\partial K_1}{\partial h_1} \right) \left(1 - \frac{dh_2}{dh_1} \right) dh_1, \quad (4-18)$$

where, since $h_1 = h_2$, and since there is complete symmetry in this case, condition $\partial K_1 / \partial \delta = \partial K_2 / \partial \delta$ is used.

For $dh_1 = dh_2$, (4-18) yields $dK_2 = 0$, and hence $K_2 = K_c$: the state remains on the fundamental equilibrium path. Suppose now, $dh_2 \neq dh_1$, and assume $dh_2 > dh_1$. Then from (4-15) and (4-18) it follows that $dK_2 < 0$, and hence dh_2 must be equal to zero, which is a contradiction. Moreover, if we assume $0 \leq dh_2 < dh_1$, we obtain $dK_2 > 0$, which is impossible, since K_2 cannot exceed its critical value K_c . Hence, as long as (4-15) holds, incremental crack extension satisfies $dh_1 = dh_2$, and the state remains on the fundamental equilibrium path AB.

As the two cracks in Fig. 4-5 grow with increasing δ , their interaction becomes more important, and when their common length is large enough, then a point may be reached at which, with h_2 and δ kept constant, an infinitesimal extension of crack 1 decreases the stress intensity factors at crack 2 and crack 1 by an equal amount. This defines a critical point on the fundamental equilibrium path. It is a stable bifurcation point, and is characterized by the condition

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} = \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} < 0, \quad h_1 = h_2. \quad (4-19)$$

To see that the state associated with (4-19) is indeed critical, observe that equilibrium requires $dK_1 = dK_2 = 0$ for $d\delta > 0$, and hence (4-16) yields

$$dh_1 = \left(\frac{\partial K_2}{\partial h_1} \frac{\partial K_2}{\partial \delta} - \frac{\partial K_1}{\partial h_1} \frac{\partial K_1}{\partial \delta} \right) d\delta / \left\{ \left(\frac{\partial K_1}{\partial h_1} \right)^2 - \left(\frac{\partial K_1}{\partial h_2} \right)^2 \right\},$$

$$dh_2 = \left(\frac{\partial K_2}{\partial h_1} \frac{\partial K_1}{\partial \delta} - \frac{\partial K_1}{\partial h_1} \frac{\partial K_2}{\partial \delta} \right) d\delta / \left\{ \left(\frac{\partial K_1}{\partial h_1} \right)^2 - \left(\frac{\partial K_1}{\partial h_2} \right)^2 \right\}. \quad (4-20)$$

Therefore uniqueness is lost when (4-19) is satisfied. On the fundamental equilibrium path in Fig. 4-6, this point is denoted by B^* . On this same path, points above B^* (but below B_u) satisfy the following conditions:

$$0 > \frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} > \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1}, \quad h_1 = h_2, \quad (4-21)$$

and hence correspond to stable states. However, each one of these points defines a stable bifurcation point from which another equilibrium path emanates.

In the immediate neighborhood of the fundamental path, these bifurcated paths characterize states with smaller total stored elastic energy per unit cell, and hence define more stable states than the corresponding ones (for the same δ) on the fundamental equilibrium path. We shall prove this assertion in the sequel, but first we shall show the existence of these stable bifurcated equilibrium paths.

To this end consider a typical point on the fundamental equilibrium path above B^* , say, point B_1 in Fig. 4-6. At this point, conditions (4-21) hold. Consider now an increase in the load parameter, $d\delta > 0$. If equal crack growth is assumed, then point B_2 on the fundamental equilibrium path will be attained and we will have

$$\begin{aligned} h_1(B_2) &= h_2(B_2) = h_1(B_1) + dh_1 = h_2(B_1) + dh_2, \\ dh_1 &= dh_2 > 0, \quad dK_1 = dK_2 = 0, \quad K_1(B_2) = K_2(B_2) = K_c; \end{aligned} \quad (4-22)$$

here $h_1(B_2)$, for example, denotes the length of crack 1 at the state corresponding to B_2 .

In addition to the equilibrium state (4-22), conditions (4-21) and (4-18) reveal that other equilibrium states with, say, $dh_2 = 0$ is now possible, because in this case, $dK_2 < 0$. This new equilibrium state is denoted by B^*_2 in Fig. 4-6. It is characterized by

$$\begin{aligned} h_1(B^*_2) &= h_1(B_1) + dh_1^*, \quad h_2(B^*_2) = h_2(B_1) = h_1(B_1), \\ dh_1^* &> 0, \quad dh_2^* = 0, \quad dK_1 = 0, \quad dK_2 < 0, \quad K_1(B^*_2) = K_c, \quad K_2(B^*_2) < K_c. \end{aligned} \quad (4-23)$$

Both states, B_2 and B_2^* , which correspond to the same value of the load parameter, $\delta = \delta(B_1) + d\delta_0$, are stable; here, $d\delta_0$ is a fixed load increment. But as we shall show below, state B_2^* on the bifurcated path corresponds to a smaller stored elastic energy, and therefore is more stable.*

To this end we observe that if there are no cracks in the solid, the total stored elastic energy per unit cell (and per unit thickness in the direction normal to Fig. 4-2) is a finite quantity proportional to the thickness of the thermal layer, δ . This is because the temperature profiles (4-1) and (4-2) involve only y/δ and therefore integration of the elastic energy density over a unit cell would always lead to a quantity of the following kind:

$$e(\delta) = A\delta, \quad (4-24)$$

where A is independent of δ , but depends on material parameters, total temperature drop, and the shape of the temperature profile. For profile (4-1), for example, we have (in plane strain, with the free surface**)

$$A = \frac{19}{48} \frac{\hat{\alpha}^2 T_0^2 E b}{(1-\nu)}, \quad (4-25)$$

and if we approximate the temperature profile (4-2) by $T = T_0[1 - (1-y/\delta)^2]$, then we obtain

$$A = \frac{1}{10} \frac{\hat{\alpha}^2 T_0^2 E b}{(1-\nu)}. \quad (4-26)$$

It is clear that in the present context other reasonable temperature profiles would always lead to an expression similar to (4-24), with A depending on the shape of the temperature profile, but not on δ .

When cracks exist, part of the total elastic energy given by (4-24) is used to generate crack surfaces, and the remaining part is stored in the body,

* The fact that the state with smaller stored elastic energy is more stable follows from Gibbs' statement of second law of thermodynamic; for the application of this to fracture problems, see Nemat-Nasser, Ref. 26.

** Without the free surface, $(1-\nu)$ must be replaced with $(1-2\nu)$.

$$\mathcal{E} = \mathcal{E}_s + S, \quad (4-27)$$

where $\mathcal{E}_s$ is the stored elastic energy per unit cell, and S is the corresponding surface energy. Since for a fixed δ the total supplied energy $\mathcal{E}$ is fixed, the state with larger total surface energy will have smaller stored elastic energy. Therefore, to prove that state B_2^* is more stable than B_2 , we shall show that $S(B_2^*) > S(B_2)$. However, since

$$\begin{aligned} S(B_2) &= S(B_1) + 2(dh_1 + dh_2)\gamma, \quad dh_1 = dh_2, \\ S(B_2^*) &= S(B_1) + 2dh_1^*\gamma, \quad dh_2^* = 0, \end{aligned} \quad (4-28)$$

to prove our assertion, we need only show that

$$(dh_1 + dh_2) = 2dh_1 < dh_1^*; \quad (4-29)$$

in (4-28) γ is surface energy per unit area and relates to the critical value of the stress intensity factor by $K_c^2 = 2\gamma E/(1-\nu^2)$, where ν is the Poisson ratio and E Young's modulus.

To prove (4-29), we recall that the load parameter at both states, B_2 and B_2^* , has the value $\delta(B_2) = \delta(B_2^*) = \delta(B_1) + d\delta_0$, where $d\delta_0$ is fixed. It then follows that

$$\frac{\partial K_1}{\partial \delta} d\delta_0 = \frac{\partial K_2}{\partial \delta} d\delta_0 \text{ at } B_1. \quad (4-30)$$

For point B_2 on the fundamental path, $dh_1 = dh_2$ and we obtain from Eq. (4-16),

$$dh_1 = - \frac{\partial K_1}{\partial \delta} d\delta_0 / \left(\frac{\partial K_1}{\partial h_1} + \frac{\partial K_1}{\partial h_2} \right). \quad (4-31)$$

For point B_2^* , on the other hand, $dh_2^* = 0$. Equations (4-16) then yield

$$dh_1^* = - \frac{\partial K_1}{\partial \delta} d\delta_0 / \frac{\partial K_1}{\partial h_1}. \quad (4-32)$$

Now, in view of (4-21), we have $dh_1^* > 2dh_1$ which is (4-29). Note that in expressions (4-30-4-32) the partial derivatives are all evaluated at state B_1 , for which $h_1 = h_2$.

The fact that the state on the bifurcated path is more stable than the corresponding one on the fundamental path, may be proved in a different but equivalent manner. For this, one may consider two neighboring states, and instead of fixing the load parameter, one chooses these two neighboring states in such a manner that they correspond to the same total surface energy. Then the state with smaller stored elastic energy would involve smaller total energy and therefore, would correspond to a smaller load parameter. Hence, it will be the one which actually will be attained. If the two neighboring states, $\hat{B}_2$ and $\hat{B}_2^*$, one on the fundamental equilibrium path and the other on the bifurcated path, have the same surface energy, we must have, for the corresponding incremental crack lengths, $dh_1 = dh_2 = \frac{1}{2}dh_1^*$, and $dh_2^* = 0$. Then from (4-24) and (4-27) we obtain

$$e_s(\hat{B}_2^*) - e_s(\hat{B}_2) = A \left[-2 \left(\frac{\partial K_1}{\partial h_1} / \frac{\partial K_1}{\partial \delta} \right) + \left(\frac{\partial K_1}{\partial h_1} + \frac{\partial K_1}{\partial h_2} \right) / \frac{\partial K_1}{\partial \delta} \right] dh_1 \quad (4-33)$$

which, in view of (4-22), yields

$$e_s(\hat{B}_2^*) < e_s(\hat{B}_2), \quad (4-34)$$

as required. Note that the actual states corresponding to a prescribed fixed load parameter may not be the same as those corresponding to a fixed surface energy.

To the second-order approximation used in arriving at (4-29), the two surface energies, one corresponding to a neighboring point on the fundamental path and the other to a point on the bifurcated path, would be equal at point B^* , i.e. we have

$$dh_1^* = 2dh_1 \quad \text{at } B^*. \quad (4-35)$$

However, if the higher-order terms are used, a conclusion similar to (4-29) will again be reached. We have circumvented the need for such a tedious consideration, by examining states in the neighborhood of point B_1 for which the strict inequality (4-21) holds; note that at point B^* conditions (4-19) hold.

4.3.4 Post-Critical Response

As discussed above, at the state corresponding to point B*, every other crack in the array of cracks shown in Fig. 4-2 stops, as the remaining ones grow with increasing load parameter δ , and at a rate initially twice as fast as before; this latter assertion follows from Eq. (4-35). To examine the growth regime pattern after point B*, we must consider three interacting cracks shown in Fig. 4-7, for which

$$h_1 = h_3, h_2 = h_2(B^*) = \text{constant}, K_1 = K_3 = K_c, K_2 < K_c. \quad (4-36)$$

However, since the spacing between cracks 1 and 3 in Fig. 4-7 is equal to $2b$, the corresponding initial interaction is weak. In fact, as it has been shown in Ref. 16, all the essential features of the crack growth regime after the bifurcation point can be established quantitatively with almost no loss in accuracy by considering only two interacting cracks for the present problem. Hence we shall pursue this approximation below.

On the bifurcated path B*B_s we have the following numerical results for temperature profile (4-1); for illustration we shall use this temperature profile:

$$dK_2 = \frac{\partial K_2}{\partial h_1} dh_1 + \frac{\partial K_2}{\partial \delta} d\delta < 0. \quad (4-37)$$

Fig. 4-7 Three interacting cracks.

Hence the stress intensity factor K_2 continues to decrease with increasing δ and h_1 . At point B_S^* the stress intensity factor K_2 is zero, and a further increase in the load parameter results in the closure of crack 2. At this point and for temperature profile (4-1), calculations show that crack 2 snaps closed while crack 1 snaps into a finitely longer length, for infinitesimally larger values of the load parameter δ ; this is illustrated in the next section. Hence point B_S^* defines a snap-through critical state. In the infinitesimally small neighborhood of this state, there are no equilibrium states corresponding to (infinitesimal) $d\delta > 0$.

From the relation between the energy release rate and the stress intensity factor in Mode I, it has been shown⁹ that

$$K_1 \frac{\partial K_1}{\partial h_2} = K_2 \frac{\partial K_2}{\partial h_1} . \quad (4-38)$$

Since at point B_S^* we have $K_2 = 0$, this point is characterized by

$$\frac{\partial K_1}{\partial h_2} = 0 . \quad (4-39)$$

Actually, since $\partial K_1 / \partial h_2$ is in general non-positive, if this quantity is plotted as a function of h_1 , the curve would be tangent to the h_1 -axis at the critical point defined by (4-39); see Nemat-Nasser, et al.⁹

For some other temperature profiles, e.g. (4-2), it happens that K_2 never reaches zero.* In this case, crack 1 and 3 of Fig. 4-7 begin to have significant interaction, and one of them stops at a certain stable bifurcation point before the stress intensity factor at crack 2 can reach zero value. For cases of this kind there is no crack closure. This has been illustrated^{9, 16} for temperature profile (4-2).

*This result is not in agreement with that presented by Bazant and Wahab in Chapter 3, for the following reasons:

- (1) three interacting cracks must be considered for the error function temperature profile in order to obtain a complete solution (see Refs. 9 and 16); only two cracks are considered in Chapter 3; and
- (2) the numerical method used in Chapter 3, does not yield sufficiently accurate results to allow obtaining correct conclusions; see also comments in Section 4. The stable bifurcation points which are the only essential states for the present problem, are not calculated in Chapter 3.

4.3.5 Effect of Small Imperfections

In the preceding discussions we consider idealized cases where the cracks are equally spaced and the material properties are homogeneous. This cannot exist in actual situations and there are always some imperfections present. It is known in structural analysis that even very small imperfections may introduce substantial reduction in the critical value of the load parameter at which the structure becomes unstable. It turns out that a similar role is played by small imperfections in the stability of a system of interacting tension cracks. The imperfection in the present case may be a small deviation from an equally spaced crack pattern (i.e. unequal crack spacing), or it may be small nonhomogeneity in material properties, e.g. the critical value of the stress intensity factor at one crack may exceed by a small amount that at the other crack in the unit cell of Fig. 4-5.

When the cracks are unequally spaced, then both Modes I and II will be involved, and this will complicate the calculations substantially. On the other hand, when equal spacing is used, but different values are assigned to the critical stress intensity factor at the two cracks within a unit cell, the effect of imperfection can be studied without additional elaborate computations. For the purpose of illustration we shall give numerical examples in Section 4.5, using the latter type of imperfection.

To discuss the effect of imperfection, it is convenient to introduce the notation

$$u = h_1 - h_2 ,$$

which measures deviation from the fundamental equilibrium path. Hence, in the u, δ -plane the fundamental equilibrium path coincides with the δ -axis; see Fig. 4-8a. At point B^* , condition (4-19) is first satisfied, and hence, this is a stable bifurcation point. At point B_u , critical condition (4-13) holds, and hence, this is an unstable bifurcation point. All points above B_u on the δ -axis in Fig. 4-8a are unstable bifurcation points; at these points, the bifurcated equilibrium path has a negative slope, the slope being zero at B_u . All points on the δ -axis between B^* and B_u are stable bifurcation points.

The path $B^*B_S^*$ in the u, δ -plane is the bifurcated path on which $h_2 = h_2(B^*) =$ constant, $K_2 < K_c$, and, as we discussed before, this is the path that actually will be followed in the absence of any imperfection.

Fig. 4-8. Post-critical response and the effect of imperfection.

Assume now that the critical stress intensity factor at crack 1 is K_c , but that at crack 2 is $K_c + \epsilon K_c$, where $\epsilon \ll 1$, e.g. $\epsilon = \text{few percent}$. Then, as δ is increased, crack 2 will have a shorter length than crack 1, (see Fig. 4-8b), and therefore, u as a function of δ will follow a path similar to that sketched in Fig. 4-8a by curve ab_s . At a certain point, say, b^* (Fig. 4-8a), on this curve, crack 2 stops growing, and crack 1 then continues to grow at a faster rate; the corresponding point, on curve 1 is denoted by b_1^* and that on curve 2 by b_2^* in Fig. 4-8b. For a certain temperature profile, for example (4-1), the stress intensity factor at crack 2 reaches zero at point b_s in Fig. 4-5a, after which crack 2 snaps closed as crack 1 extends by a finite amount. At this point, u increases by a large finite value. The crack spacing is then doubled, and the whole process repeats itself.

To give an analytic description of some of the above behaviors, one may use a perturbation approach as follows.

The crack lengths for nonzero ϵ are functions of this parameter, $h_i = h_i(\epsilon)$, $i = 1, 2$, so are the stress intensity factors $K_i = K_i(h_1, h_2; \delta, \epsilon)$. Hence, for fixed δ we may write

$$h_1(\epsilon) = h_1(0) + \Delta h_1, \quad h_2(\epsilon) = h_2(0) + \Delta h_2, \quad (4-40)$$

and

$$\begin{aligned} K_1(\epsilon) &= K_c + \frac{\partial K_1}{\partial h_1} \Delta h_1 + \frac{\partial K_1}{\partial h_2} \Delta h_2 = K_c, \\ K_2(\epsilon) &= K_c + \frac{\partial K_2}{\partial h_1} \Delta h_1 + \frac{\partial K_2}{\partial h_2} \Delta h_2 = K_c + \epsilon K_c, \end{aligned} \quad (4-41)$$

where on the fundamental path, $h_1(0) = h_2(0)$ and $K_1 = K_2 = K_c$. In Eqs. (4-40) and (4-41) all partial derivatives are evaluated on the fundamental equilibrium path, i.e. at $\epsilon = 0$. Furthermore, it is assumed that the deviations from the fundamental path, denoted by Δh_1 and Δh_2 , are small; Eqs. (4-41) apply to points below critical point b^* in Fig. 4-8a.

Equations (4-41) can be solved for Δh_1 and Δh_2 , and since on the fundamental equilibrium path $\partial K_1 / \partial h_1 = \partial K_2 / \partial h_2 < 0$ and $\partial K_1 / \partial h_2 = \partial K_2 / \partial h_1 < 0$, we obtain

$$\Delta h_1 = - [K_c \frac{\partial K_1}{\partial h_2} / \{(\frac{\partial K_1}{\partial h_1})^2 - (\frac{\partial K_1}{\partial h_2})^2\}] \epsilon > 0,$$

$$\Delta h_2 \approx [K_c \frac{\partial K_1}{\partial h_1} / \{(\frac{\partial K_1}{\partial h_1})^2 - (\frac{\partial K_1}{\partial h_2})^2\}] \epsilon < 0, \quad (4-42)$$

where, since the derivatives in (4-42) are calculated below the stable bifurcation point B^* of Fig. 4-6, conditions (4-15) are valid and hence,

$$D = (\frac{\partial K_1}{\partial h_1})^2 - (\frac{\partial K_1}{\partial h_2})^2 > 0. \quad (4-43)$$

As long as D is a relatively large finite quantity, $|\Delta h_1|$ remains of the order of ϵ , as shown by (4-42). As we increase δ , the positive quantity D tends to decrease, becoming zero at the stable bifurcation point B^* . Hence, the deviation from the fundamental equilibrium path, characterized by

$$\begin{aligned} u &= h_1(\epsilon) - h_2(\epsilon) = \Delta h_1 - \Delta h_2 \\ &\approx -\epsilon K_c [\frac{\partial K_1}{\partial h_1} + \frac{\partial K_1}{\partial h_2}] / D \\ &= \epsilon K_c / (\frac{\partial K_1}{\partial h_2} - \frac{\partial K_1}{\partial h_1}), \end{aligned} \quad (4-44)$$

tends to increase with increasing δ ; approximation (4-44) is valid only for points below the critical point b^* in Fig. 4-8a which corresponds to points b_1^* and b_2^* in Fig. 4-8b.

