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Second Order Differential Equations in Banach Space

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26

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C. C. Travis and G. F. Webb

1. Introduction

In recent years there has been an extensive effort to develop a general theory of differential equations in Banach space. Interest in this subject dates from the work of Hille and Yosida (1948) on the Cauchy problem for first order equations with unbounded operator coefficients. The associated theory of linear and nonlinear semigroups of operators has also received widespread attention and is the subject of several excellent monographs. Concurrent with the development of a theory of abstract first order equations, numerous contributions have been made to the theory of second order differential equations in Banach space. Among the earliest and most significant, we find the work of S. Kurepa, M. Sova, G. DaPrato, E. Giusti, H. Fattorini, and I. Segal.

It is our intention in this paper to survey the theory of second order differential equation in Banach space. We will not attempt to give a complete discussion of this theory but rather provide an introduction to certain selected aspects of the subject. Proofs of results presented will not be given, but references where proofs may be found will be indicated. The main part of our exposition will concern linear second order differential equations, a subject intimately linked with the theory of strongly continuous cosine families of bounded linear operators in Banach space. As a second part of our discussion, we will present results concerning nonlinear second order equations having a special semilinear form.

The organization of the paper is as follows: Section 2 concerns the basic theory of strongly continuous cosine families in Banach space,

and the fundamental generation theorem of Sova-Da Prato-Giusti-Fattorini; Section 3 concerns the problem of converting an abstract second order linear differential equation to an abstract first order differential system; Section 4 concerns perturbations of the infinitesimal generator of a strongly continuous cosine family, and the approximation theorem of Konishi-Goldstein; Section 5 concerns the special properties of compactness, uniform continuity, and inhomogeneous equations for strongly continuous cosine families; and lastly, Section 6 concerns abstract semilinear second order initial value problems in which the linear term is a cosine family generator.

We now state some notation that will be used throughout the paper. We let X be a Banach space with norm $||\cdot||$. Let $B(X;X)$ denote the Banach algebra of bounded everywhere defined linear operators from X to X with norm $||\cdot||$. If A is a linear operator from X to X , then $\rho(A)$ denotes the resolvent set of A and $R(\lambda;A)$ denotes the resolvent of A ; that is, $\lambda \in \rho(A)$ if and only if $\lambda I - A$ is one to one and onto, and $R(\lambda;A) \stackrel{\text{def}}{=} (\lambda I - A)^{-1} \in B(X;X)$. If A is a closed linear operator in X with domain $D(A)$, then $[D(A)]$ denotes the Banach space consisting of the set $D(A)$ endowed with the graph norm $||x|| = ||Ax|| + ||x||$, $x \in D(A)$. We say that the linear operator A from X to X commutes with $L \in B(X;X)$ if and only if for each $x \in D(A)$, $Lx \in D(A)$ and $ALx = LAx$. Finally, we will presume upon the reader a basic acquaintance with the theory of strongly continuous semigroups (and groups) of bounded linear operators in Banach space.

2. Basic Theory of Strongly Continuous Cosine Families

The theory of strongly continuous cosine families of bounded linear operators in Banach space is kindred in spirit to the theory of strongly continuous semigroups of bounded linear operators in Banach space, and it is equally appealing because of its great generality and simplicity. Strongly continuous cosine families of bounded linear operators are related to abstract linear second order differential equations in the same manner that strongly continuous semigroups of bounded linear operators are related to abstract linear first order differential equations. Roughly speaking, every second order differential equation of the form $u'' = Au$ which is well-posed in a certain sense gives rise to a strongly continuous cosine family of bounded linear operators with infinitesimal generator A , and conversely, every strongly continuous cosine family with infinitesimal generator A may be associated with the well-posed second order differential equation $u'' = Au$. In the discussion below, we will make these ideas more precise.

Let A be a linear, but possibly unbounded, operator from the Banach space X to itself, and let $x, y \in X$. By a linear second order initial value problem in X , we mean

$$(2.1) \quad u''(t) = Au(t), \quad t \in \mathbb{R},$$

$$(2.2) \quad u(0) = x,$$

$$(2.3) \quad u'(0) = y.$$

A solution $u(t) = u(t; x, y)$ of the initial value problem (2.1)-(2.3) is a function $u: \mathbb{R} \rightarrow X$ which is twice continuously differentiable on $\mathbb{R}$ in the norm of X , is contained in $D(A)$ for $t \in \mathbb{R}$, and satisfies (2.1) - (2.3). Notice that if (2.1) and (2.2) are to be satisfied,

then x is necessarily in $D(A)$. Since A may be unbounded, and thus $D(A)$ may not be all of X , we cannot expect to obtain a solution to the initial value problem for every $x \in X$. We say that the initial value problem (2.1) - (2.3) is well-posed provided that (2.1) - (2.3) has a unique solution for each $x \in D(A)$ and $y = 0$, and the mapping $x \rightarrow u(t; x, 0)$ has an extension which belongs to $B(X; X)$.

A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

$$(2.4) \quad C(t + s) + C(t - s) = 2C(t)C(s) \text{ for all } s, t \in \mathbb{R};$$

$$(2.5) \quad C(0) = I;$$

$$(2.6) \quad C(t)x \text{ is continuous in } t \text{ on } \mathbb{R} \text{ for each fixed } x \in X.$$

Proposition 2.1. Let the initial value problem (2.1) - (2.3) be well-posed, and for each $x \in D(A)$ and $t \in \mathbb{R}$, let $C(t)x = u(t; x, 0)$, where $u(t; x, 0)$ is the unique solution of (2.1) - (2.3) with $y = 0$. For each $t \in \mathbb{R}$, let $C(t)$ be the extension in $B(X; X)$ of $C(t)$ on $D(A)$. Then $C(t)$, $t \in \mathbb{R}$, is a strongly continuous cosine family of bounded linear operators in X .

Proposition 2.1 is easily established using the definition of a strongly continuous cosine family. Formula (2.4), which is known as the cosine identity, is a consequence of the uniqueness of solutions to the initial value problem, while (2.6) is a consequence of the definition of well-posedness. In the propositions below, we establish a converse to Proposition 2.1. We will first, however, present a further development of the elementary properties of strongly continuous cosine families.

We associate with the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ in X the strongly continuous sine family $S(t)$, $t \in \mathbb{R}$, defined by

$$(2.7) \quad S(t)x = \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R},$$

and the two sets

$$(2.8) \quad E_1 = \{ x \in X: C(t)x \text{ is once continuously differentiable in } t \text{ on } \mathbb{R} \}$$

$$(2.9) \quad E_2 = \{ x \in X: C(t)x \text{ is twice continuously differentiable in } t \text{ on } \mathbb{R} \}.$$

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, is the operator $A: X \rightarrow X$ defined by

$$(2.10) \quad Ax = d^2/dt^2 C(0)x$$

with $D(A) = E_2$.

The following basic results concerning strongly continuous cosine families were established in [27], [5], and [30].

