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HEAVY PARTICLE PRODUCTION IN HIGH ENERGY HADRON COLLISIONS

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ABSTRACT

We consider the problem of how to compute the cross section for producing heavy strongly interacting particles (quarks, gluinos, squarks ...) in high energy hadron collisions, supposing that the heavy particle masses are large compared to 1 GeV. We use heavy quark production as an example. We consider several low order graphs in the kinematic region expected to produce the bulk of the total production cross section. Based on the structure of the low order graphs, we argue that the cross section can be reliably computed in QCD by using the same factorization formula that is used for jet production and W and Z production, but inserting the appropriate parton level cross sections for the heavy particle production. We emphasize that an analysis at all orders of perturbation theory is needed to reliably establish this conjecture.

1. Introduction

The production of new heavy particles in hadron-hadron collisions is a subject of great experimental and theoretical interest, and a certain amount of theoretical uncertainty. Part of the problem in predicting cross sections for new particles is the difficulty in understanding the cross sections for old ones. Charm production, in particular, is notorious for being poorly described by the lowest order perturbative QCD mechanism [1,2,3]; the experimentally measured cross-sections appear to lie an order of magnitude above the standard theoretical predictions. This has led to a number of attempts to remedy the situation by appealing to non-perturbative phenomena [4,5,6,7,8,9].

We will here provide arguments that, in fact, the standard perturbative formula [10] of QCD should be reliable for sufficiently heavy new particles. Our conclusion is based on a critical examination of low-order diagrams for heavy quark production (considered as an example of heavy-particle production). The standard mechanism, which we will argue to be correct, is that heavy quarks are predominantly made in the hard collision of one light parton from each hadron. That is, the cross section for hadron A + hadron B → heavy quarks C and D + anything decomposes in the factorized form

$$\frac{d\sigma}{dy_C\, dy_D} \sim \sum_{a,b} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B) \times H_{ab}(y_C-Y, y_D-Y, x_A x_B s, M), \tag{1.1}$$

where the sum runs over parton types a,b and Y is the rapidity of the parton c.m. frame, $Y=\frac{1}{2}\ln(x_A/x_B)$. The kinematics is explained in more detail in the following section. The functions f are parton distribution functions and H_{ab} is a hard scattering function describing the parton level process a+b→heavy quark pair + anything. We work in the kinematic region that should produce the dominant contribution to the total heavy particle cross section.

In order for the cross-section to be calculable, the produced quarks must be *heavy* -- for two reasons. First, we neglect contributions to the cross section that are suppressed by powers of a typical hadron mass scale m divided by the heavy quark mass M. After neglecting such power-suppressed contributions in order to arrive at the factorized form (1.1), one expands the hard scattering cross section H in powers of $\alpha_s(M)$; thus, again, M must be large.

The existing literature is divided on the subject of whether the standard formula is valid when used to calculate heavy particle production. The original factorization papers [10] cover the production of light particles and make essential use of the consequent lightlike kinematics. Thus, they are not directly applicable to the case of heavy particle production, except when the heavy particle transverse momentum is much larger than its mass. There are valuable calculations available of the first order hard scattering cross sections* to produce heavy quarks [5,11,12], but these papers do not address the problem of determining the circumstances under which these calculations should provide reliable approximations to the observed cross section. Mazzanti and Wada [14] did address this problem, and made some of the points we are making in the present paper; however, they concluded that soft small-P_T physics may be expected to play a role. Much of the available theoretical analysis concerns the production of *charmed* quarks, which is reviewed in [2,3]. Here it appears that one or more of the following effects, apparently not included in the conventional factorized QCD formula, may be at work: (1) an intrinsic heavy quark component in the hadron wave function[4], (2) flavor excitation [5,14], where the hard scattering graphs include a heavy quark in the initial state, (3) diffractive production of a heavy quark pair from a gluon in one of the hadrons [7; 8], (4) recombination of a produced heavy quark with a fast quark from one of the beams [8,14], or (5) final state prebinding distortion caused by binding of the heavy quarks to slow light quarks [9].

Do these or other mechanisms actually cause the conventional QCD formula to break down for the production of very massive quarks? The conventional wisdom is perhaps best summarized in the review [13], which uses the standard QCD formula (1.1), but warns the reader that other, nonperturbative effects may play a role.

Our analysis of low order graphs will suggest that none of the mechanisms for enhancing the perturbative cross-section leads to a breakdown of the standard picture for heavy quark production. In our analysis, we ignore light quarks and concentrate on the gluon-fusion process

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* References for the first order cross sections to produce other heavy particles may be found in [13]).

[15]; light quark contributions in eq. (1.1) are expected be qualitatively similar. The basic fact of life that dominates the analysis is that in the lowest order gluon-fusion graph (fig. 1.1) for production of a heavy quark pair, the internal quark line is necessarily off shell by at least order M^2. Moreover, the production cross-section is dominated by the region of finite rapidity difference between the outgoing quarks. Therefore, in the center of mass of the heavy quark system, the lifetime of the intermediate state, containing the virtual quark, is of order $1/M$. Thus, the process is a short distance one and can legitimately be computed by perturbation theory.

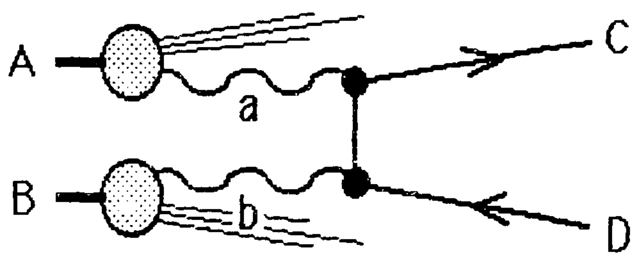

Figure 1.1 Gluon-gluon fusion.

We now briefly summarize why we believe that none of the plausible mechanisms mentioned above causes a breakdown of the standard perturbative formula for very heavy quarks. (We leave open the question of whether the charmed quark is heavy enough.) Either the alternative mechanisms give contributions that are non-leading by a power of M, or they are double-counting terms that are already in the standard formula.

Brodsky, J. C. Collins, S. D. Ellis, Gunion and Mueller have already argued that intrinsic heavy quark components are not important at high quark mass [16]. We agree entirely with their conclusion that graphs with extra collinear gluons exchanged between the hard scattering subgraph and the hadrons are non-leading by a power of the heavy mass M. We review the argument in section 3.

Combridge [5] used not only the usual graphs for the processes $gg \to Q\bar{Q}$ and $q\bar{q} \to Q\bar{Q}$, which he labelled "flavor creation", but also graphs for the processes $qQ \to qQ$ and $gQ \to gQ$, which he labelled "flavor excitation" (cf. [6]). In the flavor excitation graphs, such as fig. 1.2, the incoming heavy quark was taken to be on-shell. The biggest contribution came from the flavor excitation graphs; moreover this contribution had a dependence of the form $1/Q_0^2$ on his infra-red cut-off Q_0. However, the flavor excitation graphs should not be included in the calculation when the quarks are very heavy, since their use implies a misidentification of the hard scattering subprocess. When the flavor excitation graphs are treated as subgraphs of larger graphs like fig. 1.3, the

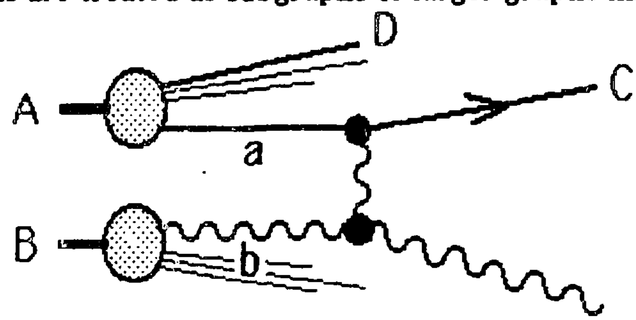

Figure 1.2 Flavor excitation.

virtuality of the heavy quark line is typically much greater than that of the exchanged gluon. Not only is it a bad approximation to put the quark on shell, but the hard scattering is partly hidden in what fig. 1.2 represents as the wave function of hadron A. The $Qg \to Qg$ subgraph is infrared sensitive, and therefore not necessarily part of the hard scattering (the part to be calculated

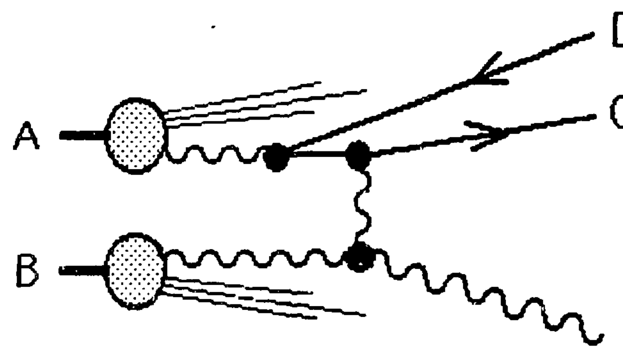

Figure 1.3 Graph containing flavor excitation subgraph and gluon fusion subgraph.

perturbatively). Mazzanti and Wada [14] recognized that the infra-red sensitive part of the excitation graphs is already included in the gluon-fusion term, but did not realize that the virtuality of the heavy quark is never much less than the virtuality of the gluon in the kinematic region that gives the dominant contribution to the total heavy quark cross section. They therefore still used an excitation term, whereas in fact there should be none. Halzen [2] has also discussed the limitations of flavor excitation.

