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THE HIGH ENERGY BEHAVIOR OF FERMION-MESON AND
MESON-MESON SCATTERING IN A SUPERSYMMETRIC FIELD THEORY

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The high energy behavior of fermion-meson and
meson-meson scattering in a supersymmetric field theory

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A review of the general methods of high energy calculations within the framework of field theory is given. These include Feynman parameter methods, the infinite momentum frame, and Mellin transform methods. A general discussion of supersymmetry is also presented. The basic aspects of the Wess-Zumino model are given. This includes a proof of renormalizability through one-loop, and a derivation of the Feynman rules. The general supersymmetry algebra is presented. The method of generating representations of this algebra through the concept of superfields is discussed.

The high-energy behavior of fermion-boson and boson-boson scattering amplitudes of a supersymmetric field theory containing a spin- $\frac{1}{2}$ fermion field, a scalar field, and a pseudoscalar field, is investigated. The results can be easily modified to apply to the Yuhawa model and the neutral version of the linear σ model. The results are also compared to those of fermion-fermion scattering in the same model.

In the leading logarithm approximation, ladders with fermions running along the sides in the t channel and mesons as rungs dominate in each order of two classes of diagrams. The sums of the dominant series give rise to fixed Regge cuts for all amplitudes in each of the three theories. All amplitudes in the supersymmetric theory possess a definite

signature factor, while the amplitudes for fermion-fermion and fermion-antifermion scattering in the Y model and the σ model lack it. The results of the supersymmetric theory are also compared to the results of the spontaneously broken non-Abelian gauge theory.

I. INTRODUCTION

The only model of high energy physics which formally reproduces all the properties of an analytic, crossing-symmetric, relativistic theory is that provided by perturbation theory within the framework of quantum field theory. Perturbation theory can serve as a theoretical laboratory in which one can test the consistency of ideas about the structure of relativistic theories.

Quantum field theory has enjoyed some success in explaining nature. On the purely theoretical side, the calculation of high energy behavior has demonstrated that certain field theories exhibit Reggeization (1), which phenomenologically is a basic feature of high energy hadronic amplitudes.

For a more direct example, QED has been established as a successful theory, at least at low energy. Here the principles are relatively simple. One starts with a simple Lagrangian and then applies quantization. In order to perform any calculations, one must expand in perturbation, which ultimately requires renormalization. But the results have been highly successful, as illustrated by the anomalous magnetic moment for the electron (2):

$$\begin{aligned} \text{Theory:} & \quad (1159655.3 \pm 2.5) \times 10^{-9} \\ \text{Experiment:} & \quad (1159657.7 \pm 3.5) \times 10^{-9} \end{aligned} \tag{1.1}$$

Another class of field theories which are being viewed favorably are the non-Abelian gauge theories. The Weinberg-Salam model (3) is a

renormalizable theory unifying weak and electromagnetic interactions. One of its successes is the prediction of the existence of neutral currents, which have been verified experimentally (4).

Non-Abelian gauge theories have also been shown to exhibit asymptotic freedom, which provides a theoretical basis for the understanding of Bjorken scaling (5). They may also provide a means for quark confinement (6). The high energy behavior of the scattering amplitude for fermion-fermion scattering in a spontaneously broken non-Abelian gauge theory (SBNGT) has been calculated independently through sixth order in the leading logarithm approximation by McCoy and Wu (7), Nieh and Yao (8), and Tyburski (9). They arrived at the interesting result that, after a complicated cancellation of the transverse-momentum ($P_{\perp}$) integration, the leading logarithm approximation gives rise to a $\ln s$ series, as opposed to the $\ln^2 s$ series expected for a renormalizable theory. This is consistent with Reggeization of the amplitude, as suggested by Grisaru, Schnitzer, and Tsao (10). Cheng and Lo (11) have recently extended the calculation to all orders. They find that the $I = 1$ (one unit of isospin exchanged) amplitude does indeed Reggeize. This absence of the $P_{\perp}$ integration corresponds to $P_{\perp}$ damping in particle-production amplitudes, in agreement with the present experimental situation for hadron production. All these favorable qualities tend to make one think that non-Abelian gauge theories, although probably not the whole answer, might embody several aspects of a correct theory of hadron interactions.

Another asymptotically free theory that has been studied in the

high energy limit is φ^3 in six dimensions $[(\varphi^3)_6]$. Brown, Gordon, Wong, and Young (12), and Muzinich and Tsao (13) found that the on-shell scattering amplitude, in the leading logarithm approximation, was the sum of a series of $\ln^2 s$, and thus lacked $P_\perp$ damping.

In the spirit of using field theory as a theoretical laboratory, it would be interesting to study the high energy behavior of a theory which has some other aspect in common with non-Abelian gauge theories. One such theory, which has been quite popular in its own right, is supersymmetry. Supersymmetry, as its name suggests, shares the common properties with non-Abelian gauge theories of a high degree of symmetry and the effects these symmetries have upon their renormalization (14). Some work in this area has already been completed by Bing-Lin Young, T. F. Wong, and J. W. Opoien (15). The model studied is the Wess-Zumino model (16), which contains a Majorana spinor, a scalar, and a pseudo-scalar. The fermion-fermion scattering amplitude was calculated to all orders in the leading logarithm approximation. No damping in transverse momentum was found, leading to a series in $\ln^2 s$. The series was summed to all orders in the leading logarithms, resulting in a fixed cut in the complex angular momentum plane.

One is naturally led to the question of what happens in boson-fermion and boson-boson scattering within this model. One is also interested in what role supersymmetry plays in these instances. In any case, it would be interesting to compare these amplitudes with the fermion-fermion amplitude found in Reference 15.

The purpose of this work is to answer these questions. For both fermion-boson and boson-boson scattering we were able to sum the leading class of diagrams to all orders. As in Reference 15, the internal meson lines play a relatively passive role in the high energy behavior, with no meson self couplings appearing in the leading diagrams. In both cases we find a series in $\ln^2 s$, a result of no $P_\perp$ damping. Order by order, the amplitudes for fermion-boson and boson-boson scattering are identical with the spin averaged amplitudes obtained in Reference 15 for fermion-fermion scattering, demonstrating that, at least in the high energy limit, supersymmetry does indeed play a large role in the relationships between scattering amplitudes (17).

As in Reference 15, we are able to compare the results in this supersymmetry model (s model) with those of similar models without the supersymmetry - the Yukawa model (Y model) and the neutral version of the linear σ model.

Although in this work we consider only the original model of supersymmetry introduced by Wess and Zumino (16), we should perhaps note that many developments have taken place since then. Relying heavily upon the concept of superfields as introduced by Salam and Strathdee (18), many people have introduced the ideas of supergravity (19) and of supersymmetric theories containing non-Abelian gauge symmetry (20). One of the hopes of supergravity is that the supersymmetry will improve the renormalizability of a theory coupling matter fields to gravity fields.

The plan of this work is as follows: Using a simple model, we discuss in Section II the general methods of high energy calculations

within the framework of field theory. These include Feynman parameter methods, the infinite momentum frame, and Mellin transform methods. We also give the results of a more realistic model for hadron interactions, massive QED. In Section III we give a general discussion of supersymmetry. We start with the Wess-Zumino model, proving its renormalizability through one-loop. We go on to discuss the general supersymmetry algebra and general means of constructing representations of this algebra. We close this section with a short summary about supersymmetry. In Section IV we discuss the model on which the calculation of Section V is based. We list the Feynman rules for the Wess-Zumino model. In Section V we discuss the calculation of fermion-boson and boson-boson scattering in a supersymmetric model to all orders in the leading logarithm approximation. We also list the modifications needed to obtain the results for the Yukawa model and for the neutral version of the linear σ model. We give the conclusions and discussion in Section VI. Appendix A lists the notational conventions we use, as well as providing a list of useful identities involving Majorana spinors. In Appendix B we derive the Feynman rules given in Section IV. Appendixes C, D, and E contain some of the formulas needed in Section V.