Proposition 2.2. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with associated sine family $S(t)$, $t \in \mathbb{R}$. The following are true:

$$(2.11) \quad C(t) = C(-t) \text{ and } S(t) = -S(-t) \text{ for all } t \in \mathbb{R};$$

$$(2.12) \quad C(s), S(s), C(t), \text{ and } S(t) \text{ commute for all } s, t \in \mathbb{R};$$

$$(2.13) \quad S(t)x \text{ is continuous in } t \text{ on } \mathbb{R} \text{ for each fixed } x \in X;$$

$$(2.14) \quad S(s+t) + S(s-t) = 2S(s)C(t) \text{ for all } s, t \in \mathbb{R};$$

$$(2.15) \quad S(s+t) = S(s)C(t) + S(t)C(s) \text{ for all } s, t \in \mathbb{R};$$

$$(2.16) \quad \text{there exist constants } M \geq 1 \text{ and } \omega \geq 0 \text{ such that}$$

$$|C(t)| \leq Me^{\omega|t|} \text{ for all } t \in \mathbb{R};$$

$$(2.17) \quad |S(t) - S(\hat{t})| \leq M \left| \int_{\hat{t}}^t e^{\omega|s|} \, ds \right| \text{ for all } t, \hat{t} \in \mathbb{R}.$$

We shall say that a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, is of type (M, ω) if it satisfies $|C(t)| \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$.

Proposition 2.3. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$. The following are true:

- (2.18) $D(A)$ is dense in X and A is a closed operator in X ;
- (2.19) if $x \in X$ and $r, s \in \mathbb{R}$, then $z \stackrel{\text{def}}{=} \int_r^s S(u) x \, du \in D(A)$ and $Az = C(s)x - C(r)x$;
- (2.20) if $x \in X$ and $r, s \in \mathbb{R}$, then $z \stackrel{\text{def}}{=} \int_0^r \int_0^s C(u)C(v) x \, du \, dv \in D(A)$ and $Az = (C(s+r)x - C(s-r)x)/2$;
- (2.21) if $x \in X$, then $S(t)x \in E_1$;
- (2.22) if $x \in E_1$, then $C(t)x \in E_1$;
- (2.23) if $x \in E_1$, then $S(t)x \in D(A)$ and $d/dt C(t)x = AS(t)x$;
- (2.24) if $x \in D(A)$, then $C(t)x \in D(A)$ and $d^2/dt^2 C(t)x = AC(t)x = C(t)Ax$;
- (2.25) if $x \in E_1$, then $\lim_{t \rightarrow 0} AS(t)x = 0$;
- (2.26) if $x \in E_1$, then $S(t)x \in D(A)$ and $d^2/dt^2 S(t)x = AS(t)x$;
- (2.27) If $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;
- (2.28) $C(t+s) - C(t-s) = 2AS(t)S(s)$ for all $t, s \in \mathbb{R}$.

Proposition 2.4. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$. Then the second order initial value problem (2.1) - (2.3) is well-posed. Moreover, if $x \in E_2$ and $y \in E_1$, then the unique solution $u(t; x, y)$ of (2.1) - (2.3) is given by $u(t; x, y) = C(t)x + S(t)y$.

Except for the claim of uniqueness of solutions to the initial value problem, Proposition 2.4 is a consequence of Proposition 2.3. The uniqueness of solutions is established by Proposition 2.4 in [29].

If $x \notin E_2$ or $y \notin E_1$, we may think of $C(t)x + S(t)y$ as a generalized solution to (2.1) - (2.3).

The following proposition is due to Sova [27], Theorem 2.24. It states that there is a unique correspondence between strongly continuous cosine families and their infinitesimal generators; that is, two different cosine families cannot have the same infinitesimal generator.

Proposition 2.4. Let $C_1(t)$, $t \in \mathbb{R}$, and $C_2(t)$, $t \in \mathbb{R}$, be strongly continuous cosine families with infinitesimal generators A_1 and A_2 , respectively. If $D(A_1) \subset D(A_2)$ and $A_1x = A_2x$ for all $x \in D(A_1)$, then $C_1(t) = C_2(t)$ for $t \in \mathbb{R}$.

As an analogue to the definition of the infinitesimal generator of a strongly continuous semigroup of operators, it is natural to try to replace definition (2.10) of the infinitesimal generator of a strongly continuous cosine family by the central difference approximation

$$\lim_{t \rightarrow 0} (C(-2t)x - 2C(0)x + C(2t)x)/4t^2,$$

of the second derivative of $C(t)$, $t \in \mathbb{R}$, at $t = 0$. In the case of an arbitrary function, it is well-known that the central difference definition of the second derivative is not equivalent to the classical definition. However, as the following proposition demonstrates, these two definitions are equivalent for functions satisfying the cosine functional equation (2.4).

Proposition 2.5. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A . Then $x \in D(A)$ if and only if $\lim_{t \rightarrow 0} (C(2t)x - x)/2t^2$ exists, and $Ax = \lim_{t \rightarrow 0} (C(2t)x - x)/2t^2$ for $x \in D(A)$.

The proof of Proposition 2.5 was first given in [27], Proposition 2.18. Another proof is presented in [30], Proposition 2.3.

The following proposition, whose proof may be found in [5], Lemma 5.6, gives a characterization of resolvent of the infinitesimal generator of a strongly continuous cosine family.

Proposition 2.6. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family of type (M, ω) with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$. Then for $\operatorname{Re} \lambda > \omega$, λ^2 is in the resolvent set $\rho(A)$ of A and

$$(2.29) \quad \lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} C(t)x \, dt, \quad x \in X,$$

$$(2.30) \quad R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt, \quad x \in X.$$

We now state a necessary and sufficient condition that an operator A be the infinitesimal generator of a strongly continuous cosine family. This condition, which is the analogue of the Hille-Yosida-Phillips-Miyadera generation theorem ([4], Theorem 13, p. 624) of operator semigroup theory, was established independently by M. Sova ([27], Theorem 3.1 and 3.2), G. DaPrato and E. Zuisti [3], and H. Fattorini ([5], Theorem 3.1).

Proposition 2.7. A closed densely defined linear operator A in X is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of type (M, ω) if and only if $R(\lambda^2; A)$ exists for $\lambda > \omega$, is strongly infinitely differentiable, and satisfies

$$(2.31) \quad \left| (d/d\lambda)^n (\lambda R(\lambda^2; A)) \right| \leq M n! / (\lambda - \omega)^{n+1}$$

for $\lambda > \omega$ and $n = 1, 2, \dots$.

We conclude this section with some simple examples of strongly continuous cosine families and their infinitesimal generators.

