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MODES AND ASSOCIATED QUASILINEAR TRANSPORT

By

G. Rewoldt, W.M. Tang, and R.J. Hastie

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Collisional Effects on Kinetic Electromagnetic Modes and Associated Quasilinear Transport

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ABSTRACT

The general procedure for the analysis of low-frequency electrostatic and electromagnetic modes in toroidal geometry is now well known. In the collisionless limit, the relevant dynamics (*e.g.*, trapped particles, resonances, *etc.*) can be treated appropriately. However, with the introduction of collisional effects, it is customary, for tractability, to employ model collision operators of varying degrees of complexity. Guided by results of earlier studies of alternative collision operators in unsheared slab geometry and in toroidal geometry, an improved model collision operator is introduced here for calculating toroidal eigenmodes. Analytic and numerical results are presented to support its relevance and to demonstrate its improvement over earlier models. The associated quasilinear particle and energy transport coefficients for each species are also calculated, and compared with the usual $D_j \simeq \kappa_j \simeq \gamma/k_{\perp}^2$ estimate.

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I. INTRODUCTION

Kinetic calculations of low-frequency modes such as trapped-electron drift modes, ion-temperature-gradient modes, and magnetohydrodynamic (MHD) ballooning modes in toroidal geometry are normally carried out using simplified model collision operators. In particular, algebraic models are used to approximate more accurate operators (e.g., Lorentz operators), in order to make tractable the analytic solution of the most comprehensive form of the linearized gyrokinetic equation. In Ref. 1, the eigenfrequencies of the corresponding modes in a simpler, unshaped slab geometry were calculated, both for Lorentz electron collision operators and for algebraic (Krook-type) electron collision operators with various conservation properties. The results have helped to identify the specific properties of a collision operator which are most important for these instabilities, and have thus served to provide guidance in choosing model collision operators in toroidal geometry. In particular, number conservation appears to be vital (especially for electromagnetic modes),¹ energy conservation appears to be considerably less important,¹ and the pitch-angle variation of the perturbed distribution function, which is important with a Lorentz operator, makes a significant quantitative difference.² The latter effect, together with the energy dependence of the collision frequency, can be readily incorporated in a Krook-type model.³

In Sec. II, an improved model collision operator for toroidal eigenfrequency eigenfunction calculations is presented. The model collision operator used previously in Refs. 3 and 4 included the pitch-angle and energy dependences just mentioned, but conserved neither particle number nor energy. In addition, when this operator is applied to a Maxwellian

distribution function, it gives a spurious nonzero result in important terms of the relevant electromagnetic equations. These deficiencies are corrected with the improved model operator specified in Sec. II. This is illustrated by analytic comparisons of the improved versus previously used collision operators. These analytic estimates serve to clarify the differences in the operators. In Sec. III and the Appendix, a comprehensive numerical model is specified for the eigenfrequency-eigenfunction calculation, which is an improvement over that of Refs. 3 and 4 by the incorporation of the improved model collision operator. In addition, the associated quasilinear particle and energy transport coefficients for each species are calculated in terms of the quantities defined for the linear calculation, using the formulation of Horton,⁵ in terms of an unspecified saturation level for the mode. Ambient gradient or mixing length type approximations for this saturated mode amplitude can then be used to make scaling estimates. In Sec. IV, numerical results are presented for the toroidal drift mode and the kinetic MHD ballooning mode for cases corresponding to the Impurity Study Experiment (ISX-B) and to the Poloidal Divertor Experiment (PDX) in its improved confinement mode of operation (H-mode). Results are summarized and conclusions are drawn in Sec. V.

II. ANALYTIC COMPARISON OF MODEL COLLISION OPERATORS

As noted in Sec. I, the results of previous studies¹ suggest that the property of number conservation is necessary in model collision operators for kinetic calculations of high- n modes in toroidal geometry. To check this, we have altered our previous model Krook collision operator for our kinetic toroidal electromagnetic eigenfrequency-eigenfunction

calculation^{3,4} to be exactly number conserving. For the usual toroidal drift modes (including both trapped-electron and ion-temperature-gradient instabilities), the conserving and nonconserving operators agree well in the electrostatic ($\beta = 0$) limit, as will be shown in Sec. IV. However, in the more general electromagnetic ($\beta \neq 0$) case, the results with the number-conserving and nonconserving operators can be quite different.

To make this more understandable, we present results of approximate analytic calculations comparing different electron collision models before presenting more exact numerical results in Secs. III and IV. Specifically, consider the linearized electromagnetic drift kinetic equation for electrons:

$$\left[-i\sigma|v_{\parallel}| \frac{\partial}{\partial s} + iC - (\omega - \omega_{de}) \right] g_{\sigma} = \frac{|e|}{T_e} F_{me} (\omega - \omega_{*e}^T) \left[\tilde{\Xi} - \frac{\sigma|v_{\parallel}|}{c} \tilde{A}_{\parallel} \right], \quad (1)$$

where $\tilde{\Xi} \equiv \tilde{\Phi} + (v_{\perp}^2/2\Omega_e c) \tilde{B}_{\parallel}$. Here, $\tilde{\Phi}$ is the perturbed electrostatic potential, $\tilde{B}_{\parallel}$ is the perturbed magnetic field component along the equilibrium magnetic field, and $\tilde{A}_{\parallel}$ is the perturbed vector potential component along the equilibrium magnetic field. Also, $\sigma = \pm 1$ is the sign of $v_{\parallel}$, $\omega_{*e}^T \equiv \omega_{*e} [1 + \eta_e (E/T_e - 3/2)]$, $\eta_e \equiv (d \ln T_e / dr) / (d \ln n_e / dr)$, s is the distance along the equilibrium magnetic field, $f_e \equiv g_{\sigma} + (|e| \tilde{\Phi} / T_e) F_{me}$ is the perturbed distribution function, and the rest of the notation is standard. We specialize here to the limit $|\omega| \ll \omega_{be}$, ω_{te} , where ω_{be} is the trapped electron bounce frequency and ω_{te} is the circulating electron transit frequency, and to the limit $\nu_{fe} \ll \omega_{be}$ (banana regime), where ν_{fe} is the effective electron collision frequency; ν_{fe} is a function of energy and pitch angle, which will be specified in the Appendix. In these limits, it is important to introduce the transformation

$$g_{\sigma} \equiv h_{\sigma} - \frac{|e|}{T_e} F_{me} \left(1 - \frac{\omega_{*e}^T}{\omega} \right) \tilde{\psi}, \quad (2)$$

where

$$\tilde{\psi} \equiv i \frac{\omega}{2c} \left[\int_{-\infty}^s ds' \tilde{A}_{\parallel}(s') - \int_s^{\infty} ds' \tilde{A}_{\parallel}(s') \right], \quad (3)$$

so that $\tilde{A}_{\parallel} = -i(c/\omega)(\partial/\partial s)\tilde{\psi}$ and $\tilde{E}_{\parallel} = -(\partial/\partial s)(\tilde{\Phi} - \tilde{\psi})$. This transformation from g_{σ} to h_{σ} removes the substantial perturbed Maxwellian part of g_{σ} before approximating the collision operator, and thus avoids a spurious dissipative contribution arising from $C \left[(|e|F_{me}/T_e)(1 - \omega_{*e}^T/\omega) \tilde{\psi} \right]$ which would be found if the approximate operator were introduced into the drift kinetic equation for g_{σ} , Eq. (1).