II. HIGH ENERGY BEHAVIOR IN PERTURBATION THEORY

A. Motivation

It is a common belief that hadron physics becomes simple at high energies, when one should be well past the bumps and dips characteristic of the resonance region. It is hoped that cross sections become smooth in the high energy regime, and that the underlying physics becomes simple enough that it can be plainly seen. With the advent of the CERN Intersecting Storage Rings and NAL accelerator, we may have finally reached the high energy region in terms of obtaining experimental data.

The most successful phenomenological model to date for two body and quasi- two body reactions at high energies is that of Regge poles, which is based upon the use of the complex angular momentum plane. From the standpoint of theory, it is most convenient to study the structure of the complex angular momentum plane at high energy. It is possible to make this study, at least to a certain extent, within the framework of perturbation theory.

One must be aware of certain reservations in making such a study. It is possible that field theory might not be relevant to hadron physics. Its relevancy at high energies might also be questionable, since even QED has been tested experimentally only at relatively low energies. In any case, since one is dealing with strongly interacting particles, and thus with large coupling constants, one must make an all-orders calculation. This is also indicated by the fact that Regge poles are connected with the bound states of a theory. In quantum field theory bound states

are described by the Bethe-Salpeter equation (21). The Bethe-Salpeter equation for spinless particles can be written in momentum space as the integral equation

$$A(p_1, p_2, p_3, p_4) = K(p_1, p_2, p_3, p_4) + \int_k K[p_1, p_2, (p_1+k), (p_2-k)] \\ \times \Delta(k-p_2) \Delta(k+p_1) A[(p_1+k), (p_2-k), p_3, p_4] \quad (2.1)$$

where $\Delta(p)$ is the propagator for the spinless particle, and the kernel K is a sum of two particle irreducible diagrams. Equation (2.1) can be represented symbolically as in Figure 1a. A formal series solution of Equation (2.1) is given diagrammatically in Figure 1b. If we take for K the exchange of a single particle, we arrive at the complete set of ladder diagrams, as illustrated in Figure 1c. In practical calculations one sums certain classes of diagrams to all orders, usually in the leading logarithm approximation (to be described below). The class of leading diagrams often turns out to be the ladders (11, 12, 15).

Any attempt to evaluate the set of ladder graphs (or any other two particle irreducible kernel K) exactly in the large s limit amounts to solving the Bethe-Salpeter equation (Equation (2.1)). Heuristically, summing over the set of leading diagrams in the leading logarithm approximation has led to interesting results (11, 12, 15).

The calculation of the leading logarithm terms in SBNGT is especially interesting (7, 8, 9, 11). In any given order the most divergent diagrams are shown to cancel. Thus other diagrams must be considered in order to calculate the leading effect.

$$A = K + K A$$

(a)

$$A = K + K K + \dots$$

(b)

$$A = \text{vertical line} + \text{vertical line} \text{ rectangle} + \dots$$

(c)

Figure 1. Bethe-Salpeter equation: (a) Exact form, (b) Formal series solution, and (c) Ladder approximation

B. General Procedure

In general, one is able to expand the scattering amplitude as a power series in the coupling constant g and in powers of $\ln s$.

$$T(s,t) \sim g^2 s^{\ell} \{ a_0(t) + g^2 [b_1(t) \ln^{\beta} s + b_2(t) \ln^{\beta-1} s + \dots] + g^4 [c_1 \ln^{2\beta} s + c_2 \ln^{2\beta-1} s + \dots] + \dots \} \quad (2.2)$$

One must be aware of the mathematical subtleties involved in making such a double expansion. This expansion is usually evaluated in the weak coupling limit, that is, $g \rightarrow 0$, in order to assure convergence. In the leading logarithm approximation the procedure is to keep only the most divergent terms in s for any given order of the coupling constant g . In this approximation (2.2) becomes

$$T(s,t) \sim g^2 s^{\ell} \{ a_0(t) + g^2 b_1(t) + g^4 c_1 \ln^{2\beta} s + \dots \} \quad (2.3)$$

One of the justifications for such an approximation is that it often leads to physically meaningful results when the summation is performed. For renormalizable theories the following values for ℓ and β can arise:

$$\ell = 1, 0, \text{ or } -1 \quad , \quad (2.4)$$

$$\beta = 1 \text{ or } 2 \quad . \quad (2.5)$$

This leads to

$$T(s,t) \sim s^{\ell} s^{\tilde{\alpha}(t)} \quad \text{for } \beta = 1 \quad , \quad (2.6)$$

which corresponds to the Regge trajectory $\alpha(t) = \ell + \tilde{\alpha}(t)$, or to

$$T(s, t) \sim s^{\ell} s^{\tilde{\alpha}(t)} (\ln s)^{\nu} \quad \text{for } \beta = 2, \quad (2.7)$$

corresponding to a fixed cut in the complex angular momentum plane.

C. Feynman Parameter Method

We shall illustrate the leading logarithm techniques, using Feynman parameter methods, within a relatively simple theory - that of φ^3 in four dimensions $[(\varphi^3)_4]$ (22). We shall concentrate on the ladder diagrams.

The Lagrangian for this theory is

$$L = \frac{1}{2} [(\partial_{\mu} \varphi)^2 - m^2 \varphi^2] + \frac{1}{3!} g \varphi^3. \quad (2.8)$$

The Feynman rules are given in Figure 2.

Let us calculate the box diagram shown in Figure 3. The amplitude is given by

$$\begin{aligned} T_4 = g^4 \int_k & \{ [(k+p_1)^2 - m^2 + i\epsilon] [(k+p_1+p_2)^2 - m^2 + i\epsilon] \\ & \times [(k+p_3)^2 - m^2 + i\epsilon] [k^2 - m^2 + i\epsilon] \}^{-1}. \end{aligned} \quad (2.9)$$

One can Feynman parameterize, using the formula (22, p. 31)

$$\frac{1}{f_1 f_2 \dots f_N} = (N-1)! \int_0^1 \frac{d\alpha_1 d\alpha_2 \dots d\alpha_N \delta(1-\sum \alpha)}{[\sum_{r=1}^N \alpha_r f_r]^N}. \quad (2.10)$$

The momentum integral can then be performed, yielding

(a) $i/(p^2 - m^2)$

(b) ig

Figure 2. Feynman rules for $(\varphi^3)_4$

Figure 3.. The box diagram

$$T_4 = g^2 \left(\frac{g^2}{16\pi^2} \right) \int_0^1 \frac{d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \delta(1-\alpha_1-\alpha_2-\beta_1-\beta_2)}{[\alpha_1 \alpha_2 s + d(\beta, t)]^2} \quad (2.11)$$

where

$$d(\beta, t) = \beta_1 \beta_2 t - m^2 [1 - \beta_1 - \beta_2 + (\beta_1 + \beta_2)^2] + i\epsilon. \quad (2.12)$$

Let us consider the limit $s \rightarrow \infty$ with t fixed. In the region $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ the integrand behaves like $1/s^2$. In the region $\alpha_1, \alpha_2 \sim 1/s$ the integrand behaves like a constant. Thus the dominant integration region should be $\alpha_1 \sim 0, \alpha_2 \sim 0$. So consider instead the integral

$$T_4 = \int_0^\epsilon d\alpha_1 d\alpha_2 \int_0^1 \frac{d\beta_1 d\beta_2 \delta(1-\beta_1-\beta_2)}{[\alpha_1 \alpha_2 s + d(\beta, t)]^2} + O\left(\frac{1}{s^2}\right) \quad (2.13)$$