Example 2.1. ([6], p. 51). Let $A \in B(X;X)$ and define $C(t) = \sum_{k=0}^{\infty} A^k t^{2k} / (2k)!$, $t \in R$. Then $C(t)$, $t \in R$, is a strongly continuous cosine family in X with infinitesimal generator A and associated sine family $S(t) = \sum_{k=0}^{\infty} A^k t^{2k+1} / (2k+1)!$, $t \in R$. If $X = R$ and $a > 0$, then $C(t) = \cosh t/a$ if $Ax = ax$ and $C(t) = \cos t/a$ if $Ax = -ax$.

Example 2.2. ([27], Example 2.27). Let $X = C_{2\pi}(R;C)$ be the Banach space of continuous, complex valued, 2π -periodic functions on R with supremum norm, and let

$$(2.32) \quad (C(t)f)(x) = (f(x+t) + f(x-t))/2, \quad f \in X, \quad t \in R, \quad x \in R.$$

Then $C(t)$, $t \in R$, is a strongly continuous cosine family in X with infinitesimal generator

$$(2.33) \quad Af = f'',$$

$$D(A) = E_2 = \{f \in X: f'' \in X\},$$

and associated sine family

$$(2.34) \quad (S(t)g)(x) = \int_{x-t}^{x+t} g(s)ds/2, \quad g \in X, \quad t \in R, \quad x \in R,$$

where

$$E_1 = \{g \in X: g' \in X\}.$$

For $f \in E_2$ and $g \in E_1$, the formula $(C(t)f)(x) + (S(t)g)(x)$ gives the classical D' Alembert solution of the 1-dimensional wave equation

$$(2.35) \quad w_{tt}(x,t) = w_{xx}(x,t), \quad t \in R, \quad x \in R,$$

$$w(x,0) = f(x), \quad w_t(x,0) = g(x).$$

3. The Problem of Conversion to a First Order System

It is natural to try to convert the linear second order initial value problem (2.1) - (2.3) to a first order differential system. Formally, this conversion can be easily accomplished by defining $A: X \times X \rightarrow X \times X$ as

$$A(x,y) = (y, Ax),$$

$$D(A) = D(A) \times X,$$

and considering the first order system

$$(3.1) \quad U'(t) = AU(t), \quad t \in \mathbb{R},$$

$$(3.2) \quad U(0) = (x,y) \in D(A) \times X,$$

where $U: \mathbb{R} \rightarrow X \times X$. The difficulty with this approach is that the initial value problem (3.1) - (3.2) may not be well-posed in the sense that A does not necessarily generate a strongly continuous group in $X \times X$. To see that this is the case, let $x \in D(A)$ and $y \in X$, and suppose that A is the infinitesimal generator of a strongly continuous cosine family. Then the solution of the initial value problem (3.1) - (3.2) must be given by $U(t) = (C(t)x + S(t)y, AS(t)x + C(t)y)$. But, since $AS(t)$ is in general unbounded, the mapping $(x,y) \rightarrow U(t)$ does not necessarily possess an extension in $B(X \times X; X \times X)$.

It is, however, always possible to convert the second order initial value problem (2.1) - (2.3) to a well-posed first order system in the Banach space $E_1 \times X$. The result is due to J. Kisynski [16] and is stated below.

Proposition 3.1. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$.

Then E_1 under the norm

$$(3.3) \quad ||x||_{E_1} = ||x|| + \sup_{0 \leq t \leq 1} ||d/dt C(t)x||$$

becomes a Banach space and $U(t)$, $t \in \mathbb{R}$, defined by

$$(3.4) \quad U(t)(x,y) = (C(t)x + S(t)y, AS(t)x + C(t)y),$$

for $(x,y) \in E_1 \times X$, is a strongly continuous group in $E_1 \times X$ with infinitesimal generator

$$A(x,y) = (y, Ax)$$

$$D(A) = D(A) \times E_1.$$

Although the conversion of the second order equation (2.1) - (2.3) to a first order system presented in Proposition 3.1 can always be achieved, it is inconvenient for some problems. In certain problems, we begin with considerable information concerning the infinitesimal generator A , but very little information about the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, or the subspace E_1 . It is desirable, therefore, to have a conversion of (2.1) - (2.3) to a first order system which only involves information about the infinitesimal generator A . The possibility of such a conversion was first investigated by H. Fattorini in [5] and [6], and later by the authors in [30]. The idea is to obtain a square root B of A and to formulate the first order system in the space $[D(B)] \times X$. We will describe this conversion in the discussion below. We first require the following results concerning the existence of a square root of the infinitesimal generator of a strongly continuous cosine family due to H. Fattorini [5], p. 95.

Proposition 3.2. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family of type (M, ω) with infinitesimal generator A . If $b \geq \omega$ and $A_b \stackrel{\text{def}}{=} A - b^2 I$, then A_b is the infinitesimal generator of a strongly continuous cosine family $C_b(t)$, $t \in \mathbb{R}$, of type $(M_1, \omega + b)$, and there exists a closed linear operator $B_b: X \rightarrow X$ such that $B_b^2 = A_b$ and B_b commutes with every operator in $B(X, X)$ which commutes with A_b .

We remark that the square root B_b of A_b is defined by Fattorini in the following way: define

$$(3.5) \quad J_b x \stackrel{\text{def}}{=} \pi^{-1} \int_0^\infty \lambda^{-1/2} R(\lambda; A_b)(-A_b) d\lambda$$

for $x \in D(A)$. Fattorini [5], p. 95, shows that the integral (3.5) exists, has a closed extension, and that $B_b \stackrel{\text{def}}{=} iJ_b$. Moreover, if $\lambda^2 \in \rho(A_b)$, then $\lambda \in \rho(B_b)$ and $R(\lambda; B_b) = (\lambda I + B_b)R(\lambda^2; A)$. Also, if $b > \omega$, then $0 \in \rho(B_b)$ and we see that it is always possible to find or translate A_b of A such that A_b has a square root with the property that $0 \in \rho(B_b)$.

We now introduce a condition similar to one used by Fattorini in [5], p. 96.

A strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$, is said to satisfy condition (F) if and only if the following are true:

- (3.6) there exists a closed linear operator B in X such that $B^2 = A$ and B commutes with every operator in $B(X; X)$ which commutes with A ;
- (3.7) $S(t)$ maps X into $D(B)$ for each $t \in \mathbb{R}$ (which implies that $BS(t) \in B(X; X)$ for each $t \in \mathbb{R}$ since B is closed);
- (3.8) $BS(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

The usefulness of condition (F) in the problem of converting the second order initial value problem (2.1) - (2.3) to a first order system can be seen in the following proposition:

Proposition 3.3. Let A and B be linear operators in X , let $B^2 = A$, let B commute with every operator in $B(X; X)$ which commutes with A , and let $0 \in \rho(B)$.