The following three collision models will be compared:

model a:

$$C = -\nu_{fe} g_{\sigma} \quad \text{for both trapped and circulating electrons.} \quad (4)$$

model b:

$$\begin{aligned} C &= -\nu_{fe} g_{\sigma} \quad \text{for trapped electrons,} \\ C &= \frac{F_{me} \int_T d^3v \nu_{fe} g_{\sigma}}{\int_C d^3v F_{me}} \quad \text{for circulating electrons.} \end{aligned} \quad (5)$$

model c:

$$\begin{aligned} C &= -\nu_{fe} h_{\sigma} \quad \text{for trapped electrons,} \\ C &= \frac{F_{me} \int_T d^3v \nu_{fe} h_{\sigma}}{\int_C d^3v F_{me}} \quad \text{for circulating electrons.} \end{aligned} \quad (6)$$

Here, $\int_T d^3v$ is the integral over the trapped-particle part of velocity space and $\int_C d^3v$ is the integral over the circulating-particle part. Model a is the non-number-conserving Krook model collision operator used in Refs. 3 and 4. Models b and c are both exactly number conserving, i.e., $\int d^3v C = \int_C d^3v C + \int_T d^3v C = 0$. In these two models, the circulating electrons do not experience collisions, but only compensate for the number density gain or loss of the trapped electrons. These forms of the collision operator simulate the

more frequent small-angle scattering suffered by trapped electrons, which is the dominant collisional process. A model collision operator of this general type was used previously in Ref. 6. Models b and c only differ in their application; to g_σ in model b and to h_σ in model c. As noted above, because g_σ contains a significant Maxwellian component, while h_σ does not, model c is expected to be more correct since it avoids the spurious dissipation generated by $C \left[(|e|F_{me}/T_e) (1 - \omega_{*e}^T/\omega) \tilde{\psi} \right]$.

Using Eqs. (2) and (3) in Eq. (1) then yields:

$$\left[-i\sigma|v_{||} \frac{\partial}{\partial s} + iC - (\omega - \omega_{de}) \right] h_\sigma = \frac{|e|}{T_e} F_{me} (\omega - \omega_{*e}^T) \left[\tilde{\Xi} - \left(1 - \frac{\omega_{de}}{\omega} \right) \tilde{\psi} \right]. \quad (7)$$

In this form, the trapped and circulating electron results for h_σ may be readily obtained using the methods of Ref. 7. The results are then used in the $\int d^3v$ moment of Eq. (7). Since the ion contribution to the perturbed current density $\tilde{j}_{||}$ is negligibly small, the resulting equation can be written in the form:

$$B \frac{\partial}{\partial s} \left(\frac{\tilde{j}_{||}}{B} \right) = -i \frac{|e|^2}{T_e} \left\{ \int d^3v F_{me} (\omega - \omega_{*e}^T) \left[\tilde{\Xi} - \left(1 - \frac{\omega_{de}}{\omega} \right) \tilde{\psi} \right] - \int_T d^3v F_{me} (\omega - \omega_{*e}^T) (\omega - \omega_{de}) \Upsilon \right\}, \quad (8)$$

where for model a:

$$\Upsilon^a = \frac{\tilde{\Xi}^{(0)}}{\omega - \omega_{de}} - \frac{\tilde{\psi}^{(0)}}{\omega} - \frac{i\nu_{fe}}{\omega} \frac{\tilde{\psi} - \tilde{\psi}^{(0)}}{\omega - \omega_{de}}, \quad (9)$$

for model b:

$$\Upsilon^b = \frac{\tilde{\Xi}^{(0)}}{\omega - \omega_{de} + i\nu_{fe}} - \frac{\tilde{\psi}^{(0)}}{\omega}, \quad (10)$$

and for model c:

$$\Upsilon^c = \frac{\tilde{\Xi}^{(0)}}{\omega - \omega_{de} + i\nu_{fe}} - \left(1 - \frac{\omega_{de}}{\omega} \right) \frac{\tilde{\psi}^{(0)}}{\omega - \omega_{de} + i\nu_{fe}}. \quad (11)$$

Here, the zero superscript denotes a time average over a trapped-particle orbit, i.e., $\bar{\psi}^{(0)} \equiv (\oint ds \bar{\psi}/v_{\parallel}) / (\oint ds/v_{\parallel})$.

In the collisionless limit, $\nu_{fe} \rightarrow 0$, all three models give the same result:

$$\Upsilon^{\nu_{fe}=0} = \frac{\bar{\Xi}^{(0)}}{\omega - \omega_{de}} - \frac{\bar{\psi}^{(0)}}{\omega}. \quad (12)$$

However, as ν_{fe} becomes large, the results differ very substantially: $\Upsilon^a \propto \nu_{fe}$, $\Upsilon^b \rightarrow \text{constant}$, and $\Upsilon^c \propto 1/\nu_{fe}$. Thus, model b overestimates the effect of trapped particles for large ν_{fe} , and model a greatly overestimates such effects, relative to the more correct model c.

III. COMPREHENSIVE NUMERICAL MODEL

In this section and in the Appendix, a comprehensive numerical model for calculating eigenfrequencies, eigenfunctions, and quasilinear transport coefficients is presented. Since the derivation of the equations exactly parallels that in Ref. 3, only the results are presented here. With regard to the specific forms of the collision models given in Sec. II, the equations of Ref. 3 basically correspond to collision model a, and the equations to be presented in this section and the Appendix correspond to collision model c. Although numerical results corresponding to collision model b will be presented for comparison purposes, the corresponding equations are not written here because they are almost identical to those of Ref. 3.

We will employ the model MHD equilibrium of Ref. 8, which has circular but nonconcentric magnetic surfaces with a Shafranov shift term proportional to $d\beta/dr$. This model MHD equilibrium has the advantage that β can be varied as a continuous parameter. The

MHD equilibrium could be extended to a more general, numerically computed one, as described in Ref. 3, but all of the results presented here are computed with the model MHD equilibrium. The calculation is, however, fully electromagnetic, retaining the parallel and perpendicular perturbed vector potential components, along with the perturbed electrostatic potential, and is valid for arbitrary ratios of the mode frequency to the particle bounce or transit frequency, and for arbitrary ratios of the perpendicular wavelength to the particle gyroradius or banana width.

We use the usual r, θ, ζ toroidal coordinate system, with θ being the poloidal angle and ζ the toroidal angle. The equilibrium magnetic field strength is $B = B_0/h(\theta)$, where $h(\theta) \equiv 1 + \epsilon \cos(\theta)$, $\epsilon \equiv r/R_0$, and R_0 is the major radius at the center of the selected magnetic surface. We employ the ballooning representation, with the perturbed electrostatic potential expressed in the form:

$$\tilde{\Phi}(r, \theta, \zeta, t) = \exp[-i\omega t + in\zeta - inq(r)\theta] \sum_{p'=-\infty}^{\infty} \hat{\phi}(\theta - 2\pi p', r) \exp[inq(r)2\pi p']. \quad (13)$$

Note that $\tilde{\Phi}$ is periodic in the toroidal and poloidal angles, while $\hat{\phi}(\theta)$ is defined in the infinite domain in θ and is not periodic. Similar forms apply for the perturbed distribution function and the parallel and perpendicular components of the perturbed vector potential, $\tilde{A}_{\parallel}$ and $\tilde{A}_{\perp}$. Adopting the Coulomb gauge, $\nabla \cdot \tilde{\mathbf{A}} = 0$, leads to $\tilde{\mathbf{A}} = \tilde{A}_{\parallel} \hat{n} + \tilde{A}_{\perp} \hat{e}_{\perp}$, where $\hat{n} \equiv \mathbf{B}/B$ and $\hat{e}_{\perp} \equiv \hat{n} \times \nabla S/|\hat{n} \times \nabla S|$, with S being the usual eikonal associated with Eq. (13) ($\mathbf{k}_{\perp} \equiv \nabla S$). The lowest order equilibrium distribution function is taken to be Maxwellian.