P. D. B. Collins and Spiller have provided a carefully constructed model for very heavy quark production based on diffractive production via pomeron exchange and recombination [8]. This is kinematically similar to flavor excitation. Consider fig. 1.4, where one of the hadrons is excited to a large enough mass to make a heavy pair by the exchange of low-momentum gluons (modelling a pomeron) from the other hadron. To make the heavy pair, there is a hard scattering, apparently inside an incoming hadron. We have analysed the complete *one* gluon exchange graph and also the graphs for pair production at *all* orders in a soft external field. We use the cancellations due to gauge invariance that always take place when there is a hard scattering.

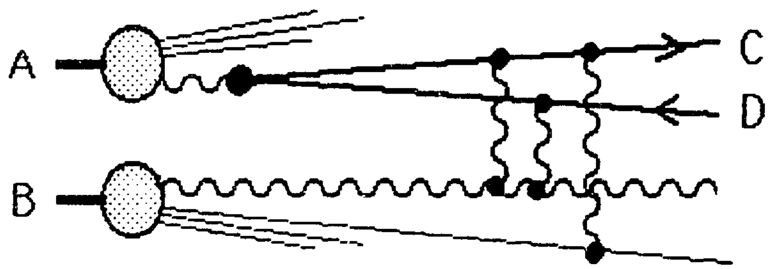

Figure 1.4 Diffractive production of heavy quarks.

The conclusion is that only a single transverse gluon (plus any number of longitudinal gluons) can be exchanged; otherwise the cross-section is suppressed by a power of $1/M$. The operator structure is such that the resulting cross section has the form of contribution to the standard formula (1.1) from gluon fusion, with the single transverse gluon that was exchanged being treated as a small x constituent of hadron B, as indicated in fig. 1.5.

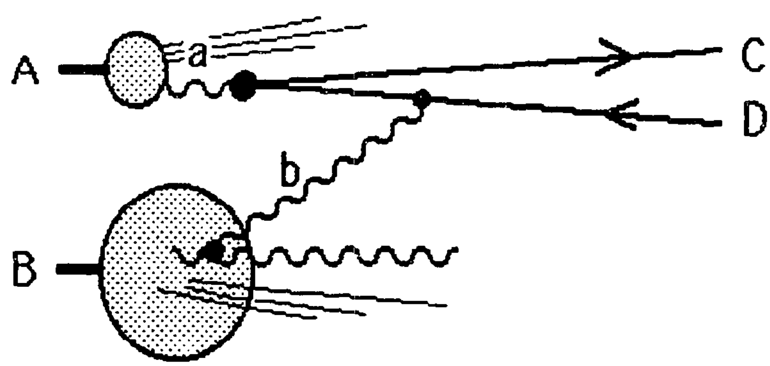

Figure 1.5 Diffractive production viewed as gluon-gluon fusion.

Brodsky and Gunion [9] have argued that "prebinding distortion" of heavy quarks attracted to partons in one of the incoming hadrons is likely to be significant for the production at low p_T of heavy quarks. However, as discussed in section 3, the only final-state interactions that do not cancel in the cross-section occur at short distances and are included in eq. (1.1) as part of the hard interaction.

The outline of the rest of our paper is as follows: In sect. 2, we review the significant properties, especially the kinematics, of the gluon-fusion graph. In sect. 3, we consider the effect of adding extra collinear gluons that couple the hadrons to the hard scattering, and we show that they do not affect the cross-section, to leading power in M^2. The possibility of diffractive production is considered in sect. 4. Our conclusions are given in sect. 5. We present in an appendix a model of pair production from an external abelian or non-abelian field (basically the Bethe-Heitler process).

2. Lowest order gluon-gluon fusion graph

In order to set some notation and to get some idea of the physics, we first study a typical tree graph for the production of heavy quarks via gluon-gluon fusion. The graph is shown in fig. 2.1.

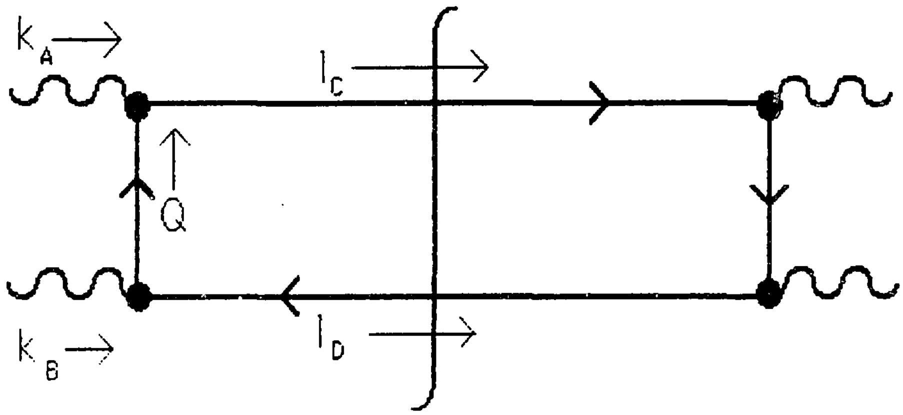

Figure 2.1. Heavy quark production via gluon-gluon fusion.

The gluon momenta are

$$k_A = (x_A P, 0, \mathbf{0})$$
$$k_B = (0, x_B P, \mathbf{0}) \,. \qquad (2.1)$$

Here $P = (s/2)^{1/2}$, and we denote momentum components by $v=(v^+,v^-,\mathbf{v})$, where $v^\pm=(v^0 \pm v^3)/2^{1/2}$ and transverse components are indicated by boldface letters. We take the gluons to have zero transverse momentum for now.

It will be useful in this and the following sections to write the heavy quark momenta in terms of their transverse components $\mathbf{l}$ and their rapidities y:

$$l_C = (\, 2^{-1/2}\, [\mathbf{l}_C^2+M^2]^{1/2} \exp(y_C),\, 2^{-1/2}\, [\mathbf{l}_C^2+M^2]^{1/2} \exp(-y_C),\, \mathbf{l}_C\,)$$
$$l_D = (\, 2^{-1/2}\, [\mathbf{l}_D^2+M^2]^{1/2} \exp(y_D),\, 2^{-1/2}\, [\mathbf{l}_D^2+M^2]^{1/2} \exp(-y_D),\, \mathbf{l}_D) \,. \qquad (2.2)$$

Let us define transverse sum and difference momenta by

$$\mathbf{l}_C = \tfrac{1}{2}\, \mathbf{q} + \boldsymbol{\Delta}$$
$$\mathbf{l}_D = \tfrac{1}{2}\, \mathbf{q} - \boldsymbol{\Delta} \,. \qquad (2.3)$$

In terms of these variables, the diquark invariant mass squared is

$$
\begin{aligned}
M_{CD}^2 &= (l_C + l_D)^2 \\
&= 2\,\{\, M^2 + \Delta^2 - q^2/4 \\
&\quad + [\,(\Delta + \tfrac{1}{2}q)^2 + M^2\,]^{1/2}\,[\,(\Delta - \tfrac{1}{2}q)^2 + M^2\,]^{1/2}\,\cosh(y_C - y_D)\,\}. \qquad (2.4)
\end{aligned}
$$

Momentum conservation for the lowest order graph, fig. 2.1, dictates that

$$
\begin{aligned}
&\mathbf{q} = \mathbf{0}, \\
&x_A = [\,(\Delta^2+M^2)/2P^2\,]^{1/2}\ \{\,\exp(+y_C) + \exp(+y_D)\,\}, \\
&x_B = [\,(\Delta^2+M^2)/2P^2\,]^{1/2}\ \{\,\exp(-y_C) + \exp(-y_D)\,\}. \qquad (2.5)
\end{aligned}
$$

The cross section is given in terms of the invariant amplitude $\mathcal{M}$ of fig. 2.1 by

$$
\frac{d\sigma}{dy_C\, dy_D\, d\mathbf{l}_C\, d\mathbf{l}_D} = \frac{1}{16\pi^2 M_{CD}^4}\, \delta^2(\mathbf{q})\, x_A\, f_{g/A}(x_A)\, x_B\, f_{g/B}(x_B)\, \langle \Sigma\, |\mathcal{M}|^2 \rangle, \qquad (2.6)
$$

where $\langle\Sigma\cdots\rangle$ denotes a sum over final state colors and spins and an average over initial state colors and spins. The function $f_{g/A}(x_A)$ gives the distribution of gluons in hadron A as a function of the momentum fraction x_A. We have associated a factor of x with $f_{g/A}(x)$ because $x\, f_{g/A}(x)$ is approximately constant for small x.

The invariant amplitude $\mathcal{M}$ is a rather complicated dimensionless function of the gluon and heavy quark momenta. It has been calculated, together with the contributions from the other lowest order graphs [5,11,12] and is reported in [13]. Here, we concentrate on the important physics issue: does M have an infrared singularity in some part of the kinematic region of interest? If it does, one may expect the lowest order diagrams to be irrelevant to physics because higher order diagrams, proportional to α_s evaluated at some small momentum scale, will be important. In the case of fig. 2.1, there is a possible singularity from the virtual heavy quark line. The denominator from this line is, from eq. (2.2) and (2.5),

$$Q^2 - M^2 = -2\, l_C \cdot k_A$$
$$= -[\, \Delta^2 + M^2]\, (\, 1 + \exp(y_D - y_C)\,). \tag{2.7}$$

The line is therefore off-shell by at least M^2.

One can immediately make a crucial observation. Because of this factor in the denominator, the production cross section for *light* quarks is large for any transverse momentum Δ, down to a lower cutoff controlled by the typical hadronic scale m. However, in the case of *heavy* quarks, the lower cutoff is the heavy quark mass M. Thus one may expect that the physics of heavy quark production will be controlled by $\alpha_s(M)$ instead of $\alpha_s(m)$.