Then the following are equivalent:

- (3.9) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, satisfying condition (F);
- (3.10) B is the infinitesimal generator of a strongly continuous group $T(t)$, $t \in \mathbb{R}$;

(3.11) $B(x,y) \stackrel{\text{def}}{=} (Bx, By)$, with $D(B) = D(B) \times D(B)$, is the infinitesimal generator of a strongly continuous group $U(t)$, $t \in \mathbb{R}$, in $X \times X$;

(3.12) $A(x,y) \stackrel{\text{def}}{=} (y, Ax)$, with $D(A) = D(A) \times D(B)$, is the infinitesimal generator of a strongly continuous group $V(t)$, $t \in \mathbb{R}$, in $[D(B)] \times X$.

The proof of Proposition 3.3 is given in [3], Proposition 2.6. In the proof that (3.9) $\Rightarrow$ (3.10), which was first given in [5], Theorem 6.6, one defines $T(t) = C(t) + BS(t)$. In the proof that (3.10) $\Rightarrow$ (3.11), which was first given in [5], Theorem 6.9, one defines $U(t)(x,y) = (C(t)x + BS(x)y, BS(t)x + C(t)y)$, where $C(t) \stackrel{\text{def}}{=} (T(t) + T(-t))/2$ and $S(t) = B^{-1}(T(t) - T(-t))/2$. In the proof that (3.11) $\Rightarrow$ (3.12), one defines $V(t)$ by

$$V(t) = \begin{bmatrix} B^{-1} & 0 \\ 0 & I \end{bmatrix} U(t) \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

In the proof that (3.12) $\Rightarrow$ (3.13), one defines $C(t)x = \pi_2 V(t)(0,x)$ and $S(t)x = \pi_1 V(t)(0,x)$, where π_1, π_2 are the projections of $[D(B)] \times X$ onto its first and second components, respectively. For $(x,y) \in D(B) \times X$, $V(t)$ is given by $V(t)(x,y) = (C(t)x + S(t)y, BS(t)Bx + C(t)y)$.

Since, for $x \in D(A)$, the set E_1 is precisely the set of $y \in X$ for which the initial value problem (2.1) - (2.3) has a twice continuously differentiable solution, the exact determination of E_1 is of considerable importance. The connection between E_1 and the square root of the infinitesimal generator of the strongly continuous cosine family is given in the proposition below.

Proposition 3.4. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A , and let B be an operator in X such that B commutes with every operator in $B(X;X)$ which commutes with A , zero is in the resolvent set of B , and $B^2 = A$. Then the following are equivalent:

$$(3.13) \quad D(B) \subset E_1;$$

$$(3.14) \quad C(t), t \in \mathbb{R}, \text{ satisfies condition (F);}$$

$$(3.15) \quad D(B) = E_1.$$

The proof that $(3.13) \Rightarrow (3.14)$ is due to the authors and also I. Mujadera (private communication), and will be published elsewhere. The proof that $(3.14) \Rightarrow (3.13)$ is found in [5], Remark 6.11, and the proof that $(3.14) \Rightarrow (3.15)$ is found in [30], Proposition 2.8.

It has been shown in [6], Theorem 2.2, that after some suitable translation of its infinitesimal generator, every strongly continuous cosine family in the Banach space L^p , $1 < p < \infty$, satisfies condition (F). A natural question to ask is whether or not every strongly continuous cosine family satisfies condition (F) (after a suitable translation of its infinitesimal generator). The answer is no and the counter example is due to J. Kisynski [16], Example 2 and B. Nagy [24], Theorem 3. Consider the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, presented in Example 2.2 of Section 2, but restricted to the Banach subspace of odd 2π -periodic complex valued functions on $\mathbb{R}$. In [16] it is shown that there does not exist a strongly continuous group $G(t)$, $t \in \mathbb{R}$, such that the representation $C(t) = (G(t) + G(-t))/2$ is valid. But if $C(t)$, $t \in \mathbb{R}$, defined on this Banach space satisfies condition (F), then by Proposition 3.3 $C(t)$, $t \in \mathbb{R}$, has such a representation, where $G(t)$, $t \in \mathbb{R}$, is the strongly continuous group defined by $G(t) \stackrel{\text{def}}{=} C(t) + BS(t)$.

There are many advantages to be gained from converting the linear initial value problem (2.1) - (2.3) to an equivalent well-posed first order system. The primary advantage being that a well-posed first order system corresponds to a strongly continuous group or semigroup of operators,

thereby making available for application the extensive theory of operator semigroups. There are some cases, however, in which it is more advantageous to treat the second order equation (2.1) - (2.3) directly, as we shall see in Section 6.

We conclude this section with a discussion of some examples.

Example 3.1. Let $C(t)$, $t \in \mathbb{R}$, be the cosine family of Example 2.2. If B is defined by $Bf = f'$, $D(B) = \{f \in C_{2\pi}(\mathbb{R}; \mathbb{C}) : f' \in C_{2\pi}(\mathbb{R}; \mathbb{C})\}$, then B satisfies the conditions (3.6), (3.7), and (3.8), and we have that the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, satisfies condition (F). Notice that $(BS(t)f)(x) = (f(x+t) - f(x-t))$ for $f \in C_{2\pi}(\mathbb{R}; \mathbb{C})$, $x \in \mathbb{R}$, and consequently that $|BS(t)| = 1$ if $t \neq 0$ and $|BS(t)| = 0$ if $t = 0$. Thus the operator $BS(t)$ is not continuous in the operator topology as a function of t on $\mathbb{R}$. Notice also that $B \neq iJ_0$, where J_0 is defined as in (3.5), since $J_0 f$ must be real-valued when f is real-valued. Thus, it is advantageous to state condition (F) in terms of an arbitrary square root B of A rather than the specific square root obtained using (3.5).

The following examples are treated by J. Goldstein in [12], section 8.

Example 3.2. Let L be a self-adjoint operator in a complex Hilbert space H such that $0 \in \rho(L)$. Define $B = iL$ and notice that $B^* = -iL^* = -B$. Thus, B is skew-adjoint and by Stone's Theorem ([Yosida], Theorem 1, p. 345), B generates a strongly continuous unitary group in H . It follows from the equivalence of (3.9) and (3.10) that $B^2 = -L^2$ is the infinitesimal generator of a strongly continuous cosine family in H satisfying condition (F).

Example 3.3. Let L be a positive self-adjoint operator in the complex Hilbert space H such that $0 \in \rho(L)$. Then L has a positive self-adjoint square root $L^{1/2}$ with the property that $0 \in \rho(L^{1/2})$. It follows from Example 3.2 that $-L$ is the infinitesimal generator of a strongly continuous cosine family in H satisfying condition (F).

4. Perturbation and Approximation Results for Strongly Continuous Cosine Families

Our objective in this section is to discuss some perturbation and approximation results for strongly continuous cosine families. A perturbation result is a result of the following type: given an infinitesimal generator A of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X , and a sequence of infinitesimal generators A_n , $n = 1, 2, \dots$, of corresponding strongly continuous cosine families $C_n(t)$, $t \in \mathbb{R}$, $n = 1, 2, \dots$, find sufficient conditions on the operators A_n , $n = 1, 2, \dots$, such that $C_n(t)$ converges in some sense to $C(t)$. One approach to establishing perturbation and approximation results for strongly continuous cosine families is to convert (2.1) - (2.3) to a well-posed first order system, and apply the perturbation and approximation theory of strongly continuous groups. This approach might be especially useful if condition (F) is satisfied. However, not every result known to be true for strongly continuous cosine families can be obtained in this manner. Our purpose here is to obtain results for strongly continuous cosine families directly.