For the approach just described, the linearized gyrokinetic equation corresponding to Eq. (7) becomes:

$$\begin{aligned} \frac{v_{\parallel}(\theta)}{qR_0} \frac{\partial}{\partial \theta} \hat{h}_{\sigma j} - [\omega - \omega_{dj}(\theta)] \hat{h}_{\sigma j}(\theta) - C(\hat{h}_{\sigma j}) = \\ - i \frac{e_j}{T_j} F_{mj} (\omega - \omega_{*j}^T) \left\{ \left[\hat{\phi} - \left(1 - \frac{\omega_{dj}}{\omega} \right) \hat{\psi} \right] J_0 - i \frac{v_{\perp}}{c} \hat{A}_{\perp} J_1 \right\} \end{aligned} \quad (14)$$

for species j ($j = e, i$), where we have included the finite Larmor radius effects through the usual Bessel functions $J_l = J_l(\theta) \equiv J_l[k_{\perp}(\theta)v_{\perp}(\theta)/\Omega_j(\theta)]$ for $l = 0$ or 1 . Now, the perturbed distribution function for species j is $\hat{f}_j = -(e_j/T_j) F_{mj} \hat{\phi} + [\hat{h}_{\sigma j} - (e_j/T_j) F_{mj} \cdot (1 - \omega_{*j}^T/\omega) J_0 \hat{\psi}] \exp[-i(k_{\perp}v_{\perp}/\Omega_j) \sin \xi]$, where ξ is the gyrophase, in the guiding center coordinate system. For the model MHD equilibrium,⁸ $k_{\perp}^2(\theta) = k_{\theta}^2 \{1 + [\hat{s}(\theta - \theta_0) - \alpha \sin \theta]^2\}$, $k_{\theta} \equiv -nq/r$, and $\omega_{dj}(\theta) = \omega_{*j}(r_{nj}/R_0)(E/T_j) \left(\left(1 + v_{\parallel}^2/v^2 \right) \{ \cos \theta + [\hat{s}(\theta - \theta_0) - \alpha \sin \theta] \sin \theta \} + (\epsilon/q^2) [2v_{\parallel}^2/v^2 + (\hat{s} - 1)v_{\perp}^2/v^2] \right)$ is the magnetic drift frequency, including also the cylindrical drifts which are of higher order in ϵ , where $\hat{s} \equiv (r/q)(dq/dr)$ measures the average shear, and $\alpha \equiv -q^2 R_0 d\beta/dr \equiv (q^2 R_0/r_{\beta}) \beta$ is a measure of the local Shafranov shift, with $\beta \equiv 8\pi \sum_j n_j T_j / B_0^2$ and $r_{\beta} \equiv -(d \ln \beta / dr)^{-1}$. For the results presented here, we take $\theta_0 = 0$, in the lowest order in the ballooning hierarchy, corresponding to the fastest growing modes. Also, $v_{\parallel}(\theta) = \sigma v [1 - \Lambda/h(\theta)]^{1/2}$ and $v_{\perp}(\theta) = v[\Lambda/h(\theta)]^{1/2}$, where the velocity space coordinates we employ are E , Λ , and $\sigma = \pm 1$, with $E \equiv m_j v^2/2$ being the single particle energy (we do not include any equilibrium electric field) and $\Lambda \equiv \mu B_0/E$, where $\mu \equiv m_j v_{\perp}^2/2B_0$ is the single particle magnetic moment. Circulating particles correspond to $0 \leq \Lambda < 1 - \epsilon$, and trapped particles to $1 - \epsilon < \Lambda \leq h(\theta)$, at some given θ .

We employ a number-conserving Krook model collision operator corresponding to model c of Sec. II: $C(\hat{h}_{\sigma j}) = -\nu_{fj}(E, \Lambda)\hat{h}_{\sigma j}$ for trapped particles and $C(\hat{h}_{\sigma j}) = F_{mj} \int_T d^3v \nu_{fj} \hat{h}_{\sigma j} / \int_C d^3v F_{mj}$ for circulating particles; the functional form for ν_{fj} is specified in the Appendix. The gyrokinetic equation, Eq. (14), is solved separately for $\hat{h}_{\sigma j}(\theta)$ for trapped and circulating particles as in Ref. 3, and the results are used in the quasineutrality condition:

$$0 = \sum_j e_j \left\{ -\frac{e_j n_j}{T_j} \hat{\phi}(\theta) + \int d^3v J_0(\theta) \left[\hat{h}_{+j} + \hat{h}_{-j} + 2\frac{e_j}{T_j} F_{mj} \left(1 - \frac{\omega_{*j}^T}{\omega} \right) J_0(\theta) \hat{\psi}(\theta) \right] \right\}, \quad (15)$$

and in the component of Ampere's law in the $\hat{e}_\perp$ direction:

$$-i \frac{k_\perp^2}{4\pi} \hat{A}_\perp(\theta) = -\frac{1}{c} \sum_j e_j \int d^3v v_\perp(\theta) J_1(\theta) \left[\hat{h}_{+j} + \hat{h}_{-j} + 2\frac{e_j}{T_j} F_{mj} \left(1 - \frac{\omega_{*j}^T}{\omega} \right) J_0(\theta) \hat{\psi}(\theta) \right], \quad (16)$$

as in Ref. 3. However, instead of using the parallel component of Ampere's law as one of our three final equations, we use the $\sum_j e_j \int d^3v$ moment of the gyrokinetic equation (14). Making use of the quasineutrality condition, the number conservation property of the collision operator, the parallel component of Ampere's law, and the relation $\hat{A}_\parallel = -i(c/\omega q R_0)(\partial/\partial\theta)\hat{\psi}$, this moment equation becomes⁷:

$$\begin{aligned} \frac{c^2}{4\pi\omega^2 q^2 R_0^2} \frac{\partial}{\partial\theta} k_\perp^2(\theta) \frac{\partial}{\partial\theta} \hat{\psi}(\theta) &= \sum_j e_j \int d^3v J_0 \left(\frac{\omega_{dj}}{\omega} - 1 \right) \hat{h}_{\sigma j} + \sum_j \frac{e_j^2}{T_j} \int d^3v F_{mj} \\ &\cdot \left(1 - \frac{\omega_{*j}^T}{\omega} \right) \left\{ \left[\hat{\phi} - \left(1 - \frac{\omega_{dj}}{\omega} \right) \hat{\psi} \right] J_0^2 + i \frac{v_\perp}{c} J_0 J_1 \hat{A}_\perp \right\}. \end{aligned} \quad (17)$$

The results for $\hat{h}_{\sigma j}$ are used in this equation also. Then the system of three linear, homogeneous, non-Hermitian, integro-differential equations is solved by a Ritz method, as outlined

in the Appendix. The resulting set of algebraic equations can be written in matrix form

as:

$$M \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\psi}_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix} = \sum_j Z_j (M_{jl}^j) \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\psi}_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix} \quad (18)$$

$$= \sum_j Z_j \begin{pmatrix} M_{l1}^{j11} & M_{l1}^{j12} & M_{l1}^{j13} \\ M_{l1}^{j21} & M_{l1}^{j22} & M_{l1}^{j23} \\ M_{l1}^{j31} & M_{l1}^{j32} & M_{l1}^{j33} \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\psi}_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \end{pmatrix}$$

Here, $M(\omega)$ is a 3×3 block matrix, with each block being an $L \times L$ matrix, where L is the number of basis functions that are used, $Z_j \equiv e_j/|e|$ is the charge number, and the detailed form of $M_{jl}^{j'm}(\omega)$ is given in the Appendix.