For a fixed value of imperfection, $\epsilon = \epsilon_0$ and for $\delta < \delta^*$, consider an increment in the load parameter, $d\delta > 0$, and calculate the corresponding increase in crack lengths, dh_1 on curve (1) and dh_2 on curve (2), of Fig. 4-8b. Since the new state must be in equilibrium, we have

$$\begin{aligned} \frac{\partial K_1}{\partial h_1} dh_1 + \frac{\partial K_1}{\partial h_2} dh_2 + \frac{\partial K_1}{\partial \delta} d\delta &= 0, \\ \frac{\partial K_2}{\partial h_1} dh_1 + \frac{\partial K_2}{\partial h_2} dh_2 + \frac{\partial K_2}{\partial \delta} d\delta &= 0, \end{aligned} \quad (4-45)$$

where the partial derivatives are calculated on curves (1) and (2). Note that, unlike the corresponding expressions in Eqs. (4-16), here $\partial K_1/\partial h_1 \neq \partial K_2/\partial h_2$, $\partial K_1/\partial h_2 \neq \partial K_2/\partial h_1$, and $\partial K_1/\partial \delta \neq \partial K_2/\partial \delta$. We now solve (4-45), and obtain

$$\frac{dh_2}{dh_1} = \left[\frac{\partial K_2}{\partial h_1} \frac{\partial K_1}{\partial \delta} - \frac{\partial K_1}{\partial h_1} \frac{\partial K_2}{\partial \delta} \right] / \left[\frac{\partial K_1}{\partial h_2} \frac{\partial K_2}{\partial \delta} - \frac{\partial K_2}{\partial h_2} \frac{\partial K_1}{\partial \delta} \right]. \quad (4-46)$$

At point b_2^* , $dh_2 = 0$ and hence, this state is characterized by

$$\frac{\partial K_2}{\partial h_1} \frac{\partial K_1}{\partial \delta} - \frac{\partial K_1}{\partial h_1} \frac{\partial K_2}{\partial \delta} = 0. \quad (4-47)$$

In the absence of imperfection, (4-47) reduces to the critical condition defined by (4-19).

Equation (4-38) remains valid provided that the quantities are evaluated at $\epsilon = \epsilon_0$, and for h_1 and h_2 , given by the corresponding points on curves (1) and (2). Hence, (4-39) defines the snap-through critical point b_s in Fig. 4-8a.

4.3.6 Summary of Basic Results

For the two interacting cracks shown in Fig. 4-5, the fundamental equilibrium path is defined by

$$h_1 = h_2, K_1 = K_2 = K_c. \quad (4-48)$$

The fundamental equilibrium path will be followed as long as

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} < \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} < 0. \quad (4-49)$$

The first critical point is a stable bifurcation point attained when

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} = \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} < 0. \quad (4-50)$$

All points on the fundamental equilibrium path for which

$$0 > \frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} > \frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} \quad (4-51)$$

holds, are stable bifurcation points. The unstable bifurcation point is first reached when

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} = 0. \quad (4-52)$$

All points on the fundamental equilibrium path for which

$$\frac{\partial K_1}{\partial h_1} = \frac{\partial K_2}{\partial h_2} > 0 \quad (4-53)$$

holds, are unstable bifurcation points. In all cases we always have on the fundamental equilibrium path,

$$\frac{\partial K_1}{\partial h_2} = \frac{\partial K_2}{\partial h_1} < 0; \quad (4-54)$$

this has been proved elsewhere.⁹ On the bifurcated path, on the other hand, $K_1 \neq K_2$ and, instead of (4-49), we have

$$K_1 \frac{\partial K_1}{\partial h_2} = K_2 \frac{\partial K_2}{\partial h_1} \leq 0, \quad (4-55)$$

the equality sign corresponding to the snap-through instability point at which, say, $K_2 = 0$ and hence, crack 2 snaps closed as crack 1 extends by a finite amount.

The introduction of a very small material imperfection results in deviation of the equilibrium path from the fundamental equilibrium path for all finite values of δ , and this deviation increases with increasing δ . Along the new equilibrium paths, the crack with the larger value of the critical stress intensity factor, say crack 2, stops at the critical state defined by

$$\frac{\partial K_2}{\partial h_1} \frac{\partial K_1}{\partial \delta} - \frac{\partial K_1}{\partial h_1} \frac{\partial K_2}{\partial \delta} = 0. \quad (4-56)$$

After this, crack 1 continues to grow with increasing δ , until the snap-through critical state defined by

$$\frac{\partial K_1}{\partial h_2} = 0 \quad (4-57)$$

is reached. An infinitesimal increase in δ then results in a snap closure of crack 2, as crack 1 extends by a finite amount.

4.4 METHODS OF ANALYSIS

As is well known, the stress field at the tip of a crack in a brittle solid has a singularity of the order of $1/\sqrt{r}$ where r measures distance from the tip of the crack. In a numerical calculation, this singularity must be removed by analytical procedures, in order to obtain a reasonable estimate of the value of the stress intensity factor which is actually the strength of the singularity. Finite-element procedures with special elements at the tip of the crack have been developed for this purpose.

The stability analysis on the other hand, requires estimates of the derivatives with respect to the crack lengths of the stress intensity factors at crack tips. These quantities correspond to the strength of singularities which are of the order of $1/r^{3/2}$. For this reason, usual finite-element methods with special elements which are designed to take care of $1/\sqrt{r}$ singularity, are not adequate, and cannot yield accurate enough estimates.

In an effort to overcome the above-mentioned difficulty, we have developed and perfected two methods which directly take care of the corresponding singularities, and therefore, yield very accurate results. These methods are briefly discussed below.

4.4.1 Analytical Method

For two interacting cracks, Fig. 4-9 shows the basic elasticity boundary value problem. For a given temperature profile the tractions on the faces of the cracks are prescribed. Therefore, the basic elasticity problem can easily be reduced to a pair of coupled singular integral equations with the distribution of

Fig. 4-9. (a) Typical unit cell consisting of two interacting edge cracks;
(b) The basic elasticity boundary-value problem.

the dislocation densities along cracks 1 and 2 being two unknown functions. These integral equations can then be solved by a collocation scheme. The method of solution is outlined in Appendix A at the end of this chapter, where the accuracy of the results is also illustrated.

We have developed a computer program based on this method, by means of which we can easily calculate the growth regime pattern of interacting edge cracks. Results presented in Figs. 4-3 and 4-4 are obtained in this manner. These results check very accurately with results obtained by a completely different

method which involves a combined analytical and finite-element scheme, as discussed below.*

4.4.2 Combined Analytical and Numerical Method

Let V denote the region occupied by the elastic body. In the absence of body forces, the field equations are,

$$\begin{aligned}\sigma_{ij,j} &= 0 \quad \text{in } V, \\ \sigma_{ij}n_j &= t_i \quad \text{on } S_t, \\ u_i &= \bar{u}_i \quad \text{on } S_u,\end{aligned}\tag{4-58}$$

where comma followed by an index denotes partial differentiation with respect to the corresponding coordinate, the summation convention on repeated indices is used, and n_j denotes the components of the exterior unit normal to the boundary of the solids; S_t is the part of the boundary on which some components of tractions, t_i , are prescribed, and S_u is the part on which some components of the displacement, $\bar{u}_i$, are given. The unit cell and the corresponding elasticity boundary value problem is shown in Fig. 4-9. The equivalent thermal tractions are

$$\begin{aligned}t_1(x_1, -b/2) &= 0, \\ t_2(x_1, -b/2) &= -\hat{\beta}(T-T_0), \quad 0 \leq x_1 \leq h_1;\end{aligned}\tag{4-59}$$

$$\begin{aligned}t_1(x_1, b/2) &= 0, \\ t_2(x_1, b/2) &= \hat{\beta}(T-T_0), \quad 0 \leq x_1 \leq h_2;\end{aligned}\tag{4-60}$$

where $\hat{\beta} = \alpha E/(1-\nu)$, α being the coefficient of thermal expansion.

According to Ref. 9, we first determine the relation between crack lengths (h_1, h_2) and the penetration depth δ/L in such a manner that the stress intensity factors of each growing crack maintain its critical value. Then we calculate the derivatives of the stress intensity factors, $\partial K_m / \partial h_n$, for the purpose of stability analysis.

*However, our numerical results are not in agreement with those given in Chapter 3.

The dominant terms of stresses around the m^{th} crack tip can be represented in the following form:

$$\sigma_{ij}^{(m)} = a_1^{(m)} \sigma_{ij}^1(r_m, \theta_m) + a_2^{(m)} \sigma_{ij}^2(r_m, \theta_m) + b_1^{(m)} \sigma_{ij}^{-1}(r_m, \theta_m) + b_2^{(m)} \sigma_{ij}^{-2}(r_m, \theta_m) + O(r_m^{3/2}), \quad (4-61)$$

where (r_m, θ_m) is a local polar coordinate whose origin is located at the m^{th} crack tip, and $\sigma_{ij}^k(r_m, \theta_m)$ and $\bar{\sigma}_{ij}^k(r_m, \theta_m)$, $k = 1, 2$, are given in Appendix B. The coefficients $b_1^{(m)}$ and $b_2^{(m)}$ can easily be determined as the Taylor expansion coefficients of applied tractions at the m^{th} crack tip, while $a_1^{(m)}$ and $a_2^{(m)}$ are determined by the method of superposition of analytical and finite-element solutions. The stress intensity factor K_m is expressed as

$$K_m = \frac{3\sqrt{2\pi}}{4} a_1^{(m)}, \quad (4-62)$$

and the load parameter δ is so determined that K_m equals the critical value at the growing crack tip.

If we differentiate Eq. (4-58) with respect to the crack length h_n , the solution of the problem, $\partial \sigma_{ij} / \partial h_n$, is obtained as

$$\frac{\partial \sigma_{ij}^{(m)}}{\partial h_n} = \begin{cases} -\frac{3}{2} a_1^{(n)} \sigma_{ij}^0 + \left(\frac{\partial a_1^{(n)}}{\partial h_n} - \frac{5}{2} a_2^{(n)} \right) \sigma_{ij}^1 + \left(\frac{\partial b_1^{(n)}}{\partial h_n} - 3b_2^{(n)} \right) \bar{\sigma}_{ij}^1 + O(r_m^{1/2}) & \text{for } m = n, \\ \frac{\partial a_1^{(m)}}{\partial h_n} \sigma_{ij}^1 + \frac{\partial b_1^{(m)}}{\partial h_n} \bar{\sigma}_{ij}^1 + O(r_m^{1/2}) & \text{for } m \neq n, \end{cases} \quad (4-63)$$

near the m^{th} crack tip, where σ_{ij}^0 has a singularity of $O(r_m^{-3/2})$ as shown in Appendix B. Here we define new variables,

$$\begin{aligned} \sigma_{nij} &= \frac{\partial \sigma_{ij}}{\partial h_n} + \frac{3}{2} a_1^{(n)} \sigma_{ij}^0, \\ u_{ni} &= \frac{\partial u_i}{\partial h_n} + \frac{3}{2} a_1^{(n)} u_i^0, \quad n = 1, 2, \end{aligned} \quad (4-64)$$

where u_i^0 represents the displacement field corresponding to the stress field σ_{ij}^0 .

These variables are chosen in such a manner that the singularity of the order $r_n^{-3/2}$ is removed from the derivative problem. With the aid of (4-64) and upon

differentiation of (4-58) with respect to h_n , we write

$$\begin{aligned}\sigma_{nij,j} - \frac{3}{2} a_1^{(n)} \sigma_{ij,j}^0 &= 0 && \text{in } V, \\ \sigma_{nij} n_j &= \frac{\partial t_i}{\partial h_n} + \frac{3}{2} a_1^{(n)} \sigma_{ij}^0 n_j && \text{on } S_t, \\ u_{ni} &= \frac{\partial \bar{u}_i}{\partial h_n} + \frac{3}{2} a_1^{(n)} u_i^0 && \text{on } S_u, \quad n = 1, 2.\end{aligned}\quad (4-65)$$

The stress distribution around the m^{th} crack tip can be expressed in the same manner as (4-61); here, this becomes

$$\sigma_{nij}^{(m)} = \alpha_{n1}^{(m)} \sigma_{ij}^1(r_m, \theta_m) + \alpha_{n2}^{(m)} \sigma_{ij}^2(r_m, \theta_m) = \left(\frac{\partial b_1^{(m)}}{\partial h_n} - 3b_2^{(m)} \right) \sigma_{ij}^1 + O(r_m). \quad (4-66)$$

The unknown coefficients $\alpha_{nk}^{(m)}$, $k, m, n = 1, 2$, can be determined by the same method used for the analysis of stress intensity factors. Comparing (4-63) and (4-66), the derivatives of the stress intensity factors are obtained as

$$\frac{\partial K_m}{\partial h_n} = \begin{cases} \frac{3\sqrt{2\pi}}{4} (\alpha_{n1}^{(n)} + \frac{5}{2} a_2^{(n)}) & \text{for } m = n, \\ \frac{3\sqrt{2\pi}}{4} \alpha_{n1}^{(m)} & \text{for } m \neq n. \end{cases} \quad (4-67)$$

In Fig. 4-10A the basic steps involved in the finite-element implementation are listed. Table 4-II compares the corresponding numerical results with those obtained with the aid of the analytical procedure of Sec. 4.4.1; see Fig. 4-10B for a typical mesh pattern. As is seen, both methods yield extremely accurate results. However, the combined analytical and finite-element procedure is an order of magnitude more efficient.

Fig. 4-10A. The finite element procedure for the analysis of the stress intensity factor.

Fig. 4-10B. Mesh subdivision for two interacting cracks in a strip of width L , where $b/L = 0.16$, $h_1/L = 0.70$, and $h_2/L = 0.58$.

TABLE 4-II

COMPARISON OF PRESENT NUMERICAL RESULTS WITH THOSE OF REF. 9;

$$h_1/b = h_2/b = 1.3229, \delta/b = 1.5.$$

	$K_1/\sqrt{2\pi b} = K_2/\sqrt{2\pi b}$	$\sqrt{\frac{b}{2\pi}} \frac{\partial K_1}{\partial h_2} = \sqrt{\frac{b}{2\pi}} \frac{\partial K_2}{\partial h_1}$	$\sqrt{\frac{b}{2\pi}} \frac{\partial K_1}{\partial h_1} = \sqrt{\frac{b}{2\pi}} \frac{\partial K_2}{\partial h_2}$
Ref. 9	0.1500	-0.3031	-0.4292
Present solution ($h_1/L=h_2/L=0.209$)	0.1509	-0.3096	-0.4403

4.5 APPLICATION TO HEAT EXTRACTION FROM HOT DRY ROCK

The theory and analytical methods which have been developed provide tools to estimate important quantities which pertain to the process of heat extraction from hot dry rock masses. In this section we discuss some of the numerical results.

4.5.1 Universal Stability Charts

For parallel edge cracks, dimensionless stability charts can be developed which provide estimates for the entire crack growth regime. These charts will only depend on the temperature profile and on the degree of heterogeneity in the material, i.e. imperfection. Figures 4-3 and 4-4 are charts of this kind for temperature profiles (4-1) and (4-2), respectively. In these figures the dimensionless parameters are

$$\Delta = \delta/b, \quad a = h/b, \quad N = K_c / [\sqrt{2\pi} \hat{\beta} T_0 b], \quad (4-68)$$

where as before, b is the crack spacing and $\hat{\beta}$ is given by (4-8).

To use these charts, we first estimate the initial minimum crack spacing from Eq. (4-6), then we calculate the initial value of N . With this value of N , we enter the chart and find the first point on the boundary between stable and unstable regions. The abscissa of this point gives the corresponding Δ , and from the chart we also obtain the corresponding a . In Figs. 4-3 and 4-4, the procedure is illustrated by taking the initial value of $N = 0.1$. At point Q_1 the crack spacing doubles, and this gives point Q_2 whose ordinate is $0.1/\sqrt{2}$. Extending Q_2 horizontally, the stability boundary is intersected at point Q_3 . At this

stage the crack spacing doubles again, and this yields point Q_4 with ordinate 0.1/2. The procedure can then be continued.

In Fig. 4-11 we have added to Fig. 4-3 new curves which correspond to the stable bifurcation points with 0, 5, 10, and 20% imperfections. These curves can be used in exactly the same manner as those of Figs. 4-3 and 4-4.

In actual applications, it is adequate to use the curve corresponding to the stable bifurcation points with 0 imperfection.

4.5.2 Crack Opening Displacement

Of considerable importance is to estimate the opening displacement of the thermally induced secondary cracks, and compare that with the opening of the

Fig. 4-11. Stability chart for temperature profile (4-1); ϵ denotes the percentage by which K_c of one crack exceeds that of the other crack.

primary crack. Our computer program can do this for any combination of material properties and temperature profiles.

A useful result would be to express everything in terms of dimensionless parameters, so that the results can be used for different material properties.

To this end we introduce the dimensionless crack opening displacement,

$$\Delta u^* = \Delta u E / \beta T_0 b(1-\nu^2), \quad (4-69)$$

and, for temperature profiles (4-1) and (4-2), obtain the results given in Tables 4-III and 4-IV, respectively. In these tables, y/h measures distance along the length of the crack from the free surface, with total crack depth being the unit of length.

TABLE 4-III

DIMENSIONLESS CRACK OPENING DISPLACEMENT Δu^* AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-1); ALL QUANTITIES ARE DIMENSIONLESS

Δ	a	N	Δu^*			
			$y/h = 0$	$y/h=0.25$	$y/h = 0.5$	$y/h=0.75$
0.33	0.256	0.264	0.988	0.862	0.724	0.531
0.50	0.398	0.251	1.106	0.978	0.841	0.635
1.00	0.835	0.212	1.056	1.012	0.967	0.812
1.973	1.718	0.150	1.000	0.996	1.005	0.942
2.970	2.652	0.106	1.000	0.994	0.998	0.972
5.00	4.626	0.052	1.000	0.996	0.996	0.969
10.00	9.625	0.013	1.000	0.998	0.998	0.931

TABLE 4-IV

DIMENSIONLESS CRACK OPENING DISPLACEMENT Δu^* AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-2); ALL QUANTITIES ARE DIMENSIONLESS

Δ	a	N	Δu^*			
			$y/h=0.0$	$y/h=0.25$	$y/h=0.50$	$y/h=0.75$
0.488	0.186	0.150	0.672	0.554	0.425	0.279
0.787	0.383	0.106	0.882	0.699	0.504	0.307
0.955	0.536	0.075	0.920	0.704	0.482	0.275
1.350	0.970	0.031	0.947	0.658	0.386	0.184
1.500	1.220	0.022	0.953	0.620	0.326	0.135
1.840	1.680	0.013	0.963	0.579	0.267	0.093
2.000	1.978	0.009	0.965	0.548	0.228	0.069

In Tables 4-V to 4-VIII, we have used the above dimensionless quantities, together with the values of the parameters given by (4-VII), in order to obtain dimensional numerical values. For the sake of comparison, we have also given in Table 4-IX, the crack opening for a single penny-shaped crack in an infinite medium, held open by fluid pressure in such a manner that the stress intensity is just at the critical value.