The perturbation results we state below are established using methods analogous to those used by R. S. Phillips in the perturbation theory of strongly continuous semigroups (see [4], Theorem 19, p. 631). The first proposition can be found in [32].

Proposition 4.1. Let A be the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, with associated sine family $S(t)$, $t \in \mathbb{R}$, and let P be a closed linear operator in X such that

$$(4.1) \quad S(t)(X) \subset D(P) \text{ for all } t \in \mathbb{R};$$

$$(4.2) \quad PS(t)x \text{ is continuous in } t \text{ on } \mathbb{R} \text{ for each fixed } x \in X.$$

Then, $A + P$ is the infinitesimal generator of a strongly continuous cosine family in X .

We remark that conditions (4.1) and (4.2) are obviously satisfied provided $P \in B(X;X)$. The stability of strongly continuous cosine family generators under bounded perturbations was also established by B. Nagy in [24], Theorem 1.

As a consequence of Proposition 4.1, the following proposition is established in [32], Corollary 1.

Proposition 4.2. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with infinitesimal generator A , let P be a closed operator in X , and let P satisfy

$$(4.3) \quad D(A) \subset D(P);$$

$$(4.4) \quad \text{for each } t \in \mathbb{R}, \text{ there exists a constant } k_t \text{ such that } \|PS(t)x\| \leq k_t \|x\| \text{ for all } x \in D(A).$$

Then, $A + P$ is the infinitesimal generator of a strongly continuous cosine family in X .

The next two results are established in [32], Corollary 2 and Corollary 3. Proposition 4.3 is similar to a result of J. Goldstein [12], Theorem 8.9, p. 91.

Proposition 4.3. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with infinitesimal generator A , and let $C(t)$, $t \in \mathbb{R}$, satisfy condition (F).

Let P be a closed linear operator in X such that

$$(4.5) \quad D(B) \subset D(P), \text{ where } B \text{ is as in (3.6), (3.7) and (3.8).}$$

Proposition 4.4. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with infinitesimal generator A , and let $C(t)$, $t \in \mathbb{R}$, satisfy condition (F). Let P be a closed linear operator in X such that

$$(4.6) \quad D(A) \subset D(P)$$

(4.7) there exist constants a and b such that $\|P x\| \leq a \|x\| + b\|B x\|$ for all $x \in D(A)$, where B is as in (3.6), (3.7) and (3.8).

Then, $A + P$ is the infinitesimal generator of a strongly continuous cosine family in X .

The following result, which uses methods similar to those used to establish the propositions above, has been obtained by T. Takenaka and N. Okazawa in [29].

Proposition 4.5. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family of type (M, ω) in X with infinitesimal generator A . Let P be a linear operator in X such that

(4.8) $D(A) \subset D(P)$ and $PR(\lambda^2; A) \in B(X; X)$ for some $\lambda < \omega$;

(4.9) there exists a constant K_0 such that $\int_0^1 \|PC(s)x\| ds \leq K_0 \|x\|$ for each $x \in D(A)$.

Let $K_\infty = \lim_{\lambda \rightarrow \infty} K_\lambda$, where

$$K_\lambda = \sup \left\{ \int_0^1 e^{-\lambda s} \|PC(s)x\| ds : \|x\| \leq 1, x \in D(A) \right\},$$

and let ε satisfy $|\varepsilon| < K_\infty^{-1}$. Then $A + \varepsilon P$ is the infinitesimal generator of a strongly continuous cosine family $C_\varepsilon(t)$, $t \in \mathbb{R}$, in X and, moreover,

(4.10) $\lim_{\varepsilon \rightarrow 0} \|C_\varepsilon(t) - C(t)\| = 0$ uniformly on finite intervals of $\mathbb{R}$.

The approximation result stated below is an analog to the Trotter-Kato semigroup approximation theorem (see [Yosida], p. 269). This result was obtained by Y. Koniski [17] and J. Goldstein [11].

Proposition 4.6. For each $n = 0, 1, 2, \dots$, let $C_n(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with infinitesimal generator A_n . Suppose there exist constants $M \geq 1$ and $\omega \geq 0$ such that $\|C_n(t)\| \leq M e^{\omega t}$ for $t \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. Then the following are equivalent:

$$(4.11) \quad \lim_{n \rightarrow \infty} C_n(t)x = C_0(t)x \text{ for each } t \in \mathbb{R} \text{ and } x \in D(A);$$

$$(4.12) \quad \lim_{n \rightarrow \infty} R(\lambda^2; A_n)x = R(\lambda^2; A_0)x \text{ for each } \lambda > \omega \text{ and } x \in X.$$

In [11], Theorem 2, it is shown that a sufficient condition for (4.11) to hold is that there exists a linear subspace D in X such that $D \subset D(A_n)$ for $n = 0, 1, 2, \dots$, A_n is the closure of its restriction to D , and $\lim_{n \rightarrow \infty} A_n x = A_0 x$ for all $x \in D$. An example is given in [11] to show that this sufficient condition is not necessary.

5. Special Properties of Strongly Continuous Cosine Families: Compactness, Uniform Continuity, Inhomogeneous Equations

In this section we discuss some special properties of strongly continuous cosine families. The first of these is compactness of the sine family, a property which is very useful in the study of semilinear second order equations. It is well-known (see, e.g., [25]) that if a strongly continuous semigroup $T(t)$, $t \geq 0$, in X is compact for some $t_0 > 0$, then it is compact for every $t \geq t_0$. To see that an analogous property does not hold for strongly continuous cosine families, consider the strongly continuous cosine family in X whose infinitesimal generator is the operator $-I$. This cosine family is given by $C(t) = (\cos t)I$ and its associated sine family is given by $S(t) = (\sin t)I$. $C(t)$ is compact at odd integer multiples of $\pi/2$, while $S(t)$ is compact at integer multiples of π . If X is an infinite dimensional space, then these are the only values of t for which $C(t)$ and $S(t)$ are compact. The following properties are established in [31], Lemmas 2.1 and 2.2 and Proposition 2.3.

Proposition 5.1. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with associated sine family $S(t)$, $t \in \mathbb{R}$. The following are true:

- (5.1) if $C(t)$ is compact for t in an interval of positive length, then $C(t)$ is compact for all $t \in \mathbb{R}$, and in particular, the identity is compact and X is necessarily finite dimensional;
- (5.2) if $S(t)$ is compact for t in an interval of positive length, then $S(t)$ is compact for all $t \in \mathbb{R}$.