In this paper, we will discuss the region of heavy quark momenta that (at least in a diagrammatic analysis) provides the largest part of the total cross section for production of heavy quarks. We choose to consider the case that $s \gg 4M^2$, since this is the most common situation in practical cases. (The case $M^2 \sim s$ is simpler and is covered by the original factorization proofs [10], modified by the treatment of initial state interactions as given in [17;18].) The dominant region, as we will now argue, is that in which the heavy quark transverse momenta l_C and l_D are of order M and their rapidity difference $y_C - y_D$ is of order 1.

We have already seen that the dominant region can include only tranverse momenta are not much smaller than M. Because of the factor $1/M_{CD}^4$ in the formula (2.6) for the cross-section, the transverse momenta must also be not much larger than M: from eq. (2.4), M_{CD}^2 is proportional to the quark transverse momenta squared when these are large. Notice also from eq. (2.5) that when Δ^2 is allowed to grow beyond order M^2, x_A and x_B grow. However, there is no corresponding growth in the factors $x_A f_{g/A}(x_A)$ or $x_B f_{g/B}(x_B)$ that appear in the cross-section. (A similar power counting argument applied to higher order graphs also implies that the production of heavy quarks as fragments of high p_T jets is not a dominant part of the total cross-section for producing heavy quarks. Of course, finding heavy quarks in high p_T jets is itself interesting [19; 20].)

We now note that M_{CD}^2 also gets large when the rapidity difference Δy between the two heavy quarks is large. Examination of the dependence of the matrix element $\mathcal{M}$ on the rapidities shows that $\mathcal{M}$ does not get large when Δy gets large. (This is related to the spin ½ nature of the exchanged heavy quark when one heavy quark is produced going forward in the subprocess center-of-mass frame and the other is produced going backward. It might not hold for cases in which a spin 1 heavy particle can be exchanged.) Therefore the dominant part of the total heavy quark production cross section comes from the region of rapidities $|y_C - y_D| \lesssim 1$.

We may summarize the results of these arguments by saying that the dominant region for production of heavy quarks is where the diquark mass not too much larger than 2M. Since we assume that M is much smaller than P, this implies that one or both of x_A and x_B must be small. These arguments, which are based on power counting, appear to generalize to higher-order graphs.

We shall want to integrate the cross section over a region of heavy quark momenta of order $\Delta k^\mu \sim M$ in order to average over any small scale features in the cross section such as quarkonium resonances.

3. Absorption of gluons collinear to A by the heavy quarks.

Let us consider first the absorption by one of the heavy quarks of a transversely polarized gluon with momentum collinear to hadron A. The emission of such a collinear gluon may be treated analogously.

This case is important for the following reason. Suppose that we were attempting to perturbatively calculate the cross section for production of *light* quarks, for example, the production of strange quarks in a proton-proton collision. We would then find that graphs in which a constituent gluon from hadron A is absorbed by one of the produced quarks gives a contribution of order 1 as $P \to \infty$. These graphs involve small transverse momenta and thus values of α_s that are *not* small. Therefore these graphs, and an infinite number of similar graphs, could not be neglected compared to the lowest order graphs. One would be left with the conclusion that strange quarks may be present in the proton, but that the probability of finding one cannot be calculated perturbatively. (One then says that the quarks are present "intrinsically" [4]). This issue was investigated with respect to heavy quarks by Brodsky, Collins, Ellis, Gunion, and Mueller [16]. They concluded that heavy quark production via higher order graphs involving more than one collinear gluon in the hadron is suppressed by a power of the heavy quark mass M, compared to the lowest order graph. We agree with this conclusion. The calculation is brief, and we present it below for completeness.

Let us consider the graph of fig. 2.1 with a transversely polarized collinear gluon carrying momentum $p = (x' P, 0, \mathbf{0})$ added. An example is illustrated in fig. 3.1.

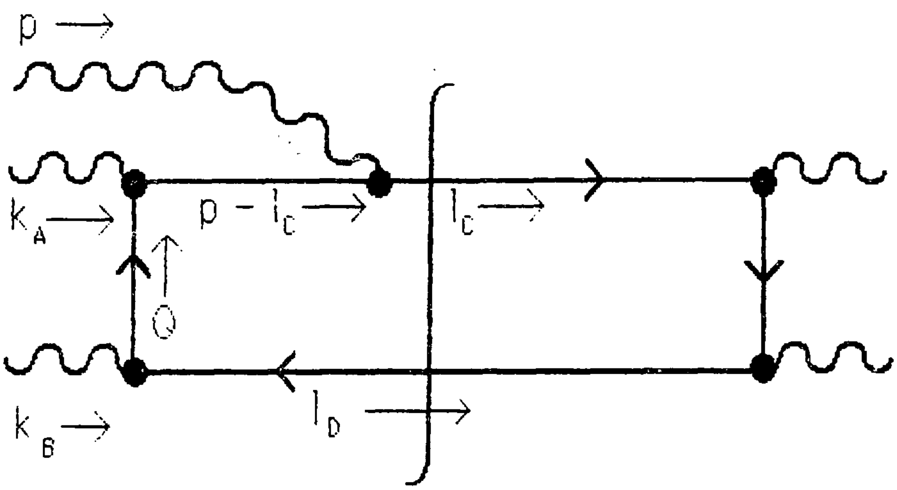

Figure 3.1. Lowest order gluon-gluon fusion diagram with added collinear gluon.

At the point of attachment of the collinear gluon p in fig. 3.1, there is a numerator factor

$$\varepsilon\cdot J = -\,\varepsilon\cdot J \sim l_C \sim M. \qquad (3.1)$$

Thus there is an enhancement from the numerator of order M. From the denominator there is a suppression factor

$$1/((l_C-p)^2 - M^2) = 1/(-2\, p\cdot l_C) = 1/(-2p^+ l_C^-)$$
$$= 1/\{-(x'/z_C)\,[\Delta^2+M^2]\}. \qquad (3.2)$$

where z_C is the momentum fraction carried by the heavy quark, $z_C = l_C^+/P$. We consider here the case in which the added gluon contributes a substantial portion of the total +momentum of the heavy quark: $x'/z_C \sim 1$. Then the added denominator factor is of order M^{-2}. Thus there is a net suppression factor $1/M$.

One must also consider the soft gluon case $x' \ll z_C$. When $x'/z_C <$ (1GeV)/M, the graph of fig. 3.1 is not suppressed because the heavy quark line with momentum $p - l_C$ is no longer far enough off shell. That is, the heavy quark travels for a long time before it absorbs the second gluon. More precisely, the initial hard interaction is constrained to take place in a time

$\delta t \lesssim 1/[\Delta^2+M^2]^{1/2}$ in the heavy quark c.m. frame. (Recall that we integrate over a range of heavy quark momenta to insure this.) The second gluon is absorbed at a time $t'-t \gtrsim z_C/(x' [\Delta^2+M^2]^{1/2}) \gg \delta t$ later. However, although interactions that happen a long time after the hard interaction are not suppressed in individual graphs, they may be expected to cancel because of unitarity, just as they do in e^+-e^- annihilation and the Drell-Yan process (see, for example, [21;17]). We thus conclude that the attachnment of extra gluons always leads to suppression.

There is a potential problem with the above reasoning concerning graphs with extra gluons in the case that M^2/s is very tiny, so that the parton level process involves gluons with very small momentum fractions x. The problem is that the graphs with extra gluons are suppressed by a factor m^2/M^2, but contain extra factors of the probability to find such a low x gluon in the hadron. When x is very small, this probability is large. Thus the suppression can be overcome. This issue is discussed in [22]. It may be important for $M<10$ Gev at the Superconducting Super Collider. This effect is just as important for jet production with transverse momenta of a similar size as M.

4. Gluon fragmentation

Let us now consider a process that can produce heavy quarks with rapidities in the hadron A fragmentation region. As we shall see, heavy quarks can be produced in this region even when the momentum transfer to the quarks is very small. This process would thus seem to be a strong candidate for nonperturbative effects.

In the fragmentation process to be studied, a gluon constituent from hadron A fragments into a heavy quark pair, as illustrated in fig. 4.1. During this process, the fragmenting system scatters from the gluon field of hadron B. Thus the heavy quarks can emerge on shell into the final state. At lowest order, there are three graphs that contribute to the shaded blob in fig. 4.1. These are shown in fig. 4.2.

This set of graphs is the lowest order example of diffractive excitation of the heavy quark component of the hadron wave function, which has been frequently used as a mechanism to describe charm production (see, for instance Halzen's review [2]). Diffractive excitation has been studied in great detail as a mechanism for very heavy quark production by P. D. B. Collins and Spiller [8]. Their model is essentially that of fig. 4.1, but with Pomeron exchange between the heavy quark system and the oppositely moving hadron replacing the single gluon exchange depicted in fig. 4.1, and with the addition of recombination of a heavy quark with fast moving light quarks from the hadron to form the observed heavy hadrons in the final state.

We recognize that Figs. 4.1 and 4.2 contain as subgraphs the lowest order graphs for heavy quark production via gluon-gluon fusion, as discussed in sect. 2. However the present case is apparently quite different because the gluon with momentum q is not necessarily on shell with zero transverse momentum, nor is it necessarily transversely polarized.

