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MASTER

SINGLE PASS COLLIDER MEMO CN-340

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TITLE: Relationship of Field Components and the
 Matched Dispersion Function in Arc Achromats

Summary

The general integral condition connecting the field, its derivative and the resulting η function derived for any lattice is applied to the achromats of the SLC Arcs. This condition can be combined with the non-dispersive condition in Ref. 1 to give a simple parameterization of second-order achromats constructed of combined function magnets.

Introduction

In general the equation satisfied by the dispersion function $D(s)$ is given by

$$D'' + k D = \frac{1}{\rho} \quad (1)$$

where D'' is the second derivative of D with respect to path length s of the central trajectory and where k and ρ can be functions of s . The coefficient k is a linear monomial expression of the magnetic field gradient and ρ is obtained from the magnetic rigidity of the beam and the strength of the magnetic dipole. The initial conditions placed on $D(s)$ and $D'(s)$ uniquely define the dispersion function.

By rewriting this equation and integrating over the path length between two arbitrary points $s = 0$ to $s = L$ we obtain

$$\int_0^L \left(\frac{1}{\rho} - k D \right) ds = D'(L) - D'(0) \quad (2)$$

If the specific boundary condition $D'(L) = D'(0)$ is applied then $D(s)$ satisfies

the integral condition

$$\int_0^L \left(\frac{1}{\rho} - k D \right) ds = 0 \quad (3)$$

Matched Dispersion Function

A specific case where equation (3) is satisfied is that of the *matched* dispersion function η for which the integral is taken over a lattice cell and for which the boundary conditions become

$$\eta(L_{cell}) = \eta(0) \quad (4)$$

and

$$\eta'(L_{cell}) = \eta'(0) \quad (5)$$

For this function equation (3) becomes

$$\int_0^{L_{cell}} \left(\frac{1}{\rho} - k \eta \right) ds = 0 \quad (6)$$

This condition on the dipole and quadrupole fields and the η function produced by them is valid for a cell of an arbitrary lattice. A similar and independent finding has been recently reported in Ref. 2 from a somewhat different point of view.

Application to Achromats

Though equation (6) can be applied to any lattice here we will evaluate this integral for the cases specified in Ref. 1, where the lattices consist of combined function magnets of equal length l and where ρ and k are piecewise constant. Then for focusing magnets:

$$\frac{1}{\rho} = \frac{B_o}{p_o} ; \quad k = \frac{B_o'}{p_o} \quad k > 0 \quad (7)$$

and for defocusing magnets:

$$\frac{1}{\rho} = \frac{B_o}{p_o} ; \quad k = \frac{B_o'}{p_o} ; \quad k < 0 \quad (8)$$

Performing the integration in equation (6) we find

$$\int_0^{L_{cell}} \left(\frac{1}{\rho} - k \eta \right) ds = \frac{l}{p_o} [2B_o - |B_o'|(\bar{\eta}_F - \bar{\eta}_D)] = 0 \quad , \quad (9)$$

where $\bar{\eta}_F$ and $\bar{\eta}_D$ are the average values of the η function in the focusing and defocusing magnets, respectively.

Equation (9) is satisfied if

$$\frac{|B_o'|}{2B_o} (\bar{\eta}_F - \bar{\eta}_D) = 1 \quad , \quad (10)$$

Evaluation of the average η

Consider half of a symmetric cell of length $L/2$ starting with half of a focusing magnet with length l , followed by a drift of length d and half of a defocusing magnet also with length l . Both magnets are assumed to be combined function magnets, i.e. they produce dipole, quadrupole and sextupole fields. (The latter will be considered later).

Introduce the notations

$$G = 1/|k|\rho \quad . \quad (11)$$

$$\lambda = 1/\sqrt{|k|} \quad . \quad (12)$$

Consider now the region of the focusing half a magnet $0 < s < l$, where both λ and G are constant ($k > 0$). By symmetry the initial conditions for the η function and its derivative are

$$\eta(0) = \eta_{maz} \quad , \quad (13)$$

$$\eta'(0) = 0 \quad . \quad (14)$$

The solution of equation (1) can be immediately written as

$$\eta(s) = \eta_{maz} \cos \frac{s}{\lambda} + G(1 - \cos \frac{s}{\lambda}) \quad , \quad (15)$$

$$\eta'(s) = \frac{G - \eta_{maz}}{\lambda} \sin \frac{s}{\lambda} \quad . \quad (16)$$

In the drift region $l < s < l + d$, where both k and G are zeros the solution of equation (1) is

$$\eta(s) = \alpha(s - l) + \beta \quad , \quad (17)$$

$$\eta'(s) = \alpha \quad (18)$$

where the constants α and β are to be determined.