Proposition 5.2. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family in X with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$. The following are equivalent:

(5.3) $S(t)$ is compact for every $t \in \mathbb{R}$;

(5.4) $R(\lambda; A)$ is compact for some λ , and hence every λ , in $\rho(A)$.

In the example below, we demonstrate that the Cauchy problem for the wave equation

$$\begin{aligned} (5.5) \quad w_{xx}(x, t) &= w_{xx}(x, t), \quad 0 \leq x \leq \pi, \quad t \in \mathbb{R}, \\ w(0, t) &= w(\pi, t) = 0, \quad t \in \mathbb{R}, \\ w(x, 0) &= f(x), \quad w_t(x, 0) = g(x), \quad 0 \leq x \leq \pi, \end{aligned}$$

gives rise to a compact sine family in $L^2[0, \pi]$.

Example 5.1. If we write equation (5.5) abstractly in $X = L^2[0, \pi]$, it becomes

$$\begin{aligned} (5.6) \quad u''(t) &= Au(t), \quad t \in \mathbb{R}, \\ u(0) &= f, \quad u'(0) = g, \end{aligned}$$

where $u: \mathbb{R} \rightarrow X$, and A is the operator from X to X defined by

$$(5.7) \quad Af = f''$$

$D(A) = \{f \in X: f, f' \text{ are absolutely continuous,}$

$$f'' \in X, f(0) = f(\pi) = 0\}.$$

A can also be written as

$$Af = - \sum_{n=1}^{\infty} n^2 (f, f_n) f_n, \quad f \in D(A),$$

where $f_n(s) = (\sqrt{2}/\pi) \sin nx$, $n = 1, 2, \dots$, is the orthonormal set of eigenvectors of A . Since $-A$ is positive and self-adjoint in X , we have from Example 3.3 in Section 3 that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X satisfying condition (F).

It is easily seen that $C(t)$, $t \in \mathbb{R}$, is given by

$$C(t)f = \sum_{n=1}^{\infty} \cos nt (f, f_n) f_n, \quad f \in X,$$

and the associated sine family is given by

$$S(t)f = \sum_{n=1}^{\infty} (\sin nt/n) (f, f_n) f_n, \quad f \in X.$$

The resolvent of A is given by

$$R(\lambda; A)f = \sum_{n=1}^{\infty} (f, f_n) f_n / (\lambda + n^2),$$

for $f \in X$ and $-\lambda \neq 1^2, 2^2, \dots$. The compactness of $R(\lambda; A)$ follows from the fact that the eigenvalues of $R(\lambda; A)$ are $\lambda_n = 1/(\lambda + n^2)$, $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \lambda_n = 0$.

The next property of strongly continuous cosine families we discuss is the uniform continuity of $C(t)$, $t \in \mathbb{R}$; that is, the continuity of $C(t)$ as a function in t from $\mathbb{R}$ to $B(X; X)$. As in semigroup theory, this property is equivalent to the boundedness of the infinitesimal generator (see [4], Corollary 9, p. 621).

Proposition 5.3. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$. The following are equivalent:

$$(5.8) \quad \lim_{t \rightarrow \infty} C(t) = I \text{ in } B(X, X);$$

$$(5.9) \quad \lim_{t \rightarrow 0} S(t)/t = I \text{ in } B(X, X);$$

$$(5.10) \quad A \in B(X, X);$$

$$(5.11) \quad C(t)(X) \subset E_1 \text{ for all } t \text{ in an interval of positive length};$$

$$(5.12) \quad \text{there is an interval } [a, b] \text{ of positive length such that for all } t \in [a, b], S(t)(X) \subset D(A) \text{ and } AS(t) \text{ is strongly continuous.}$$

The proof of Proposition 5.3 is given in [31], Proposition 4.1.

The equivalence of (5.8) and (5.10) was first established by S. Kurepa in [20].

The last property of strongly continuous cosine families we treat in this section is the regularity of solutions to the inhomogeneous initial value problem

$$(5.13) \quad \begin{aligned} u''(t) &= Au(t) + f(t), \quad t \in \mathbb{R}, \\ u(0) &= x, \quad u'(0) = y, \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine family and f is a given continuous function from R to X . The inhomogeneous linear equation (5.13) has implications in the investigation of nonlinear second order abstract equations, as we will see in the next section. We define a function u from R to X to be a strong solution of (5.13) provided that u is twice continuously differentiable on R , $u(t) \in D(A)$ for $t \in R$, and (5.13) is satisfied. The following proposition is established in [31],

Proposition 2.4:

Proposition 5.4. Let $C(t)$, $t \in R$, be a strongly continuous cosine family with infinitesimal generator A and let $f: R \rightarrow X$ be continuous. If u is a strong solution of (5.13), then

$$(5.14) \quad u(t) = C(t)x + S(t)y + \int_0^t S(t-s) f(s) ds.$$

Equation (5.14), however, is more general than equation (5.13), and every function of the form (5.14) need not be a strong solution of (5.13). For this reason, we will define functions of the form (5.14) as mild solutions of equation (5.13). In the discussion below, we will state conditions on f that guarantee that every mild solution is a strong solution.

Proposition 5.5. Let $C(t)$, $t \in R$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in R$, and let f be continuous from R to X . The initial value problem (5.13) has a strong solution for every $x \in D(A)$ and $y \in E_1$ if and only if the X -valued function

$$(5.15) \quad g(t) = \int_0^t S(t-s) f(s) ds, \quad t \in R,$$

is twice continuously differentiable in t on R . A sufficient condition that the function g defined by (5.15) be twice continuously differentiable in t on R is that f be once continuously differentiable in t on R .

Proposition 5.6. Let $C(t)$, $t \in \mathbb{R}$, be a strongly continuous cosine family with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$, and let f be continuous from $\mathbb{R}$ to X . The initial value problem (5.13) has a strong solution for every $x \in D(A)$ and $y \in E_1$ if and only if the X -valued function

$$(5.16) \quad g(t) = \int_0^t S(t-s) f(s) ds, \quad t \in \mathbb{R},$$

has the property that $g(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Ag(t)$ is continuous as a function in t from $\mathbb{R}$ to X . A sufficient condition that the function g defined by (5.16) have the property that $g(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Ag(t)$ is continuous as a function in t from $\mathbb{R}$ to X is that $f(t) \in D(A)$ for all $t \in \mathbb{R}$ and $Af(t)$ is continuous as a function in t from $\mathbb{R}$ to X .

Propositions 5.6 and 5.7 are analogous to results concerning the inhomogeneous first order equation from semigroup theory (see [25]). The proof of these propositions is given in [31], Proposition 3.4, Corollary 3.5, Proposition 3.6, Corollary 3.7.

6. Abstract Second Order Semilinear Equations

In this section we will discuss second order semilinear initial value problems. For the sake of simplicity, we will restrict our attention to autonomous equations. We will consider equations of the form

$$(6.1) \quad u''(t) = Au(t) + F(u(t), u'(t)),$$

$$(6.2) \quad u(0) = x, u'(0) = y,$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X , F is a mapping from $X \times X$ to X , and the unknown solution u maps some interval about 0 to X .