TABLE 4-V

CRACK OPENING DISPLACEMENT AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-1); $\gamma = 100 \text{ J/m}^2$ AND $T_0 = 100^\circ\text{C}$; OTHER MATERIAL PARAMETERS ARE GIVEN IN EQ. (4-7).

δ (m)	h (m)	b (m)	crack opening (mm)			
			$y/h=0$	$y/h=0.25$	$y/h=0.50$	$y/h=0.75$
0.0034	0.0026	0.0104	0.01	0.01	0.01	0.01
0.0062	0.0049	0.0124	0.01	0.01	0.01	0.01
0.0174	0.0145	0.0174	0.02	0.02	0.02	0.01
0.0684	0.0596	0.0347	0.03	0.03	0.03	0.03
0.2062	0.1841	0.0694	0.07	0.07	0.07	0.07
1.4441	1.3362	0.2888	0.28	0.28	0.28	0.27
43.50	41.86	4.35	4.24	4.24	4.24	3.95

TABLE 4-VI

CRACK OPENING DISPLACEMENT AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-2); $\gamma = 1000 \text{ J/m}^2$ AND $T_0 = 100^\circ\text{C}$; OTHER MATERIAL PARAMETERS ARE GIVEN IN EQ. (4-7).

δ (m)	h (m)	b (m)	crack opening (mm)			
			$y/h=0$	$y/h=0.25$	$y/h=0.50$	$y/h=0.75$
0.034	0.026	0.104	0.100	0.087	0.073	0.054
0.062	0.049	0.124	0.134	0.118	0.102	0.077
0.174	0.145	0.174	0.179	0.172	0.164	0.138
0.684	0.596	0.347	0.339	0.337	0.340	0.319
2.062	1.841	0.694	0.677	0.673	0.676	0.658
14.441	13.362	2.888	2.819	2.807	2.808	2.731

TABLE 4-VII

CRACK OPENING DISPLACEMENT AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-2); $\gamma = 100 \text{ J/m}^2$ and $T_0 = 100^\circ\text{C}$; OTHER MATERIAL PARAMETERS ARE GIVEN IN EQ. (4-7).

δ (m)	h (m)	b (m)	crack opening (mm)			
			$y/h=0$	$y/h=0.25$	$y/h=0.5$	$y/h=0.75$
0.034	0.026	0.104	0.100	0.087	0.073	0.054
0.062	0.049	0.124	0.134	0.118	0.102	0.077
0.174	0.145	0.174	0.179	0.172	0.164	0.138
0.684	0.596	0.347	0.339	0.337	0.340	0.319
2.062	1.841	0.694	0.677	0.673	0.676	0.658
14.441	13.362	2.888	2.819	2.807	2.808	2.731

TABLE 4-VIII

CRACK OPENING DISPLACEMENT AT STABLE BIFURCATION STATES FOR TEMPERATURE PROFILE (4-2); $\gamma = 1000 \text{ J/m}^2$ and $T_0 = 100^\circ\text{C}$; OTHER MATERIAL PARAMETERS ARE GIVEN IN EQ. (4-7).

δ (m)	h (m)	b (m)	crack opening (mm)			
			$y/h=0$	$y/h=0.25$	$y/h=0.5$	$y/h=0.75$
0.169	0.064	0.347	0.227	0.187	0.144	0.094
0.546	0.266	0.694	0.597	0.473	0.341	0.208
1.326	0.744	1.388	1.246	0.954	0.652	0.372
10.830	7.782	8.023	7.415	5.152	3.025	1.438
24.650	20.049	16.434	15.288	9.944	5.232	2.162
83.738	76.457	45.510	42.761	25.704	11.850	4.135

4.6 THERMALLY INDUCED ROCK FRAGMENTATION

When the two horizontal principal tectonic stresses have values close to each other, then one expects that secondary thermally induced cracks, if they do form, should extend into the rock perpendicular to the face of the primary crack, forming a system of essentially parallel edge cracks. On the other hand, if the two horizontal principal tectonic stresses are far apart in value, or if there are strong local heterogeneity and anisotropy, then the thermally induced cracks may tend to curve out of their own plane, joining adjacent cracks, and causing rock fragmentation.

TABLE 4-IX

CRACK OPENING DISPLACEMENT FOR A SINGLE PENNY-SHAPED CRACK IN AN INFINITE MEDIUM;
 $E = 4 \times 10^{10} \text{ N/m}^2$, $\nu = 0.22$, AND a IS THE RADIUS OF THE CRACK

Radius (m)	Crack opening at $K=K_c$ for $\gamma = 100 \text{ J/m}^2$ (mm)				Crack opening at $K=K_c$ for $\gamma = 1000 \text{ J/m}^2$ (mm)			
	$r/a=0.0$	$r/a=0.25$	$r/a=0.50$	$r/a=0.75$	$r/a=0.0$	$r/a=0.25$	$r/a=0.50$	$r/a=0.75$
10	0.27	0.26	0.23	0.17	0.85	0.82	0.73	0.56
20	0.38	0.37	0.33	0.25	1.20	1.16	1.04	0.79
30	0.46	0.45	0.40	0.31	1.47	1.43	1.27	0.97
50	0.60	0.58	0.52	0.40	1.90	1.84	1.65	1.26
100	0.85	0.82	0.73	0.56	2.69	2.61	2.33	1.78
200	1.20	1.16	1.04	0.79	3.81	3.69	3.30	2.52
300	1.47	1.43	1.28	0.97	4.67	4.52	4.04	3.09
400	1.70	1.65	1.47	1.12	5.39	5.22	4.67	3.56
600	2.09	2.02	1.81	1.38	6.60	6.39	5.72	4.37
800	2.41	2.33	2.09	1.59	7.62	7.38	6.60	5.04
1000	2.69	2.61	2.33	1.78	8.53	8.25	7.38	5.64

In Fig. 4-12 it is assumed that the principal stress σ_2 exceeds substantially the principal stress σ_1 , in such a manner that the thermally induced cracks tend to curve away from their collinear extension.

A significant problem pertaining to heat extraction from hot rock masses is to estimate the size of these fragments. A partial answer to this question is provided by our estimate of the minimum initial crack spacing [see Sec. 4-2, Table 4-I and Eqs. (4-6) and (4-9)], and by the stability theory outlined in Sec. 4-3. From the estimate of the initial crack spacing we obtain the approximate initial size of the fragments. Then if a pattern similar to Fig. 4-12 develops, with the maximum depth before two adjacent cracks join being given by the critical value of the depth at which the configuration of parallel edge cracks becomes unstable, one can obtain an estimate of the depth of the fragments which should be of the order of the crack spacing.

The above comments are intuitive, and must be supported by theoretical and experimental work. Little theoretical work on crack curving, however, exists; relevant papers are listed in the references at the end of this chapter. The latest work in this area is due to Karihaloo and the present writers,^{25,51} a summary of which is presented below.

Fig. 4-12. Possible crack branching when $\sigma_2 > \sigma_1$.

4.6.1 Crack Curving or Kinking

Here we derive the conditions necessary for a slight deviation from straightness of a crack in the presence of an in-plane stress and under mixed-mode loading, the deviation occurring with a finite initial slope (crack kinking) or without it (crack curving). The influence of the in-plane stress on the stability of the crack growth is also examined.

The problem of slightly out-of-plane quasi-static growth of a semi-infinite straight crack under mixed-mode loading was originally studied by Bani-chuk,²⁰ and, more recently, reconsidered by Cotterell and Rice.²⁴ The latter authors also examined the stability of the crack growth under the influence of a non-singular stress acting parallel to the crack plane. In both Ref. 20 and Ref. 24, it was shown that the deviation of the crack that always occurred at a (small) finite angle from the initial straight path was a result of the Mode II component of the external loading. It was further shown in Ref. 24 that the straight path is stable for the in-plane stress $T < 0$, but unstable otherwise.

There is, however, experimental evidence pointing to the tendency of a straight crack to deviate from its plane even under strict Mode I loading. This experimental observation cannot be explained by the theoretical results obtained in Refs. 20 and 24. The present work is an attempt at explaining theoretically the observed behavior. To this end, allowance has been made for second-order effects in crack geometry (crack curvature).

The problem of a slightly curved crack is, in the limit as the portion of the loaded crack surfaces shrinks to the tip of the straight crack, analogous to that for a semi-infinite straight crack. In such a situation it is expedient²⁰ to rewrite the boundary conditions along the straight crack and its (small) curved extension through their counterparts along the tangent to the extended crack tip whose coordinates are $\{x, y(x)\}$ as measured from the tip of the initial straight crack, see Fig. 4-13.

Further, let $h(x)$ be the perpendicular distance between this tangent and any point $\{x, y(x)\}$ on the straight crack and its curved extension ($y(x) \equiv 0, x \leq 0$).

Fig. 4-13. Geometry of the straight crack and its out-of-plane extension, and the external loading.

To within the terms of first order of smallness in $y(x)$, the stress intensity factors K_I and K_{II} at the tip of the curved crack, in terms of the normal traction, T_n , and the tangential traction, T_s , at the point $\{x, y(x)\}$, are²⁴

$$\begin{aligned} \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} &= \left(\frac{2}{\pi L}\right)^{1/2} \int_0^L \left[\begin{Bmatrix} T_n - 2h'(x)T_s - h(x)T_s' \\ T_s - h(x)T_n' \end{Bmatrix} \right] \left(\frac{L-x}{x}\right)^{1/2} \\ &\quad + \frac{y'(0)}{2} \begin{Bmatrix} T_s \\ T_n \end{Bmatrix} \left(\frac{x}{L-x}\right)^{1/2} dx, \end{aligned} \quad (4-70)$$

where $y'(0)$ is the initial angle of deviation from straightness, and L is the length of the straight crack and the horizontal projection of the curved extension (ℓ).

In order to include the second-order terms in crack geometry (curvature terms) in the spirit of Hertzian contact problems, we assume that the equation to the curved crack extension can be expanded in Taylor's series about the tip

of the original straight crack and retain only the first three terms in the expansion

$$y(x) = y(0) + y'(0)x + y''(0) \frac{x^2}{2}. \quad (4-71)$$

We further assume that the (small) curvature is constant ($y''(0) = 1/R$). Note that $y(0) \equiv 0$, and that for smooth crack curving $y'(0) = 0$, while for crack kinking $y'(0)$ is finite, but small.

To further simplify the expressions (4-70) it is convenient to express the tractions in the direction of x (horizontal) and y (vertical) axes, bearing in mind that the former is rotated by $y'(\ell)$ relative to the tangent to the curved crack tip. Moreover, as the initial growth of the two crack tip occurs independently, we can let $L \rightarrow \infty$.

The tractions on the crack extension from the origin to a point $\{x, y(x)\}$ can be derived from the stress field existing along the prolongation ($x > 0$) of the straight crack (Irwin-Williams solution) by allowing for the variation in the stress field due to crack deviation from straightness.

The resulting expression for the stress intensity factor K_{II} is

$$K_{II} = \sqrt{\frac{2}{\pi}} \int_0^\ell \frac{T_x + y'(\ell) T_y}{(\ell - x)^{1/2}} dx + \sqrt{\frac{2}{\pi}} \int_0^\ell \frac{h(x)}{2(\ell - x)^{3/2}} [T_y + \{h'(x) - h'(\ell) T_x\}] dx. \quad (4-72)$$

On substituting (4-71) into (4-72) and performing the indicated integration the following expression for K_{II} results:

$$K_{II} = (k_{II} + \frac{\ell}{2R} k_I - \frac{4}{3} \frac{\ell}{R} \sqrt{\frac{2\ell}{\pi}} T) + \left(-\frac{k_I}{2} - 2\sqrt{\frac{2\ell}{\pi}} T - \frac{15}{16} k_{II} \frac{\ell}{R} \right) y'(0), \quad (4-73)$$

in which only the terms to within the second-order of smallness in extended crack geometry have been retained. Here, k_I and k_{II} are the stress intensities at the tip of the original straight crack and T corresponds to the non-singular stress

term appearing in the Irwin-Williams solution and acting parallel to the crack plane at its tip.

As in Refs. 20 and 24, we assume that the path taken by the crack is that along which the local stress field is of Mode I type (i.e., $K_{II} = 0$ at the advancing tip). We are, therefore, able to specify the initial angle at which the crack deviates from its straight path

$$y'(0) = - \left(\frac{\ell}{R} + 2 \frac{k_{II}}{k_I} - \frac{8}{3} \frac{\ell}{R} \sqrt{\frac{2\ell}{\pi}} \frac{T}{k_I} \right) / \left(1 - 4 \sqrt{\frac{2\ell}{\pi}} \frac{T}{k_I} - \frac{15}{8} \frac{k_{II}}{k_I} \frac{\ell}{R} \right). \quad (4-74)$$

If, on the other hand, it is assumed that the crack deviates gradually from straightness, i.e. $y'(0) = 0$, the condition $K_{II} = 0$ leads to

$$\frac{\ell}{R} = - 2 \frac{k_{II}}{k_I} / \left(1 - \frac{8}{3} \sqrt{\frac{2\ell}{\pi}} T \right). \quad (4-75)$$

By including second-order geometric effects in the spirit of Hertzian contact problems within the framework of linear elastic fracture mechanics, we have established conditions necessary for a slight deviation of a quasi-statically growing crack from its straight path, the deviation occurring either through gradual curving (4-75) or through kinking (4-74). An examination of the necessary conditions (4-74) and (4-75) enables us to draw the following conclusions:

(i) Crack kinking is possible even in the absence of a Mode II type loading ($k_{II} = 0$). This contradicts the conclusion arrived at in Ref. 24, but seems to be borne out by experimental evidence.

(ii) The presence of a Mode II type loading component can cause the crack to deviate by smooth curving.

(iii) It is possible to avoid crack kinking, but not crack curving, in the presence of Mode II loading by applying a large in-plane tensile stress

$$T = \frac{3}{4} \sqrt{\frac{\pi}{2\ell}} \frac{R}{\ell} k_{II} + \frac{3}{8} \sqrt{\frac{\pi}{2\ell}} k_I. \quad (4-76)$$

It is interesting to note that such a procedure is, in fact, attempted in practice.

(iv) Both in crack kinking and in crack curving situations, the crack growth is unstable under $T > 0$, but stable otherwise. An identical conclusion was arrived at by the first-order analysis adopted in Ref. 24.

(v) Since a large in-plane tensile stress T has to be used to prevent crack kinking when $k_{II} \neq 0$, there is a need to exercise caution in the selection of its magnitude in order to simultaneously avoid unstable crack growth.

PART II

HYDRAULIC FRACTURING OF CONNECTED PENNY-SHAPED CRACKS

4.1 INTRODUCTION

In order to generate more surface area and improve the efficiency of an hydraulically fractured rock system, one proposal has been to grow a second coaxial penny-shaped crack to connect with the initial crack. The second crack would double the effective surface area; the idea is attractive since the cracks can probably be generated from a single drill pipe. The objective of the present analysis is to analyze such a multiple crack problem in order to establish parameter limits for the design of considered geothermal systems.

The problem will be slightly simplified in order to reduce the amount of analysis, since we are interested in the general features of the results. The two crack system will be approximated by an infinite array of co-axial penny-shaped cracks equally spaced at distance b apart. The considered cracks will have alternating radii, a_1 and a_2 , as shown in Fig. 4-14. We use this model to show the effects of interaction between cracks 1 and 2; the results can also be compared for certain cases with the single crack, when all cracks have equal lengths but the interaction effects are not considered.

4.2 DERIVATION OF BASIC EQUATIONS

We consider an array of cracks in which alternate coaxial cracks will have a radius of a_1 , a_2 , respectively (see Fig. 4-14 for the geometry and coordinate system). The boundary conditions for the considered cracks are as follows:

$$\sigma_{zz}(r,0) = f(r) \quad , \quad 0 \leq r \leq a_1 \quad , \quad (4-77)$$

$$\sigma_{rz}(r,0) = 0 \quad , \quad 0 \leq r \leq \infty \quad (4-78)$$

Fig. 4-14. Geometry and coordinate system for an array of penny-shaped cracks.

$$\sigma_{zz}(r,b) = f(r), \quad 0 \leq r \leq a_2 \quad (4-79)$$

$$\sigma_{rz}(r,b) = 0, \quad 0 \leq r < \infty. \quad (4-80)$$

We note that these boundary conditions only approximate the case for two parallel cracks in which both normal and shear stresses will be found to be present on the crack surfaces; however, it is expected that the trend for the results will be realistically described by the present model.

Following Collins⁴⁸ and the displacement $\underline{U}$ at any point is given by

$$\begin{aligned} \underline{U} = & (3-4\nu)k \sum_{n=-\infty}^{\infty} (\phi_{1n} + \phi_{2n}) - \sum_{n=-\infty}^{\infty} \{ (z-2nb)\nabla\phi_{1n} + (z-2nb-b)\nabla\phi_{2n} \} \\ & + \sum_{n=-\infty}^{\infty} (\nabla\psi_{1n} + \nabla\psi_{2n}), \end{aligned} \quad (4-81)$$

$$\frac{\partial \psi_{jn}}{\partial z} = - (1 - 2\nu) \phi_{jn}, \quad j = 1, 2, \quad (4-82)$$

where ν is Poisson's ratio and ϕ_{jn} , ψ_{jn} are axially symmetric harmonic functions which can be represented as

$$\phi_{jn}(r,z) = \frac{1}{2i} \int_{-a_j}^{a_j} \frac{g_j(t)dt}{[r^2 + (z-2nb+it)^2]^{1/2}}, \quad g_j(-t) = -g_j(t), \quad j = 1, 2 \quad (4-83)$$

$$\psi_{jn}(r,z) = -\frac{(1-2\nu)}{2i} \int_{-a_j}^{a_j} g_j(t) \ln[\{r^2 + (z-2nb+it)^2\}^{1/2} + z-2nb+it] dt \quad (4-84)$$

$j = 1, 2$

It can be easily shown upon calculation of the shear stresses that the representations (4-81), (4-83), (4-84) automatically satisfy Eqs. (4-78), (4-80). The remaining boundary conditions, (4-77), (4-79), in terms of the displacement potentials are

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{\partial \phi_{1n}}{\partial z} + 2nb \frac{\partial^2 \phi_{1n}}{\partial z^2} + \frac{\partial \phi_{2n}}{\partial z} + (2n+1)b \frac{\partial^2 \phi_{2n}}{\partial z^2} \right\} = \frac{1}{2\mu} f(r), \quad 0 \leq r \leq a_1, \quad z = 0 \quad (4-85)$$

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{\partial \phi_{1n}}{\partial z} + (2n-1)b \frac{\partial^2 \phi_{1n}}{\partial z^2} + \frac{\partial \phi_{2n}}{\partial z} + 2nb \frac{\partial^2 \phi_{2n}}{\partial z^2} \right\} = \frac{1}{2\mu} f(r), \quad 0 \leq r \leq a_2, \quad z = 0. \quad (4-86)$$

Equations (4-85), (4-86), when the techniques of Collins are applied, may be reduced to the following pair of Fredholm integral equations in terms of the unknown functions, $g_j(t)$, $j = 1, 2$:

$$g_1(t) - \int_{-1}^1 g_1(s) M_1(R_1 t, R_1 s) R_1 ds - \int_{-1}^1 g_2(s) M_2(R_1 t, R_2 s) R_2 ds = R_1 b F_1(t), \quad -1 \leq t \leq 1 \quad (4-87)$$

$$g_2(t) - \int_{-1}^1 g_2(s) M_1(R_2 t, R_2 s) R_2 ds - \int_{-1}^1 g_1(s) M_2(R_2 t, R_1 s) R_1 ds = R_2 b F_2(t), \quad -1 \leq t \leq 1, \quad (4-88)$$

where

$$F_1(t) = \frac{1}{\pi\mu} \int_0^t \frac{\rho F(R_1 b \rho) d\rho}{(t^2 - \rho^2)^{1/2}}, \quad F_2(t) = \frac{1}{\pi\mu} \int_0^t \frac{\rho F(R_2 b \rho) d\rho}{(t^2 - \rho^2)^{1/2}} \quad (4-89)$$

$$M_1(t, s) = \frac{1}{\pi} \int_0^\infty \frac{e^{-\xi}}{\text{sh}(\xi)} [1 + \xi + \xi \coth[\xi]] [1 - \cos(\xi(t-s))] d\xi \quad (4-90)$$

$$M_2(t, s) = \frac{1}{\pi} \int_0^\infty \frac{1}{\text{sh}(\xi)} [1 + \xi \coth[\xi]] [1 - \cos(\xi(t-s))] d\xi. \quad (4-91)$$

The kernels $M_j(t, s)$, $j = 1, 2$ have had their infinite series replaced by their sums which can be expressed in a closed form, and the length scale has been normalized by letting $R_j = a_j/b$.