One is now able to perform the α_1 and α_2 integrals with the result

$$\begin{aligned} \int_0^\epsilon d\alpha_1 d\alpha_2 \frac{1}{[\alpha_1 \alpha_2 s + d(\beta, t)]^2} &= \int_0^\epsilon d\alpha_2 \frac{\epsilon}{d(\beta, t) [\epsilon \alpha_2 s + d(\beta, t)]} \\ &= \frac{1}{s d(\beta, t)} \ln \left[\frac{\epsilon^2 (-s - i\epsilon)}{d(\beta, t)} \right] \\ &\rightarrow \frac{1}{d(\beta, t)} \frac{\ln(-s - i\epsilon)}{s} \end{aligned} \quad (2.14)$$

where we have introduced the symbol $\rightarrow$ for $\lim_{s \rightarrow \infty}$. We obtain $\ln(-s-i\epsilon)$ upon the careful handling of the limit $\epsilon \rightarrow 0^+$. Therefore (2.13) becomes

$$T_4 = g^2 \frac{\ln(-s-i\epsilon)}{s} K(t) \quad (2.15)$$

where

$$\begin{aligned} K(t) &= \frac{g^2}{16\pi^2} \int_0^1 \frac{d\beta_1 d\beta_2 \delta(1-\beta_1-\beta_2)}{d(\beta, t)} \\ &= \frac{g^2}{8\pi^2} \frac{1}{\sqrt{t(t-4m^2)}} \ln \frac{\sqrt{t-4m^2} + \sqrt{t}}{\sqrt{t-4m^2} - \sqrt{t}} \end{aligned} \quad (2.16)$$

We find it convenient to represent $K(t)$ as given in Equation (2.16) by the diagram shown in Figure 4. This diagram is obtained from that given in Figure 3 by the shrinking of the lines labeled by α_1 and α_2 . These are the Feynman parameters we set approximately equal to zero in the integrand of (2.11).

We now proceed to the n -rung ladder illustrated in Figure 5.

One can make a similar Feynman parameterization as in the case of the box diagram. This enables the momentum integrations to be performed, with the result

$$T_{2n} = g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (n-1)! \int_0^1 \frac{\Pi d\alpha d\beta \delta(1-\Sigma\alpha-\Sigma\beta) [c(\alpha, \beta)]^{n-2}}{[\alpha_1 \dots \alpha_n s + d(\alpha, \beta, t)]^n} \quad (2.17)$$

Figure 4. The contracted diagram associated with Figure 3

Figure 5. The n-rung diagram

The α 's are the parameters of the rungs of the ladder and the β 's are the parameters for the sides. The polynomials C and D can be obtained by the graphical cutting rules (22, pp. 31-36). Again we make the approximation of setting the α 's equal to zero everywhere except in the coefficient of s. One can then perform the α integrals, obtaining

$$T_{2n} = \frac{g^2}{s} \frac{[\ln(-s-i\epsilon)]^{n-1}}{(n-1)!} \left\{ \left(\frac{g^2}{16\pi^2} \right)^{n-1} (n-2)! \right. \\ \left. \times \int_0^1 \frac{\pi d\beta \delta(1-\Sigma\beta) [C(0,\beta)]^{n-2}}{[d(0,\beta,t)]^{n-1}} \right\} \quad (2.18)$$

In the derivation of (2.18) the formula (12)

$$\int_0^1 \frac{d\alpha_1 d\alpha_2 \dots d\alpha_n}{\alpha_1 \alpha_2 \dots \alpha_n s + m^2} \rightarrow \frac{1}{n!} \frac{\ln^n(s)}{s} \quad (2.19)$$

has been used. If one shrinks the lines in Figure 5 which are labeled by α 's, the parameters we set approximately equal to zero, one obtains the diagram shown in Figure 6. This resembles a product of diagrams like that shown in Figure 4. This suggests that the term in the brackets {} in Equation (2.18) is factorizable and is simply $[K(t)]^{n-1}$ where $K(t)$ is given in (2.16). That is, we assert that

Figure 6. The contracted diagram associated with Figure 5

$$\left(\frac{g^2}{16\pi^2} \right)^{n-1} (n-2)! \int_0^1 \frac{\pi d\beta \delta(1-\Sigma\beta) [C(0,\beta)]^{n-2}}{[d(0,\beta,t)]^{n-1}} = [K(t)]^{n-1} . \quad (2.20)$$

This can indeed be proven (22). Substituting (2.20) into (2.18) yields

$$T_{2n} = g^2 \frac{[\ln(-s-i\epsilon)]^{n-1}}{n(n-1)!} [K(t)]^{n-1} . \quad (2.21)$$

The sum over all the ladders can then be performed, giving the result

$$\begin{aligned} T &= \sum_{n=1}^{\infty} T_{2n} \\ &= g^2 \frac{1}{s} \sum_{n=1}^{\infty} [K(t) \ln(-s-i\epsilon)]^{n-1} / (n-1)! \\ &= g^2 \frac{1}{s} \exp [K(t) \ln(-s-i\epsilon)] \\ &= g^2 (-s)^{\alpha(t)} \end{aligned} \quad (2.22)$$

where

$$\alpha(t) = -1 + K(t) . \quad (2.23)$$

We can substitute for $K(t)$ in (2.23), using (2.16), to obtain

$$\begin{aligned}
\alpha(t) &= -1 + \frac{g^2}{8\pi^2} \frac{1}{\sqrt{t(t-4m^2)}} \ln \frac{\sqrt{t-4m^2} + \sqrt{t}}{\sqrt{t-4m^2} - \sqrt{t}} \\
&= -1 + \frac{g^2}{4\pi^2} \frac{1}{\sqrt{t(t-4m^2)}} \tanh^{-1} \left[\frac{1}{\sqrt{1-4m^2/t}} \right] . \quad (2.24)
\end{aligned}$$

We have thus obtained a Regge pole with a trajectory given by (2.24).

D. Infinite Momentum Frame

A calculational technique which is sometimes useful in calculations at high energy is that of the infinite momentum frame (23). This method is based upon the introduction of Sudakov variables (24). Given the momentum labeling as in Figure 7, the Sudakov variables are defined by

$$\begin{aligned}
p_+ &= p^0 + p^3 , \\
p_- &= p^0 - p^3 , \\
p_\perp &= (p^1, p^2) . \quad (2.25)
\end{aligned}$$

Then

$$(p \cdot q) = \frac{1}{2}(p_+ q_- + p_- q_+) - p_\perp \cdot q_\perp . \quad (2.26)$$

In the following discussion we shall assume equal mass particles. In order to take the high energy limit, it is convenient to choose the

Figure 7. Momentum assignments for the general two-body scattering amplitudes

reference frame in which the following relation hold (the Breit frame):

$$\begin{aligned}
 p_1 &= (E, -r_x, -r_y, \lambda\eta) \quad , \\
 p_2 &= (E, r_x, r_y, -\lambda\eta) \quad , \\
 p_3 &= (E, r_x, r_y, \lambda\eta) \quad , \\
 p_4 &= (E, -r_x, -r_y, -\lambda\eta) \quad .
 \end{aligned}
 \tag{2.27}$$

In this reference frame, if we take the limit $s \rightarrow \infty$ with t fixed, the Sudakov variables (2.25) take the form

$$\begin{aligned}
 p_{1+} &= p_{3+} = p_{2-} = p_{4-} = \sqrt{s} + 0\left(\frac{1}{\sqrt{s}}\right) \quad , \\
 p_{1-} &= p_{3-} = p_{2+} = p_{4+} = \frac{m^2}{\sqrt{s}} + 0\left(\frac{1}{s\sqrt{s}}\right) \quad , \\
 p_{1\perp} &= p_{4\perp} = (-r_x, -r_y) \quad , \\
 p_{2\perp} &= p_{3\perp} = (+r_x, +r_y) \quad .
 \end{aligned}
 \tag{2.28}$$

Here

$$m^2 = m^2 + \vec{r}^2 \tag{2.29}$$

and

$$4\vec{r}^2 = -t \quad . \tag{2.30}$$

We thus see that this special choice of variables can lead to simplifications in evaluating leading effects.

E. The Mellin Transform

The Mellin transform is another calculational technique useful in the calculation of the leading behavior of diagrams (25). This method is powerful enough that, in some cases, one is able to calculate all terms in the asymptotic behavior of a graph.

The Mellin transform is defined by the integral

$$F(\alpha) = \int_0^{\infty} f(s) s^{-\alpha-1} ds \quad (2.31)$$

and its inverse

$$f(s) = \frac{1}{2\pi i} \int_C F(\alpha) s^{\alpha} d\alpha \quad (2.32)$$

where the contour C is parallel to the imaginary α axis and $F(\alpha)$ is analytic along C .