We seek mild solutions of (6.1) - (6.2), that is, solutions of the integral equation

$$(6.3) \quad u(t) = C(t)x + S(t)y + \int_0^t S(t-s) F(u(s), u'(s)) ds.$$

We lose no generality by considering solutions to (6.3) since if $F(u(t), u'(t))$ is continuous in t , then by virtue of Proposition 5.4, every solution of the initial value problem (6.1) - (6.2) is also a solution of (6.3). Moreover, equation (6.2) is easier to work with than (6.1) - (6.2) because of the nice properties of the bounded operators $C(t)$, $t \in \mathbb{R}$, and $S(t)$, $t \in \mathbb{R}$, as opposed to the unbounded operator A in equation (6.1). This approach to problem (6.1) - (6.2) traces back to the work of I. Segal in [26]. In [26] cosine family theory was not employed, but rather (6.1) - (6.2) was converted to a first order system and the theory of strongly continuous groups was applied.

We will allow for a variety of hypotheses in the semilinear initial value problem (6.1) - (6.2): (i) hypotheses on the cosine family $C(t)$, $t \in \mathbb{R}$, such as condition (F) or compactness of the sine family; (ii) hypotheses on F such as continuity conditions or the assumption that F depends only on its first variable; (iii) hypotheses on the initial values

such as membership in the set $D(A)$ or E_1 . Many subtleties arise from these various hypotheses with regard to the existence, uniqueness, continuous dependence upon initial values, smoothness, and continuability of solutions. We will give a sampling of results that can be obtained. The proofs can be found in [30], Section 3. In what follows, we suppose that $C(t)$, $t \in \mathbb{R}$, is a strongly continuous cosine family in X with infinitesimal generator A and associated sine family $S(t)$, $t \in \mathbb{R}$.

Proposition 6.1. Let D be an open subset of $X \times X$ and let $F: D \rightarrow X$ satisfy

$$(6.4) \quad || F(x,y) - F(\hat{x},\hat{y}) || \leq L(r) (||x - \hat{x}|| + ||y - \hat{y}||)$$

for $(x,y), (\hat{x},\hat{y}) \in D$ satisfying $||x||, ||\hat{x}||, ||y||, ||\hat{y}|| \leq r$,

where $L(r)$ is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$. For each $(x,y) \in D$ such that $x \in E_1$, there exist $t_0 > 0$ and a unique continuously differentiable function $u: (-t_1, t_1) \rightarrow X$ satisfying (6.3). Further, if $D = X \times X$ and $L(r)$ is constant, then t_1 can be taken as $+\infty$.

The method of proof employed to obtain the local existence and uniqueness of solutions claimed in Proposition 6.1 is the classical method of successive integrations. Regarding the continuous dependence of solutions on initial values, we have the following result.

Proposition 6.2. Suppose the hypothesis of proposition 6.1 and let $t_1 > 0$.

There exist constants $K = K(t_0)$ and $\gamma = \gamma(t_1)$ such that if u and $\hat{u}$ satisfy

$$(6.3) \text{ for } |t| < t_1 \text{ with } u(0) = x \in E_1, u'(0) = y, \hat{u}(0) = \hat{x} \in E_1,$$

$\hat{u}'(0) = \hat{y}, (x,\hat{y}), (\hat{x},\hat{y}) \in D$, then for $|t| < t_1$,

$$(6.5) \quad || u(t) - \hat{u}(t) || + || u'(t) - \hat{u}'(t) || \leq K(||x - \hat{x}|| + ||A(x - \hat{x})|| + ||y - \hat{y}||) e^{\gamma|t|}.$$

The proof of Proposition 6.2 uses Gronwall's Lemma and the estimates in (2.16) and (2.17). With additional hypothesis on F , we can claim that mild solutions are strong solutions.

Proposition 6.3. Suppose the hypotheses of Proposition 6.1 and, in addition, that F is continuously differentiable on D and satisfies

$$(6.6) \quad \begin{aligned} || F_1(x,y) - F_1(\hat{x},\hat{y}) || &\leq L(r) (|| x - \hat{x} || + || y - \hat{y} ||), \\ || F_2(x,y) - F_2(\hat{x},\hat{y}) || &\leq L(r) (|| x - \hat{x} || + || y - \hat{y} ||), \end{aligned}$$

for $(x,y), (\hat{x},\hat{y}) \in D$ satisfying $|| x ||, || y ||, || \hat{x} ||, || \hat{y} || \leq r$, where $L(r)$ is a nondecreasing function from R^+ to R^+ and F_1 and F_2 denote the derivative of F with respect to its first and second variable, respectively. Then the mild solution u of equation (6.3), whose existence is established by Proposition 6.1, is a strong solution of (6.1) - (6.2), provided $x \in D(A)$ and $y \in E_1$.

The next result concerns the continuability of local solutions of equation (6.3).

Proposition 6.4. Suppose the hypotheses of Proposition 6.1 and suppose $D = X \times X$. If $x \in E_1$, $y \in X$, and u is a solution of (6.3) noncontinuable to the right on $[0,b]$, then either $b = +\infty$ or $\lim_{t \rightarrow b^-} (|| u(t) || + || u'(t) ||) = +\infty$. An analogous result holds for solutions noncontinuable to the left. If F does not depend on its second variable, then we obtain existence of mild solutions under weaker assumptions on the initial values.

Proposition 6.5. Let D be an open subset of X and let $F: D \rightarrow X$ be such that

$$(6.7) \quad || F(x) - F(\hat{x}) || \leq L(r) || x - \hat{x} ||$$

for $x, \hat{x} \in D$ satisfying $|| x ||, || \hat{x} || \leq r$, where $L(r)$ is a nondecreasing function from R^+ to R^+ . For each $x \in D$ and $y \in X$, there exist $t_1 > 0$ and an unique continuous function $u: (-t_1, t_1) \rightarrow X$ satisfying

$$(6.8) \quad u(t) = C(t)x + S(t)y + \int_0^t S(t-s) F(u(s)) ds.$$

Further, if $D = X$ and L is a constant, then t_1 can be taken as $+\infty$.

If $D = X$, $x \in E_1$, $y \in X$, and u is a solution of (6.8) noncontinuable to the right on $[0,b]$, then either $b = +\infty$ or $\lim_{t \rightarrow b^-} || u(t) || = +\infty$.

An analogous result holds for solutions noncontinuable to the left.

If we know that F does not depend on its second variable and that $S(t)$, $t \in \mathbb{R}$, is a compact sine family, then the Lipschitz continuity condition (6.7) on F can be weakened to only continuity. As is to be expected, we lose the uniqueness of solutions.