Since the cracks are connected together by means of a single source of liquid, the loading on the crack faces is assumed to be uniform pressure given by $\sigma_{zz} = -p$; hence we have

$$F_1(t) = -\frac{pt}{\pi\eta}, \quad F_2(t) = -\frac{pt}{\rho\eta}. \quad (4-92)$$

If we define

$$g_j = pb_j(t)/\rho\mu, \quad j = 1, 2 \quad (4-93)$$

then Eqs. (4-92), (4-93) may be written as follows:

$$\frac{1}{R_2} B_1(t) - \int_{-1}^1 B_1(s) M_1(R_1 t, R_1 s) \frac{R_1}{R_2} ds - \int_{-1}^1 B_2(s) M_2(R_1 t, R_2 s) ds = -\frac{R_1}{R_2} t, \quad -1 \leq t \leq 1 \quad (4-94)$$

$$\frac{1}{R_1} B_2(t) - \int_{-1}^1 B_2(s) M_1(R_2 t, R_2 s) \frac{R_2}{R_1} ds - \int_{-1}^1 B_1(s) M_2(R_2 t, R_1 s) ds = -\frac{R_2}{R_1} t, \quad -1 \leq t \leq 1. \quad (4-95)$$

The Eqs. (4-94), (4-95) are prepared in a form such that numerical calculations can be easily performed. Once they are solved for $B_j(s)$, then the physical quantities may be computed. The dimensionless stress intensity factors are defined as follows:

$$N_j = -\frac{2p}{\mu} \frac{B_j(1)}{\sqrt{\pi R_j}}, \quad j = 1, 2. \quad (4-96)$$

The derivatives of N_1, N_2 with respect to R_1 are in the form

$$\frac{\partial N_j}{\partial R_1} = -\frac{N_j}{2R_1} - \frac{2p}{\mu} \frac{D_j(1)}{\sqrt{\pi R_1}} + \frac{N_j}{p} \frac{\partial p}{\partial R_1}, \quad D_j(t) = \frac{\partial B_j(t)}{\partial R_1}, \quad j = 1, 2 \quad (4-97)$$

where the unknown functions D_1, D_2 have to satisfy the following equations:

$$\begin{aligned} \frac{1}{R_2} D_1(t) - \int_{-1}^1 D_1(s) M_1(R_1 t, R_1 s) \frac{R_1}{R_2} ds - \int_{-1}^1 D_2(s) M_2(R_1 t, R_2 s) ds \\ = -\frac{t}{R_2} + \frac{1}{R_2} \int_{-1}^1 B_1(s) L_1(R_1 t, R_1 s) ds + \frac{1}{R_1} \int_{-1}^1 B_2(s) \frac{1}{2}(R_1 t, R_2 s) ds, \\ -1 \leq t \leq 1, \end{aligned} \quad (4-98)$$

$$\frac{1}{R_1} D_2(t) - \int_{-1}^1 D_2(s) M_1(R_2 t, R_2 s) \frac{R_2}{R_1} ds - \int_{-1}^1 D_1(s) M_2(R_2 t, R_1 s) ds$$

$$= \frac{1}{R_1} \int_{-1}^1 B_1(s) \frac{1}{3} (R_2 t, R_1 s) ds, \quad -1 \leq t \leq 1. \quad (4-99)$$

The functions L_1 , L_2 , and L_3 are given as

$$L_1(t, s) = \frac{1}{\pi} \int_0^\infty \frac{e^{-\xi}}{\text{sh}(\xi)} [1 + \xi + \xi \coth(\xi)] \{1 - \cos(\xi(t-s)) + \xi(t-s) \sin(\xi(t-s))\} d\xi, \quad (4-100)$$

$$L_2(t, s) = \frac{1}{\pi} \int_0^\infty \frac{1}{\text{sh}(\xi)} [1 + \xi \coth(\xi)] \xi t \sin(\xi(t-s)) d\xi, \quad (4-101)$$

$$L_3(t, s) = \frac{1}{\pi} \int_0^\infty \frac{1}{\text{sh}(\xi)} [1 + \xi \cosh(\xi)] \{1 - \cos(\xi(t-s)) - \xi s \sin(\xi(t-s))\} d\xi. \quad (4-102)$$

4.3 EVALUATION OF PHYSICAL QUANTITIES

The crack opening displacements may be calculated from Eq. (4-87) and appropriate evaluation of the potential functions as

$$w_j(bp) = - \frac{\Delta pb}{\pi \mu} (1 - \nu) \int_0^1 \frac{B_j(t) dt}{(t^2 - \rho^2)^{1/2}}, \quad 0 \leq \rho \leq 1, \quad j = 1, 2. \quad (4-103)$$

The volume of liquid in each crack is computed from

$$\bar{V}_j = \int_0^{a_j} w_j(r) 2\pi r dr = - \frac{8pb^3}{\mu} (1 - \nu) R_j^2 \int_0^1 t B_j(t) dt, \quad j = 1, 2. \quad (4-104)$$

Let V_1, V_2 be the dimensionless volumes defined as follows:

$$V_j = \frac{\bar{V}_j}{b^3(1-\nu)} = \frac{p}{\mu} V_j^*, \quad V_j^* = 8 R_j^2 \int_{-1}^1 t B_j(t) dt. \quad (4-105)$$

The term $p^{-1} \partial p / \partial R_1$ is computed from the condition that, at any instant

$$\partial(V_1 + V_2) / \partial R_1 = 0 \quad (4-106)$$

which leads to

$$\frac{1}{p} \frac{\partial p}{\partial R_1} = - \frac{1}{(V_1^* + V_2^*)} \left\{ \frac{2V_1^*}{R_1} + 8R_1^2 \int_0^1 D_1(t) dt + 8R_2^2 \int_0^1 D_2(t) dt \right\}. \quad (4-107)$$

the derivatives of N_1, N_2 with respect to R_2 can be computed by the same method as used to calculate those with respect to R_1 .

The numerical analysis required for the solution of these equations and for the determination of the physical quantities is straightforward (see e.g. Stroud and Secrest⁴⁹), where Gaussian quadrature formulae is used to replace integrals by their sum to obtain matrices for the determination of the unknown quantities.

4.4 GROWTH AND STABILITY OF COAXIAL PENNY-SHAPED CRACKS

In previous articles by the authors and their coworkers the growth and stability of interacting cracks have been investigated. During the process of crack growth with all cracks growing equally a stable bifurcation point may be reached, when

$$\frac{\partial N_1}{\partial R_1} = \frac{\partial N_2}{\partial R_1} < \frac{\partial N_1}{\partial R_2} = \frac{\partial N_2}{\partial R_2}. \quad (4-108)$$

In this state an increase in load will cause growth in every other crack with no further growth in the remaining cracks. An unstable critical point may also be reached when

$$\frac{\partial N_1}{\partial R_1} = 0 \quad \text{or} \quad \frac{\partial N_2}{\partial R_2} = 0. \quad (4-109)$$

At this point crack growth in every other crack can occur with no increase in the load.

The stability of an array of coaxial penny-shaped cracks can be discussed by considering the results shown in Fig. 4-15 which shows curves that give the values for the derivatives of the stress intensity factors as a function of the ratio of crack radius to crack spacing, R . The unstable critical point is reached when $R_1 = R_2 = 0.827$. Some of the data for this figure is also given in Table 4-X. For this table the volume V and stress intensity factors N_1, N_2 are given with respect to p/μ ; therefore, with $N_1 = N_2 = N_c$ the pressure p/μ would vary.

Fig. 4-15. Curves showing volume, stress intensity factor, and derivatives of stress intensity factor versus $R = R_1 = R_2$.

TABLE 4- X
GENERAL DATA

$R_1 = R_2$	$\frac{\mu}{p} \frac{V}{1-v}$	$\frac{\mu}{p} (N_1 = N_2)$	$\frac{\mu}{p} \left(\frac{\partial N_1}{\partial R_1} = \frac{\partial N_2}{\partial R_2} \right)$	$\frac{\mu}{p} \left(\frac{\partial N_1}{\partial R_2} = \frac{\partial N_2}{\partial R_1} \right)$	
0.2	0.041	0.491	-2.37	-3.75	a = radius
0.4	0.290	0.618	-1.20	-2.75	b = spacing
0.6	0.808	0.655	-0.48	-2.05	R = radius/b
0.8	1.581	0.668	-0.04	-1.81	V = Total Volume/b ³
*0.82692	1.704	0.670	0.00	-1.79	N = K/μ√b
					K = 11m√2π(r-a)σ _{zz}

TABLE 4-XI
POST CRITICAL BEHAVIOR ($R_1 = 0.827$)

p/μ	R_2	$V/(1-\nu)$	$V_1/(1-\nu)$	$V_2/(1-\nu)$	N_1	N_2
1.493	0.827	2.545	1.272	1.272	1	1
1.241	1.000	2.799	0.716	2.083	0.615	1
1.114	1.200	3.650	0.302	3.348	0.311	1
1.070	1.400	4.985	0.055	4.930	0.113	1
1.056	1.600	6.686	-0.077	6.763	-0.003	1

Table 4-XI, which shows the post critical behavior, also gives the values of p/μ with $N_c = 1$. The results show that as the critical point (4-109) is exceeded, every other one of the cracks will stop growing, while the active cracks will grow at a faster rate; furthermore, the fluid of the stopped crack will tend to be transferred into the growing crack; i.e. as the total volume of the system is increased, the volume of the stopped crack will decrease, eventually becoming zero. The significance of these results is that when the geometry of the crack array is such that the unstable critical point has been achieved, then for connected cracks any further growth will occur in only one of the cracks, and the fluid volume of the inactive crack will tend to be transferred to the growing crack as the total fluid volume increases. Thus, care must be taken to ensure that the cracks are not grown beyond the critical radius $R = a/b = 0.827$; otherwise, continued crack growth, which can occur naturally through thermal expansion and cracking, may cause a loss of fluid volume from the stopped crack and its transfer to the growing crack. The introduction of an additional crack will then have little effect in improving the reservoir efficiency.

We also note in Fig. 4-15 that Eq. (4-108) is always satisfied, and a stable bifurcation will be achieved for all values of R . Therefore, for connected cracks of equal radius, if the pressure in the cracks is large enough to cause crack growth, one crack will always grow at the expense of the other, and it would be impossible to grow two cracks simultaneously. The proper method of growing more than one crack on a single borehole would be to grow the cracks sequentially so that only one crack at a time is active. All cracks that are intended to be inactive would have to be sealed to prevent their further growth. When the cracks have been grown, then the seals can be removed and the fluid circulated between the cracks. At this stage the stress intensity factor must be well below its critical value in order to prevent further crack growth.

Since the critical value of stress intensity factor N_c is an important consideration in the crack growth calculation, we have computed curves that show the relationship between pressure and crack radius for the cases of a single penny-shaped crack in an infinite medium and an infinite array of penny-shaped cracks in an infinite medium. The case, which is often considered as most realistic for the practical usage, is for two parallel cracks having equal radii, and results for this case should fall between the two given in Figs. 4-16 and 4-17. The values of fracture surface energy used for these figures are $\gamma = 100$ and 1000 J/m^2 . It is seen that a higher value of pressure is required to achieve a critical value of stress intensity factor for multiple cracks than for a single crack.

Fig. 4-16. Curves of water pressure, as functions of the radius, that will maintain the critical stress intensity factor at the edge of a penny-shaped crack in an infinite medium. The curves are obtained for material with $E = 4.05 \times 10^{10} \text{ N/m}^2$, $\nu = 0.22$ and the indicated values of γ .

4.5 FRACTURE OPENING BY COOLING

As a crack is cooled the crack opening displacement will tend to grow with time. A recent Los Alamos report⁵ calculated the time required to open a penny-shaped crack when the internal pressure P on the crack is less than the normal earth stress σ . This calculation was made for a single crack in the material,

Fig. 4-18. Curves for deriving time for penny-shaped cracks to open by cooling; the water pressure p is less than the tectonic pressure σ .

where $\kappa = \lambda/\rho c$, λ is the thermal conductivity, T_R , T_W are the temperatures of rock and water, and t is time. The displacement w is a function of $\sigma - P$, which for a crack array can not be put into a closed form. Using the following values for the constants:

$$\begin{aligned} E &= 8 \times 10^4 \text{ MPa} \\ &\quad (\text{modulus of elasticity}) \\ \alpha &= 8 \times 10^{-6} \text{ K}^{-1} \\ &\quad (\text{coefficient of thermal expansion}) \\ \nu &= 0.25 \\ &\quad (\text{Poisson's ratio}) \\ c &= 1000 \text{ J kg}^{-1} \text{ K}^{-1} \\ &\quad (\text{specific heat}) \\ \rho &= 2700 \text{ kg m}^{-3} \\ &\quad (\text{density}) \\ \lambda &= 2.7 \text{ W/m} \cdot \text{K} \\ &\quad (\text{thermal conductivity}) \end{aligned}$$

Curves for deriving the time for an unpressurized fracture to open by cooling can be obtained. These are shown in Fig. 4-18 for two values of $R = a/b$. The figure shows the trend that for an array of cracks having equal radii, the wider the spacing between the cracks the sooner the cracks will tend to open by cooling.

APPENDIX A BASIC EQUATIONS

The tractions, equivalent to the thermal stresses, acting on the crack faces may be expressed as

$$\sigma_{22}(0,y) = \int_0^{h_1} D_1(t) \frac{\pi}{2b} G_1\left(\frac{\pi t}{2b}, \frac{\pi y}{2b}\right) dt + \int_0^{h_2} D_2(t) \frac{\pi}{2b} G_2\left(\frac{\pi t}{2b}, \frac{\pi y}{2b}\right) dt, \quad (4-112)$$

and

$$\sigma_{22}(b,y) = \int_0^{h_1} D_1(t) \frac{\pi}{2b} G_2\left(\frac{\pi t}{2b}, \frac{\pi y}{2b}\right) dt + \int_0^{h_2} D_2(t) \frac{\pi}{2b} G_1\left(\frac{\pi t}{2b}, \frac{\pi y}{2b}\right) dt, \quad (4-113)$$

where $x_1 = y$ and $x_2 = x$. The functions $D_1(t)$, $D_2(t)$ represent dislocation densities and because of symmetry on $x = 0$, b , the shear stress terms have been omitted; the kernels G_1 G_2 are defined as follows:

$$G_1(t,y) = 2\coth(y+t) - (y+3t) \operatorname{cosech}^2(y+t) + 4yt \coth(y+t) \operatorname{cosech}^2(y+t) \\ - 2 \coth(y-t) + (y-t) \operatorname{cosech}^2(y-t), \quad (4-114)$$

$$G_2(t,y) = 2\tanh(y+t) + (y+3t) \operatorname{sech}^2(y+t) - 4yt \tanh(y+t) \operatorname{sech}^2(y+t) \\ - 2\tanh(y-t) - (y-t) \operatorname{sech}^2(y-t). \quad (4-115)$$

To solve (4-112) and (4-113) subject to boundary conditions

$$\sigma_{22}(0,y) = -\hat{\beta} T_0 f(y/\delta), \quad \sigma_{22}(b,y) = -\hat{\beta} T_0 f(y/\delta), \\ \sigma_{12}(0,y) = 0, \text{ for } 0 < y < h_1, \quad \sigma_{12}(b,y) = 0, \text{ for } 0 < y < h_2, \quad (4-116)$$

we set

$$D_1(t) = -\frac{2b}{\pi} \hat{\beta} T_0 \frac{A_1(s)}{(1-s^2)^{1/2}}, \quad A_1(s) = A_1(-s), \quad 0 < s < 1, \\ D_2(t) = -\frac{2b}{\pi} \hat{\beta} T_0 \frac{A_2(s)}{(1-s^2)^{1/2}}, \quad A_2(s) = A_2(-s), \quad 0 < s < 1, \\ a_1 = \frac{h_1}{b}, \quad a_2 = \frac{h_2}{b}, \quad \Delta = \frac{\delta}{b}, \quad (4-117)$$

and obtain, from (4-112) and (4-113),

$$\frac{1}{2} \int_{-1}^1 \frac{A_1(s)}{(1-s^2)^{1/2}} a_1 G_1\left(\frac{\pi a_1 |s|}{2}, \frac{\pi a_1 x}{2}\right) ds + \frac{1}{2} \int_{-1}^1 \frac{A_2(s)}{(1-s^2)^{1/2}} x \\ a_2 G_2\left(\frac{\pi a_2 |s|}{2}, \frac{\pi a_1 x}{2}\right) ds = f\left(\frac{a_1 x}{\Delta}\right), \quad 0 < x < 1, \quad (4-118)$$

$$\frac{1}{2} \int_{-1}^1 \frac{A_1(s)}{(1-s^2)^{1/2}} a_1 G_2\left(\frac{\pi a_1 |s|}{2}, \frac{\pi a_2 x}{2}\right) ds + \frac{1}{2} \int_{-1}^1 \frac{A_2(s)}{(1-s^2)^{1/2}} a_2 G_1\left(\frac{\pi a_2 |s|}{2}, \frac{\pi a_2 x}{2}\right) ds \\ = f\left(\frac{a_2 x}{\Delta}\right), \quad 0 < x < 1. \quad (4-119)$$

Using the numerical integration scheme described in Erdogan, Gupta and Cook,¹¹ we obtain

$$\sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} A_1(s_i) a_1 G_1(t_{1i}, y_{1k}) + \sum_{j=1}^{n_2} \frac{\pi}{2n_2+1} A_2(s_j) a_2 G_2(t_{2j}, y_{1k}) = f\left(\frac{2y_{1k}}{\pi\Delta}\right),$$

$$k = 1, 2, \dots, n_1, \quad (4-120)$$

$$\sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} A_1(s_i) a_1 G_2(t_{1i}, y_{2k}) + \sum_{j=1}^{n_2} \frac{\pi}{2n_2+1} A_2(s_j) a_2 G_1(t_{2j}, y_{2k}) = f\left(\frac{2y_{2k}}{\pi\Delta}\right)$$

$$k = 1, 2, \dots, n_2 \quad (4-121)$$

where n_1, n_2 are the number of integration points in crack h_1, h_2 respectively, and

$$s_i = \cos\left(\frac{2i-1}{4n_1+2} \pi\right), \quad s_j = \cos\left(\frac{2j-1}{4n_2+2} \pi\right),$$

$$y_{1k} = \frac{\pi a_1}{2} \cos\left(\frac{k\pi}{2n_1+1}\right), \quad y_{2k} = \frac{\pi a_2}{2} \cos\left(\frac{k\pi}{2n_2+1}\right),$$

$$t_{1i} = \frac{\pi a_1}{2} s_i, \quad t_{2j} = \frac{\pi a_2}{2} s_j. \quad (4-122)$$

Equations (4-120), (4-121) provide a system of (n_1+n_2) linear algebraic equations to determine the unknown functions A_1, A_2 .