In particular, the function

$$f(s) = \frac{s^{\alpha_0} (\ln s)^{n-1}}{\Gamma(n)} \quad (2.33)$$

has the Mellin transform

$$F(\alpha) = \frac{1}{(\alpha - \alpha_0)^n} \quad (2.34)$$

We illustrate the Mellin transform method with the n -rung diagram of Figure 5, again within the $[(\varphi^3)_4]$ model. In order to implement the Mellin transform technique, we find it convenient to rewrite Equation (2.19) as

$$T_{2n} = g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (-i)^n \int_0^\infty \frac{\Pi d\alpha d\beta}{c^2} e^{iD/c} \quad (2.35)$$

Here

$$D = \alpha_1 \dots \alpha_n s + d(\alpha, \beta, t) \quad (2.36)$$

This can be seen to be equivalent to (2.17) by rescaling the parameters

$$\begin{aligned} \alpha_i &\rightarrow \rho \bar{\alpha}_i, \\ \beta_j &\rightarrow \rho \bar{\beta}_j, \end{aligned} \quad (2.37)$$

with the constraint

$$\rho = \sum \alpha_i + \sum \beta_j \quad (2.38)$$

The equivalence is established upon the performance of the ρ integral when one makes use of the formula

$$\int_0^\infty d\rho \rho^{-\lambda-1} e^{i\rho(x+i\epsilon)} = e^{-i\pi\lambda/2} x^\lambda \Gamma(-\lambda) \quad (2.39)$$

plus the fact that C is of degree $n-1$ and D is of degree n in the Feynman parameters.

The Mellin transform of (2.35) is given by

$$L_{2n}(\lambda, t) = g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (-i)^n \int_0^\infty ds \int_0^\infty \frac{\Pi d\alpha d\beta}{c^2} s^{-\lambda-1} e^{iD/C} \quad (2.40)$$

The s integral can be performed with the aid of Equation (2.39). This gives

$$L_{2n}(\lambda, t) = g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (-i)^n e^{-i\pi\lambda/2} \Gamma(-\lambda) \int_0^\infty d\alpha_1 \dots d\alpha_n \Pi d\beta \\ \times (\alpha_1 \dots \alpha_n)^\lambda [C(\alpha, \beta)]^{-2-\lambda} e^{id(\alpha, \beta, t)/C} \quad (2.41)$$

This integral is convergent for $\text{Re } \lambda > -1$, but becomes divergent at $\lambda = -1$ due to the α parameter integrations. The integral can be analytically continued to the region $\text{Re } \lambda < -1$ by the method of integration by parts. This yields

$$L_{2n}(\lambda, t) = g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (-i)^n e^{-i\pi\lambda/2} (-1)^n \frac{\Gamma(-1)}{(\lambda+1)^n} \\ \times \int_0^\infty d\alpha_1 \dots d\alpha_n \Pi d\beta (\alpha_1 \dots \alpha_n)^{\lambda+1} \frac{\partial^n}{\partial \alpha_1 \dots \partial \alpha_n} \\ \times (c^{-2-\lambda} e^{id/C}) \quad (2.42)$$

The integral is now convergent for $\text{Re } \lambda > -2$. To calculate the leading behavior from the multiple pole at $\lambda = -1$, one simply sets $\lambda = -1$ in all factors except the term $(\lambda + 1)^{-n}$. One can then perform the α integrations. Upon this integration, the terms c and d factorize, becoming

$$c(0, \beta) = \prod_i (\beta_i + \beta_{n+i-1}) \quad (2.43)$$

and

$$d(0, \beta, t) = \sum_{j=1}^{n-1} \left[t \frac{\beta_j (\beta_{n+j-1})}{\beta_j + \beta_{n+j-1}} - (m^2 - i\epsilon) (\beta_j + \beta_{n+j-1}) \right] \quad (2.44)$$

One now needs the result

$$\begin{aligned} \int_0^\infty dydz (y+z)^{-1} \exp i \left[\frac{tyz}{y+z} - (m^2 - i\epsilon)(y+z) \right] &= \\ &= \int_0^1 dydz \int_0^\infty dp \delta(1-y-z) \exp ip \left[\frac{tyz}{y+z} - (m^2 - i\epsilon)(y+z) \right] \\ &= i \int_0^1 dydz \delta(1-y-z) [yzt - m^2 + i\epsilon]^{-1} \\ &= i \left(\frac{16\pi^2}{g^2} \right) K(t) \end{aligned} \quad (2.45)$$

where $K(t)$ is given in (2.16).

The β integrals can now be performed, giving the result

$$\begin{aligned}
L_{2n}(\lambda, t) &= g^2 \left(\frac{g^2}{16\pi^2} \right)^{n-1} (-1)^n (i)^n \frac{i}{(\lambda+1)^n} \left[i \left(\frac{16\pi^2}{g^2} \right) K(t) \right]^{n-1} \\
&= g^2 \frac{K(t)^{n-1}}{(\lambda+1)^n}
\end{aligned} \tag{2.46}$$

With the aid of (2.33) and (2.34), the inverse Mellin transform of (2.46) can be taken. This yields

$$\begin{aligned}
T_{2n} &= \frac{1}{2\pi i} \int_C L_{2n}(\lambda, t) s^\lambda d\lambda \\
&= \frac{1}{2\pi i} \int_C g^2 K(t)^{n-1} \frac{s^\lambda}{(\lambda+1)^n} d\lambda \\
&= g^2 \frac{[\ln(-s-i\epsilon)]^{n-1}}{(n-1)!} K(t)^{n-1},
\end{aligned} \tag{2.47}$$

which agrees with Equation (2.21) (the term becomes $\ln(-s-i\epsilon)$ when one is careful about the limit $\epsilon \rightarrow 0^+$).

F. Massive QED

Another model which could give some information about high energy hadron physics is the massive QED model (26). This can be taken as essentially a field theory of quarks coupled by a massive gluon. Although a realistic theory would have to incorporate internal symmetries, the massive QED model could perhaps be relevant.

In this model, the most divergent diagrams are again the ladders. In order to restore gauge invariance, though, one must also include all

twisted ladders. The sum of all ladders plus twisted ladders is called a tower. The amplitude for a tower turns out to be (26)

$$T_{\text{tower}} = i s \frac{s^a}{(\ln s)^2} F(t) \quad (2.48)$$

The value of a is model dependent, but in general it is independent of t . For instance, in spinor electrodynamics, $a = (11/32)\pi\alpha^2$, while in scalar electrodynamics, $a = (5/32)\pi\alpha^2$. The presence of the $1/\ln^2 s$ term represents a fixed Regge cut in the complex angular momentum plane.

One notices immediately that (2.48) violates the Froissart bound (since a is, in general, positive). This implies that in order to obtain the correct asymptotic behavior one must also include graphs which, in any given order, are less than leading. The authors in Reference 26 include all repeated exchanges of towers. In such a model one is able to make several predictions for the high energy region:

- a) $\sigma_T \sim (\ln s)^2$;
- b) The first zero of $d\sigma/dt$ occurs at $t = -(\beta/R_0 \ln s)$ where β is the first zero of $J_1(\beta\pi)$. Thus there is strong shrinkage.
- c) $\sigma_{el}/\sigma_T = \frac{1}{2}$;
- d) $\text{Re } T/\text{Im } T \rightarrow 0$;
- e) One is also able to make predictions for multiplicities in particle production. The predictions are

$$\langle n(s) \rangle \sim s^{a/(1+a)} \ln s \quad (2.49)$$

Several of these results are consistent with the data at ISR and NAL energies (27). All measured hadron cross sections are rising,

except for $\bar{p}p$, which, however, does not decrease anymore and has leveled out. All differential cross sections, except again for $\bar{p}p$, show shrinkage. The ratio $\text{Re } T/\text{Im } T$ is approaching zero for all amplitudes. However, the ratio σ_{el}/σ_T is far from the value $\frac{1}{2}$, ranging from ~ 0.1 for K^-p to ~ 0.2 for pp at 40 GeV/c.