In addition to computing the linear eigenfrequencies and eigenfunctions of the instabilities, the corresponding particle and thermal transport is also estimated. To do this, standard quasilinear procedures⁵ for calculating the relevant fluxes are employed, which contain contributions both from the perturbed $\mathbf{E} \times \mathbf{B}$ drift and from the perturbed magnetic field. These involve an estimate (ϕ_0) for the saturated amplitude of the perturbed electrostatic potential for the mode. One such estimate is that from the familiar ambient gradient or mixing-length approximation, which gives results such as $|e\phi_0/T_e| \simeq C_\phi/|k_\theta r_{pe}|$, where C_ϕ is a dimensionless constant and $r_{pe} \equiv -(d \ln p_e/dr)^{-1}$, with $p_e \equiv n_e T_e$. Note that this amplitude is finite even when $dn_e/dr = 0$ [$r_{ne} \equiv -(d \ln n_e/dr)^{-1} = \infty$]. We will leave $C_\phi = |e\phi_0 k_\theta r_{pe}/T_e|$ unspecified in the results to be presented in Sec. IV. To make these

calculated transport coefficients fall in the typical experimental range requires $C_\phi \sim 0.03$ to 0.1. In particular, by extending the results in Eqs. (129) and (130) of Ref. 5 to apply to the present calculation, the particle and thermal fluxes are obtained in the form:

$$\Gamma_j = \frac{cT_e}{|e|B_0} k_B n_e \left(\frac{e\phi_0}{T_e} \right)^2 \frac{1}{\sum_l |\hat{\phi}_l|^2} \text{Im} \left(\hat{\phi}_1^*, \dots, \hat{\psi}_1^*, \dots, \hat{A}_{\perp 1}, \dots \right) \cdot \left(M_{l'l}^j \right) \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\psi}_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix}, \quad (19)$$

and

$$Q_j = \frac{cT_e}{|e|B_0} k_B n_e \left(\frac{e\phi_0}{T_e} \right)^2 \frac{1}{\sum_l |\hat{\phi}_l|^2} \text{Im} \left(\hat{\phi}_1^*, \dots, \hat{\psi}_1^*, \dots, \hat{A}_{\perp 1}, \dots \right) \cdot \left(N_{l'l}^j \right) \begin{pmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\psi}_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix}, \quad (20)$$

where $N_{l'l}^j$ is the same as $M_{l'l}^j$ in Eq. (18) and the Appendix, except for an extra factor of E/T_j in the energy integrations in the Appendix. The corresponding transport coefficients are defined as $D_j \equiv -\Gamma_j / (dn_j/dr)$ and $\kappa_j \equiv -Q_j / [n_j (dT_j/dr)]$. We can also compute the surface-integrated fluxes. For the present model MHD equilibrium these are just $\int dA \Gamma_j = (2\pi)^2 r R_0 \Gamma_j$ and $\int dA Q_j = (2\pi) r R_0 Q_j$. Note that the particle fluxes given by Eq. (19)

are automatically ambipolar, since

$$\sum_j e_j \Gamma_j \propto \text{Im} \sum_j Z_j \left(M_{\nu 1}^j \right) \begin{pmatrix} \phi_1 \\ \vdots \\ \psi_1 \\ \vdots \\ \hat{A}_{\perp 1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \end{pmatrix},$$

by virtue of Eq. (18).

IV. NUMERICAL RESULTS

We will present results for two sets of input parameters, one corresponding to ISX-B, and one to PDX in the H-mode. These are the same parameters that were used for two of the cases in Ref. 4. For the ISX-B case, the local parameters are: $\epsilon = 0.10$, $T_e = 1.26$ keV, $T_i = 1.16$ keV, $q = 1.20$, $(r/q)(dq/dr) = 0.62$, $n_e = 6.57 \times 10^{13} \text{ cm}^{-3}$, $\eta_e = \eta_i = 1.09$, $k_{\theta} \rho_i = 0.15$, $\nu_e^* = 0.145$, $\tau_{ne}/r = 0.95$, and $B_0 = 12.8 \text{ kG}/(\beta/3.9\%)^{1/2}$, with varying β . We will first present results from the comprehensive numerical calculation for this case for the three collision models a, b, and c, considered in Sec. II, and specified by Eqs. (3), (4), and (5), respectively. The results are shown in Fig. 1 for the real frequency, ω_r , and the growth rate, γ , in units of ω_{*e} versus the collisionality parameter $\nu_e^* \propto n_e$ for $\beta = 3.9\%$ for the kinetic MHD ballooning mode. Model a, which does not conserve particle number, shows a considerably stronger dependence on ν_e^* than model b, which is number conserving. In turn, model c, which also eliminates the spurious contribution from a Maxwellian distribution function to the collision term in the gyrokinetic equation, gives results for ω which are essentially independent of ν_e^* , as is expected for this mode. These results are in accord with the conclusions reached in Sec. II.

Collision models a and c are further compared in Figs. 2 and 3 for these ISX-B parameters. Here, γ/ω_{*e} and ω_r/ω_{*e} are displayed as functions of β . Curve (i) is the usual compressionless ideal-MHD result for the high- n ballooning mode, shown for comparison. Curve (ii) is the corresponding fully kinetic result in either the collisionless limit or with collision model c (the two curves are indistinguishable). Curve (iii) is the result for the toroidal drift mode in the collisionless limit. As noted earlier, "toroidal drift modes" include the trapped-electron as well as the toroidal η_i instabilities. Curve (iv) is the corresponding result with collision model c. Curve (v) is the result with the non-number-conserving collision model a. Curves (i), (ii), (iii), and (v) were previously shown in Ref. 4. Curve (v) corresponds to a (spurious) hybrid mode which is formed between the toroidal drift mode and the kinetic MHD ballooning mode as an artifact of collision model a. With the more exact collision model c, the two modes remain distinct [curves (iv) and (ii)]. Note that curves (iv) and (v) coalesce at $\beta = 0$ in Figs. 2 and 3, showing that collision model a is a good approximation in the electrostatic limit, as is expected from the results of Ref. 2.

The corresponding transport coefficients $D_e = D_i$, κ_e , and κ_i for the two modes for this ISX-B case are shown in Figs. 4 and 5. For these results, collision model c is used, corresponding to curves (iv) and (ii) of the preceding figures. In all of these results for this case, as β varies, we keep n_j , T_j , and τ_β , and $k_\theta \rho_i$ fixed, and vary $B_0 \propto \beta^{-1/2}$ and $\alpha \equiv -q^2 R_0 (d\beta/dr) \equiv (q^2 R_0 / \tau_\beta) \beta \propto \beta$. An alternative choice, such as keeping B_0 , τ_β , and $k_\theta \rho_i$ fixed with $n_j \propto \beta^{1/2}$, $T_j \propto \beta^{1/2}$, and $\alpha \propto \beta$, would give very similar results, as will be seen when we consider the PDX H-mode case. Thus, this choice is arbitrary to a certain