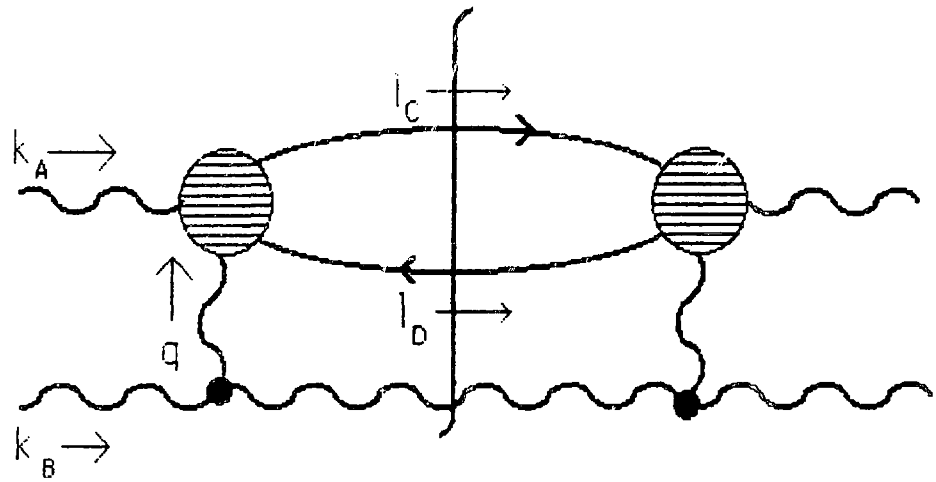

Figure 4.1. Gluon fragmention

Let us consider the kinematics of fig. 4.1. The incoming parton momenta are

$$k_A = (x_A P, 0, \mathbf{0})$$
$$k_B = (0, x_B P, \mathbf{0}). \tag{4.1}$$

The outgoing heavy quark momenta are

$$l_C = (2^{-1/2} [\mathbf{l}_C^2+M^2]^{1/2} \exp(y_C), 2^{-1/2} [\mathbf{l}_C^2+M^2]^{1/2} \exp(-y_C), \mathbf{l}_C)$$
$$l_D = (2^{-1/2} [\mathbf{l}_D^2+M^2]^{1/2} \exp(y_D), 2^{-1/2} [\mathbf{l}_D^2+M^2]^{1/2} \exp(-y_D), \mathbf{l}_D), \tag{4.2}$$

where, since we have the fragmentation region in mind, we imagine that the rapidities are large. That is, $x_A P$, $2^{-1/2} [\mathbf{l}_C^2+M^2]^{1/2} \exp(y_C)$ and $2^{-1/2} [\mathbf{l}_D^2+M^2]^{1/2} \exp(y_D)$ are of order P.

We can now use this kinematic information to help analyze the subgraphs shown in fig. 4.2. Consider first the off-shell parton momenta, for instance k_A+q in subgraph (a).

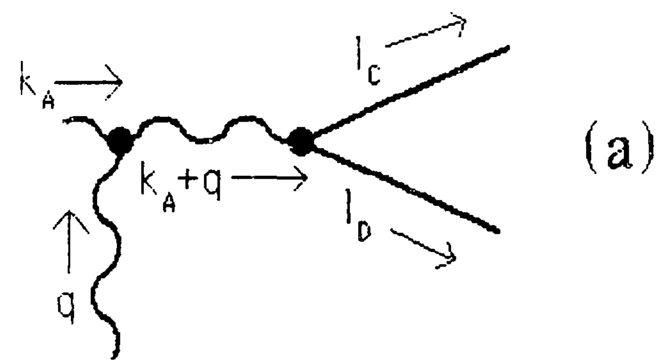

Figure 4.2 (a). Subgraph for fig. 4.1

We have, using eq. (2.4),

$$
\begin{aligned}
(k_A+q)^2 &= (l_C+l_D)^2 \\
&= 2\,\{\,M^2 + [\,l_C^2 + M^2\,]^{1/2}\,[\,l_D^2 + M^2\,]^{1/2}\,\cosh(y_C - y_D) - l_C \cdot l_D\,\} \\
&> 4\,M^2. \qquad (4.3)
\end{aligned}
$$

Thus the gluon must be off-shell by at least $4M^2$. Furthermore, the denominator $1/(k_A+q)^2$ favors transverse momenta l_C, l_D that are small compared to $s^{1/2}$, but there is a lower cutoff at $l^2 \sim M^2$. The

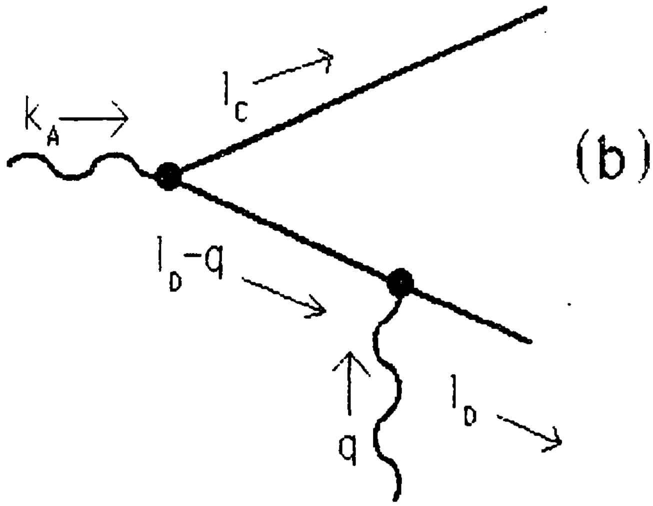

Figure 4.2(b). Another contribution to the subgraph in fig. 4.1.

situation with respect to the heavy quark momentum l_D-q in fig. 4.2(b) is similar:

$$
\begin{aligned}
(l_D - q)^2 - M^2 &= (k_A - l_C)^2 - M^2 \\
&= -2 k_A \cdot l_C \\
&= -2\,(l_C^+ + l_D^+)\, l_C^- \\
&= -\,(l_C^2 + M^2 + [l_C^2 + M^2]^{1/2}\,[l_D^2 + M^2]^{1/2} \exp(y_D - y_C)), \qquad (4.4)
\end{aligned}
$$

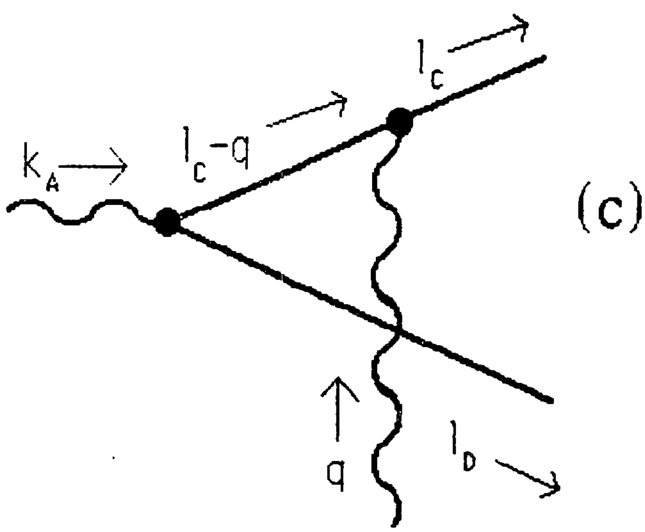

Fig. 4.2(c). The third contribution to the subgraph in fig. 4.1.

For the final subgraph, fig. 4.2(c), one finds

$$(\mathbf{l}_C-q)^2 - M^2 \approx -(\mathbf{l}_D{}^2 + M^2 + [\mathbf{l}_D{}^2+M^2]^{1/2}\,[\mathbf{l}_C{}^2+M^2]^{1/2}\exp(y_C-y_D)). \qquad (4.5)$$

Thus in all cases there is a propagator that is off-shell by $\sim M^2$, and the propagator favors heavy quark transverse momenta that are small compared to $s^{1/2}$ but not smaller than M.

We now turn our attention to the momentum q of the exchanged gluon. The transverse components of q are given by the transverse momenta of the heavy quarks:

$$\mathbf{q} = \mathbf{l}_C+\mathbf{l}_D \qquad (4.6)$$

Now, consider the plus component of q. The on-shell quark momentum k_B-q has a minus component that is approximately x_BP, a transverse component $-\mathbf{q}$, and thus a plus component $\mathbf{q}^2/2x_BP$. Thus

$$q^+ \approx -\mathbf{q}^2/2x_BP. \qquad (4.7)$$

Finally, the minus component of q is determined by the minus components of the heavy quark momenta (recall that $k_A{}^-=0$):

$$q^- = 2^{-1/2}\,[\mathbf{l}_C^2+M^2]^{1/2}\exp(-y_C) + 2^{-1/2}\,[\mathbf{l}_D^2+M^2]^{1/2}\exp(-y_D), \qquad (4.8)$$

which is small because of the factors exp(-y). Since q^+ and q^- are so small, one finds that

$$q^2 \approx -\mathbf{q}^2 \qquad (4.9)$$

whenever $M \ll P$.

The next step of the analysis depends on our choice of gauge. We choose Feynman gauge in this paper. In Feynman gauge, the numerator associated with the exchanged gluon in each of the subgraphs of fig. 4.2 is proportional to P^2, with negligible $\mathbf{q}$ dependence. The denominator contributes a factor $|\mathbf{q}|^{-2}$ to the amplitude. Thus each term in the cross section is proportional to the very singular factor $|\mathbf{q}|^{-4}$.