Let N_1, N_2 be the nondimensional stress-intensity factors at the crack tips $(0, h_1), (b, h_2)$, respectively, defined as follows:

$$N_1 = \frac{1}{\beta T_0 b^{1/2}} \lim_{y \rightarrow h_1^+} (y-h_1)^{1/2} \tau_{xx}(0, y) = (2a_1)^{1/2} A_1(1), \quad (4-123)$$

$$N_2 = \frac{1}{\beta T_0 b^{1/2}} \lim_{y \rightarrow h_2^+} (y-h_2)^{1/2} \tau_{xx}(b, y) = (2a_2)^{1/2} A_2(1). \quad (4-124)$$

The approximate polynomial solutions of $A_1(1), A_2(1)$ can be obtained by using the method described in Krenk⁴¹ as follows:

$$A_1(1) = \frac{2}{2n_1+1} \sum_{i=1}^{n_1} (-1)^{i+1} \operatorname{cosec}\left(\frac{2i-1}{4n_1+2} \pi\right) A_1(s_i) + (-1)^{n_1} \frac{A_1(0)}{2}, \quad (4-125)$$

$$A_2(1) = \frac{2}{2n_2+1} \sum_{j=1}^{n_2} (-1)^{j+1} \operatorname{cosec}\left(\frac{2j-1}{4n_2+2} \pi\right) A_2(s_j) + (-1)^{n_2} \frac{A_2(0)}{2}. \quad (4-126)$$

The derivative of the stress-intensity factors N_1, N_2 with respect to a_1 are obtained by differentiating (4-123), (4-124) with respect to a_1 ,

$$\frac{\partial N_1}{\partial a_1} = \frac{N_1}{2a_1} + (2a_1)^{1/2} \frac{\partial A_1(1)}{\partial a_1}, \quad (4-127)$$

$$\frac{\partial N_2}{\partial a_1} = (2a_2)^{1/2} \frac{\partial A_2(1)}{\partial a_1}. \quad (4-128)$$

The values of $\frac{\partial A_1(1)}{\partial a_1}$, $\frac{\partial A_2(1)}{\partial a_1}$ can be determined by using (4-125), (4-126) respectively. The derivatives $\frac{\partial A_1(s_i)}{\partial a_1}$, $\frac{\partial A_2(s_j)}{\partial a_1}$ can be found by differentiating (4-120), (4-121) with respect to a_1 , which may be written as

$$\begin{aligned} & \sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} \frac{\partial A_1(s_i)}{\partial a_1} a_1 G_1(t_{1i}, y_{1k}) + \sum_{j=1}^{n_2} \frac{\pi}{2n_2+1} \frac{\partial A_2(s_j)}{\partial a_1} a_2 G_2(t_{2j}, y_{1k}) \\ &= \frac{\partial f}{\partial a_1} \left(\frac{2y_{1k}}{\pi \Delta} \right) - \sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} A_1(s_i) L_1(t_{1i}, y_{1k}) - \sum_{j=1}^{n_2} \frac{\pi}{2n_2+1} A_2(s_j) \frac{a_2}{a_1} \\ & \quad L_3(t_{2j}, y_{1k}), \quad k = 1, 2, \dots, n_1, \end{aligned} \quad (4-129)$$

$$\begin{aligned} & \sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} \frac{\partial A_1(s_i)}{\partial a_1} a_1 G_2(t_{1i}, y_{2k}) + \sum_{j=1}^{n_2} \frac{\pi}{2n_2+1} \frac{\partial A_2(s_j)}{\partial a_1} a_2 G_1(t_{2j}, y_{2k}) \\ &= - \sum_{i=1}^{n_1} \frac{\pi}{2n_1+1} A_1(s_i) L_2(t_{2i}, y_{2k}), \quad k = 1, 2, \dots, n_2, \end{aligned} \quad (4-130)$$

where the functions L_1, L_2, L_3 are defined as follows:

$$\begin{aligned}
L_1(t,y) = & 12t + 2 \coth(y+t) - 4(y+2t) \coth^2(y+t) \\
& + 2(y^2+10yt+3t^2) \coth(y+t) \operatorname{cosech}^2(y+t) \\
& - 4yt(y+t) \operatorname{cosech}^2(y+t) [3 \coth^2(y+t)-1] \\
& - 2\coth(y-t) + 4(y-t) \coth^2(y-t) - 2(y-t)^2 \coth(y-t) \operatorname{cosech}^2(y-t)
\end{aligned}
\tag{4-131}$$

$$\begin{aligned}
L_2(t,y) = & 12t + 2 \tanh(y+t) - (y+8t) \tanh^2(y+t) - 2t(5y+3t) \tanh(y+t) \operatorname{sech}^2(y+t) \\
& - 4yt^2 \operatorname{sech}^2(y+t) [1-3\tanh^2(y+t)] - 2\tanh(y-t) + (y-4t) \tanh^2(y-t) \\
& - 2t(y-t) \tanh(y-t) \operatorname{sech}^2(y-t) ,
\end{aligned}
\tag{4-132}$$

$$\begin{aligned}
L_3(t,y) = & -3y \tanh^2(y+t) - 2y(y+5t) \tanh(y+t) \operatorname{sech}^2(y+t) \\
& - 4y^2 t \operatorname{sech}^2(y+t) [1-3\tanh^2(y+t)] + 3y \tanh^2(y-t) \\
& + 2y(y-t) \tanh(y-t) \operatorname{sech}^2(y-t) ,
\end{aligned}
\tag{4-133}$$

The derivatives of N_1 , N_2 with respect to a_2 can be determined in the same way as those with respect to a_1 .

4.A.1 ON THE ACCURACY OF THE RESULTS

Lacking a rigorous error estimate, we have tried to check the accuracy of our numerical results in three different ways: (1) we have applied the method to problems with known solutions; (2) since a collocation method is used, we have obtained the same results using several significantly different numbers of collocation points; and (3) we have used the identity

$$N_1 \partial N_1 / \partial a_2 = N_2 \partial N_2 / \partial a_1 . \tag{4-134}$$

The first check is that for the single edge crack solved by Koiter,⁵² in which the value of stress intensity factor is given exactly as $2N=1.5861$. Table A-I shows values for stress intensity factor which we have obtained by our analysis for a uniform load and for different numbers of collocation points. The results for forty points appear to differ by about 0.3% from the exact value. Also shown in Table 4- XII are values for the stress intensity factors using two interpolation points. These values were also tested for three and four interpolation points and the results are the same.

TABLE A-I
SINGLE EDGE CRACK

<u>Number of points</u>	<u>2N (Eq. 4-125)</u>	<u>2N (Interpolation)</u>
40	1.5811	1.5818
60	1.5827	1.5832
80	1.5835	1.5839
100	1.5840	1.5844

Another check was made by specializing the problem to an infinite array of edge cracks of equal lengths. For this case the error function profile was used. The results are given in Table A-II for $\Delta \equiv \delta/b = 1.5$ and $a_1 = a_2 = 1.3229$. It can be seen from this table that while the stress intensity factor is essentially unchanged the derivatives do change significantly with the number of collocation points. One might infer that the error in the derivatives is the same for different crack configurations. However, the accuracy of the derivatives will also be seen to depend upon the relative crack length as well as the crack lengths.

TABLE A-II
INFINITE ARRAY OF CRACKS
($a_1 = a_2 = 1.3229$, $\Delta = 1.5$)

<u>Number of points</u>	<u>$N_1 = N_2$</u>	<u>$\partial N_1 / \partial a_2$</u>	<u>$\partial N_2 / \partial a_2$</u>
30	0.1501	-0.2966	-0.4380
40	0.1500	-0.2984	-0.4338
50	0.1500	-0.3011	-0.4288
60	0.1500	-0.3016	-0.4299
70	0.1500	-0.3031	-0.4292

For equal cracks, when $a_1 = a_2$ is less than 10, 50 collocation points seem to give better than three significant figure accuracy in the dimensionless stress intensity factors $N_1 = N_2$, and in the crack lengths $a_1 = a_2$. In our calculations the critical points are established using 80 collocation points for each crack. Thus the numerical results are accurate to at least four significant figures for $N_1 = N_2$ and $a_1 = a_2$.

The accuracy in the calculations of the derivatives, $\partial N_i / \partial a_j$, is considerably lower than that for N_i and a_i , and becomes worse as the cracks become more unequal. This feature is seen in Table A-II where 40 collocation points were used and where $a_1 = 3.883$, $N_2 = N_c = 0.15$. Equation (4-134) provides an excellent check for the consistency of the results when the cracks are unequal, as long as N_1 is positive. The error $\epsilon = |N_1 \partial N_1 / \partial a_2 - N_2 \partial N_2 / \partial a_1|$ is given in Table A-III for three cases and it appears to be consistent in magnitude.

It should be pointed out that the stability calculations require quantities that are not usually calculated in fracture analysis, namely the derivatives of the stress intensity factors with respect to crack lengths. Consequently, there is a requirement for higher accuracy than is usually obtained by numerical techniques. The authors doubt that twice direct numerical differentiations of the strain energy with respect to the crack lengths, with crude finite-element mesh patterns, can possibly yield adequately accurate results. On the other hand, the new combined analytical and finite-element approach, outlined in Section 4.4 of Part I, provides a very powerful tool for this kind of calculations; see Table 4-II of Section 4.4 of Part I.

TABLE A-III

ERROR NEAR CRITICAL POINT

($a_1 = 3.883$, $N_2 = N_c = 0.15$, 40 collocation points)

Δ	A_2	N_1	$\partial N_1 / \partial a_1$	$\partial N_1 / \partial a_2$	$\partial N_2 / \partial a_1$	$\partial N_2 / \partial a_2$	Error* ϵ
4.65	4.076946	0.10817	0.08457	-0.24798	-0.17813	-0.17690	0.00010
4.95	4.430745	0.05665	0.12901	-0.10210	-0.03788	-0.30976	0.00010
5.25	4.701950	0.02163	0.12208	-0.02794	-0.00375	-0.31827	0.00004

* $\epsilon = |N_1 \partial N_1 / \partial a_2 - N_2 \partial N_2 / \partial a_1|$.

APPENDIX B

Various stress fields used in Section 4.4 are summarized below.

$$\sigma_{ij}^0 = -\frac{1}{8} r_m^{-3/2} \begin{cases} 2 \cos(\theta_m/2) - 3 \sin \theta_m \sin(5\theta_m/2), & i=j=1, \\ 2 \cos(\theta_m/2) + 3 \sin \theta_m \sin(5\theta_m/2), & i=j=2, \\ 3 \sin \theta_m \cos(5\theta_m/2), & i=1, j=2; i=2, j=1. \end{cases} \quad (4-135)$$

$$\sigma_{ij}^1 = \frac{3}{4} r_m^{-1/2} \cos(\theta_m/2) \begin{cases} 1 - \sin(\theta_m/2) \sin(3\theta_m/2), & i = j = 1, \\ 1 + \sin(\theta_m/2) \sin(3\theta_m/2), & i = j = 2, \\ \sin(\theta_m/2) \cos(3\theta_m/2), & i = 1, j = 2; i = 2, j = 1. \end{cases} \quad (4-136)$$

$$\sigma_{ij}^2 = \frac{15}{4} r_m^{1/2} \cos(\theta_m/2) \begin{cases} 1 + \sin^2(\theta_m/2), & i = j = 1, \\ 1 - \sin^2(\theta_m/2), & i = j = 2, \\ -\sin(\theta_m/2) \cos(\theta_m/2), & i = 1, j = 2; i = 2, j = 1. \end{cases} \quad (4-137)$$

$$\bar{\sigma}_{ij}^1(r_m, \theta_m) = \begin{cases} 2 + \text{constant}, & i = j = 1, \\ 2, & i = j = 2, \\ 0, & i \neq j. \end{cases} \quad (4-138)$$

$$\bar{\sigma}_{ij}^2(r_m, \theta_m) = \begin{cases} 6 r_m \cos \theta_m, & i = j = 1, \\ 6 r_m \cos \theta_m, & i = j = 2, \\ -6 r_m \sin \theta_m, & i \neq j. \end{cases} \quad (4-139)$$

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CHAPTER 5

LABORATORY STUDIES OF HYDRAULIC AND THERMAL FRACTURING

by

J. Dundurs*

5.1 EXPERIMENTAL TECHNIQUES

The laboratory experiments on hydraulic and thermal fracturing are done in transparent epoxy blocks. The blocks are cast from Shell Epon 828 resin with phthalic anhydride hardener and cured, using a cooling program that is common for work in three-dimensional photoelasticity. Subsequently, the block faces are dressed flat and polished. The sizes used are either 150- or 200-mm cubes.

Techniques have been developed for drilling and reaming deep 1.5-mm holes into the blocks, and cementing stainless steel tubing in the holes. This is done by forcing epoxy cement into the gap between the hole and tube by means of compressed air. The diametral clearance between the hole and the tube is typically 0.1 mm. If it is desired to control the plane in which the hydraulic fracture develops initially, it is also possible to prefracture the bottom of a drilled hole before the cementing operation. This is done with a wedge-shaped tool using a detergent solution as a lubricant. The operation requires some experience and skill, which must be gained by practicing with many shallow holes. The initial fractures typically achieved with this technique are penny-shaped with a 5- to 10-mm diameter.

The hydraulic fracturing is done with mercury by connecting the tubes to small plunger pumps and squeezing the fluid into either the initially precracked bottom of the hole, or simply the uncased bottom of the hole (typically 5 mm long). The pumps are motor driven and can be run at various speeds. This allows one to make the cracks grow at any desired rate. There are two reasons for using mercury as the fluid. One is that the mercury-filled cracks are perfectly visible in the transparent blocks. The other is the very high bulk modulus of mercury which helps to avoid catastrophic cracking.

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5.2 HYDRAULIC FRACTURES ORIGINATING FROM THE SAME BOREHOLE

A possible technique for achieving a large size geothermal reservoir in hot dry rock would be to fracture at several places from a slanted borehole, which would hopefully lead to several vertical fractures in parallel planes, and then to intersect the set of fractures formed with a second borehole. The critical question then is whether the fractures generated from the same borehole would stay parallel, or whether they would have the tendency to join and thus destroy the desired geometry.

It was not deemed possible to make several fractures from the same hole in the laboratory models. This was due to the small inner diameter of the tubes (~0.7 mm) which made localized and controlled rupturing of the casing, and building the required stoppers, an impossible task. It was felt, however, that the question could be answered by fracturing from two holes. The model used for this purpose is shown in Fig. 5-1 (the drawing is to scale and not all dimensions are given; the dimensions shown are in millimetres). The directions in which the blocks were compressed externally are also shown in Fig. 5-1. The vertical externally applied compression was 2 MPa and the horizontal 1.5 MPa. Neither of the hole bottoms was prefractured.

The lower hole was pressurized first. The initial hydraulic fracture was achieved at approximately 20 MPa, and it was in a slanted plane containing the casing and opening against the direction of no external compression in the horizontal direction. As the fracture was expanded, it quickly curved, however, to go into a vertical plane, so as not having to open against the vertically applied compression. Its growth was stopped before its top edge reached the halfway point between the two holes.

While maintaining the lower hole under pressure, the upper hole was fractured next. The initial fracture was achieved at about the same hydraulic pressure as before, and it started at the upper end of the uncased hole. The initial fracture was again in a plane containing the borehole, but at about 45° with respect to the horizontally applied compression. During its subsequent growth, the fracture curved sharply, however, to align itself so that it would not have to open against the externally applied compressions. While the first fracture grew more toward the top of the block, the second fracture primarily expanded downward.

There was no noticeable interaction between the cracks, tending to make them join, and their planes during their major growth phases were practically parallel.

Fig. 5.1. Test block with slanted boreholes.

At the very end of the test, the lower fracture was extended some more, and it eventually intersected the "wrongly" oriented leg of the upper fracture which had formed during its initiation stage. The final configuration of the fractures achieved is shown in Fig. 5-2.

5.3 THERMAL CRACKING

The blocks that have been used to study the interaction between hydraulic fractures can later be used to investigate thermal cracking. It was found that an epoxy block at room temperature requires a very cold fluid to achieve thermal cracking. Consequently the experiments have been done with liquid nitrogen.

A selected block with hydraulic fractures is first cleaned of the mercury filling the cracks. It is generally not possible to remove the mercury completely, and small islands of it remain inside the cracks. They show up as blots on the photographs. A metal cup is then attached to the tube, liquid nitrogen is poured into the cup, and the cup sealed. As vapor pressure in the cup rises, the liquid nitrogen is forced into the hydraulic fracture, where it flushes into vapor coming out of the other hole.

Fig. 5.2a. Front view of two hydraulic fractures from slanted boreholes.

Fig. 5-2b. Side view of the hydraulic fractures from slanted boreholes.

Fig. 5-2c. Other side of view.

Figure 5-3 shows the typical thermal cracks achieved. The block shown contained two hydraulic fractures. One was a large penny-shaped crack with a diameter of about 100 mm. As the connection between the two cracks was very small, an exit hole was drilled directly to the large penny-shaped crack. The distance between the bottom end of the cased hole and the exit hole was 80 mm. The thermal cracking is quite rapid but the growth of the fractures can be observed by eye. It can be seen from Fig. 5-3 that the thermal cracks achieved are very deep (typically 20 to 30 mm), and that they exhibit a characteristic, although complex pattern. Invariably, the thermal cracks curve in two directions. First, if viewed from the face of the hydraulic fracture, the thermal cracks grow on what appears to be circular paths. In addition, the thermal cracks also curve in the other direction. It also is seen that the pattern of the thermal cracks becomes coarser as they grow deeper.

Fig. 5-3a. Thermal cracks viewed from the face side of the hydraulic fracture.

Fig. 5-3b. Opposite view from the face side of the hydraulic fracture.

Fig. 5-3c. Side view of the thermal cracks. The penny-shaped hydraulic fracture is perpendicular to the figure.

CHAPTER 6

SEISMIC DETECTION OF WATER-FILLED CRACKS

by

J. D. Achenbach* and K. Viswanathan **

6.1 ABSTRACT

Analytical studies are presented in this Chapter which bear on the feasibility of mapping a hydraulically-induced crack by measuring the reflections, transmissions and diffractions of mechanical signals emitted by a point source. Recent results on the direct problem of wave diffraction by empty and water-filled flat cracks are briefly summarized. Particular attention is devoted to an approach whereby the interaction of the crack faces due to the presence of water in the crack, and due to contact between the rough faces of the crack, is accounted for by a relation between the tractions on the crack faces and the time-rates of change of the crack-opening-displacements. This relation takes viscous effects into account. Parametrical studies are presented for the reflection and transmission coefficients, where the angle of incidence and a parameter describing the relative resistance to normal and tangential crack-opening displacements are taken as the variables. The effects of scattering by the edge of the crack are studied on the basis of a far-field integral representation for the scattered field, in conjunction with the assumption that the local separation of the crack faces computed on the basis of reflection and transmission considerations, produces an acceptable approximation to the crack-opening displacements. This approximation is valid if the wavelengths are small as compared to the cross-sectional dimensions of the crack. The simple integrals for the scattered far-field, suggest the use of Fourier-type integrals to solve the inverse problem. The use of two such inversion integrals is explored in some detail. The integrals are applied to both

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the scattered far-field of longitudinal motion and of transverse motion. Some examples are discussed. In particular it is shown how part of the edge of an elliptical crack can be constructed by applying an inversion integral to scattering data from two observation points and twenty source positions. In the last section of the Chapter we briefly consider a one-dimensional formulation for the case that the crack is not distinct, but rather is represented by an inhomogeneity of finite width which is characterized by different wave velocities than in the adjoining materials.