extent. In Fig. 4, as β increases from zero, the transport coefficients for the toroidal drift mode increase strongly, as is expected since D_j and κ_j vary roughly as B_0^{-2} at fixed n_j , T_j , τ_β , $k_\theta \rho_i$, and $(e\phi_0 k_\theta r_{pe}/T_e)^2$, as can be shown from Eqs. (19) and (20). Then, just before the toroidal drift mode becomes stable at $\beta \simeq 1.7\%$, the transport coefficients start decreasing again. For the kinetic MHD ballooning mode in Fig. 5, the transport coefficients also rise with β from $\beta_{c1} \simeq 1.1\%$, though somewhat less steeply than for the toroidal drift mode, before turning over for β slightly less than $\beta_{c2} \simeq 6.1\%$. For both of these modes, $\kappa_e \simeq \kappa_i > D_e = D_i$, for the present values of $\eta_i = \eta_e = 1.09$. For a very weak density gradient case with $\eta_i \gg 1$ in the electrostatic limit, $\kappa_i > \kappa_e \simeq D_e = D_i$ was found.⁹ The electromagnetic transport coefficients differ from the corresponding electrostatic results, shown for comparison in Fig. 4 for the toroidal drift mode. The electrostatic results were obtained by suppressing the vector potential perturbations in the stability calculation, and evaluating only the electrostatic contributions to the quasilinear fluxes; i.e., only the $M_{\mu l}^{j11}$ part of $M_{\mu l}^j$ was kept in Eqs. (18), (19), and (20). Also, α was set to zero in evaluating quantities for the model MHD equilibrium. In the $\beta \rightarrow 0$ limit, the full electromagnetic fluxes asymptote to the electrostatic values, but for quite low β values the electromagnetic structure of the toroidal drift mode is seen to give significantly enhanced transport over the electrostatic limit. In Figs. 4 and 5 also, the estimate γ/k_θ^2 for the transport coefficients, calculated from the full electromagnetic growth rate, is plotted for comparison. For the toroidal drift mode in Fig. 4, the γ/k_θ^2 values will be comparable to the D_j and κ_j values for $C_\phi \sim 0.2$, while for the kinetic MHD ballooning mode in Fig. 5, the values are comparable for $C_\phi \sim 1$. However, for both instabilities, the functional dependences of the

curves on β are seen to be different, *e.g.*, the γ/k_θ^2 curves turn over at lower β values. These comparisons illustrate the limitations of the usual $\gamma/k_\perp^2 \sim \gamma/k_\theta^2$ estimates for transport coefficients.

We have also considered a PDX H-mode case where the local parameters are: $\epsilon = 0.15$, $T_e = 0.78$ keV, $T_i = 1.05$ keV, $q = 1.27$, $(r/q)(dq/dr) = 0.71$, $n_e = 4.9 \times 10^{13} \text{ cm}^{-3}$, $\eta_e = 3.3$, $\eta_i = 3.7$, $k_\theta \rho_i = 0.30$, $\nu_e^* = 0.24$, $r_{ne}/r = 1.9$, and $B_0 = 16.65 \text{ kG}/(\beta/1.3\%)^{1/2}$, with varying β . The growth rate, γ/ω_{*e} , and the real frequency, ω_r/ω_{*e} , are shown versus β in Figs. 6 and 7, respectively, with the curves labeled in the same way as in Figs. 2 and 3. The results are qualitatively the same as in the ISX-B case. The corresponding transport coefficients are shown for the two modes in Figs. 8 and 9, for n_j , T_j , r_β , and $k_\theta \rho_i$ fixed, with $B_0 \propto \beta^{-1/2}$ and $\alpha \propto \beta$. The electrostatic results for the toroidal drift mode are again plotted in Fig. 8, as are γ/k_θ^2 for both modes in Figs. 8 and 9. The results are generally similar to those for the ISX-B case, except that now, when $\eta_i = 3.7$ and $\eta_e = 3.3$, $\kappa_i > \kappa_e \simeq D_e = D_i$ for the toroidal drift mode, while $\kappa_i \gtrsim \kappa_e \simeq D_e = D_i$ for the kinetic MHD ballooning mode. For an alternative way of varying β , with $B_0 = 16.65 \text{ kG}$, $k_\theta \rho_i = 0.15$, and r_β held fixed, and with $n_j = 4.9 \times 10^{13} \text{ cm}^{-3} \times (\beta/1.3\%)^{1/2}$, $T_e = 0.78 \text{ keV} \times (\beta/1.3\%)^{1/2}$, $T_i = 1.05 \text{ keV} \times (\beta/1.3\%)^{1/2}$, and $\alpha \propto \beta$, the transport coefficients are shown in Fig. 10. The results in Figs. 8, 9, and 10 are very close; we conclude that the particular combination of n_j , T_j , and B_0 variation causing the β variation is not critical in cases like this.

V. CONCLUSIONS

We have extended our previous toroidal electromagnetic eigenfrequency-eigenfunction calculation to implement an improved model collision operator and to calculate also the associated quasilinear transport coefficients for the eigenmode. In toroidal geometry, the absence of the number conservation property for the model collision operator leads to a large overestimate of the effect of collisions at finite β , though in the electrostatic limit, at $\beta = 0$, the property makes very little difference. It is also important to ensure that, when operating on a Maxwellian distribution function, the collision operator gives no contribution. Including both properties in a model collision operator, we find that the kinetic MHD ballooning mode is essentially independent of the trapped electron collisionality in the banana regime. This is in agreement with the observation of Ref. 10 that trapped particle effects are negligible for the ideal MHD ballooning mode. The toroidal drift mode and the kinetic MHD ballooning mode can form a spurious hybrid mode when using a non-number-conserving collision operator. When a more correct, number-conserving operator is used, they remain separate modes.

The associated quasilinear transport coefficients, $D_e = D_i$, κ_e , and κ_i , are calculated in terms of an unspecified saturation level $|e\phi_0 k_\theta r_{pe}/T_e|$ for the mode. The results show that the usual estimate for anomalous transport, $D_j \simeq \kappa_j \simeq \gamma/k_\perp^2$, can sometimes be quite misleading. For instance, the ISX-B case has $\kappa_e \simeq \kappa_i > D_e = D_i$ for $\eta_i = \eta_e = 1.09$, while the PDX H-mode case, with $\eta_i = 3.7$ and $\eta_e = 3.3$, has $\kappa_i > \kappa_e \simeq D_e = D_i$ for the toroidal drift mode at lower β values, while $\kappa_i \gtrsim \kappa_e > D_e = D_i$ for the kinetic MHD ballooning mode at higher β values. Also, the variation of γ/k_θ^2 with β is quite different from that of

the transport coefficients, if we assume that $k_{\theta\rho_i}$ and $|e\phi_0 k_{\theta r_{pe}}/T_e|$ are fixed, independent of β .

We have presented results here from an improved toroidal linear eigenmode calculation, along with those from an associated quasilinear transport coefficient calculation. These constitute a useful, though cumbersome, tool for analyzing present and future tokamak experiments. We plan to apply it to further cases of interest, including tokamak edge region cases, to study the effect of a separatrix on the modes considered here, with an appropriate extension of the model MHD equilibrium. It could also be used to study the effects of α -particles on these modes.