Now we consider the question of cancellations among the three subgraphs. For this purpose, we use the soft gluon approximation [23;24]. We begin with a factor $J\cdot\varepsilon$ at the upper quark gluon vertex in fig. 4.2(c), say. Here J^μ is the current, which lies mostly in the μ="+" direction, and ε^μ is the gluon polarization vector, which lies mostly in the μ= "-" direction. In the soft gluon approximation, one makes the replacement

$$J\cdot\varepsilon \rightarrow J\cdot q\ v\cdot\varepsilon\ /\ v\cdot q \qquad (4.10)$$

where v=(1,0,0). This approximation is exact if J^μ has only a + component. Notice that if the soft gluon approximation were valid for the attachment of the gluon with momentum q to the three subgraphs, then the three subgraphs would cancel each other by a Ward identity. Thus we must ask under what conditions the soft gluon approximation is useful. The required condition is that q^- (= $v\cdot q$) not be too small. Specifically, we need

$$q^- \gg \mathbf{q}^2/P,\ \ \mathbf{q}\cdot\mathbf{l}/P. \qquad (4.11)$$

When this condition is satisfied, the heavy quark denominator $[(l_C-q)^2 - M^2]$ in fig. 4(c) is dominated by $l_C^+q^-$. The factor $1/q^-$ that thus appears is just the $1/q^-$ in the soft gluon approximation (4.10). Recalling from eq. (4.8) that q^- is of order M^2/P, we see that the soft approximation will be valid, and thus some cancellation will occur if

$$\mathbf{q}^2 \ll M^2. \qquad (4.12)$$

We must now examine how much cancellation occurs. The difference between the exact vertex and the soft approximation vertex is

$$\begin{aligned}\Delta\, J\cdot\varepsilon &= J\cdot\varepsilon - J\cdot q\ v\cdot\varepsilon\ /\ v\cdot q \\ &= J^-\varepsilon^+ - \mathbf{J}\cdot\boldsymbol{\varepsilon} - J^-q^+\ v\cdot\varepsilon\ /\ v\cdot q + \mathbf{J}\cdot\mathbf{q}\ v\cdot\varepsilon\ /\ v\cdot q \end{aligned} \qquad (4.13)$$

Of the four terms here, the first is suppressed by a factor $1/P^4$ compared to $J\cdot\varepsilon = J^+\varepsilon^-$ because both J^- and ε^+ are suppressed by $1/P^2$ compared to J^+ and ε^-; the second term is suppressed by $1/P^2$ because $\mathbf{J}$ and $\boldsymbol{\varepsilon}$ are suppressed by $1/P$ compared to J^+ and ε^-; the third term is suppressed by a factor of at least $1/P^2$ because J^- is suppressed by $1/P^2$ compared to J^+; finally the fourth term is not suppressed by any powers of P at all. Thus a complete leading twist approximation in the $P \to \infty$ limit (with masses and transverse momenta fixed) consists of making the replacement

$$J \cdot \varepsilon \to \textit{soft approximation} + \textit{correction}$$
$$= \quad J \cdot q \; v \cdot \varepsilon \,/\, v \cdot q \qquad + \mathbf{J} \cdot \mathbf{q} \; v \cdot \varepsilon \,/\, v \cdot q. \tag{4.14}$$

As we remarked earlier, the sum over graphs (a), (b), and (c) of the *soft approximation* vanishes. Thus in each graph we need only retain the *correction* term. Noting that **J** is proportional to the transverse momenta of the heavy quarks, we read off from Eq. (4.14) and from Eq (4.8) for q^- that the cancellation of the soft approximation terms on *both* sides of the final state cut creates a suppression factor that behaves as a function of **q** like

$$\frac{[(\textit{transverse momentum}) \cdot \mathbf{q}]^2}{P^2 \left([\mathbf{l}_C^2 + M^2]^{1/2} \exp(-y_C) + 2^{-1/2} [\mathbf{l}_D^2 + M^2]^{1/2} \exp(-y_D) \right)^2} . \tag{4.15}$$

Thus we reach an important conclusion. In individual graphs there is a denominator $[1/q^\mu q_\mu]^2 \sim [1/\mathbf{q}^2]^2$ that favors small q. When one integrates the differential cross section over **q** there would be a quadratic divergence if one were to neglect the lower cutoff at $\mathbf{q}^2 \sim (300 \text{ MeV})^2$, where cancellations due to the color singlet nature of the hadrons begin. Thus, only contributions from **q**~300 MeV would be important in the **q**-integrated cross section. However, in the sum of graphs there is a cancellation of this very singular behavior. The quadratic sensitivity to the lower cutoff becomes only logarithmic (cf. ref. [14]). That is to say, the **q**-integrated cross section obtains contributions equally from all **q** between 300 Mev and M.

We come now to the crucial point of this paper. We have seen that the entire region $(300 \text{ MeV})^2 \le \mathbf{q}^2 \le M^2$ contributes at leading twist to the **q**-integrated cross section. But the physics near the lower end of this range is presumably nonperturbative and thus not calculable. We must therefore ask what is happening in fig. 4.1 when $\mathbf{q}^2 \ll M^2$. It will prove helpful to consider this problem from the point of view of a reference frame boosted along the z-axis until the heavy quarks are approximately at rest, so that the rapidities y_C and y_D are near to zero. In such a frame, the minus component of the momentum of the exchanged gluon (eq. (4.8)),

$$q^- = 2^{-1/2}\,[l_C^2+M^2]^{1/2}\exp(-y_C) + 2^{-1/2}\,[l_D^2+M^2]^{1/2}\exp(-y_D),$$

is large, of order M at least. The plus component of q is small and the transverse components are, we assume, small. Therefore the momentum of the exchanged gluon is seen by the heavy quark pair as being collinear to hadron B.

This observation suggests the proper approximations to make. We begin by writing the cross section in the form

$$\frac{d\sigma}{dy_C\, dy_D\, dl_C\, dl_D} = P^{-1}\int dq^+ \frac{1}{16\pi^2\,[M_{CD}^2-q^2]^2}\, x_A\, f_{g/A}(x_A) \qquad (4.16)$$
$$\times (2\pi)^{-4}\,(v\cdot q)^2\, A(x_A,q^\mu)^{\alpha\beta}\, B(q^\mu)_{\alpha\beta}$$

Here A is the factor corresponding to the upper subdiagram in fig. 4.1. The factor B includes the Bethe-Salpeter wave function for hadron B (not shown in the figure), the quark line and the propagator for the exchanged gluon. The delta function that fixes q^+ has been absorbed into B. The momentum fraction x_A is determined by $x_A = (l_C^+ + l_D^+ - q^+)/P$.

We notice immediately that q^+ is negligible in comparison to the plus components of the particles in A, so that we can set q^+ to zero in the function $A(x_A,q^\mu)^{\alpha\beta}$ and in the kinematics of the upper subgraph.

The quantity B is the contribution from the particular Feynman diagram shown in fig. 4.1 to the quantity

$$\mathcal{B}(q^\mu)_{\alpha\beta} = \int d^4x\ \exp(-iq\cdot x)\,\langle P_B\,|\,A_\beta(x)\,A_\alpha(0)\,|\,P_B\rangle \qquad (4.17)$$

Our discussion of the soft approximation causes us to realize that we can replace $A^{\alpha\beta}\ B_{\alpha\beta}$ by

$$A^{\alpha\beta}\ (v\cdot q)^{-2}\,(v\cdot q\ g_{\alpha\rho}-q_\alpha v_\rho)\,(v\cdot q\ g_{\beta\sigma}-q_\beta v_\sigma)\,B^{\rho\sigma}$$
$$= A^{\alpha\beta}\ C_{\alpha\beta}. \qquad (4.18)$$

Here $C_{\alpha\beta}$ is a contribution to

$$C(q^\mu)_{\alpha\beta} = (v\cdot q)^{-2} \int d^4x \, \exp(-iq\cdot x)\, v^\lambda v^\tau$$
$$\times \langle P_B |\, [\, \partial_\lambda A_\beta(x) - \partial_\beta A_\lambda(x)\,]\; [\, \partial_\tau A_\alpha(0) - \partial_\alpha A_\tau(0)\,] \,| P_B \rangle. \quad (4.19)$$

Let us suppose that we are interested in the cross section integrated over the small $\mathbf{q}$ region, $\mathbf{q}^2 \ll M^2$. We can then neglect $\mathbf{q}$ in $A(x_A, q^\mu)^{\alpha\beta}$ and in the kinematics of the upper subgraph. That is, we can replace $C(q^+, q^-, \mathbf{q})_{\alpha\beta}$ by $\delta(\mathbf{q}) \int d\boldsymbol{\kappa}\; C(q^+, q^-, \boldsymbol{\kappa})_{\alpha\beta}$. This leaves us with the factor (including some of the kinematic factors from eq.(4.16)),

$$\delta(\mathbf{q})\;\; P^{-1} \int dq^+\, (2\pi)^{-4}\, (v\cdot q)^2 \int d\boldsymbol{\kappa}\; C(q^+, q^-, \boldsymbol{\kappa})_{\alpha\beta}. \quad (4.20)$$

There is some bookkeeping to do with the indices. First, according to eqs. (4.14) and (4.18), the leading contribution comes from transverse values for the Lorentz indices α, β. Also, $C_{\alpha\beta}$, integrated over $\boldsymbol{\kappa}$, must be invariant under rotations about the z axis. Thus we replace $C_{\alpha\beta}$ by $-\frac{1}{2}\, \delta^T_{\alpha\beta}\, C^\gamma{}_\gamma$, where $\delta^T_{\alpha\beta}$ is 1 for $\alpha=\beta=1$ or 2, zero otherwise. Similarly, C must be invariant under SU(3) transformations of its color indices, which we have suppressed. Thus we may replace C by $(1/3)\; 1_{color}\, \mathrm{Tr}(C)$. With these changes, the factor (4.20) becomes

$$\delta(\mathbf{q})\, (1/3)\; 1_{color}\, \tfrac{1}{2}\, \delta^T_{\alpha\beta}$$
$$\times (2\pi)^{-4}\, (v\cdot q)^2\, P^{-1} \int dq^+ \int d\boldsymbol{\kappa}\; \mathrm{Tr}(-C(q^+, q^-, \boldsymbol{\kappa})^\gamma{}_\gamma) \quad (4.21)$$