Proposition 6.6. Let the sine family $S(t)$ be compact for each $t \in \mathbb{R}$ and let $F: D \rightarrow X$ be continuous where D is an open subset of X . For each $x \in D$ and $y \in X$ there exist $t_1 > 0$ and a continuous function $u: (-t_1, t_1) \rightarrow X$ satisfying (6.8). Further, if $D = X$, F maps closed bounded sets into bounded sets, $x \in E_1$, $y \in X$, and u is a solution of (6.8) noncontinuable to the right on $[0, b]$, then either $b = +\infty$ or $\lim_{t \rightarrow b} \|u(t)\| = +\infty$. An analogous result holds for a solution noncontinuable to the left.

Proposition 6.6 is established by an application of the Schrauder Fixed Point Theorem. Another weakening of the hypothesis on F can be achieved if we suppose that $C(t)$, $t \in \mathbb{R}$, satisfies condition (F).

Proposition 6.7. Let $C(t)$, $t \in \mathbb{R}$, satisfy condition (F) and let B be a square root of A satisfying (3.6) - (3.8). Let D be an open subset of $[D(B)] \times X$ and let $F: D \rightarrow X$ be such that

$$(6.9) \quad \|F(x, y) - F(\hat{x}, \hat{y})\| \leq L(r) (\|B(x - \hat{x})\| + \|y - \hat{y}\|)$$

for $(x, y), (\hat{x}, \hat{y}) \in D$ satisfying $\|Bx\|, \|B\hat{x}\|, \|y\|, \|\hat{y}\| \leq r$, where $L(r)$ is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$. For each $(x, y) \in D$, there exist $t_1 > 0$ and a unique continuously differentiable function $u: (-t_1, t_1) \rightarrow X$ satisfying (6.3). Further, if $D = D(B) \times X$ and $L(r)$ is constant, then t_1 can be taken as $+\infty$.

The proof of Proposition 6.7 uses the fact that when $C(t)$, $t \in \mathbb{R}$, satisfies condition (F), the second order equation (6.1) - (6.2) can be converted to a well-posed first order system in the product space.

$[D(B)] \times X$. The condition (6.9) on F gives rise to a Lipschitz condition on the nonlinear part of the new first order system, and the proof proceeds by the method of successive integrations. We observe that there is no loss of generality in considering equation (6.1) - (6.2) even in the case where it is necessary to translate the infinitesimal generator to obtain a cosine family satisfying condition (F). We replace A by $A - bI$ and $F(u(t), u'(t))$ by $F(u(t), u'(t)) + bu(t)$, and the hypothesis of Proposition (6.7) will still be satisfied. It is possible to obtain results under the hypothesis of Proposition (6.7) which are analogous to the results of Propositions 6.2, 6.3, and 6.4, but we omit their statement here. We remark that by virtue of Proposition 3.4, we have that $E_1 = D(B)$ whenever condition (F) is satisfied.

REFERENCES

1. V. Barbu, A class of boundary problems for second order abstract differential equations, J. Fac. Sci. Univ. Tokyo I A Math. 19 (1972), 295-319.
2. V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International, Leyden, The Netherlands, 1975.
3. G. Da Prato and E. Giusti, Una caratterizzazione dei generatori di funzioni coseno astratte, Boll. Unione Mat. Italiana 22 (1967), 357-362.
4. N. Dunford and J. T. Schwartz, Linear Operators, Part I: General Theory, Interscience, New York, 1958.
5. H. O. Fattorini, Ordinary differential equations in linear topological spaces, I, J. Differential Equations 5 (1968), 72-105.
6. H. O. Fattorini, Ordinary differential equations in linear topological spaces, II, J. Differential Equations 6 (1969), 50-70.
7. H. O. Fattorini, Uniformly bounded cosine functions in Hilbert space, Indiana Univ. Math. J. 20 (1970), 411-425.
8. E. Giusti, Funzioni coseno periodiche, Boll. Unione Mat. Italiana 22 (1967), 478-485.
9. J. Goldstein, Semigroups and hyperbolic equations, J. Functional Analysis 4 (1969), 50-70.
10. J. Goldstein, On a connection between first and second order differential equations in Banach spaces, J. Math., Anal., Appl. 30 (1970), 246-251.
11. J. Goldstein, On the convergence and approximation of cosine functions, Aequationes Math. 11 (1974), 201-205.
12. J. Goldstein, Semigroups of Operators and Abstract Cauchy Problems, Monograph, Tulane University, New Orleans, 1970.
13. T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.
14. J. Kisynski, On operator-valued solutions of d'Alembert's functional equation, I., Colloquium Math. 23 (1971), 107-114.
15. J. Kisynski, On operator-valued solutions of d'Alembert's functional equation, II., Studia Math. 42 (1971), 43-66.
16. J. Kisynski, On cosine operator functions and one-parameter groups of operators, Studia Math. 44 (1972), 93-105.
17. Y. Konishi, Cosine functions of operators in locally convex spaces, J. Fac. Sci. Univ. Tokyo I A Math. 18 (1971/72), 443-463.

REFERENCES (CONT.)

13. S. G. Krein, Linear Differential Equations in Banach Spaces, Amer. Math. Soc. Translations of Math. Monographs, Vol. 29, Providence, 1971.
19. S. Kurepa, On some functional equations in Banach spaces, Studia Math. 19 (1960), 149-158.
20. S. Kurepa, A cosine functional equation in Banach algebras, Acta. Sci. Math. Szeged 23 (1962), 255-267.
21. V. Lakshmikantham, Differential Equations in Abstract Spaces, Academic Press, New York, 1972.
22. B. Nagy, On the generators of cosine operator functions, Publicationes Mathematicae 21 (1974), 151-154.
23. B. Nagy, On cosine operator functions in Banach spaces, Acta Scientiarum Mathematicarum Szeged 35 (1974), 281-289.
24. B. Nagy, Cosine operator functions and the abstract Cauchy problem, Periodica Mathematica Hungarica Vol. 7 (3) (1976), 15-18.
25. A. Pazy, Semi-Groups of Linear Operators and Applications to Partial Differential Equations, Lecture Notes, University of Maryland, 1974.
26. I. Segal, Non-linear semi-groups, Ann. Math. 78 (1963), 339-364.
27. M. Sova, Cosine operator functions, Rozprawy Matematyczne 49 (1966), 1-47.
28. M. Sova, Semigroups and cosine functions of normal operators in Hilbert Spaces, Casopis Pest. Mat. 93 (1968), 437-458.
29. T. Takenaka and N. Okazawa, A Phillips-Miyadera type perturbation theorem for cosine functions of operators, to appear Tohoku Math. J.
30. C. Travis and G. Webb, Cosine families and abstract nonlinear second order differential equations, to appear.
31. C. Travis and G. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, to appear.
32. C. Travis and G. Webb, Perturbation of strongly continuous cosine family generators, to appear.
33. C. Travis and G. Webb, An abstract second order semilinear volterra integrodifferential equation, to appear.
34. K. Yosida, Functional Analysis, Fourth Ed., Springer-Verlag, New York, 1974.