Finally, in the region of the defocusing half a magnet $l + d < s < L/2$, both λ and G are also constant ($k < 0$). By symmetry the boundary conditions for η function and its derivative at $L/2$ are

$$\eta(L/2) = \eta_{min} \quad , \quad (19)$$

$$\eta'(L/2) = 0 \quad . \quad (20)$$

the solution of equation (1) is similarly

$$\eta(s) = \eta_{min} \cosh \frac{s - L/2}{\lambda} + G(\cosh \frac{s - L/2}{\lambda} - 1) \quad , \quad (21)$$

$$\eta'(s) = \frac{G + \eta_{min}}{\lambda} \sinh \frac{s - L/2}{\lambda} \quad . \quad (22)$$

Integrating equations (15) and (21) over half of the focusing (defocusing) magnet we obtain

$$\bar{\eta}_F = (\eta_{maz} - G) \frac{\lambda}{l} \sin \frac{l}{\lambda} + G \quad , \quad (23)$$

$$\bar{\eta}_D = (\eta_{min} + G) \frac{\lambda}{l} \sinh \frac{l}{\lambda} - G \quad , \quad (24)$$

Next we determine η_{maz} and η_{min} by matching η and η' on both sides of the interfaces at $s = l$ and $s = l + d$, which eliminates α and β . The resulting linear algebraic system of equations is then solved for η_{maz} and η_{min} giving

$$\eta_{maz} = G \left(1 + \frac{2 \sinh \frac{l}{\lambda}}{\sin \frac{l}{\lambda} \cosh \frac{l}{\lambda} - \cos \frac{l}{\lambda} \sinh \frac{l}{\lambda} + \frac{d}{\lambda} \sin \frac{l}{\lambda} \sinh \frac{l}{\lambda}} \right) \quad . \quad (25)$$

$$\eta_{min} = -G \left(1 - \frac{2 \sin \frac{l}{\lambda}}{\sin \frac{l}{\lambda} \cosh \frac{l}{\lambda} - \cos \frac{l}{\lambda} \sinh \frac{l}{\lambda} + \frac{d}{\lambda} \sin \frac{l}{\lambda} \sinh \frac{l}{\lambda}} \right) \quad (26)$$

Inserting these expressions into equations (23) and (24) it follows that equation (10) indeed is satisfied exactly.

The nondispersive condition specified in Ref. 1. is reproduced here for the reader's convenience:

$$\frac{B_o'' \bar{\eta}}{B_o'} = 1 \quad (27)$$

This relationship and that given in equation 10 have been examined for the two achromats cited in that reference.

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This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

In Table 1 are collected the values of B_o , B'_o and B''_o for two examples of lattice parameters at 50 GeV. The first one represents the SLC Arc lattice (108° phase shift per cell) and the second is similar but with 135° phase shift per cell. In both cases $l = 125.15\text{cm}$ and $d = 9.32\text{cm}$. The calculated values for η_{max} , η_{min} , $\bar{\eta}_F$, $\bar{\eta}_D$ and left sides of both equations (10) and (27) are also given.

Table 1

Magnet	B_o	B'_o	B''_o	$\eta_{max,min}$	$\bar{\eta}$	$\frac{B''_o}{B_o} \bar{\eta}$	$\frac{ B'_o }{2B_o} (\bar{\eta}_F - \bar{\eta}_D)$
	kg	kg/cm	kg/cm ²	cm	cm		
phase/cell 108°							1.000
FFOC108	5.970	7.014	1.621	4.743	4.330	1.001	
DFOC108	5.970	-7.014	-2.710	2.273	2.627	1.015	
phase/cell 135°							1.000
FFOC135	5.970	8.034	2.367	3.757	3.392	1.000	
DFOC135	5.970	-8.034	-4.310	1.600	1.906	1.023	

As demonstrated equation (10) is satisfied exactly. Equation (27) is satisfied to within a few percent using the value for the sextupole strength as determined by TRANSPORT.

Combining equations (10) and (27) gives an expression for the fields which is independent of η and correct to a few percent.

Finally we note that equations (9) through (26) can be rewritten to include the more general case where the magnets are of differing lengths and differing gradients. This has been done but is not reproduced here. Calculations for such an achromat have been made with results similar to those shown in Table 1. That is, equation (10) is satisfied exactly, equation (27) is good to a few percent and the calculated values for η agree exactly with those determined by TRANSPORT.

Conclusion

The relationships given in equations (10) and (27) along with the analytic expression for the matched η function can be useful for parameterization of second-order achromats constructed of combined function magnets.

REFERENCES

1. J. J. Murray, T. Fieguth and S. Kheifets, CN-338, July 23, 1986
2. M. Sands and W. Weng, CN-333, July 20, 1986