The second line of (4.21) is a particular lowest order contribution to the quantity

$$x_B\, f_{g/B}(x_B), \quad (4.22)$$

where $x_B = q^-/P = v\cdot q/P$ is the gluon momentum fraction and $f_{g/B}(x_B)$ is the gluon distribution function as defined in [25]:

$$f_{g/B}(x) = (2\pi x P)^{-1} \int dy^+ \exp(-ixPy^+) \quad (4.23)$$
$$\times \langle P |\, F_a(y^+, 0, \mathbf{0})^{-\nu}\, \mathcal{P} \exp\!\left(ig \int_0^{y^+} dz^+\, A_c^+(z^+, 0, \mathbf{0})\, T_c\right) F_a(0)_\nu{}^-\, | P \rangle .$$

Here $F^a_{\mu\nu}$ is the field strength tensor $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$, T_C is the generator matrix for the adjoint representation of SU(3), and $\mathcal{P}$ denotes path ordering of the line integral of the potential. Also, the factor

$$(1/3)\ \mathrm{Tr}_{color}(1_{color} + \delta^T_{\alpha\beta} A^{\alpha\beta}) \tag{4.24}$$

is a particular contribution to the lowest order invariant amplitude squared, $|\mathcal{M}|^2$, discussed in sect.1.

We have now reached a result that would have seemed surprising at first. When the "diffractive" gluon fragmentation graphs are integrated over the momentum transfer **q**, the contribution from the dangerous region of low momentum transfer, $q^2 \ll M^2$, factorizes. The cross section takes the form, under an integral over transverse momentum,

$$\frac{d\sigma}{dy_C\, dy_D\, dl_C\, dl_D} = \frac{1}{16\pi^2 M_{CD}^4}\, \delta^2(\mathbf{q})\, x_A f_{g/A}(x_A)\ x_B f_{g/B}(x_B)\ |\mathcal{M}|^2$$

$$+ \text{ hard scattering correction.} \tag{4.26}$$

The first term, corresponding to small **q**, has exactly the form of the gluon-gluon fusion graphs, eq. (2.6). The exchanged gluon appears as a constituent gluon in hadron B, no different from any other constituent gluon except that it carries a very low momentum fraction x_B, which is of order M^2/s when the heavy quarks are produced in the fragmentation region of hadron A.

When we integrate over all momentum transfers **q**, there will be leading contributions from the region $q^2 \sim M^2$. In this region, the various small **q** approximations will not be valid, so there will be a computable correction to eq. (2.6). This is a hard scattering correction to the Born graphs for gluon-gluon fusion. It is proportional to $\alpha_s(M^2)$ since only $q^2 \sim M^2$ will contribute importantly to the integral.

One can also discuss the cross section for measured **q**, although this requires more analysis, which is outside the scope of this paper. We may nevertheless list some conjectures concerning the **q**-distribution. We expect that the $\mathbf{q}$-distribution of the heavy quark pair is similar to the $\mathbf{q}$-distribution of a muon pair produced by the Drell-Yan process at a similar value of s with the dimuon invariant mass approximately equal to the diquark invariant mass, ie. about M. Thus we expect that the factor $\delta(\mathbf{q})$ becomes essentially the convolution of the transverse momentum distributions of the gluons in the two hadrons. The width of this distribution is small compared to M, but it depends on the renormalization scale of the problem, in this case M^2, and gets slowly wider as M^2 increases. The distribution has a $1/\mathbf{q}^2$ tail that connects smoothly to the large $\mathbf{q}^2$ perturbative region.

In the one gluon exchange example analyzed above, we do not see the higher order terms in $f_{g/B}(x)$, eq. (4.23), that arise from the $A_\mu{}^2$ term in $F_{\mu\nu}$ and the exponential of the line integral of A_μ. We have not investigated the factorization of $f_{g/B}(x)$ from the cross section in higher order graphs in this paper. In the Appendix, however, we examine the pair production cross section in a model in which the quantum gluons produced by hadron B are replaced by a non-abelian external potential $A_\mu(x)$ and the incoming parton from hadron A has very large energy in the rest frame of the external potential. In this simple model, it is easy to work to all orders in the external potential. One finds that the quantity $f_{g/B}(x)$ given in eq. (4.23) does factor from the cross section.

5. Conclusions

We have argued that the cross section for the production of very heavy quarks or other heavy particles carrying color factorizes in QCD. That is, the cross section decomposes in the form

$$\frac{d\sigma}{dy_C\, dy_D} = \sum_{a,b} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B) \times H_{ab}(y_C - Y,\, y_D - Y,\, x_A x_B s,\, M), \tag{5.1}$$

where the sum runs over parton types a,b and Y is the rapidity of the parton c.m. frame, $Y = \frac{1}{2}\ln(x_A/x_B)$. Here the f's are parton distribution functions and , H_{ab} is a hard scattering function describing the process a+b→heavy quark pair + X. It is assumed that the rapidities are in the region $|y_C - y_D| \leq 1$, which we expect to contribute most of the total cross section. The hard scattering function is perturbatively calculable in an expansion in powers of $\alpha_s(M)$: potential singularities in H have been factored into the parton distribution functions. Corrections to this formula are suppressed by powers of (hadron mass scale/M).

We have by no means proved this result in this paper, but we believe that the analysis given here should make the result plausible. We are arguing that heavy quark production is not essentially different from the Drell-Yan process, $A+B \to \mu^+\mu^- X$, with a dimuon mass Q on the order of the heavy quark mass M. It is not a simple matter to establish that the Drell-Yan cross section factorizes, given the difficulties with initial state interactions, but we believe that the arguments for factorization at all orders of perturbation theory in the Drell-Yan case are now reasonably complete [17,18,26]. In the heavy quark case, there are additional difficulties because the produced heavy quarks carry color and can therefore interact with soft gluons, but we expect that these difficulties can be rather easily overcome because the heavy quarks are in the final state, and final state interactions are easily disposed of by using unitarity.

In designing experiments to look for heavy particle production one must know how reliable the standard predictions are. Thus it will be important to test the predictions by analyzing data on the production of hadrons containing the b and t quarks. It would also be useful to have charm production data over a wider kinematic range. If the standard formula (5.1) holds up both to further theoretical scrutiny and to expeı imental tests, then one will be confident of predictions for production of other, more exotic, heavy particles.

In both the heavy quark process and the Drell-Yan process the case of $M^2 \ll s$ or $Q^2 \ll s$ is of practical interest. In this case it is helpful to view the process from the heavy quark pair (or dimuon) c.m. frame. When $M^2 \ll s$ the spectator partons from each hadron are much more energetic than the active partons. However, the spectators effectively do not interact with the active partons [17,18], so this is not expected to be a problem.

When $M^2 \ll s$, the values of momentum fractions x of the incoming partons are very small. The predictions depend on the gluon distribution function in extreme regions of x -- where data from deeply inelastic scaling violations is sparse. Thus more effort is needed to measure these distributions, in hadron-hdaron collisions as well as in deeply inelastic lepton scattering. There are also potential problems with the theory for sufficiently small values of x, such as might be obtained at the Superconducting Super Collider, as discussed in sect. 3.

One may also draw some phenomenological conclusions based on the lowest order analysis of this paper.

1) Most of the cross section comes form the region $|y_C - y_D| \lesssim 1$, $|\mathbf{k}_C - \mathbf{k}_D| \lesssim M^2$, where the invariant mass M_{CD} of the heavy quark pair is not too far above the threshold, $M_{CD} = 2M$. (In the production of heavy quarks, the necessity of having a spin-½ exchange keeps the produced heavy quarks close in rapidity. But if in production of some more exotic particles exchange of a heavy spin-1 particle is allowed, then the cross section will be larger and the produced particles will not be closely correlated in rapidity.)

2) There are quarkonium resonance peaks in this region, but the individual peaks are washed out by the integrations over heavy quark momenta needed to form the cross section (5.1). The cross section to form individual resonances is sensitive to soft gluon effects and is may not be calculable in a simple way.

3) The transverse momenta of the heavy quarks are typically of order M. (Schematically, $d\sigma/d\mathbf{k} \sim 1/(\mathbf{k}^2+M^2)^2$.)

4) The distribution in the net transverse momentum $\mathbf{q} = \mathbf{k}_C + \mathbf{k}_D$ of the

pair should be similar to the transverse momentum distribution of a Drell-Yan pair (or W or Z boson) of similar mass, as discussed in [27,28,29]. This the width of this distribution is quite small compared to M, but nevertheless it is large compared to 1 GeV when M is large enough.

5) The rapidity distribution of the heavy quarks should have a broad peak in the central rapidity region and should fall off in the fragmentation regions of the incoming hadrons, following the distributions of the gluons (or, sometimes, light quarks) that collide to make the heavy quarks. This behavior is *not* observed in the distribution of charmed quarks [2], presumably because slow charmed quarks are able to recombine with fast spectator quarks to form fast charmed particles (cf. [30]). In contrast, a produced top quark will generally have about 40 GeV of transverse momentum and will not easily form a bound state with a fast light quark having small transverse momentum.