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APPENDIX: DETAILS OF NUMERICAL MODEL

For electrons, the form of ν_{fe} used is:

$$\nu_{fe}(E, \Lambda) = \left(\nu_f^l + \nu_f^h \frac{\nu_e^*}{\tilde{v}^3 Z_f} [Z_f + H_{ee}(\tilde{v})] \right) \cdot \left(1 + \frac{\nu_e^*}{\tilde{v}^3 Z_f} [Z_f + H_{ee}(\tilde{v})] \right)^{-1}, \quad (A1)$$

where

$$\nu_f^l = \frac{\nu_e \epsilon}{\tilde{v}^3} \frac{Z_f + H_{ee}(\tilde{v})}{|(1 - \epsilon) - \Lambda|^2} \frac{0.111(|\omega|/\nu_e Z_f) + 1.31}{11.79(|\omega|/\nu_e Z_f) + 1},$$

and

$$\nu_f^h = 1.08 \nu_e \frac{Z_f + H_{ee}(1)}{1 + H_{ee}(1)}.$$

Here, $Z_f \equiv \sum_i Z_i^2 n_i / n_e$ is the effective charge, $\sum_i$ implies a sum over all ionic species, $Z_j \equiv e_j / |\epsilon|$ is the particle charge number, $\nu_e = 1221.8 n_e \lambda_e / T_e^{3/2}$, $\lambda_e = 15.94 - (1/2) \ln(n_e / T_e^2)$, n_e is in units of 10^{13} cm^{-3} , T_e is in units of keV, ν_e is in units of sec^{-1} , $\nu_e^* \equiv \nu_e Z_f / (\epsilon^{3/2} \bar{\omega}_{te})$, $\bar{\omega}_{te} \equiv v_e / q R_0$, $\bar{\omega}_{be} \equiv \epsilon^{1/2} \bar{\omega}_{te}$, $v_j \equiv (2T_j / m_j)^{1/2}$, $\hat{v} \equiv v / v_j$, $H_{ee}(\hat{v}) \equiv (\pi \hat{v})^{-1} \exp(-\hat{v}^2) + [1 - 1/(2\hat{v}^2)] \text{erf}(\hat{v})$, erf is the usual error function, and $H_{ee}(1) = 0.6239$. As shown in Ref. 2, the collision frequency ν_f^l will reproduce the results from Lorentz operator calculations in the electrostatic limit in the banana regime, $\nu_e^* < 1$, in the limit $\nu_e Z_f \ll |\omega| \ll \bar{\omega}_{be}$, as well as in the limit $|\omega| \ll \nu_e Z_f$, with interpolation in between. For the electrostatic limit in the regime $\bar{\omega}_{be} < \nu_e Z_f$, the collision frequency ν_f^h reproduces the effect of using the Braginskii transport coefficients¹¹ in an unsheared slab collisional drift wave calculation. For the results presented in Sec. IV, $\nu_e^* < 1$, so that $\nu_{fe} \simeq \nu_f^l$.

For the ions, the collision operator is the same as for the electrons, except that the collision frequency (for species i) is:

$$\nu_{fi} = \frac{\nu_i(\hat{v}) \epsilon}{\hat{v}^3} \frac{1}{|(1 - \epsilon) - \Lambda|^2} \frac{0.111(|\omega|/\nu_i) + 1.31}{11.79(|\omega|/\nu_i) + 1}, \quad (\text{A2})$$

where

$$\begin{aligned} \nu_i(\hat{v}) &\equiv 1221.8 Z_i^2 \frac{(m_e/m_i)^{1/2}}{T_i^{3/2}} \sum_j n_j Z_j^2 \lambda_{ij} H_{ij}(\hat{v}), \\ \lambda_{ij} &\equiv 18.39 - \ln \left[\frac{Z_i Z_j (m_i + m_j)}{m_i T_j + m_j T_i} \left(\frac{n_i Z_i^2}{T_i} + \frac{n_j Z_j^2}{T_j} \right)^{1/2} \right], \\ H_{ij}(\hat{v}) &\equiv H_{ee} \left[\left(\frac{m_j T_i}{m_i T_j} \right)^{1/2} \right], \end{aligned}$$

n_i and n_j are in units of 10^{13} cm^{-3} , T_i and T_j are in units of keV, ν_i is in units of sec^{-1} , and $\hat{v} \equiv v/v_j$. The expression for ν_{fi} reproduces the effects of using a Lorentz operator in

the limit $\nu_i < |\omega|$ as well as in the limit $\nu_i > |\omega|$, with interpolation in between, as shown in Ref. 2. Finite gyroradius modifications to the ion collision operator are negligibly small for the modes of interest (for $k_\theta \rho_i \lesssim 1$) and are not considered here. For the results presented in Sec. IV, ion collisions are a small effect, *i.e.*, $\nu_{fi} \ll |\omega|$.

Equations (15), (16), and (17) are solved by a Ritz method, after substituting the solutions for $\hat{h}_{\sigma j}$ as in Ref. 3. We decompose $\hat{\phi}(\theta)$ in terms of chosen basis functions:

$$\hat{\phi}(\theta) = \sum_{l=0}^{L-1} \hat{\phi}_l h_l(\theta), \quad (A3)$$

where $h_l(\theta) \equiv H_l(\delta^{1/2}\theta) \exp(-\delta\theta^2/2) M_l^{-1/2}$, H_l is the Hermite polynomial of order l , $M_l \equiv (\pi/\delta)^{1/2} 2^l l!$, and $\delta > 0$ is an adjustable real parameter. Also, $\hat{A}_\perp(\theta)$ is decomposed in the same fashion, and $\hat{\psi}(\theta)$ as $\hat{\psi}(\theta) = (|\omega|/\omega) \sum_{l=0}^{L-1} \hat{\psi}_l h_l(\theta)$. Then, the operator $[T_e/(|e|^2 n_e)] \int_{-\infty}^{\infty} d\theta h_{l'}(\theta)$ is applied to the quasineutrality condition and to the perpendicular component of Ampere's law, and the operator $[\omega T_e/(|\omega||e|^2 n_e)] \int_{-\infty}^{\infty} d\theta h_{l'}(\theta)$ is applied to the gyrokinetic moment equation, for $0 \leq l' \leq L-1$. The extra factors of $\omega/|\omega|$ are included to avoid spurious contributions from the first term in Eq. (17) to the quasilinear fluxes in Eqs. (19) and (20). The resulting set of algebraic equations is written in matrix form as Eq. (18), in which