6.2 INTRODUCTION

Crack detection by wave propagation methods has been successful in the field of quantitative nondestructive evaluation of materials.¹ The difficulties in a geological setting are, however, much greater, primarily because wave propagation in rock and rock formations is more complicated than in metals, and water-filled cracks in rock may be weak reflectors of incident waves. In addition, there are

limited options for the placement of sources and receivers in boreholes, and the shape of the crack may be quite irregular.

For reference purposes a schematic view of the wellbore system at the Fenton Hill site taken from Ref. 2 is shown in Fig. 6-1. As reported in Ref. 2, a transmitter-receiver pair was lowered in EE-1, while a single receiver was lowered in GT-2B. Signals generated by the transmitter in EE-1 were thus detected by receivers in both EE-1 and GT-2B (it appears, however, that the reflected signals have not been used). With the receiver in position, the transmitter was moved from an inclination of 45° above the receiver to 45° below it. Next the receiver was moved to a second position, and the procedure was repeated.

Fig. 6-1. Elevation view schematic of the GT-2B/EE-1 fracture and wellbore system (from Ref. 2).

Fig. 6-2. Geometry of transmitter, crack and receiver for analytical work.

In the analytical work reported here, the Fenton Hill configuration is modeled as shown in Fig. 6-2. The crack is assumed to be flat, and it is located in the (xy) -plane. The positions of source and receiver are defined by coordinates (x_S, y_S, z_S) and (x_Q, y_Q, z_Q) , respectively. Both the direct problem and the inverse problem have been investigated. The solution to the direct problem (the field when an incident wave is reflected, transmitted and diffracted by a known crack) is a necessary preliminary to the solution of the inverse problem, which is the problem of inferring the geometrical characteristics of an unknown crack from the reflected, transmitted and/or diffracted fields.

The hot dry rock in which the hydraulically induced fracture is located has been modeled as an isotropic, homogeneous, linearly elastic solid. This is clearly a rather simple model for the presumably anisotropic and inhomogeneous material behavior that characterizes rock, at least on a small scale. It is, however, assumed that the local anisotropies average out on a larger scale, and that the wavelengths are long enough so that local inhomogeneities can be ignored. Probably, the best argument for the isotropic, homogeneous, linearly elastic model is, however, that the propagation of waves in such a solid is relatively simple, and amenable to analytical approaches. In contrast, the analytical possibilities for anisotropic and inhomogeneous media are rather limited.

The equations governing elastodynamic theory and several pertinent mathematical techniques to obtain solutions, have been discussed in detail by Achenbach.³

The hydraulically induced fracture (or zone of fractures) can be modeled in several ways. From the theoretical point of view a single flat crack is a planar surface across which the displacement can be discontinuous. This implies that the presence of a crack can be modeled by appropriate boundary conditions on a

plane surface. Thus, for an empty flat crack which covers an area S of the (xy) -plane, the conditions on the stress components are

$$\tau_{zy} = \tau_{zx} = \tau_z = 0.$$

A crack filled with an inviscid fluid can be modeled by assuming that the shear stresses vanish, but that the normal displacement and the normal stress are continuous across the surface of the crack.

The effects on an incident wave field on both an empty crack and a water-filled crack have been analyzed and results are discussed in Section 6.3. The exact mathematical formulation of the elastodynamic fields generated by the presence of such cracks is rather complicated. Diffraction of an incident wave by a crack is a mixed boundary value problem whose exact solution satisfies one or more generally singular integral equations for the displacement discontinuities across the crack faces. The system of integral equations for a flat empty crack of finite dimensions was recently derived by Budiansky and Rice.⁴ Numerical solutions have only been obtained for a two-dimensional geometry. Only for a semi-infinite crack can an analytical solution conveniently be obtained, as discussed in Section 6.3.

The difficulties with exact solutions have motivated the development of approximate methods. In Section 6.4 particular attention is devoted to an approximate method which is valid at high frequencies. This method is based on elastodynamic ray theory. For the harmonic components of a pulse, elastodynamic ray theory provides a convenient method to trace the amplitude and position of a disturbance as it propagates along a ray. In a homogeneous, isotropic, linearly elastic solid the rays are straight lines which are normal to the wave front. An unbounded solid can support rays of longitudinal and transverse wave motion. A surface of a solid can, in addition, support rays of surface-wave motion. The interaction of incident rays with the edge of a crack is described by geometrical diffraction theory (GDT). With the aid of GDT it is relatively simple to write expressions for the fields on diffracted rays, provided that $\omega a/c_T$ is sufficiently larger than unity, where ω is the circular frequency, a is a characteristic dimension of the crack, and c_T is the velocity of transverse waves. Results that are already available show that a uniform asymptotic version of GDT provides surprisingly good results at relatively small values of the frequency and relatively close to the crack.

In Section 6.5 we try to come to a more realistic description of the interaction of the crack faces due to the presence of water in the crack, and due to contact between the rough faces of the crack. The interaction is accounted for by a relation between the tractions on the crack faces and the time-rates of change of the crack-opening displacements. Viscous effects are taken into account by this model. Parametrical studies are presented for the reflection and transmission coefficients, and diffraction effects are studied on the basis of "physical elastodynamics." This approach produces simple expressions for the diffracted fields, which suggest the use of Fourier type integrals for the inverse problems. The use of two such integrals is explored in some detail. An example problem for detection of an elliptical crack edge has been worked out.

In the last section of the report we briefly consider a one-dimensional formulation for the case that the crack is not distinct, but is rather represented by an inhomogeneity of finite width which is characterized by different wave velocities than in the adjoining material. It is shown that a simple inversion procedure which yields the width of the zone of inhomogeneity can be based on an analysis of the reflected signals. No such simple procedure can be worked out for the transmitted waves.

6.3 DIFFRACTION BY A LARGE CRACK: EXACT ANALYSIS

An exact mathematical analysis of reflection, transmission and diffraction of elastic waves by empty or fluid-filled cracks is very complicated. The one and only problem that is amenable to a reasonably simple exact analytical approach is two-dimensional in nature, and it concerns diffraction by a semi-infinite crack. The geometry is shown in Fig. 6-3. A source of axially symmetric wave motion and a point of observation are located at (x_0, z_0) and (x, z) , respectively. The crack is defined by $z = 0, x < 0$. The source produces a pulse at time $t = 0$. For $t > 0$ the pattern of wave fronts on the observation side of the crack is sketched in Fig. 6-3. The pattern of wave fronts on the source side of the crack is more complicated, and it is not shown. Clearly there will be a sequence of signal arrivals at the observation point Q. The corresponding arrival times can easily be calculated. For small times the diffraction pattern produced by a semi-infinite crack will be the same as for a crack of finite dimensions.

The conditions on the faces of a water-filled semi-infinite crack are:

Fig. 6-3. Wave fronts of transmitted and diffracted waves at shadow-side of crack.

$$z = 0, x < 0 : \quad \tau_z^+ = \tau_z^- \quad (6.1)$$

$$u_z^+ = u_z^- \quad (6.2)$$

$$\tau_{zx}^+ = \tau_{zx}^- = 0 \quad (6.3)$$

where + and - defines the field variables at $z = 0^+$ and $z = 0^-$, respectively. Equations (6.1) - (6.3) imply the assumption that the fluid is inviscid, that no cavitation takes place and that the faces of the crack do not interact.

In the usual manner the total field can be expressed as the superposition of the incident field and the scattered field

$$\underline{u}^t = \underline{u}^i + \underline{u}^{sc}. \quad (6.4)$$

For the scattered field, Eqs. (6.1) - (6.3) then imply the following boundary conditions at $z = 0, x < 0$:

$$(\tau_z^+)^{sc} = (\tau_z^-)^{sc} \quad (6.5)$$

$$(u_z^+)^{sc} = (u_z^-)^{sc} \quad (6.6)$$

$$(\tau_{zx}^+)^{sc} = (\tau_{zx}^-)^{sc} = -(\tau_{zx})^i. \quad (6.7)$$

These conditions define an anti symmetric problem relative to $z = 0$. The initial-value, mixed boundary-value problem defined by the elastodynamic field equations and Eqs. (6.5) - (6.7) can be solved. Briefly summarized, the method of solution involves application of the one-sided Laplace transform over time, the two-sided Laplace transform over x , an application of the Wiener-Hopf technique, and an evaluation of the inverse transforms by means of the Cagniard-de Hoop method. The details have been worked out by Harris.⁵ In Ref. 5, curves are presented for the displacement components versus time at a specific point of observation Q . Harris⁵ has also derived first terms in wave front expansions which have a relatively simple form.

6.4 RAY-THEORY SOLUTIONS

At high frequencies the diffraction of elastic waves by cracks can be analyzed conveniently on the basis of elastodynamic ray theory. The basic concepts of elastodynamic ray theory have been presented by Karal and Keller.⁶ For time-harmonic wave motion, ray theory provides a method to trace the amplitude of a disturbance as it propagates along a ray. In a homogeneous, isotropic, linearly elastic solid the rays are straight lines, which are normal to the wave front. An unbounded solid can support rays of longitudinal and transverse wave motion. These rays are denoted as L-rays and T-rays, respectively. The free surface of a solid can, in addition, support rays of surface-wave motion, which are denoted as R-rays.

In analogy with geometrical optics, the simplest theory for diffraction of elastic waves by cracks may be called geometrical elastodynamics (GE). In GE, a fluid-filled crack acts as a screen, which creates a zone of transmitted waves, and zones of reflected waves. The zone of transmitted waves is bounded by all rays passing through the source point and the edge of the crack. The geometrical reflections of these rays bound the zone(s) of reflected rays. The displacement field according to GE, which is denoted by u^g is of the same order of magnitude as the incident field. The GE-field is, however, physically unrealistic, because

of discontinuities at the boundaries of the shadow zone and the zone(s) of reflected waves.

The geometrical theory of diffraction (GTD) provides a first correction to GE. This correction is valid for $\omega a/c_L \gg 1$, and at points where $S/a > 1$. Here ω is the circular frequency, a is a length dimension of the crack, c_L is the velocity of longitudinal waves, and S is the distance from a crack edge. The correction provided by GTD is of order $(\omega a/c_L)^{-1/2}$. The basic ideas of GTD were introduced by Keller.⁷ They were extended to elastodynamics by Achenbach et al.^{8,9}

Within the context of the GTD theory of Refs. 8 and 9, the diffracted field at a point of observation Q is comprised of contributions corresponding to "primary" diffracted body-wave rays, which are directly generated by incident body-wave rays, and contributions corresponding to "secondary" diffraction body-wave rays. The latter are generated by rays traveling via the crack faces. Thus, the diffracted displacement field u^d at Q can be represented by

$$u^d = \sum u_{\beta}^{\alpha} + \sum u_{\beta\gamma}^{\alpha}, \quad (6.8)$$

where u_{β}^{α} and $u_{\beta\gamma}^{\alpha}$ represent the primary and secondary diffractions, respectively. In u_{β}^{α} , the symbol α defines the incident ray, i.e., $\alpha = L$ or $\alpha = T$, while β defines the diffracted ray, $\beta = L$ or $\beta = T$. In $u_{\beta\gamma}^{\alpha}$ the symbol β defines the crack-face ray, i.e., $\beta = RS$ (surface-symmetric), $\beta = RA$ (surface-antisymmetric) or $\beta = TH$ (horizontally polarized transverse). The symbol γ defines the body-wave rays generated by diffraction of a crack-face ray; thus $\gamma = L$ or $\gamma = T$. The summations in Eq. (6.8) are carried out over all rays of a particular type passing through Q .

Results obtained on the basis of Eq. (6.8) have been presented in Refs. 10, 11, and 12. In Ref. 10 results obtained by elastodynamic ray theory have been compared with results obtained by numerical solution of a governing singular integral equation. The results presented in Ref. 12 are concerned with a water-filled crack.

6.5 VISCOUS INTERACTION OF THE CRACK FACES

In this section a better approximation to the conditions on the faces of the crack when the crack interacts with wave motions in the adjoining rock masses is attempted. It may be expected that the crack faces are quite rough, and

Fig. 6-4. Cross-section of water-filled crack.

consequently incident waves may give rise to contact and sliding of contact areas, even though the crack is filled with hot water. A cross-section of the water-filled crack is sketched in Fig. 6-4.

As a simple but quite reasonable model for the resistance to relative tangential motion of the crack-faces we propose the relation

$$\tau_{xz}^+ = \kappa_T [\dot{u}_x^+ - \dot{u}_x^-]. \quad (6.9)$$

Here it is assumed that the shear stress τ_{xz} is continuous, i.e.,

$$\tau_{xz}^+ = \tau_{xz}^-, \quad (6.10)$$

where the + and - notation defines a field variable at $z = 0^+$ and $z = 0^-$ (upper and lower crack-face), respectively. Equations (6.9) and (6.10) imply that the inertia of the fluid layer is neglected which is acceptable for wavelengths that are large as compared to the thickness of the fluid layer. The viscosity implied by κ_T and the time derivatives of the displacements at the upper and lower crack-faces includes the viscosity of the fluid as well as damping effects due to sliding of contact areas.

A similar relation is proposed for the normal stress and the normal displacements, i.e.,

$$\tau_z^+ = \tau_z^- \equiv \kappa_N [\dot{u}_z^+ - \dot{u}_z^-]. \quad (6.11)$$

As the crack-faces come closer or separate, fluid is displaced. The fluid flows in between contact areas, and it is pushed into (or drawn from) pores in the rock. The operative mechanism is thought to be similar to that of a piston in a cylinder with a perforated bottom, hence (6.11) is expected to be a reasonable approximation.

Equations (6.9) - (6.11) can of course easily be extended to the three-dimensional case of a flat crack by adding

$$\tau_{yz}^+ = \tau_{yz}^- \equiv \kappa_T [\dot{u}_y^+ - \dot{u}_y^-] . \quad (6.12)$$

6.5.1 Reflection and Transmission of Plane Waves

A good deal of insight into the interaction of an incident wave with a fluid-filled crack modeled by the boundary conditions (6.9) - (6.12) can be obtained by considering the local reflection and transmission of a plane longitudinal wave. This problem is two-dimensional in nature, and the geometry is shown in Fig. 6.5a. Since the material properties are the same on both sides of the crack, the problem statement and the computation of reflection and transmission coefficients can be simplified by taking advantage of symmetry considerations. As sketched in Figs. 6-5a, b, and c the problem of a longitudinal wave of amplitude A incident

Fig. 6-5. Propagation vectors of incident, reflected and transmitted waves.

from $z \leq 0$ can be decomposed into a symmetric and an antisymmetric problem. For the symmetric problem longitudinal waves of amplitudes $\frac{1}{2} A$ are incident from both $z > 0$ and $z < 0$ as shown in Fig. 6-5b. For the antisymmetric problem longitudinal waves are incident as shown in Fig. 6-5c. By superimposing the two problems the incident waves in $z > 0$ cancel, and the incident waves in $z < 0$ add up to a single wave of amplitude A .

On the basis of considerations of symmetry and antisymmetry the boundary conditions at $z = 0$ may be written as

symmetric problem:

$$\text{at } z = 0 : \tau_{zx} = 0 ; \tau_z = 2\kappa_N \dot{u}_z ; \quad (6.13)$$

antisymmetric problem:

$$\text{at } z = 0 : \tau_z = 0 ; \tau_{zx} = 2\kappa_T \dot{u}_x . \quad (6.14)$$

The incident longitudinal wave is taken as

$$\underline{u}^i = \frac{1}{2} A \underline{p} \exp [ik_L(x \sin \theta_0 + z \cos \theta_0) - i\omega t] \quad (6.15)$$

where $\underline{p}$ is the propagation vector, i.e.,

$$\underline{p} = \sin \theta_0 \underline{i}_x + \cos \theta_0 \underline{i}_z . \quad (6.16)$$

For both the symmetric and the antisymmetric problem, the reflected wave is composed of a longitudinal wave and a transverse wave. For the symmetric problem we have

$$\begin{aligned} \underline{u}^r = & \frac{1}{2} A R_L^S \underline{p}_L \exp[ik_L(x \sin \theta_L - z \cos \theta_L) - i\omega t] \\ & + \frac{1}{2} A R_T^S \underline{d}_T \exp[ik_T(x \sin \theta_T - z \cos \theta_T) - i\omega t] \end{aligned} \quad (6.17)$$

where R_L^S and R_T^S are the reflection coefficients, and

$$\vec{p}_L = \sin\theta_L \vec{i}_x - \cos\theta_L \vec{i}_z ; \vec{d}_T = \cos\theta_T \vec{i}_x + \sin\theta_T \vec{i}_z . \quad (6.18a,b)$$

In Eqs. (6.15) - (6.17) k_L and k_T are the wavenumbers of longitudinal and transverse waves, respectively. The wavenumbers are related to the circular frequency ω by the relations $\omega = k_L c_L = k_T c_T$, where c_L and c_T are the velocities of longitudinal and transverse waves, respectively.

An expression completely analogous to Eq. (6.16) holds for the antisymmetric problem. For that case we use the notations R_L^{AS} and R_T^{AS} for the reflection coefficients.