If the experimental data on charm production, taken at face value, are correct and if QCD is a correct theory of strong interactions, then our results suggest that the higher twist corrections to heavy quark production are large. Such large corrections need explanation. Now, to obtain the total cross-section for the production of charm (or other heavy quarks) we have a situation analogous to the total cross-section for e^+e^- annihilation to hadrons, where a low-order perturbative calculation is valid beyond a GeV or two of energy. However, in order to have prominent jets in the e^+e^- annihilation final state, one must go to much higher energies, above 10 GeV. It is only then that one can say that the distribution of hadrons in the final state is a reasonbable facsimile of the distribution of the partons that are produced in perturbation theory. In other words, the 'higher twist' corrections to the distribution of hadrons are very much larger than those to the total cross section. In the hadroproduction of charm, one might expect the higher twist corrections to the charm distribution to be similarly large, while the total charm cross section may be less affected. Models that include large final-state interactions [9; 8] have an obvious role to play here. There is also the possibility of substantial higher-order, leading twist, corrections -- analogous to the large 'K-factor' for the Drell-Yan process, except larger because of the higher color charges involved. It would be useful to know the sizes of the higher-order corrections.

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APPENDIX: Heavy quark production in an external field

In this Appendix, we consider a simple model in which factorization for heavy quark pair production can be investigated beyond lowest order perturbation theory.

First, we choose to work in the rest frame of the target hadron, B. In this frame, the beam hadron, A, has very large momentum in the + direction. We first note an elementary kinematical fact: in this frame the heavy quarks are highly relativistic. Indeed, the minus component, l^-, of the momentum of one of the heavy quarks cannot be larger than the minus component of the momentum of the entire system: $l^- < (P_B^- + P_A^-) \approx P_B^- = m_B/2^{1/2}$. Thus the plus component of the heavy quark's momentum, $l^+ = (M^2+l^2)/2l^-$, cannot be smaller than $M^2/2^{1/2}m_B$. We conclude that l^+ is much larger than the heavy quark mass, M, so that each heavy quark is necessarily highly relativistic.

We are thus led to consider the following physical picture. A highly relativistic parton from hadron A splits into a highly relativistic heavy quark pair in the presence of hadron B, which is at rest. (In general, there will be other debris from hadron A around, but we ignor that complication here). We replace the target hadron and its sea of gluons by an external field $A_\mu(x)$. We shall suppose that the field has a limited spatial extent $r_0 \sim 1$ fm. Since high momentum gluons are to be dealt with perturbatively as part of the hard scattering cross section, we suppose that the external field contains only gluons of moderate momentum; that is, $A_\mu(x)$ is a smooth function of x, with no wiggles of characteristic size much smaller than r_0.

In order to see the physics in the simplest posssible context, let us begin by replacing QCD by the abelian theory, QED. We also replace the incoming photon from hadron A and the heavy quarks by spin-0 particles. We let the incoming parton then have charge 0 the heavy quark and antiquark have charges +e and -e, respectively and denote the momenta by

$$\begin{aligned} k_A &= (P,0,0) \\ l_C &= (z_C P, [\,\mathbf{l}_C^2 + M^2]/2z_C P, \mathbf{l}_C) \\ l_D &= (z_D P, [\,\mathbf{l}_D^2 + M^2]/2z_D P, \mathbf{l}_D). \end{aligned} \tag{A.1}$$

The amplitude for pair production from an external field in the $P \to \infty$ limit can be taken from [31], where the relevant theory is explained in detail.

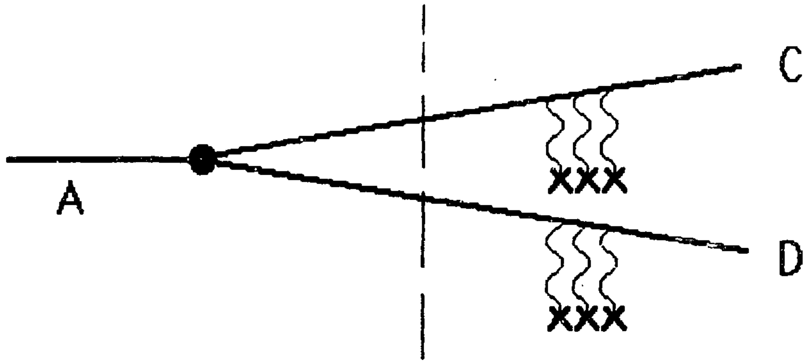

Figure A.1 Pair production in an external field.

The result, illustrated by fig. A.1, is

$$\langle C,D\,|S|\,A\rangle = 2\pi\,\delta(\,[1-z_C-z_D]\,P)\,G\,(2\pi)^{-2}\int d\boldsymbol{k}$$
$$\times\,[\,0-(\boldsymbol{k}^2+M^2)/2z_C P-(\boldsymbol{k}^2+M^2)/2z_D P\,]^{-1}$$
$$\times\,\{\,F_+(\boldsymbol{l}_C-\boldsymbol{k})\,F_-(\boldsymbol{l}_D+\boldsymbol{k})-(2\pi)^4\,\delta(\boldsymbol{l}_C-\boldsymbol{k})\,\delta(\boldsymbol{l}_D+\boldsymbol{k})\,\}. \qquad \text{(A.2)}$$

Here G is a coupling constant. The first factor is the energy denominator (more accurately, the k^- denominator) for the heavy quark intermediate state indicated by the dashed vertical line in fig. A.1. Notice that this k^- denominator is small, of order 1/P; this indicates that the heavy quarks live a long time before they encounter a non-zero external field. The functions F in the second factor are amplitudes for heavy quark scattering from the external field it the eikonal approximation:

$$F_\pm(\boldsymbol{q}) = \int d\boldsymbol{x}\,\exp\,(i\,\boldsymbol{q}\cdot\boldsymbol{x}\,)\,\exp\,[\,\pm\,i\,\chi(\boldsymbol{x})\,], \qquad \text{(A.3)}$$

where

$$\chi(\boldsymbol{x}) = -\,e\int_{-\infty}^{\infty} dx^+\;A^-(\,x^+,0,\boldsymbol{x}\,)\,. \qquad \text{(A.4)}$$

This result can be rewritten in terms of Fourier transforms as

$$\langle C,D\,|S|\,A\rangle = -\,\delta(1-z_C-z_D)\,2G\,z_C z_D\int d\boldsymbol{x}_C\int d\boldsymbol{x}_D\,\exp[\,-i\,(\boldsymbol{l}_C\cdot\boldsymbol{x}_C+\boldsymbol{l}_D\cdot\boldsymbol{x}_D)\,]$$
$$\times\,K_0(M\,|\,\boldsymbol{x}_C-\boldsymbol{x}_D\,|\,)\,\{\,\exp[\,i\,\chi(\boldsymbol{x}_C)-i\,\chi(\boldsymbol{x}_D)\,]-1\,\}. \qquad \text{(A.5)}$$

where K_0 is a modified Bessel function, which falls off exponentially when its argument is large. The physical interpretation is evident. The heavy quarks diverge somewhat in transverse position during the long time after they are produced. The amplitude for them to have a transverse separation $\mathbf{x}_C - \mathbf{x}_D$ is $K_0(M\,|\,\mathbf{x}_C - \mathbf{x}_D\,|\,)$. Then the heavy quarks pass through the external field, travelling along straight line paths parallel to the z-axis. They acquire phases $\exp[\, i\,\chi(\mathbf{x}_C)\,]$ and $\exp[\, -\, i\,\chi(\mathbf{x}_D)\,]$ respectively.

We can now take note of an essential piece of physics: when M is large, the transverse separation between the heavy quarks is small (because of the Bessel function). Thus the eikonal factors $\chi(\mathbf{x})$ tend to cancel. We can make this cancellation explicit by defining

$$\mathbf{b} = z_C\mathbf{x}_C + z_D\mathbf{x}_D\,, \quad \mathbf{r} = \mathbf{x}_C - \mathbf{x}_D\,, \tag{A.6}$$

so that

$$\mathbf{x}_C = \mathbf{b} + z_D\,\mathbf{r}\,, \qquad \mathbf{x}_D = \mathbf{b} - z_C\,\mathbf{r}\;. \tag{A.7}$$

(The factors of z in (A.6) make $\mathbf{b}$ the transverse postion of the "center of P^+", which has useful properties under Lorentz transformations [31]). Then we can make the replacements

$$\chi(\,\mathbf{b} + z_D\,\mathbf{r}\,) = \chi(\,\mathbf{b}\,) + z_D\,\mathbf{r}\cdot\partial\,\chi(\,\mathbf{b}\,) + O(r^2/r_o^2)\quad,$$

$$\chi(\,\mathbf{b} - z_C\,\mathbf{r}\,) = \chi(\,\mathbf{b}\,) - z_C\,\mathbf{r}\cdot\partial\,\chi(\,\mathbf{b}\,) + O(r^2/r_o^2)\quad. \tag{A.8}$$

The two $\chi(\,\mathbf{b}\,)$ terms in the exponent of eq. (A.5) cancel exactly, while the $\mathbf{r}\cdot\partial\,\chi(\,\mathbf{b}\,)$ terms add. Expanding the exponential then gives

$$\langle\, C,D\,|S|\,A\,\rangle = -\,\delta(1-z_C-z_D)\,2G\,z_C\,z_D\int d\mathbf{x}_C\int d\mathbf{x}_D\,\exp[\,-i\,(\mathbf{l}_C\cdot\mathbf{x}_C + \mathbf{l}_D\cdot\mathbf{x}_D)\,]$$

$$\times\,K_0(M\,|\,\mathbf{r}\,|\,)\,\{\,i\,\mathbf{r}\cdot\partial\,\chi(\,\mathbf{b}\,) + O(r^2/r_o^2)\quad\}\;. \tag{A.9}$$

The omitted terms here have more powers of r/r_o, where r_o represents the characteristic dimension of the external field, and thus more powers of $1/Mr_o$. Thus, *to leading power in 1/M, only one photon from the external* *field participates in the pair production.* This is just what is needed to make factorization work in this simple case.