$$\begin{aligned} M_{l'l}^{jm'm}(\omega) = & \frac{n_j}{n_e} \frac{T_e}{T_j} Z_j \delta_{ll'} \delta_{m1} \delta_{m'1} - \delta_{m2} \delta_{m'2} \frac{\bar{\omega}_{ji}^2 Z_j^2}{|\omega|^2 \beta} \left(\frac{T_e}{T_i} \right)^2 \\ & \cdot \frac{n_j}{n_e} \frac{T_j}{T_e Z_j} \int_{-\infty}^{\infty} d\theta b_i(\theta) \frac{\partial}{\partial \theta} h_l(\theta) \frac{\partial}{\partial \theta} h_{l'}(\theta) + \delta_{m3} \delta_{m'3} \frac{2T_e}{\beta |e|^2 B_0^2 n_e} \\ & \cdot \frac{n_j T_j}{Z_j} \int_{-\infty}^{\infty} d\theta k_\perp^2(\theta) h_l(\theta) h_{l'}(\theta) - (\delta_{m2} \delta_{m'1} + \delta_{m1} \delta_{m'2}) \\ & \cdot \frac{\omega}{|\omega|} \frac{n_j}{n_e} \frac{T_e}{T_j} Z_j \int_{-\infty}^{\infty} d\theta h_{l'}(\theta) h_l(\theta) \left\{ \left[1 - \frac{\omega_{*j}}{\omega} (1 - \eta_j) \right] \Gamma_0 - \frac{\omega_{*j} \eta_j}{\omega} \right. \\ & \cdot \left. [\Gamma_0 + b_j(\theta) (\Gamma_1 - \Gamma_0)] \right\} - i (\delta_{m2} \delta_{m'3} + \delta_{m3} \delta_{m'2}) \frac{\omega}{|\omega|} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{n_j T_e}{n_e T_j} Z_j \frac{v_j}{c} \int_{-\infty}^{\infty} d\theta h_{\nu}(\theta) h_l(\theta) \frac{1}{2} k_{\perp} \rho_j \left\{ \left[1 - \frac{\omega_{*j}}{\omega} (1 - \eta_j) \right] (\Gamma_1 - \Gamma_0) \right. \\
& - \frac{\omega_{*j} \eta_j}{\omega} [\Gamma_1 - 2\Gamma_0 + 2b_j(\theta) (\Gamma_0 - \Gamma_1)] \left. \right\} - \delta_{m2} \delta_{m'2} \\
& \cdot \frac{\omega^2}{|\omega|^2} \frac{n_j T_e}{n_e T_j} Z_j \frac{\bar{\omega}_{dj}}{\omega} \int_{-\infty}^{\infty} d\theta h_{\nu}(\theta) h_l(\theta) \left(\left(1 - \frac{\omega_{*j}}{\omega} \right) C(\theta) \Gamma_0 \right. \\
& + \left\{ \left[1 - \frac{\omega_{*j}}{\omega} (1 - \eta_j) \right] D(\theta) - \frac{\omega_{*j} \eta_j}{\omega} C(\theta) \right\} [\Gamma_0 + b_j(\theta) (\Gamma_1 - \Gamma_0)] \\
& - \frac{\omega_{*j} \eta_j}{\omega} D(\theta) [\Gamma_0 (2 - 4b_j + 2b_j^2) + \Gamma_1 (3b_j - 2b_j^2)] \left. \right) \\
& + \delta_{m2} \delta_{m'2} \frac{\omega^2}{|\omega|^2} \frac{n_j T_e}{n_e T_j} Z_j \int_{-\infty}^{\infty} h_{\nu}(\theta) h_l(\theta) \left\{ \left[1 - \frac{\omega_{*j}}{\omega} (1 - \eta_j) \right] \Gamma_0 \right. \\
& - \frac{\omega_{*j} \eta_j}{\omega} [\Gamma_0 + b_j(\theta) (\Gamma_1 - \Gamma_0)] \left. \right\} - \frac{n_j T_e}{n_e T_j} \frac{Z_j}{4\pi^{3/2}} \int_0^{1-\epsilon} d\Lambda \frac{|\omega_t| q R_0}{v} \\
& \cdot \int_{-\infty}^{\infty} d\kappa \int_0^{\infty} d\hat{E} \hat{E}^{1/2} e^{-\hat{E}} \left\{ \omega - \omega_{*j} [1 + \eta_j (\hat{E} - 3/2)] \right\} \sum_{\sigma=\pm 1} (\omega - \kappa \omega_t)^{-1} \\
& \cdot \left\{ \int_{-\infty}^{\infty} d\theta \frac{h_{\nu}(\theta) \exp [i\kappa \hat{\theta}(\theta) + i\sigma w_d(\theta)]}{h(\theta) [1 - \Lambda/h(\theta)]^{1/2}} \left[\delta_{m'1} J_0(\theta) - \delta_{m'2} \left(1 - \frac{\omega_{dj}(\theta)}{\omega} \right) \right. \right. \\
& \cdot J_0(\theta) \frac{\omega}{|\omega|} - i\delta_{m'3} \frac{v_{\perp}(\theta)}{c} J_1(\theta) \left. \right] \left\{ \left(\int_{-\infty}^{\infty} d\theta' \frac{h_l(\theta') \exp [-i\kappa \hat{\theta}(\theta') - i\sigma w_d(\theta')] }{h(\theta') [1 - \Lambda/h(\theta')]^{1/2}} \right. \right. \\
& \cdot \Upsilon_l^m \left. \right) - \frac{n_j T_e}{n_e T_j} \frac{Z_j}{2\pi^{1/2}} \int_{-\infty}^{\infty} d\theta \int_{1-\epsilon}^{\Lambda(\theta)} d\Lambda \left\{ h(\theta) [1 - \Lambda/h(\theta)]^{1/2} \right\}^{-1} \int_0^{\infty} d\hat{E} \\
& \cdot \hat{E}^{1/2} e^{-\hat{E}} \left\{ \omega - \omega_{*j} [1 + \eta_j (\hat{E} - 3/2)] \right\} \left(\delta_{m'1} J_0(\theta) \right. \\
& - \delta_{m'2} \left(1 - \frac{\omega_{dj}(\theta)}{\omega} \right) J_0(\theta) \frac{\omega}{|\omega|} - i\delta_{m'3} \frac{v_{\perp}(\theta)}{c} J_1(\theta) \left. \right) \sum_{p=-\infty}^{\infty} \left[\exp [ip\hat{\theta}(\theta)] \right. \\
& \cdot \left(\frac{\Psi_{l+}^{pm} \exp [i w_d(\theta)]}{\omega + i\nu_{fj} - \omega_d^{(0)} - p\omega_b} + \frac{\Psi_{l-}^{pm} \exp [-i w_d(\theta)]}{\omega + i\nu_{fj} - \omega_d^{(0)} + p\omega_b} \right) \\
& - \frac{(-1)^p i \sin [\Gamma \hat{\theta}(\theta) + w_d(\theta)]}{\cos \pi \Gamma} \left(\frac{\Psi_{l+}^{pm}}{\omega + i\nu_{fj} - \omega_d^{(0)} - p\omega_b} \right. \\
& \left. \left. - \frac{\Psi_{l-}^{pm}}{\omega + i\nu_{fj} - \omega_d^{(0)} + p\omega_b} \right) \right].
\end{aligned}$$

(A4)

Here, $\delta_{ll'} = 1$ for $l = l'$ and $\delta_{ll'} = 0$ for $l \neq l'$; the subscript i refers to a single, arbitrarily chosen ion species (for the results presented in Sec. IV, there is only one ion species); $\Gamma_{0,1} \equiv I_{0,1}[b_j(\theta)] \exp[-b_j(\theta)]$, $b_j(\theta) \equiv k_{\perp}^2(\theta) \rho_j^2(\theta)/2$, $I_{0,1}$ is the usual modified Bessel function, $C(\theta) \equiv \cos \theta + [\hat{s}(\theta - \theta_0) - \alpha \sin \theta] \sin \theta + \epsilon/q^2$, $D(\theta) \equiv \cos(\theta) + [\hat{s}(\theta - \theta_0) - \alpha \sin \theta] \sin \theta + (\hat{s} - 1)\epsilon/q^2$, and $\hat{E} \equiv E/T_j$. For circulating particles, $\tau_i \equiv (2\sigma q R_0/v) \int_0^\pi d\theta \{h(\theta)[1 - \Lambda/h(\theta)]^{1/2}\}^{-1}$, $\omega_i \equiv 2\pi/\tau_i$, $\hat{\theta}(\theta) \equiv \pi \int_0^\theta d\theta' \{h(\theta')[1 - \Lambda/h(\theta')]^{1/2}\}^{-1} / \int_0^\pi d\theta' \{h(\theta')[1 - \Lambda/h(\theta')]^{1/2}\}^{-1}$, and $w_d(\theta) \equiv - \int_0^\theta d\theta' \omega_d(\theta') q R_0 / [v_{\parallel}(\theta') |h(\theta')|]$. Also, for circulating particles,