III. SUPERSYMMETRY

A. Wess-Zumino Model

Supersymmetry is an interesting concept which has emerged within the last few years. The remarkable aspect of supersymmetry is the property of allowing fields of different spin to appear in the same irreducible multiplet. The first successful attempt to apply this symmetry to conventional field theory in $3 + 1$ dimensions was achieved by Wess and Zumino (16) who adapted the idea from its original string context (28). They constructed a Lagrangian from scalar fields $A(x)$, $F(x)$, pseudoscalar fields $B(x)$, $G(x)$, and a Majorana spinor field $\psi(x)$. The particular form of the Lagrangian they chose is the following:

$$\mathcal{L} = \mathcal{L}_0 + \frac{g}{2} \mathcal{L}_1 + m \mathcal{L}_m \quad (3.1)$$

where

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2} [(\partial_\mu A)^2 + (\partial_\mu B)^2 + i\bar{\psi}\not{\partial}\psi + F^2 + G^2], \\ \mathcal{L}_1 &= F(A^2 - B^2) + 2GAB - \bar{\psi}(A - i\gamma_5 B)\psi, \\ \mathcal{L}_m &= FA + GB - \frac{1}{2}\bar{\psi}\psi. \end{aligned} \quad (3.2)$$

The supersymmetry was imposed by assuming that the fields transformed infinitesimally as

$$\begin{aligned} \delta A(x) &= \bar{\psi}(x)\epsilon, \\ \delta B(x) &= i\bar{\psi}(x)\gamma_5\epsilon, \\ \delta F(x) &= i\partial_\mu \bar{\psi}(x)\gamma^\mu\epsilon, \end{aligned} \quad (3.3)$$

$$\delta G(x) = - \partial_\mu \bar{\psi}(x) \gamma^\mu \gamma_5 \epsilon$$

$$\delta \psi(x) = - i \gamma_\mu \partial^\mu [A(x) + i \gamma_5 B(x)] \epsilon + [F(x) + i \gamma_5 G(x)] \epsilon .$$

Here ϵ is an infinitesimal parameter. It is a Majorana spinor which anti-commutes with all other fermionic numbers and operators. With these variations, one finds that

$$\delta \mathcal{L}_0 = i \partial^\mu [\bar{\psi} \gamma_\mu (F + i \gamma_5 G) \epsilon - i (\partial_\mu A) \psi \epsilon + (\partial_\mu B) \psi \gamma_5 \epsilon] ,$$

$$\delta \mathcal{L}_1 = i \partial^\mu [\bar{\psi} \gamma_\mu (A + i \gamma_5 B)^2 \epsilon] , \quad (3.4)$$

$$\delta \mathcal{L}_m = i \partial^\mu [\bar{\psi} \gamma_\mu (A + i \gamma_5 B) \epsilon] .$$

Since each of these terms transforms like a total derivative, the action constructed from $\mathcal{L}$ is invariant. In the derivation of the Equations in (3.4) and in much of what follows, use is made of several identities listed in Appendix A.

The equation of motion for the F and G fields are

$$F + mA + \frac{g}{2} (A^2 - B^2) = 0 ,$$

$$G + mB + gAB = 0 . \quad (3.5)$$

Therefore the F and G fields are dependent fields. The remaining fields obey the following equations of motion:

$$\square A = mF + g(FA + GB - \frac{1}{2} \bar{\psi} \psi)$$

$$\square B = mG + g(GA - FB + \frac{1}{2} i \bar{\psi} \gamma_5 \psi) \quad (3.6)$$

$$i \not{\partial} \psi = m\psi + g(A - i \gamma_5 B) \psi$$

The invariance of the action under supersymmetry implies the

existence of a conserved current,

$$J^\mu = \gamma^\lambda [\partial_\lambda (A - i\gamma_5 B)] \gamma^\mu \psi + im\gamma^\mu (A + i\gamma_5 B) \psi \\ + i \frac{g}{2} \gamma^\mu (A + i\gamma_5 B)^2 \psi. \quad (3.7)$$

With the aid of the Equations in (3.5), this can be rewritten as

$$J^\mu = \gamma^\lambda [\partial_\lambda (A - i\gamma_5 B)] \gamma^\mu \psi - i(F + i\gamma_5 G) \gamma^\mu \psi \quad (3.8)$$

That this current is indeed conserved, i.e., that

$$\partial_\mu J^\mu = 0, \quad (3.9)$$

can be easily verified using the equations of motion given in (3.6). It is also clear that the associated supersymmetry charge, $Q = \int d^3 x J_0$, generates the transformations listed in (3.3).

As stated earlier, the F and G fields are dependent fields. They could be eliminated from the Lagrangian using their equations of motion given in (3.5). However, the same effect can be achieved by functional methods, which we now illustrate.

All n -point Green's functions involving A , B , and ψ fields can be generated from the vacuum transition amplitude in the presence of external sources. This generating functional can be written as

$$Z(J_A, J_B, \bar{\eta}) \sim \int \mathcal{D}F \mathcal{D}G \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \left[\int d^4 x \left\{ \mathcal{L}_0 + \frac{g}{2} \mathcal{L}_1 \right. \right. \\ \left. \left. + im\bar{\psi}\psi + J_A A + J_B B + \bar{\eta}\psi \right\} \right]. \quad (3.10)$$

One does not need to integrate over ψ since $\psi = (\bar{\psi}C)^T$.

Define

$$J_F \equiv \frac{g}{2}(A^2 - B^2) + mA, \quad (3.11)$$

$$J_G = g AB + mB.$$

Expression (3.10) then becomes

$$\begin{aligned} Z(J_A, J_B, \bar{\eta}) \sim \int \mathcal{D}F \mathcal{D}G \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \left[\int d^4x \left\{ \frac{1}{2} [(\partial_\mu A)^2 + (\partial_\mu B)^2 \right. \right. \\ \left. \left. + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - g\bar{\psi}(A - i\gamma_5 B)\psi \right\} + \frac{1}{2}(F^2 + G^2) \right. \\ \left. + F J_F + G J_G + J_A A + J_B B + \bar{\eta}\psi \right]. \end{aligned} \quad (3.12)$$

The F and G integrals can be performed using the formula

$$\begin{aligned} \int \mathcal{D}\xi e^{i \int dx dy [\frac{1}{2} \xi(x) A(x, y) \xi(y) + j(x) \delta(x-y) j(y)]} = \\ = \det A^{-\frac{1}{2}} e^{-\frac{1}{2} \int dx dy j(x) A^{-1}(x, y) j(y)}. \end{aligned} \quad (3.13)$$

In this case $A = \delta(x-y)$, so $\det A = 1$. Thus (3.12) becomes

$$\begin{aligned} Z(J_A, J_B, \bar{\eta}) \sim \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \left[\int d^4x \left\{ \frac{1}{2} [(\partial_\mu A)^2 + (\partial_\mu B)^2 \right. \right. \\ \left. \left. + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - g\bar{\psi}(A - i\gamma_5 B)\psi - \frac{1}{2}(J_F^2 + J_G^2) \right. \right. \\ \left. \left. + J_A A + J_B B + \bar{\eta}\psi \right\} \right]. \end{aligned} \quad (3.14)$$

From the Equations in (3.11) it follows that

$$J_F^2 + J_G^2 = \left(\frac{g}{2}\right)^2 (A^2 + B^2)^2 + m^2(A^2 + B^2) + gmA(A^2 + B^2). \quad (3.15)$$

Thus

$$Z(J_A, J_B, \bar{\eta}) \sim \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \int d^4x \frac{1}{2} [(\partial_\mu A)^2 - m^2 A^2]$$

$$\begin{aligned}
& + [(\partial_\mu B)^2 - m^2 B^2] + [i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi] \\
& - g\bar{\psi}(A - i\gamma_5 B)\psi - mgA(A^2 + B^2) \\
& - \frac{g^2}{4} (A^2 + B^2)^2 \quad . \quad (3.16)
\end{aligned}$$

Therefore it is clear that the following Lagrangian is equivalent to the Lagrangian given in Equation (3.1):

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}[(\partial_\mu A)^2 - m^2 A^2] + \frac{1}{2}[(\partial_\mu B)^2 - m^2 B^2] \\
& + \frac{1}{2}[i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi] - \frac{g}{2}\bar{\psi}(A - i\gamma_5 B)\psi \\
& - \frac{mg}{2} A(A^2 + B^2) - \frac{g^2}{8} (A^2 + B^2)^2 \quad . \quad (3.17)
\end{aligned}$$

This is the form of the Lagrangian we shall work with in the leading logarithm calculation of Sections IV and V. The Lagrangian given in Equation (3.16) can also be derived from Equation (3.1) by substituting for the F and G fields using the equations of motion given in (3.5)

The supersymmetry invokes certain restrictions upon the Lagrangian. For instance, in Equation (3.17) the masses are all equal, as are the coupling constants. We shall see in the following section that the relations between the masses and couplings play a significant role in the renormalization of the theory.