The external field appears in eq. (A.9) through the quantity $\partial_i \chi(\mathbf{b})$. This quantity is

$$\begin{aligned} \partial_i \chi(\mathbf{b}) &= -e \int dx^+ \ \partial_i A^-(x^+, 0, \mathbf{b}) \\ &= -e \int dx^+ [\partial_i A^-(x^+, 0, \mathbf{b}) + \partial_+ A^i(x^+, 0, \mathbf{b})] \\ &= e \int dx^+ \ F_i{}^-(x^+, 0, \mathbf{b}), \end{aligned} \tag{A.10}$$

where the second equality follows by integration by parts. In the standard factorization formula, (4.26), the gluons from hadron B (here represented by the external field) appear in the form of the distribution function $f_{g/B}(x)$. Eq. (4.26) for this function, adapted to abelian gauge theory, would lead one to expect the external field factor (A.10) to be

$$e \int dx^+ \exp(i\, q^- x^+)\, F_i{}^-(x^+, 0, \mathbf{b})$$

here, where

$$q^- = (\mathbf{l}_C^2 + M^2)/2z_C P + (\mathbf{l}_D^2 + M^2)/2z_D P$$

is the minus component of the momentum transfer from the external field. When $P \to \infty$, one has $q^- \to 0$. Thus one can indeed set q^- to zero here as long as the external field falls off quickly as $x^+ \to \infty$, as we have supposed in our model. (This assumption does not model the hadronic case accurately, however; the hadronic gluon distribution has a significant q^- dependence for small momentum fraction $x = q^-/2^{-1/2} m_B$. Thus the factor $\exp(i\, q^- x^+)$ cannot be neglected in the hadronic case.)

Thus in the abelian external field model one does find factorization, including the expected form of the "gluon distribution function."

Let us now examine the more complicated case of an external non-abelian field. Let the incoming parton transform according to the $\mathcal{A}$ representation of the SU(3) group, while the outgoing heavy scalar quarks transform according to the $\mathcal{C}$ and $\mathcal{D}$ representations, respectively. Then there are two contributions to the scattering amplitude. In the first, the incoming parton A scatters from the external field, then breaks up into the two heavy quarks. In the second, the breakup comes first, then the scattering from the external field occurs. The scattering matrix in the $P\to\infty$ limit is

$$\begin{aligned}\langle C,D|S|A\rangle = {} & \delta(1-z_C-z_D)\,2\,z_C z_D \int d\mathbf{x}_C \int d\mathbf{x}_D \exp[\,i\,(\mathbf{l}_C\cdot\mathbf{x}_C+\mathbf{l}_D\cdot\mathbf{x}_D)\,] \\ & \times K_0(M\,|\,\mathbf{r}\,|) \\ & \times \{\, [\mathcal{P}\exp[\,i\,\chi_A(\mathbf{b})]]_{a'a}\, G_{a'cd} \\ & \quad - G_{ac'd'}\,[\mathcal{P}\exp[\,i\,\chi_C(\mathbf{x}_C)]]_{cc'}\,[\mathcal{P}\exp[\,i\,\chi_B(\mathbf{x}_D)\,]]_{dd'}\,\}\,. \end{aligned} \tag{A.11}$$

Here

$$\chi_A(\mathbf{b}) = g\int dx^+\; A_m^-(\,x^+,0,\mathbf{b})\,t^A_m\,, \tag{A.12}$$

where the t^A_m, $m=1...8$, are the generating matrices for the A representation of SU(3); the other χ's are defined similarly. The symbols $\mathcal{P}$ indicate path ordering of the exponentials, with the highest values of x^+ to the left. The array G_{acd} denotes the coupling of the initial parton (with color a) to the heavy scalar quarks (with colors c and d, respectively).

Let us now analyze eq. (A.11) in the $M\to\infty$ limit. As in the abelian case analyzed above, when M is large, only small values of $\mathbf{r} = \mathbf{x}_C - \mathbf{x}_D$ are important. If one simply substitutes $\mathbf{r} = \mathbf{0}$ (that is $\mathbf{x}_C = \mathbf{x}_D = \mathbf{b}$, in eq. (A.11)), then the two terms on the right hand side of the equation cancel each other because SU(3) invariance requires that the coupling G obey

$$\begin{aligned} G_{a'c'd'}\,[\mathcal{P}\exp[\,-i\,\chi_A(\mathbf{b})]]_{a'a}\,[\mathcal{P}\exp[\,i\,\chi_C(\mathbf{b})]]_{cc'}\,[\mathcal{P}\exp[\,i\,\chi_B(\mathbf{b})\,]]_{dd'} \\ = G_{acd} \end{aligned} \tag{A.12}$$

In order to obtain the first non-vanishing term in the expansion of the right hand side of (A.11) in powers of r, we use the expansion

$$\begin{aligned} \mathcal{P}\exp[\,i\,g\int dx^+\; A_m^-(\,x^+,0,\mathbf{b}+z\,\mathbf{r}\,)\,t_m\,] \\ = \{\,1 + ig\,z\,r^j\,\mathcal{F}_j^m(\mathbf{b})\,t_m + O(r^2/r_o^2)\,\} \\ \times \mathcal{P}\exp[\,i\,g\int dx^+\; A_m^-(\,x^+,0,\mathbf{b}\,)\,t_m\,]\,, \end{aligned} \tag{A.13}$$

where

$$\mathcal{F}_j^{m}(\mathbf{b}) = \int_{-\infty}^{\infty} dy^+ \; \mathcal{P}\exp[\, i\, g \int_{y^+}^{\infty} dx^+ \; A_a^-(x^+, 0, \mathbf{b})\, T_a\,]_{mn} F_{j+}^{n}(y^+, 0, \mathbf{b})\,. \tag{A.14}$$

Here the T_a are the generating matrices for the 8 representation of SU(3) and

$$F_{\mu\nu}^{n} = \partial_\mu A_\nu^{n} - \partial_\nu A_\mu^{n} + g\, f_{nab}\, A_\mu^{a} A_\nu^{b} \tag{A.15}$$

is the usual non-abelian field strength tensor.

Using the expansion (A.13), the expression (A.11) for the pair production amplitude becomes

$$\begin{aligned}
\langle C,D\,|S|\,A\,\rangle = & -\delta(1-z_C-z_D)\, 2\, z_C z_D \int d\mathbf{x}_C \int d\mathbf{x}_D \, \exp[\, i\,(\mathbf{l}_C\cdot\mathbf{x}_C + \mathbf{l}_D\cdot\mathbf{x}_D)\,] \\
& \times K_0(M\,|\,\mathbf{r}\,|\,)\,\{\, ig\, z_D\, r^j\, \mathcal{F}_j^{m}(\mathbf{b})\, [t^C_m\,]_{cc'}\, \delta_{dd'} \\
& \qquad - ig\, z_C\, r^j\, \mathcal{F}_j^{m}(\mathbf{b})\, [t^D_m\,]_{dd'}\, \delta_{cc'} + O(r^2/r_o^2)\,\} \\
& \times [\mathcal{P}\exp[\, i\, \chi_C(\mathbf{b})]]_{c'c''}\, [\mathcal{P}\exp[\, i\, \chi_D(\mathbf{b})\,]]_{d'd''}\, G_{ac''d''} \\
= & -\delta(1-z_C-z_D)\, 2\, z_C z_D \int d\mathbf{x}_C \int d\mathbf{x}_D \, \exp[\, i\,(\mathbf{l}_C\cdot\mathbf{x}_C + \mathbf{l}_D\cdot\mathbf{x}_D)\,] \\
& \times K_0(M\,|\,\mathbf{r}\,|\,)\,\{\, ig\, (x_C-b)^j\, \mathcal{F}_j^{m}(\mathbf{b})\, [t^C_m\,]_{cc'}\, G_{a'c'd} \\
& \qquad + ig\, (x_D-b)^j\, \mathcal{F}_j^{m}(\mathbf{b})\, [t^D_m\,]_{dd'}\, G_{a'cd'} + O(r^2/r_o^2)\,\} \\
& \times [\mathcal{P}\exp[\, i\, \chi_A(\mathbf{b})]]_{a'a}\,,
\end{aligned} \tag{A.16}$$

where we have used eqs. (A.7) and (A.12) to obtain the second equality.

Let us first notice the factor $[\mathcal{P}\exp[\, i\, \chi_A(\mathbf{b})]]_{a'a}$. This factor is a unitary matrix, and cancels when we square the scatering amplitude and sum over colors *a*.

We thus reach an important conclusion. In the $Mr_o \to \infty$ limit, the dependence of the pair production cross section on the external field occurs only through the factor $\mathcal{F}_j^{m}(\mathbf{b})$ defined in eq. (A.13). This factor is a complicated non-linear function of the external potential $A_\mu(x)$. However, it is just the factor that occurs in the standard definition (4.23) of the gluon distribution function $f_{g/B}(x)$, except that the quantum operator $A_\mu(x)$ in $f_{g/B}(x)$ is replaced by an external potential in our model.

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