$$\begin{aligned} \Upsilon_l^m(\theta) \equiv & \left[\delta_{m1} J_0(\theta) - \delta_{m2} \left(1 - \frac{\omega_{dj}(\theta)}{\omega} \right) \frac{\omega}{|\omega|} J_0(\theta) - i \delta_{m3} \frac{v_{\perp}(\theta)}{c} J_1(\theta) \right] h_l(\theta) \\ & + \left\{ \omega - \omega_{sj} \left[1 + \eta_j (\hat{E} - 3/2) \right] \right\}^{-1} \left\{ 1 - [\epsilon(1 + \cos \theta)/(1 + \epsilon \cos \theta)]^{1/2} \right\}^{-1} \\ & \cdot \frac{1}{2\pi^{1/2}} \int_0^\infty d\hat{E}' (\hat{E}')^{1/2} e^{-\hat{E}'} \left\{ \omega - \omega_{sj} \left[1 + \eta_j (\hat{E}' - 3/2) \right] \right\} \\ & \cdot \int_{1-\epsilon}^{h(\theta)} d\Lambda' \frac{i\nu_{fj}(E', \Lambda')}{h(\theta)[1 - \Lambda'/h(\theta)]^{1/2}} \sum_{p=-\infty}^{\infty} \left[\exp[ip\hat{\theta}(\theta)] \right. \\ & \cdot \left(\frac{\Psi_{l+}^{pm} \exp[iw_d(\theta)]}{\omega + i\nu_{fj} - \omega_d^{(0)} - p\omega_b} + \frac{\Psi_{l-}^{pm} \exp[-iw_d(\theta)]}{\omega + i\nu_{fj} - \omega_d^{(0)} + p\omega_b} \right) \\ & - \frac{(-1)^p i \sin[\Gamma\hat{\theta}(\theta) + w_d(\theta)]}{\cos \pi \Gamma} \left(\frac{\Psi_{l-}^{pm}}{\omega + i\nu_{fj} - \omega_d^{(0)} - p\omega_b} \right. \\ & \left. \left. - \frac{\Psi_{l+}^{pm}}{\omega + i\nu_{fj} - \omega_d^{(0)} + p\omega_b} \right) \right]. \end{aligned} \quad (A5)$$

For trapped particles, $\tau_b \equiv (2qR_0/v) \int_0^{\theta_1} d\theta \{h(\theta)[1 - \Lambda/h(\theta)]^{1/2}\}^{-1}$, $\omega_b \equiv 2\pi/\tau_b$, $\hat{\theta}(\theta) \equiv \pi \int_0^\theta d\theta' \{h(\theta')[1 - \Lambda/h(\theta')]^{1/2}\}^{-1} / \int_0^{\theta_1} d\theta' \{h(\theta')[1 - \Lambda/h(\theta')]^{1/2}\}^{-1}$, $w_d(\theta) \equiv - \int_{\theta_2}^\theta d\theta' [\omega_d(\theta') - \omega_d^{(0)}] / [v_{\parallel}(\theta') |h(\theta')|]$, $\omega_d^{(0)} \equiv (qR_0/\tau_b) \int_{\theta_1}^{\theta_2} d\theta \omega_d(\theta) / [v_{\parallel}(\theta) |h(\theta)|]$, $\Gamma \equiv$

$(\omega + i\nu_{jj} - \omega_d^{(0)})/\omega_b$, $\theta_1 \equiv -\theta_r + 2\pi N$, $\theta_2 \equiv \theta_r + 2\pi N$, $\theta_r \equiv \arccos[(\Lambda - 1)/\epsilon]$, N is chosen such that $\theta_1 \leq \theta \leq \theta_2$, and

$$\Psi_{i\sigma}^{pm} \equiv \frac{\omega_b q R_0}{2\pi v} \int_{\theta_1}^{\theta_2} d\theta \frac{h_i(\theta) \exp[-ip\hat{\theta}(\theta) - i\sigma w_d(\theta)]}{h(\theta)[1 - \Lambda/h(\theta)]^{1/2}} \cdot \left[\delta_{m1} J_0(\theta) - \delta_{m2} \left(1 - \frac{\omega_d(\theta)}{\omega} \right) \frac{\omega}{|\omega|} J_0(\theta) - i\delta_{m3} \frac{v_{\perp}(\theta)}{c} J_1(\theta) \right]. \quad (A6)$$

The matrix M in Eq.(18) has $3L$ eigenvalues λ_i ($i = 1, \dots, 3L$), and $3L$ corresponding eigenvectors (and eigenfunctions). Equation (18) can be satisfied only if $\det(M) = 0$, and this will happen only if at least one of the λ_i 's is zero, since $\det(M) = \prod_{i=1}^{3L} \lambda_i$. Now, each of the λ_i 's is a function of ω . Therefore, the eigenvalue condition on ω is $\lambda_i(\omega) = 0$ for a single λ_i corresponding to a chosen eigenfunction. The eigenvalue of ω can be found by standard root-finding methods, such as the method of false position.¹² The number of basis functions, L , must be chosen sufficiently large that the eigenvalue ω is independent of L to some acceptable accuracy.

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FIGURE CAPTIONS

FIG. 1. Comparison of mode frequency ω for the kinetic MHD ballooning mode in the ISX-B case versus collisionality parameter ν_e^* for the model collision operators a, b, and c. The real frequency, ω_r , in units of ω_{*e} , is shown by the solid lines and the growth rate, γ , in units of ω_{*e} , by the dashed lines.

FIG. 2. Growth rate γ in units of ω_{*e} versus β (and $\alpha \propto \beta$) for the ISX-B case. Curve (i) is the usual compressionless ideal-MHD result for the high- n ballooning mode, shown for comparison. Curve (ii) is the corresponding fully kinetic result in either the collisionless limit or with collision model c (the two curves are indistinguishable). Curve (iii) is the result for the toroidal drift mode in the collisionless limit. Curve (iv) is the corresponding result with collision model c. Curve (v) is the result for a (spurious) hybrid mode which is an artifact of the non-number-conserving collision model a.

FIG. 3. Real frequencies, ω_r , corresponding to Fig. 2, with the curves labeled in the same way. The real frequency corresponding to curve (i) is just $\omega_r = 0$.

FIG. 4. Transport coefficients, $D_e = D_i$, κ_e , and κ_i , in units of $(e\phi_0 k_{\theta} r_{pe}/T_e)^2 \text{ cm}^2 \text{ sec}^{-1}$, corresponding to the toroidal drift mode [curve (iv)] in Figs. 2 and 3, shown as solid lines. The corresponding results in the electrostatic limit, $D_e^{ES} = D_i^{ES}$, κ_e^{ES} , and κ_i^{ES} , are shown as dashed lines. For comparison, γ/k_{θ}^2 is also shown in units of $\text{cm}^2 \text{ sec}^{-1}$.

FIG. 5. Transport coefficients, $D_e = D_i$, κ_e , and κ_i , corresponding to the kinetic MHD ballooning mode [curve (ii)] in Figs. 2 and 3. For comparison, γ/k_{θ}^2 is also shown.

FIG. 6 Growth rate γ in units of ω_{*e} versus β (and $\alpha \propto \beta$) for the PDX H-mode case. The curves are labeled as in Fig. 2.

FIG. 7 Real frequencies, ω_r , corresponding to Fig. 6, with the curves labeled in the same way.

FIG. 8 Transport coefficients corresponding to the toroidal drift mode [curve (iv)] in Figs. 6 and 7, with the curves identified as in Fig. 4. Here, n_j , T_j , r_β , and $k_\theta \rho_i$ are held fixed, with $B_0 \propto \beta^{-1/2}$ and $\alpha \propto \beta$.

FIG. 9 Transport coefficients corresponding to the kinetic MHD ballooning mode [curve (ii)] in Figs. 6 and 7, with the curves identified as in Fig. 5. Here, n_j , T_j , r_β , and $k_\theta \rho_i$ are held fixed, with $B_0 \propto \beta^{-1/2}$ and $\alpha \propto \beta$.

FIG. 10 Transport coefficients, $D_e = D_i$, κ_e , and κ_i , as in Figs. 8 and 9, except with B_0 , r_β , and $k_\theta \rho_i$ held fixed, and with $n_j \propto \beta^{1/2}$, $T_j \propto \beta^{1/2}$, and $\alpha \propto \beta$.

Fig. 1

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Fig. 2

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Fig. 3

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Fig. 4

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Fig. 5

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Fig. 6

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Fig. 7

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Fig. 8

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Fig. 9

Fig. 10

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