The similarity of the Lagrangian given in Equation (3.17) to the Yukawa model and to the neutral version of the linear σ model will be discussed in Section IV.

B. Renormalization of the Wess-Zumino Model Through One Loop

The Lagrangian given in Equation (3.16) is clearly renormalizable.

The renormalization of this Lagrangian is simplified from the case where the coupling constants and masses are independent. Iliopoulos and Zumino (29) demonstrated that the only counterterm needed to renormalize this Lagrangian is the wavefunction renormalization. A common wavefunction renormalization constant Z applies to each of the fields A, B , and ψ .

The improved renormalizability is already illustrated on the one-loop level, as we shall see below. For instance, the quadratic divergence of the mass renormalization for the A and B fields cancels among the various diagrams contributing to it. Similarly, the logarithmic divergence of the vertex correction to the Yukawa interaction also cancels between the two diagrams where a scalar or a pseudoscalar is exchanged, leaving a finite vertex correction.

Following Iliopoulos and Zumino (29), we define

$$\begin{aligned}
 m_R &\equiv Zm, \\
 g_R &\equiv Z^{3/2}g, \\
 A_R &\equiv Z^{-\frac{1}{2}}A, \\
 B_R &\equiv Z^{-\frac{1}{2}}B, \\
 \psi_R &\equiv Z^{-\frac{1}{2}}\psi.
 \end{aligned}
 \tag{3.18}$$

In terms of these quantities, the Lagrangian given by Equation (3.17) becomes

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}[(\partial_\mu A_R)^2 - m_R^2 A_R^2] + \frac{1}{2}[(\partial_\mu B_R)^2 - m_R^2 B_R^2] \\
& + \frac{1}{2}[i\bar{\psi}_R \not{\partial} \psi_R - m_R \bar{\psi}_R \psi_R] - \frac{1}{2}g_R \bar{\psi}_R (A_R - i\gamma_5 B_R) \psi_R \\
& - \frac{1}{2}m_R g_R A_R (A_R^2 + B_R^2) - \frac{1}{8}g_R^2 (A_R^2 + B_R^2)^2 \\
& + \frac{1}{2}[(Z-1)(\partial_\mu A_R)^2 - (Z^{-1}-1)m_R^2 A_R^2] \\
& + \frac{1}{2}[(Z-1)(\partial_\mu B_R)^2 - (Z^{-1}-1)m_R^2 B_R^2] \\
& + \frac{1}{2}[(Z-1)i\bar{\psi}_R \not{\partial} \psi_R] - \frac{1}{2}(Z^{-1}-1)m_R g_R^2 A_R (A_R^2 + B_R^2) \\
& - \frac{1}{8}(Z^{-1}-1)g_R^2 (A_R^2 + B_R^2)^2.
\end{aligned} \tag{3.19}$$

For the rest of this subsection we will work only with renormalized quantities and drop the subscript R for the fields, masses, and coupling constants.

The counter terms present in the Lagrangian of Equation (3.19) require the following Feynman rules in addition to those given in Figure 8:

$$\text{---} \times \text{---} \quad i(Z-1)p^2 - i(Z^{-1}-1)m^2$$

$$\text{~~~~~} \times \text{~~~~~} \quad i(Z-1)p^2 - i(Z^{-1}-1)m^2$$

$$\text{---} \times \text{---} \quad i(Z-1)\not{p}$$

(a)		$i/(p-m)$	
(b)		$1/(p^2-m^2)$	$[i/(p^2-\mu^2)]$
(c)		$1/(p^2-m^2)$	$[i/(p^2-\mu^2)]$
(d)		$-ig$	
(e)		$-g\gamma_5$	$[+g\gamma_5]$
(f)		$-3img$	$[-24i\lambda g]$
(g)		$-img$	$[-8i\lambda g]$
(h)	8	$-3ig^2$	$[-24i\lambda g^2]$
(i)		$-ig^2$	$[-8i\lambda g^2]$
(j)			$[8i\lambda]$

Figure 8. Feynman rules for the s , Y , and σ models

$$-3 \operatorname{img}(Z^{-1} - 1)$$

$$-\operatorname{img}(Z^{-1} - 1)$$

$$-3ig^2(Z^{-1} - 1)$$

$$-ig^2(Z^{-1} - 1)$$

We define

$$I_1 \equiv -i \int_k \frac{1}{(k^2 - m^2)^2} \quad (3.20)$$

We will show that assuming

$$Z = 1 - g^2 I_1 \quad (3.21)$$

will remove all divergences to the one-loop level. From Equation (3.21) we obtain

$$(Z^{-1} - 1) = g^2 I_1 \quad (3.22)$$

Equation (3.22) is valid to the order of g^2 .

We will use dimensional regularization for divergent integrals so that all ordinary operations, such as shift of variables, remain valid.

All subtractions will be performed at the origin in momentum space, $p^\mu = 0$.

Let us examine the pseudoscalar self-energy diagrams. To the one-loop level this is given by

where

$$\begin{aligned}
 a &= \frac{1}{2} g^2 \int_k \frac{1}{(k^2 - m^2)}, \\
 b &= \frac{3}{2} g^2 \int_k \frac{1}{(k^2 - m^2)}, \\
 c &= m^2 g^2 \int_k \frac{1}{(k^2 - m^2) [(k-p)^2 - m^2]}, \\
 d &= \frac{g^2}{2} \int_k \text{Tr} \left\{ \gamma_5 \frac{1}{(k-m)} \gamma_5 \frac{1}{(k-p-m)} \right\}
 \end{aligned}$$

$$= -2g^2 \left[\int_k \frac{1}{(k^2 - m^2)} + \int_k \frac{(p \cdot k) - p^2}{(k^2 - m^2) [(k-p)^2 - m^2]} \right],$$

$$e = -i(p^2 + m^2) I_1. \quad (3.23)$$

Adding the first four (non-counter) terms of (3.23), we obtain

$$a + b + c + d = g^2 \int_k \frac{m^2 - 2(p \cdot k) + 2p^2}{(k^2 - m^2) [(k-p)^2 - m^2]}. \quad (3.24)$$

Note that the quadratic divergence has cancelled among the diagrams, leaving an integral which is only logarithmic divergent.