The boundary conditions (6.13) and (6.14) at $z = 0$ yield equations for the angles θ_L and θ_T , and for the reflection coefficients. Because the phases of the incident and reflected waves must be the same at $z = 0$, we must have $k_L \sin \theta_0 = k_L \sin \theta_L = k_T \sin \theta_T$, which implies the usual result that

$$\theta_L = \theta_0, \sin \theta_T = (c_T/c_L) \sin \theta_L .$$

With these results and (6.13) - (6.14), two sets of linear algebraic equations for R_L^S , R_T^S and R_L^{AS} , R_T^{AS} , respectively, are obtained. The reflection and transmission coefficients for the problem of a single incident longitudinal wave, as shown in Fig. 6-5, then follow as

$$R_L = R_L^S + R_L^{AS} ; R_T = R_T^S + R_T^{AS}$$

$$T_L = R_L^S - R_L^{AS} ; T_T = R_T^S - R_T^{AS} .$$

Explicit expressions for the reflection and transmission coefficients are

$$R_L^S = P_S/F_S, \quad (6.19)$$

$$R_T^S = Q_S/F_S, \quad (6.20)$$

$$R_L^{AS} = P_{AS}/F_{AS}, \quad (6.21)$$

$$R_T^{AS} = Q_{AS}/F_{AS}, \quad (6.22)$$

where

$$P_S = \sin 2\theta_0 (\nu C \sin 2\theta_T + 2D \sin \theta_T) - [(\lambda/\mu) + 2\cos^2 \theta_0] C - 2D \cos \theta_0 \nu \cos 2\theta_1,$$

$$Q_S = 2[(\lambda/\mu) + 2\cos^2 \theta_0] C \sin 2\theta_0,$$

$$F_S = \sin 2\theta_0 (\nu C \sin 2\theta_T + 2D \sin \theta_T) + [(\lambda/\mu) + 2\cos^2 \theta_0] C + 2D \cos \theta_0 \nu \cos 2\theta_T,$$

$$P_{AS} = (C \sin 2\theta_0 - 2\sin \theta_0) \nu \sin 2\theta_T - \{(\lambda/\mu) + 2\cos^2 \theta_0\} (2\cos \theta_T + \nu C \cos 2\theta_T),$$

$$Q_{AS} = Q_S,$$

and

$$F_{AS} = (C \sin 2\theta_0 + 2\sin \theta_0) \nu \sin 2\theta_T + \{(\lambda/\mu) + 2\cos^2 \theta_0\} (2\cos \theta_T + \nu C \cos 2\theta_T).$$

In these expressions:

$$\nu = c_L / c_T,$$

$$C = \mu / c_L \kappa_T,$$

and

$$D = \kappa_N / \kappa_T.$$

A useful check on the magnitudes of the reflection and transmission coefficients can be obtained by energy considerations. The rate at which energy is communicated per unit area equals the power per unit area. A representation of the intensity of an harmonic wave is given by the power per unit area, time-averaged over the period $T = 2\pi/\omega$. For the incident longitudinal wave we find (see Achenbach,³ p. 181)

$$\langle P_L \rangle_i = \frac{1}{2} (\lambda + 2\mu) \frac{\omega^2}{c_L^2} A^2.$$

The analogous quantities for the transmitted waves are

$$\langle P_L \rangle_t = \frac{1}{2} (\lambda + 2\mu) \frac{\omega^2}{c_L^2} (T_L)^2 A^2$$

$$\langle P_T \rangle_t = \frac{1}{2} \mu \frac{\omega^2}{c_T^2} (T_T)^2 A^2.$$

Similarly for the reflected waves we have

$$\begin{aligned}\langle P_L \rangle_r &= \frac{1}{2} (\lambda + 2\mu) \frac{\omega^2}{c_L} (R_L)^2 A^2 \\ \langle P_T \rangle_r &= \frac{1}{2} \mu \frac{\omega^2}{c_T} (R_T)^2 A^2.\end{aligned}$$

Now let us consider a beam of incident L-waves of cross-sectional area ΔS_L . The corresponding beams of transmitted and reflected L-waves and T-waves have cross-sectional areas ΔS_L and ΔS_T , respectively. The geometry is shown in Fig. 6-5. If no energy were dissipated in the interface the average energy transmission across ΔS_L must equal the sum of the average energy transmissions across the areas ΔS_L and ΔS_T of the reflected and transmitted waves. In the present problem there is energy dissipation due to the viscous behavior represented by Eq. (6.9).

The rate of energy dissipation per unit area of the interface, time-averaged over a period T , is

$$\langle D \rangle = \frac{1}{T} \int_0^T [\tau_z \Delta \dot{u}_z + \tau_{zx} \Delta \dot{u}_x] dt, \quad (6.24)$$

where the stresses and displacements are the actual real-valued field variables. From the expressions for the transmitted and reflected waves it follows that

$$\begin{aligned}\Delta u_x/A &= (T_L \sin \theta_L + T_T \cos \theta_T - R_L \sin \theta_L - R_T \cos \theta_L - \sin \theta_L) \exp[i(k_L x \sin \theta_L - \omega t)] \\ \Delta u_z/A &= (T_L \cos \theta_L - T_T \sin \theta_T + R_L \cos \theta_L - R_T \sin \theta_T - \cos \theta_L) \exp[i(k_L x \sin \theta_L - \omega t)].\end{aligned}$$

An expression of the kind (6.24) can conveniently be evaluated by using the relation that

$$\langle \text{Re}(F) \text{Re}(f) \rangle = \frac{1}{2} \text{Re}(Ff^*),$$

where f^* is the complex conjugate of f . It can then be verified that $\langle D \rangle$ reduces to

$$\begin{aligned} \langle D \rangle = & \frac{1}{2} \omega^2 A^2 \{ \kappa_N [(1 - R_L - T_L) \cos \theta_L + (R_T + T_T) \sin \theta_T]^2 \\ & + \kappa_T [(1 + R_L - T_L) \sin \theta_L + (R_T - T_T) \cos \theta_T]^2 \}. \end{aligned}$$

A check on the reflection and transmission coefficients is then provided by the relation

$$\langle P_L \rangle_i \cos \theta_0 - \{ \langle P_L \rangle_t + \langle P_L \rangle_r \} \cos \theta_L - \{ \langle P_T \rangle_t + \langle P_T \rangle_r \} \cos \theta_T = \langle D \rangle.$$

Reference 13 lists the following input parameters as corresponding to conditions in situ at Fenton Hill:

$$c_L = 6.0 \times 10^3 \text{ m/s} ; c_T = 3.5 \times 10^3 \text{ m/s}$$

$$\lambda = 3.1 \times 10^{10} \text{ Pa} ; \mu = 3.3 \times 10^{10} \text{ Pa}.$$

In addition, a shear viscosity is given as $\eta = 0.0014 \text{ poise} = 0.00014 \text{ Pa.s}$. Assuming an average crack-opening of 1 cm, this value of η corresponds to $\kappa_T = 0.014 \text{ Pa.s/m}$, which corresponds to $\mu/c_L \kappa_T = .393 \times 10^9$ (dimensionless). These numbers were used as input parameters in numerical computations of the reflection and transmission coefficients. The coefficients were computed as functions of the angle of incidence θ_0 and the dimensionless ratio κ_N/κ_T .

The results of the numerical computations are shown in Figs. 6-6a-d. The angle of incidence varies from $\theta_0 = 0$ to $\theta_0 = \pi/2$, and κ_N/κ_T varies from 10^5 to 10^{15} . For $\kappa_N/\kappa_T = 10^5$ very little wave motion is transmitted at all angles of incidence, and the water-filled crack acts essentially as an empty stress-free crack. It is noted that significant changes of the reflection and transmission coefficients take place in the range $10^7 < \kappa_N/\kappa_T < 10^{10}$. As κ_N/κ_T approaches 10^{10} the resistance to normal crack opening displacements becomes very large. Thus, for normal incidence ($\theta_0 = 0$) $|R_L|$ decreases to zero, while T_L increases to unity as κ_N/κ_T increases, which implies that all wave motion is transmitted. Of course for oblique incidence $R_L \neq 0$ and $T_L < 1$ when $\kappa_N/\kappa_T > 10^{10}$ because of the weak resistance to tangential crack-opening displacements.

Fig. 6-6a. Reflection coefficient of reflected longitudinal wave as a function of angle of incidence θ_0 and κ_N/κ_T .

Fig. 6-6b. Reflection coefficient of reflected transverse wave as a function of angle of incidence θ_0 and κ_N/κ_T .

Fig. 6-6c. Transmission coefficient of transmitted longitudinal wave as a function of angle of incidence θ_0 and κ_N/κ_T .

Fig. 6-6d. Transmission coefficient of transmitted transverse wave as a function of angle of incidence θ_0 and κ_N/κ_T .

Fig. 6.7 shows the fraction of energy dissipated by the viscosity effects at the crack faces. The time-averaged energy dissipation per unit area has been rendered dimensionless by dividing by $\frac{1}{2}(\omega^2/c_L)(\lambda+2\mu)A^2 \cos\theta_0$.

6.5.2 The Scattered Field

If it is possible to vary both the vertical and the horizontal positions of transmitters and receivers in two planes (preferably parallel), then the region between the two planes can be logged on the basis of reflection and transmission data. On the other hand, if the positions of transmitters and receivers can only be varied along two boreholes, then the location of points on the crack edge not on the lines connecting positions in the boreholes must be inferred from data on diffraction (or scattering) of signals by the edge of the crack.

To study scattering of mechanical signals, the total displacement field is usually expressed as $\underline{u}^t = \underline{u}^i + \underline{u}^{sc}$, where $\underline{u}^i$ represents the incident wave, and $\underline{u}^{sc}$ defines the scattered displacement field. Figure 6-8 shows a flat crack of Area A, a source point S and a point of observation Q. Using a well-known

Fig. 6-7. Dimensionless time-averaged energy-dissipation per unit area as a function of angle of incidence θ_0 and κ_N/κ_T ;
 $\langle \bar{D} \rangle = \langle D \rangle / \frac{1}{2}(\omega^2/c_L)(\lambda+2\mu) A^2 \cos\theta_0$.

Fig. 6-8. Flat crack, with source point S and point of observation Q.

where 0 is the origin of the coordinate system, and $\underline{x} = \underline{x}$ defines a point on the crack surface. For economy of presentation we have used indicial notation and the summation convention is invoked. In Eq. (6.25) $\underline{n}$ is the outward normal (pointing from the A^+ to the A^- face). Also $\Delta u_i^{SC}(\underline{x})$ is the crack-opening displacement defined by $\Delta u_i^{SC}(\underline{x}) = (u_i^{SC})^{A^+} - (u_i^{SC})^{A^-}$, and $\tau_{ij;m}^G$ is a tensor of rank three, which represents the stress-components due to a unit load applied in the x_m direction at the point defined by $\underline{x} = \underline{x}_Q$.

Let us now assume that the origin 0 is close to the crack while the source S and the observation point Q are far away, see Fig. 6-8. Thus if $\underline{x}_S, \underline{x}_Q$, and $\underline{x}$ denote the position vectors of S, Q and any point on the crack, and $x_S = |\underline{x}_S|$, $x_Q = |\underline{x}_Q|$ and $x = |\underline{x}|$, then $x_S, x_Q \gg x$. Defining the unit vector $\hat{x}_Q = \underline{x}_Q/x_Q$, we can write $|\underline{x}_Q - \underline{x}| \approx x_Q - (\hat{x}_Q \cdot \underline{x})$, $x_Q \gg x$. The expression for $\tau_{ij;m}^G$ then simplifies considerably. An explicit expression is

$$\tau_{ij;m}^G = ik_L b_{ij;m}^{G;L}(\hat{x}_Q) G_L(x_Q) \exp(-ik_L \hat{x}_Q \cdot \underline{x}) + ik_T b_{ij;m}^{G;T}(\hat{x}_Q) G_T(x_Q) \exp(-k_T \hat{x}_Q \cdot \underline{x})$$

where $\hat{x}$ is the unit vector in the direction of $\underline{x}$, and $x = |\underline{x}|$. Also

$$b_{ij;m}^{G;L}(\hat{x}) = (\lambda + 2\mu)^{-1} (2\mu \hat{x}_i \hat{x}_j + \lambda \delta_{ij}) \hat{x}_m,$$

$$b_{ij;m}^{G;T}(\hat{x}) = \delta_{im} \hat{x}_j + \delta_{jm} \hat{x}_i - 2\hat{x}_i \hat{x}_j \hat{x}_m,$$

$$G_\beta(x) = (1/4\pi x) \exp(ik_\beta x) \quad (\beta = L, T),$$

$$\text{and } k_L = \omega/c_L, \quad k_T = \omega/c_T.$$

representation theorem, the scattered field at Q can be expressed in the form

$$u_m^{SC}(\underline{x}_Q) = \int_{A^+} \tau_{ij;m}^G(\underline{x}_Q - \underline{x}) \Delta u_i^{SC}(\underline{x}) n_j dA(\underline{x}) \quad (6.25)$$

where 0 is the origin of the coordinate system, and $\underline{x} = \underline{x}$ defines a point on the crack surface. For economy of presentation we have used indicial notation and the summation convention is

invoked. In Eq. (6.25) $\underline{n}$ is the outward normal (pointing from the A^+ to the A^- face). Also $\Delta u_i^{SC}(\underline{x})$ is the

crack-opening displacement defined by $\Delta u_i^{SC}(\underline{x}) = (u_i^{SC})^{A^+} - (u_i^{SC})^{A^-}$, and $\tau_{ij;m}^G$ is a tensor of rank three, which represents the stress-components due to a unit load applied in the x_m direction at the point defined by $\underline{x} = \underline{x}_Q$.

Let us now assume that the origin 0 is close to the crack while the source S and the observation point Q are far away, see Fig. 6-8. Thus if $\underline{x}_S, \underline{x}_Q$, and $\underline{x}$ denote the position vectors of S, Q and any point on the crack, and $x_S = |\underline{x}_S|$, $x_Q = |\underline{x}_Q|$ and $x = |\underline{x}|$, then $x_S, x_Q \gg x$. Defining the unit vector $\hat{x}_Q = \underline{x}_Q/x_Q$, we can write $|\underline{x}_Q - \underline{x}| \approx x_Q - (\hat{x}_Q \cdot \underline{x})$, $x_Q \gg x$. The expression for $\tau_{ij;m}^G$ then simplifies considerably. An explicit expression is

$$\tau_{ij;m}^G = ik_L b_{ij;m}^{G;L}(\hat{x}_Q) G_L(x_Q) \exp(-ik_L \hat{x}_Q \cdot \underline{x}) + ik_T b_{ij;m}^{G;T}(\hat{x}_Q) G_T(x_Q) \exp(-k_T \hat{x}_Q \cdot \underline{x})$$

where $\hat{x}$ is the unit vector in the direction of $\underline{x}$, and $x = |\underline{x}|$. Also

$$b_{ij;m}^{G;L}(\hat{x}) = (\lambda + 2\mu)^{-1} (2\mu \hat{x}_i \hat{x}_j + \lambda \delta_{ij}) \hat{x}_m,$$

$$b_{ij;m}^{G;T}(\hat{x}) = \delta_{im} \hat{x}_j + \delta_{jm} \hat{x}_i - 2\hat{x}_i \hat{x}_j \hat{x}_m,$$

$$G_\beta(x) = (1/4\pi x) \exp(ik_\beta x) \quad (\beta = L, T),$$

$$\text{and } k_L = \omega/c_L, \quad k_T = \omega/c_T.$$

Substitution of the expression for $\tau_{ij;m}^G$ in Eq. (6.25) yields the scattered field as

$$u_m^{SC}(\underline{x}_Q) = -ik_L b_{ij;m}^{G;L}(\hat{\underline{x}}_Q) G_L(\underline{x}_Q) n_j I_i^L(\hat{\underline{x}}_Q) - ik_T b_{ij;m}^{G;T}(\hat{\underline{x}}_Q) G_T(\underline{x}_Q) n_j I_i^T(\hat{\underline{x}}_Q), \quad (6.26)$$

where for $\beta = L$ or $\beta = T$,

$$I_i^\beta(\hat{\underline{x}}_Q) = \int_{A^+} \exp(-ik_\beta \underline{x}_Q \cdot \underline{X}) \Delta u_i^{SC}(\underline{X}) dA(\underline{X}). \quad (6.27)$$

In the simplest approximation, which in analogy with "physical optics" may be called "physical elastodynamics," only the leading contributions arising from the incident wave and the specularly reflected and transmitted body waves are included in Δu^{SC} . Thus, the effect of the crack edge on the displacement discontinuity of the crack faces is neglected. This is acceptable if the wavelength is sufficiently smaller than the cross-sectional dimensions of the crack.

On the faces of the crack the spherical wave emitted at S is locally approximated by a plane wave:

$$u^i = -A \hat{\underline{x}}_S G_L(x_S) \exp(-ik_L \hat{\underline{x}}_S \cdot \underline{X}), \quad x_S \gg X.$$

By using the foregoing results we then find

$$\Delta u^{SC} \sim A \Delta U G_L(x_S) \exp(-ik_L \hat{\underline{x}}_S \cdot \underline{X}), \quad (6.28)$$

where

$$\Delta U(\theta_0) = (T_L - 1) p + T_T d - R_L p_L - R_T d_T. \quad (6.29)$$

Here p , p_L and d_T are defined by (6.16) and (6.18a,b), respectively, and

$$\underline{d} = -\cos\theta_T \underline{i}_x + \sin\theta_T \underline{i}_z,$$

while θ_0 is defined by

$$\theta_0 = \cos^{-1}(\hat{x}_S \cdot \underline{N}),$$

where $\underline{N}$ is a unit vector normal to the crack-plane. Substitution of (6.28) into (6.25) yields

$$\frac{\{u_m^{sc}(\underline{x}_Q)\}^\beta}{G_L(\underline{x}_S)G_\beta(\underline{x}_Q)} = -A \alpha_m^\beta(\underline{x}_S, \underline{x}_Q) I(k_\beta), \quad (6.30)$$

where $\beta = L$ for the longitudinal wave and $\beta = T$ for the transverse wave, and

$$\alpha_m^\beta(\underline{x}_S, \underline{x}_Q) = \Delta U_i(\theta_0) b_{ij;m}^{G;\beta}(\hat{x}_Q) n_j.$$

Also

$$I(k_\beta) = ik_\beta \int_{A^+} \exp(-ik_\beta \underline{g} \cdot \underline{X}) dA(\underline{X}), \quad (6.31)$$

where

$$\underline{g} = \hat{x}_S + \hat{x}_Q \text{ for } \beta = L; \text{ and } \underline{g} = (c_T/c_L)\hat{x}_S + \hat{x}_Q \text{ for } \beta = T. \quad (6.32)$$

The relative strength of the far-field L- and T- signals is governed by α_m^β , which depends only on the positions of S and Q and on the elastic constants. The far-field dependence on the crack occurs only through the function $I(k_\beta)$, which depends on the wavenumber k_β and the vector $\underline{g}$. The general form of Eq. (6.30) shows that $I(k_\beta)$ governs the amplitude modulations of the scattered fields.

Various alternate expressions can be written for the integral $I(k_\beta)$. In Fig. 6-9 the coordinate system ξ is defined, with origin O' in the plane of the crack, and with the ξ_3 -axis normal to that plane. Let $\overline{OO'} = \underline{x}_0$. Then Eq. (6.31) can be rewritten as

$$I(k_\beta) = ik_\beta \exp(-ik_\beta \underline{g} \cdot \underline{x}_0) \iint_{A^+} \exp(-ik_\beta \underline{g} \cdot \underline{\xi}) d\xi_1 d\xi_2. \quad (6.33)$$

Fig. 6-9. Coordinates in the plane of the crack.

This kind of representation seems to be analogous to the method of equivalent currents, which has been explored by several authors in electromagnetic scattering theory.

Let us consider still another system $\bar{\xi}$ defined by (see Fig. 6-9)

$$\bar{\xi}_1 = q_1 \xi_1 + q_2 \xi_2 ; \bar{\xi}_2 = -q_2 \xi_1 + q_1 \xi_2 ; \bar{\xi}_3 = \xi_3 \quad (6.35)$$

Note that the $\bar{\xi}_1$ -axis is parallel to the projection of q on the crack-plane. The q_i used here are defined in the ξ -system. In the $\bar{\xi}$ -system Eq. (6.34) can be reduced to

$$I(k_\beta) = - (q_1^2 + q_2^2)^{-1} \exp(-ik_\beta q \cdot x_0) \int_C \exp(-ik_\beta \bar{\xi}_1) d\bar{\xi}_2 \quad (6.36)$$

Equation (6.36) can be evaluated in an approximate manner by the use of the method of stationary phase. The points of stationary phase are those points that have the shortest connection from the source via a point on the crack edge to the observer. They may be called the flash points, and they are indicated by $E(\zeta_1, \zeta_2, 0)$ and $F(\eta_1, \eta_2, 0)$ in Fig. 6-9. The contribution from $E(\zeta_1, \zeta_2, 0)$ is

$$I(k_\beta) \sim - \frac{(2\pi)^{1/2} e^{-i\pi/4}}{(q_1^2 + q_2^2)^{1/2}} k_\beta^{-1/2} \exp[-ik_\beta (\zeta_1 + q \cdot x_0)] \quad (6.37)$$

Upon application of the 2-D divergence theorem $I(k_\beta)$ can be reduced to

$$I(k_\beta) = - \frac{\exp(-ik_\beta q \cdot x_0)}{q_1^2 + q_2^2} \int_C \underline{v} \cdot q e^{-ik_\beta q \cdot \xi} ds \quad (6.34)$$

where $\underline{v}$ is the outward unit normal to the crack edge in the plane of the crack, and s is arc length measured along the crack edge C . By means of Eq. (6.34) the scattered fields are expressed as radiation generated by a superposition of sources over the edge of the crack.

where γ is the curvature of the crack edge at point E. A similar contribution arises from $F(\eta_1, \eta_2, 0)$. Details of the derivation of Eqs. (6.34) and (6.37) can be found in Achenbach et al.¹⁴

It is of interest to note that for a crack of elliptical shape defined by

$$\xi_1^2/a^2 + \xi_2^2/b^2 = 1; \quad \xi_3 = 0,$$

Eq. (6.33) can be evaluated analytically to give

$$I(k_L) = -2\pi i \frac{ab}{\zeta_0} J_1(k_L \zeta_0) e^{-ik_L g \cdot X_0}, \quad (6.38)$$

where $J_1(k_L \zeta_0)$ is the cylindrical Bessel function and ζ_0 is defined by

$$\zeta_0 = \zeta_1 = -\eta_1 = (a^2 q_1^2 + b^2 q_2^2)^{1/2}. \quad (6.39)$$

The simple result given by (6.38) in conjunction with (6.30) has been shown to give quite good agreement with experimental results. It has also proven to be very useful in practical inversion procedures based on the application of adaptive learning networks.