Equation (3.24) can be expanded about $p^\mu = 0$ to obtain

$$\begin{aligned} a + b + c + d &= g^2 \left\{ m^2 \int_k \frac{1}{(k^2 - m^2)^2} + \frac{1}{2} p^\mu p^\nu \left[\int_k \frac{-2g_{\mu\nu}(m^2)}{(k^2 - m^2)^3} \right. \right. \\ &\quad \left. \left. + \int_k \frac{2k^\mu k^\nu g_{\mu\nu}(m^2)}{(k^2 - m^2)^4} - \int_k \frac{8k_\mu k_\nu}{(k^2 - m^2)^3} + \int_k \frac{4g_{\mu\nu}}{(k^2 - m^2)^2} \right] + O((p^2)^2) \right\} \\ &= g^2 \left\{ m^2 \int_k \frac{1}{(k^2 - m^2)^2} + \frac{1}{2} p^2 \left[\int_k \frac{-2}{(k^2 - m^2)^3} + \int_k \frac{2k^2}{(k^2 - m^2)^4} \right. \right. \\ &\quad \left. \left. + 2 \int_k \frac{1}{(k^2 - m^2)^2} \right] + O((p^2)^2) \right\} \quad (3.25) \end{aligned}$$

In deriving Equation (3.25), we have used the result, valid in dimensional regularization, that (30)

$$\int_k \frac{k_\mu k_\nu}{(k^2 - m^2)^3} = \frac{1}{4} \int_k \frac{g_{\mu\nu}}{(k^2 - m^2)^2} \quad (3.26)$$

The terms of $O((p^2)^2)$ in Equation (3.25) are all finite. Using definition (3.20), we can write Equation (3.25) as

$$a + b + c + d = g^2 \{ i I_1 (m^2 + p^2) + p^2 \int_k \frac{m^2}{(k^2 - m^2)^3} + O((p^2)^2) \}. \quad (3.27)$$

If we now add the term e , given by Equation (3.23), we obtain

$$(a + b + c + d + e) = g^2 \{ p^2 \int_k \frac{m^2}{(k^2 - m^2)^3} + O((p^2)^2) \}. \quad (3.28)$$

Equation (3.28) now contains no divergences.

We next consider the scalar meson self-energy. This is given by

where

$$\begin{aligned}
 f &= \frac{3}{2} g^2 \int_k \frac{1}{(k^2 - m^2)} \\
 g &= \frac{1}{2} g^2 \int_k \frac{1}{(k^2 - m^2)} \\
 h &= \frac{9}{2} m^2 g^2 \int_k \frac{1}{(k^2 - m^2) [(k-p)^2 - m^2]} , \\
 i &= \frac{1}{2} m^2 g^2 \int_k \frac{1}{(k^2 - m^2) [(k-p)^2 - m^2]} , \\
 j &= -\frac{1}{2} g^2 \int_k \text{Tr} \left\{ \frac{1}{(k-m)} \frac{1}{(k-p-m)} \right\} \\
 &= -\frac{1}{2} g^2 \int_k 4 \left\{ \frac{1}{(k^2 - m^2)} - 2 g^2 \frac{[2m^2 + p \cdot k] - p^2}{(k^2 - m^2) [(k-p)^2 - m^2]} \right\} , \\
 k &= -i(p^2 + m^2)I_1 .
 \end{aligned} \tag{3.29}$$

Adding the terms $f - j$, we obtain

$$f + g + h + i + j = g^2 \int_k \frac{m^2 - 2(p \cdot k) + 2p^2}{(k^2 - m^2) [(k-p)^2 - m^2]} . \tag{3.30}$$

Again the quadratic divergence has cancelled among the various diagrams. Equation (3.30) is identical to Equation (3.23). Since the counterterm i is identical to the counterterm e for the pseudoscalar self-energy, i will cancel all divergences of Equation (3.30). We obtain

$$f + g + h + i + j + k = g^2 \left\{ p^2 \int_k \frac{m^2}{(k^2 - m^2)^3} + O((p^2)^2) \right\} \tag{3.31}$$

Equation (3.31) is identical to Equation (3.28). All divergences are removed.

The fermion self-energy is given by

where

$$l = -g^2 \int_k \frac{\gamma_5 (k+m) \gamma_5}{(k^2 - m^2) [(k-p)^2 - m^2]}$$

$$= g^2 \int_k \frac{k-m}{(k^2 - m^2) [(k-p)^2 - m^2]},$$

$$m = g^2 \int_k \frac{k+m}{(k^2 - m^2) [(k-p)^2 - m^2]},$$

$$n = i (Z - 1) \not{p} \quad . \quad (3.32)$$

The non-counter terms are given by

$$m + n = 2 g^2 \int_k \frac{k}{[k^2 - m^2] [(k-p)^2 - m^2]} \quad . \quad (3.33)$$

We expand Equation (3.33) in a power series about $p^\mu = 0$. Note that $m + n \big|_{p^\mu = 0} = 0$.

$$\begin{aligned} m + n &= 0 + 4 g^2 p^\mu \gamma^\nu \int_k \frac{k_\nu k_\mu}{(k^2 - m^2)^3} + O(p^2) \\ &= g^2 \not{p} \int_k \frac{1}{(k^2 - m^2)^2} + O(p^2) \\ &= i g^2 \not{p} I_1 + O(p^2) \quad . \end{aligned} \quad (3.34)$$

The terms of $O(p^2)$ are convergent. Adding the expression for n to (3.34) yields

$$m + n = O(p^2) \quad . \quad (3.35)$$

Therefore the divergences for the fermion self-energy are removed.

Let us now consider the trilinear couplings. The radiative corrections to the fermion-scalar vertex are given by

where

$$\begin{aligned} p &= -g^3 \int_{\ell} \frac{\gamma_5 (k_2 + \ell + m) (k_1 + \ell + m) \gamma_5}{D} \\ &= -g^3 \int_{\ell} \frac{(k_2 + \ell - m) (k_1 + \ell - m)}{D} \quad , \\ q &= g^3 \int_{\ell} \frac{(k_2 + \ell + m) (k_1 + \ell + m)}{D} \quad , \\ r &= -mg^3 \int_{\ell} \frac{\ell + m}{D} \quad , \\ s &= -3mg^3 \int_{\ell} \frac{\ell - m}{D} \quad . \end{aligned} \quad (3.36)$$

The denominator D is defined as

$$D = [\ell^2 - m^2] [(k_1 + \ell)^2 - m^2] [(k_2 + \ell)^2 - m^2] \quad (3.37)$$

There are no counterterms for this vertex. None are needed, as we shall see that $p + q + r + s$ is finite.

$$p + q + r + s = 2 g^3 m \int_{\ell} \frac{(k_1 + k_2 + m)}{D} \quad (3.38)$$

The integral in Equation (3.38) is convergent. The logarithmic divergence has cancelled between the terms p and q , i.e., between the diagrams with particles A and B exchanged.

Similarly, the $B \bar{\psi}\psi$ vertex is given by

with

$$t = - (i \gamma_5) g^3 \int_{\ell} \frac{(k_2 + \ell + m) (k_1 + \ell - m)}{D},$$

$$u = i \gamma_5 g^3 \int_{\ell} \frac{(k_2 + \ell - m) (k_1 + \ell + m)}{D},$$

$$v = - i m g^3 \gamma_5 \int_{\ell} \frac{\ell + m}{D},$$

$$w = - i m g^3 \gamma_5 \int_{\ell} \frac{-\ell + m}{D}.$$

The denominator D is given by Equation (3.37). Adding these terms yields

$$t + u + v + w = - (i \gamma_5) 2 m g^3 \int \frac{k_1 - k_2 + m}{D} \quad (3.39)$$

This integral is convergent. Thus no counterterms are needed. Again the logarithmic divergence cancelled between the diagrams exchanging the A and B fields.