The asymptotic expression of the Bessel function $J_1(k_L \zeta_0)$ for $k_L \gg 1$ may be written as

$$J_1(k_L \zeta_0) \sim -\frac{1}{2} \left(\frac{2}{\pi k_L \zeta_0} \right)^{1/2} \{ e^{-i(k_L \zeta_0 - 3\pi/4)} + e^{i(k_L \zeta_0 - 3\pi/4)} \}, \quad (6.40)$$

and it may be checked that the first term in (6.40) corresponds to the contribution from E stated by (6.37), while the second term corresponds to the contribution from F.

6.5.3 Inversion Integrals

The simple forms of $I(k_g)$ in Eqs. (6.36) and (6.37) suggest simple Fourier-type inversion integrals to recover the size, shape and orientation of a crack from the far-field data. The following inversion integrals have been investigated in some detail by Achenbach et al.¹⁴

$$f_1^*(\underline{\lambda}) = I_1\{f(k_\beta)\} = \int_{-\infty}^{\infty} e^{ik_\beta g \cdot \underline{\lambda}} f(k_\beta) dk_\beta \quad (6.41)$$

$$f_2^*(\underline{\lambda}) = I_2\{f(k_\beta)\} = \int_{-\infty}^{\infty} k_\beta^{1/2} e^{ik_\beta g \cdot \underline{\lambda}} f(k_\beta) dk_\beta \quad (6.42)$$

where $\underline{\lambda}$ defines any test-point in the medium. It is expedient to select $\underline{\lambda}$ along g .

Application of the operator of Eq. (6.41) to the expression for $I(k_L)$ in Eq. (6.36) gives

$$I_1^*(\underline{\lambda}) = - \frac{2\pi}{(q_1^2 + q_2^2)} \int_C \delta[g \cdot (\underline{\lambda} - \underline{x}_0) - \bar{\xi}_1] \frac{d\bar{\xi}_2}{d\bar{\xi}_1} d\bar{\xi}_1 \quad (6.43)$$

where we have used the relation

$$\int_{-\infty}^{\infty} e^{ik_L \phi} dk_L = 2\pi \delta(\phi).$$

Evaluating the integral in Eq. (6.43) by the sifting property of the δ -function we obtain

$$I_1^*(\underline{\lambda}) = \begin{cases} \frac{2\pi}{(q_1^2 + q_2^2)} \left[\left(\frac{d\bar{\xi}_2}{d\bar{\xi}_1} \right)^A + \left(\frac{d\bar{\xi}_2}{d\bar{\xi}_1} \right)^B \right]_{\bar{\xi}_1 = \kappa}, & (\eta_1 < \kappa < \zeta_1) \\ 0, & (\kappa < \eta_1; \kappa > \zeta_1) \end{cases} \quad (6.44)$$

$$(6.45)$$

where

$$\kappa = g \cdot (\underline{\lambda} - \underline{x}_0). \quad (6.46)$$

Here the range of variation of $\bar{\xi}_1$, on C is given by $\eta_1 < \bar{\xi}_1 < \zeta_1$, where η_1 and ζ_1 were defined earlier in the context of the local co-ordinate system. In Eq. (6.44) A and B are the points at which the plane $\bar{\xi}_1 = \kappa$ intersects the crack edge C when $\eta_1 < \kappa < \zeta_1$. At the extreme positions $\kappa = \eta_1$ and $\kappa = \zeta_1$ the gradient

terms in Eq. (6.44) become infinite, usually, in the inverse square-root sense, as will later be illustrated by the example of an elliptical crack.

In practice, we can choose λ along the q -direction from 0 and determine the finite layer normal to q which contains the crack. The singular behavior of the gradient terms will be the principal test for identification of the end-planes of the layers.

The inverse operator of Eq. (6.41) when applied to Eq. (6.38) gives

$$I_1^*(\lambda) = \begin{cases} \frac{2\pi ab}{\zeta_0^2} \frac{\kappa}{(\zeta_0^2 - \kappa^2)^{1/2}}, & |\kappa| < \zeta_0 \\ 0, & |\kappa| > \zeta_0 \end{cases}$$

which clearly exhibits the inverse square root singularity at $\kappa = \pm \zeta_0$, where κ is defined in Eq. (6.46). Conversely it is not difficult to verify that the planes

$$\kappa = \pm \zeta_0$$

touch the ellipse at the following points in the ξ -system:

$$\left(\pm \frac{q_1 a^2}{\zeta_0}, \pm \frac{q_2 b^2}{\zeta_0}, 0 \right).$$

Next we will consider application of the operator of Eq. (6.42) to the expression for $I(k_\beta)$ in Eq. (6.37). The result is

$$I_2^*(\lambda) = - \frac{(2\pi)^{3/2} e^{-i\pi/4}}{(q_1^2 + q_2^2)(\gamma_E)^{1/2}} \delta(\kappa - \zeta_1) - \frac{(2\pi)^{3/2} e^{i\pi/4}}{(q_1^2 + q_2^2)(\gamma_F)^{1/2}} \delta(\kappa - \eta_1). \quad (6.47)$$

At the extreme positions $\kappa = \eta_1$ and $\kappa = \zeta_1$, delta-function behavior occurs. By virtue of these results and Eq. (6.30) it then follows that the application of the inversion integral defined by Eq. (6.42) to the terms controlling amplitude

modulation of the scattered field of either longitudinal or transverse motion yields two planes which touch the edge of the crack. An approximation to the crack can be obtained by taking a sufficient number of observation points. In band-limited observations the range of k_L that can be covered will be finite, say $k_- \leq |k_\beta| \leq k_+$. The corresponding effects on the sharpness of the δ -function behavior which govern our inverse procedure have been discussed by Achenbach et al.¹⁴

6.6 INVERSE PROBLEM

The amplitudes of signals reflected and transmitted by fluid-filled cracks generally are larger than the amplitudes of signals produced by scattering of elastic waves. Thus, reflected or transmitted signals passing through an observation point will be relatively easy to detect, and their arrival times can be used to infer the location of the crack-plane. Many properly placed observation points are, however, required to infer the two-dimensional extent of a crack solely from reflection and transmission data. Hence it is more efficient to use scattering data (when available) to locate the crack-edge, because every set of high-frequency data is related to a specific tangent to the edge of the crack.

In this section we first briefly discuss how reflection and transmission data for a fluid-filled crack can be used to infer the location of the crack-plane. We then proceed with applications of the inversion procedure of the previous section to infer the location of the crack-edge from either or both longitudinal and transverse high-frequency scattering data.

6.6.1 Reflection and Transmission Data

Let us assume that the plane of the crack is located in between the two points S and Q. A receiver is placed at Q, and both a transmitter and a receiver are placed at S. From the travel times and the known velocity of longitudinal waves the normal distance a from S to the crack plane can be computed. The first transmitted signal arriving at Q is a longitudinal signal which travels along the ray $\overline{SQ}$, as discussed earlier in this paper. The second signal arriving at Q is a transverse wave which has traveled as a longitudinal wave from S to the crack-plane. The rays along which these three signals travel are located in a plane through $\overline{SQ}$ normal to the plane of the crack. This plane is shown in Fig. 6-10. From the reflected and transmitted waves we know a , ℓ and $\ell^* = b + c$. From the geometry shown in Fig. 6-10 the following relations are evident:

Fig. 6-10. Ray-geometry in plane through S and Q normal to the plane of the crack.

$$a = b \cos \theta_L$$

$$b = b \cos \theta_S + c \cos \theta_Q$$

$$\theta_S + \theta_Q = \theta_L - \theta_T$$

$$b \sin \theta_S = c \sin \theta_Q$$

In addition we have Snell's law

$$c_L \sin \theta_T = c_T \sin \theta_L$$

$$g^* = b + c.$$

Thus, there are three known quantities (a , g and g^*), and six unknowns (b , c , θ_S , θ_Q , θ_L and θ_T). Because there are six equations it should be possible to solve for θ_S and θ_L , which will define the plane of the crack.

6.6.2 Scattering Data

The inversion problem becomes relatively simple if some apriori information, such as the orientation of the plane of the crack or of a plane of symmetry of the crack is available. The general three-dimensional problem will, however, also be considered in some detail.

Plane of the crack is known apriori. In this case the x_i coordinate system can be taken to coincide with the ξ_i system. In Eqs. (6.33) - (6.36) we then have $X_0 = 0$. For a large crack the scattered longitudinal and transverse waves from the closest flash-points can presumably be detected, in particular if no reflected waves pass through the point of observation. Then, if we select a test point λ in the plane of the crack (which is known), the application of the operator I_2 defined by (6.42) to $\{u_m^{sc}(x_Q)\}^B / G_L(x_S) G_L(x_Q)$ should produce a delta-function type behavior for $g \cdot \lambda = \tau_1$, as follows from the first of (6.47). This relation defines a line normal to the projection of g on the crack plane. Note that we have assumed a large crack, so that the contributions from the two flash points can easily be separated. By taking several points of observation (or several source points), i.e. several vectors g , an envelope of the crack edge can be constructed.

An iteration procedure which improves the accuracy was discussed in some detail by Achenbach et al.¹⁴

Source and receiver in a plane of symmetry of the crack. Another simplification occurs if it can be assumed apriori that both the source and the receiver are located in a plane of symmetry of the crack. The inversion procedure then becomes essentially two-dimensional. Numerical examples have been discussed by Achenbach et al.¹⁴ The latter reference also includes an application of the inversion method to experimental data.

3-D construction of the crack-edge location. As an example we construct part of the edge of an elliptical crack from theoretical scattering data provided by Eqs. (6.37) and (6.30). For any given pair of source and observation points, the inverse operator I_2 of Eq. (6.42) leads to a pair of end-planes touching the crack edge. The plane of the crack and segments of the crack-edge will be constructed by using the far-field diffraction data from 20 source locations S_k , $k = 1, 20$ and two observation points Q_n , $n = 1, 2$. In the (x, y, z) -system shown in Fig. 6-2, the positions of these points are defined by

$$S_k: \{65 + (11-k)x3, -40 \cos 30^\circ, -40 \sin 30^\circ\}, \quad k = 1-20; \quad (6.48)$$

$$Q_n: \{55 + 10n, 30 \cos 60^\circ, 30 \sin 60^\circ\}, \quad n = 1, 2. \quad (6.49)$$

A tentative origin of the x_i -system (see Fig. 6-2) is selected at (33.836, .955, 1.5).

For a source at S_k and an observer at Q_n the end-plane produced by application of I_2 to the contribution from the single flash-point is defined by $\Omega(k, n)$. For large k , each set of planes $\Omega(k, n)$ for a given n , forms a prismatic surface, which will touch the crack-edge C at a set of points. These points span a polygon, which approximates a segment $C(n)$ of the edge C . The intersection of the two prismatic surfaces for $n = 1$ and $n = 2$, respectively, will lead to points on C common to $C(1)$ and $C(2)$. We can use these points to generate the crack-plane. The above points are obtained as follows. The prismatic surface formed by the first set of planes $\Omega(k, 1)$ will be intersected by the various individual planes of the set $\Omega(k, 2)$ along a set of polygons $\Gamma(k, 2)$, which we initially determine. The points where the polygons of this set intersect in 3-D space constitute points for an approximate determination of the crack-plane. These points are easily found by testing where any polygon $\Gamma(k, 2)$ with a given k is intersected by the remaining planes of $\Omega(k, 2)$. The plane of the crack can thus be determined, and

once this has been achieved, the intersection of this plane with the set of planes $\Omega(k,n)$ gives the desired tangent lines enveloping the crack edge.

The example we consider is that of an elliptical crack of semi-major and semi-minor axes 30 and 10, respectively. Thus in the (x,y,z) -coordinate system the ellipse is defined by

$$x^2/A^2 + y^2/B^2 - 1 = 0 \quad (6.50)$$

$$z = 0 \quad (6.51)$$

where $A = 30$, $B = 10$.

Table 6.1 presents points computed on the edge of the crack from the scattered fields of longitudinal motion for the sources and observers stated by (6.48) - (6.49). The errors ERR1 and ERR2 denote the values of the left-hand sides of Eqs. (6.50) and (6.51), respectively for these points. The analogous quantities computed from the scattered field of transverse motion are listed in Table 6.2. Figure 6.11 shows the x and y coordinates tabulated in Tables 6.1 and 6.2, plotted together with Eq. (6.50), which defines the actual crack-edge. The points computed from the scattered transverse and longitudinal waves cover different but overlapping parts of the edge. It is noted that for the sources and observers given by (6.48) and (6.49), a larger amount of the edge is recovered from the scattered field of transverse motion.

6.7 INVERSE PROBLEM FOR AN EMBEDDED LAYER

A well-known inverse problem in the mathematical literature is concerned with the detection of a single layer whose mechanical properties are slightly different from those of the adjoining media. The normal reflection of a plane wave from such a layer carries sufficient information for the inverse problem. On the other hand a simple inversion procedure based on transmitted waves does not seem to be possible.

The problem of normal transmission and reflection by a layer is one-dimensional in nature. Considering a harmonic wave of the form $u(x,t) = u(x)\exp(-i\omega t)$, the governing equation for $u(x)$ is

$$\frac{d^2 u}{dx^2} + \omega^2 s_L^2 [1 + \epsilon \alpha(x)] u = 0, \quad \epsilon \ll 1. \quad (6.52)$$

TABLE 6-1
POINTS ON THE CRACK-EDGE COMPUTED FROM THE SCATTERED LONGITUDINAL WAVES

x	y	z	ERR1	ERR2
29.983430	-.332047	.000445	-.000002	.000445
29.979304	-.371568	.000237	.000001	.000237
29.974336	-.414121	.000011	.000005	.000011
29.968365	-.460023	-.000234	.000008	-.000234
29.961198	-.509633	-.000502	.000012	-.000502
29.952600	-.563351	-.000796	.000016	-.000796
29.942288	-.621626	-.001118	.000020	-.001118
29.929918	-.684959	-.001472	.000025	-.001472
29.915072	-.753908	-.001862	.000030	-.001862
29.897243	-.829093	-.002294	.000035	-.002294
29.875816	-.911199	-.002772	.000041	-.002772
29.850043	-1.000976	-.003303	.000047	-.003303
29.819021	-1.099244	-.003896	.000054	-.003896
29.781655	-1.206891	-.004556	.000062	-.004556
29.736626	-1.324871	-.005295	.000072	-.005295
29.682344	-1.454209	-.006123	.000082	-.006123
29.616898	-1.595999	-.007052	.000095	-.007052
29.537984	-1.751419	-.008097	.000111	-.008097
29.442814	-1.921751	-.009273	.000131	-.009273

where $s_L (= 1/c_L)$ is the wave slowness, and $\alpha(x) \equiv 0$ for $|x| > a$.

Equation (6.52) applies to the detection of a hydraulic fracture if there is no distinct single crack but rather a zone of multiple fractures saturated with water. This zone is represented by $|x| < a$, and $\epsilon\alpha(x)$ represents the deviation from the value of s_L^2 in the adjoining unfractured rock masses. In field work at Fenton Hill it has been found that travel times increase by approximately 3% due to pressurization. This indicates a deviation from c_L which is small enough to justify the assumption that $\epsilon \ll 1$.

The total field is written as

$$u^t(x) = u^i + u^{sc}, \quad (6.53)$$

and the reflection and transmission coefficients are defined as

TABLE 6-2
POINTS ON THE CRACK-EDGE COMPUTED FROM THE SCATTERED TRANSVERSE FIELD

x	y	z	ERR1	ERR2
29.906451	-.789641	-.000733	.000008	-.000733
29.893171	-.843792	-.001522	.000011	-.001522
29.877722	-.902628	-.002355	.000012	-.002355
29.859676	-.966745	-.003236	.000013	-.003236
29.838503	-1.036838	-.004166	.000013	-.004166
29.813543	-1.113719	-.005146	.000012	-.005146
29.783963	-1.198346	-.006177	.000010	-.006177
29.748710	-1.291854	-.007258	.000006	-.007258
29.706438	-1.395594	-.008385	.000002	-.008385
29.655409	-1.511183	-.009552	-.000004	-.009552
29.593356	-1.640576	-.010745	-.000011	-.010745
29.517292	-1.786147	-.011945	-.000018	-.011945
29.423225	-1.950812	-.013120	-.000025	-.013120
29.305747	-2.138179	-.014219	-.000030	-.014219
29.157419	-2.352763	-.015164	-.000028	-.015164
28.967813	-2.600278	-.015832	-.000014	-.015832
28.722026	-2.888036	-.016031	.000024	-.016031
28.398279	-3.225487	-.015456	.000107	-.015456
27.963967	-3.624939	-.013612	.000272	-.013612

$$R = \lim_{x \rightarrow \infty} e^{iks_L x} u^s / |u^i| ,$$

and

$$T = \lim_{x \rightarrow \infty} e^{-iks_L x} u^s / |u^i| ,$$

respectively. Explicit expressions for R and T are available. For the reflection coefficient one obtains to first order in ϵ :

$$R = -\epsilon \frac{i\omega}{2c_L} \int_{-\infty}^{\infty} e^{2i\omega s_L x} \alpha(x) dx. \quad (6.54)$$

Fig. 6-11. Points on crack-edge computed on basis of scattered far-field longitudinal wave (—○—) and transverse wave (—△—) as compared to actual crack-edge (—).

Thus

$$R \sim O(\epsilon).$$

The complete expression for T is more complicated, but the important observation is that

$$T \sim O(\epsilon^2).$$

It is relatively simple to recover $\alpha(x)$ from the reflection coefficient. Let us define

$$\theta(\omega, \epsilon) = - \frac{2ic_L}{\omega} \frac{1}{\epsilon} R(\omega, \epsilon).$$

Equation (6.54) may then be rewritten as

$$\Theta(\omega, \epsilon) = \int_{-\infty}^{\infty} e^{2i\omega s_L x} \alpha(x) dx.$$

It is noted that this is a Fourier-type integral, which can be inverted to yield

$$\alpha(x) = \frac{1}{\pi c_L} \int_{-\infty}^{\infty} e^{-2i\omega s_L x} \Theta(\omega, \epsilon) d\omega. \quad (6.55)$$

Thus, if the reflection coefficient is known, the zone of inhomogeneity can be obtained from Eq. (6.55). Unfortunately at the Fenton Hill site it seems to be rather difficult to measure reflected waves. On the other hand a simple analysis of the kind presented here, resulting in Eq. (6.55) does not exist for transmitted waves.

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