Next consider the A^3 vertex, given by

We find it convenient to define

$$\begin{aligned}
 D_1 &\equiv [\ell^2 - m^2] [(\ell + k_1)^2 - m^2] \quad , \\
 D_2 &\equiv [\ell^2 - m^2] [(\ell + k_2)^2 - m^2] \quad , \\
 D_3 &\equiv [\ell^2 - m^2] [(\ell - k_1 + k_2)^2 - m^2] \quad .
 \end{aligned} \tag{3.40}$$

Using the relations of (3.40) and (3.37), the above diagrams can be written as

$$\begin{aligned}
 x.1 &= \frac{1}{2} mg^3 \int_{\ell} \frac{1}{D_3} \quad , \\
 x.2 &= \frac{1}{2} mg^3 \int_{\ell} \frac{1}{D_2} \quad , \\
 x.3 &= \frac{1}{2} mg^3 \int_{\ell} \frac{1}{D_1} \quad , \\
 y.1 &= \frac{9}{2} mg^3 \int_{\ell} \frac{1}{D_3} \quad , \\
 y.2 &= \frac{9}{2} mg^3 \int_{\ell} \frac{1}{D_2} \quad , \\
 y.3 &= \frac{9}{2} mg^3 \int_{\ell} \frac{1}{D_1} \quad , \\
 z &= m^3 g^3 \int_{\ell} \frac{1}{D} \quad , \\
 aa &= m^3 g^3 \int_{\ell} \frac{1}{D} \quad , \\
 ab &= - g^3 \operatorname{tr} \int_{\ell} \frac{\ell (\ell + k_1) (\ell + k_1 + m) (\ell + k_2 + m)}{D} \quad , \\
 ac &= - 3 i mg l_1 \quad .
 \end{aligned} \tag{3.41}$$

These diagrams are at most logarithmically divergent. Expanding in a power series about the origin in momentum space, $k_1^\mu = 0$ and $k_2^\mu = 0$, we obtain

$$x.1 + x.2 + x.3 + y.1 + y.2 + y.3 + z + aa + ab = 3\text{img } I_1 + 0(k) \quad (3.42)$$

The terms of $0(k)$ are all convergent. Thus the counterterm ac eliminates the divergence.

Consider the AB^2 vertex

where

$$\begin{aligned}
 ad &= \frac{3}{2} mg^3 \int_{\ell} \frac{1}{D_3} , \\
 ae &= \frac{3}{2} mg^3 \int_{\ell} \frac{1}{D_3} , \\
 af &= mg^3 \int_{\ell} \frac{1}{D_2} , \\
 ag &= mg^3 \int_{\ell} \frac{1}{D_1} , \\
 ah &= m^3 g^3 \int_{\ell} \frac{1}{D} , \\
 ai &= 3 m^3 g^3 \int_{\ell} \frac{1}{D} , \\
 aj &= g^3 \text{ tr } \int_{\ell} \frac{(\ell + m) \gamma_5 (\ell + k_1 + m) (\ell + k_2 + m) \gamma_5}{D} , \\
 ak &= -i mg I_1 .
 \end{aligned} \tag{3.43}$$

If we expand in a power series, we obtain

$$ad + ae + af + ag + ah + ai + aj = mg I_1 + O(k). \tag{3.44}$$

The terms of $O(k)$ are convergent. The divergence is removed by the counterterm ak .

We must also consider the quartic vertices. The most severe divergence we will encounter is logarithmic. Therefore, when expanding about the origin in momentum space, only the first term ($p_i = 0$) will be divergent. In the following, let us consider only the divergent diagrams and set the external momentum equal to zero. This will give us the correct divergence for the diagram.

Consider the vertex A^4 :

The factor of three comes from the crossed diagrams, all of which are equal for zero external momentum. We obtain

$$\begin{aligned}
 a_l &= g^4 \operatorname{tr} \int_{\ell} \frac{(\ell + m)(\ell + m)(\ell + m)(\ell + m)}{(\ell^2 - m^2)^4} + O(p^2) \\
 &= -4 g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_m &= \frac{9}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_n &= \frac{1}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_o &= -3 i g^2 I_1 .
 \end{aligned} \tag{3.45}$$

We obtain the result

$$3 a_l + 3 a_m + 3 a_n + a_o = O(p^2) . \tag{3.46}$$

Therefore, all divergences are removed.

The case for the B^4 vertex is entirely analagous

$$\begin{aligned}
 & \text{Shaded circle} = \text{Four-point vertex} + 3 \text{ (Square loop)} + 3 \text{ (Two internal vertices)} + 3 \text{ (Wavy line)} + \text{Cross vertex} \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{ap} \qquad \qquad \qquad \text{aq} \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{ar} \qquad \qquad \qquad \text{as}
 \end{aligned}$$

Keeping only the divergent terms, we obtain

$$\begin{aligned}
 a_p &= -g^4 \operatorname{tr} \int_{\ell} \frac{\gamma_5 (\ell + m) \gamma_5 (\ell + m) \gamma_5 (\ell + m) \gamma_5 (\ell + m)}{(\ell^2 - m^2)^4} + O(p^2) \\
 &= -4g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_q &= \frac{9}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_r &= \frac{1}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + O(p^2) , \\
 a_s &= -3 i g^2 I_1 .
 \end{aligned} \tag{3.47}$$

We obtain

$$3 a_p + 3 a_q + 3 a_r + a_s = O(p^2) . \tag{3.48}$$

The counterterm removes the divergence.

Lastly, consider the $A^2 B^2$ vertex:

The diagram shows the decomposition of a four-point vertex (two wavy lines, two dashed lines) into a sum of eight diagrams:

- $=$ (Diagram 1: wavy line exchange) $+$ (Diagram 2: box diagram, labeled at)
- $+$ (Diagram 3: box diagram, labeled au) $+$ (Diagram 4: box diagram, labeled av)
- $+$ (Diagram 5: loop diagram, labeled ax) $+$ (Diagram 6: loop diagram, labeled ay)
- $+$ (Diagram 7: loop diagram, labeled az)

The labels at , au , av , ax , ay , and az are positioned below their respective diagrams. The coefficient 2 is placed before the diagram labeled ay .

Evaluating the above diagrams yields

$$\begin{aligned}
 a_t &= g^4 \operatorname{tr} \int_{\ell} \frac{\gamma_5 (\ell + m) \gamma_5 (\ell + m) (\ell + m) (\ell + m) \times 0 (p^2)}{(\ell^2 - m^2)^4} \\
 &= -4 g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^4} + 0 (p^2) \\
 a_u &= g^4 \operatorname{tr} \int_{\ell} \frac{\gamma_5 (\ell + m) (\ell + m) \gamma_5 (\ell + m) (\ell + m)}{(\ell^2 - m^2)^4} \\
 &= 4 g^4 \operatorname{tr} \int_{\ell} \frac{1}{(\ell^2 - m^2)^4} + 0 (p^2) \\
 a_v &= g^4 \operatorname{tr} \int_{\ell} \frac{\gamma_5 (\ell + m) \gamma_5 (\ell + m) (\ell + m)}{(\ell^2 - m^2)^4} + 0 (p^2) \\
 &= -4 g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + 0 (p^2) , \\
 a_w &= \frac{3}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + 0 (p^2) , \\
 a_x &= \frac{3}{2} g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + 0 (p^2) , \\
 a_y &= g^4 \int_{\ell} \frac{1}{(\ell^2 - m^2)^2} + 0 (p^2) , \\
 a_z &= -3ig^2 I_1 . \tag{3.49}
 \end{aligned}$$

Adding these terms, we arrive at the result

$$a_t + a_u + a_v + a_w + a_x + 2a_y + a_z = 0 (p^2) . \tag{3.50}$$

Therefore the divergences for the corrections to that $A^2 B^2$ vertex have also been eliminated.

This completes the analysis of renormalization of the Wess-Zumino model through one loop. We have explicitly verified that the counter-terms in Equation (3.19) are sufficient to remove all divergences through the one loop level.

C. Graded Lie Algebras

In Subsection A we introduced the concept of a supersymmetry generator within the context of the Wess-Zumino model. In this subsection we construct the algebra formed by the closure of this generator with the Poincare algebra. The supersymmetry generator as given in Subsection A is the charge Q associated with the conserved current of Equation (3.7). From this we see that Q is a Majorana spinor which has anticommutation relations with the fermion fields. For this reason we expect the enlarged algebra to contain anticommutators as well as commutators. A generalized Lie algebra of this type is called a graded Lie algebra (31, 32). One of the interesting features of graded Lie algebras is their ability to allow particles of different spin (i.e., fermions and bosons) to appear within the same irreducible multiplet.

We will use the supersymmetry charge Q of Subsection A to guide us in the construction of the algebra. Then the Wess-Zumino model will be a specific realization of the supersymmetry group. Part of this subsection and part of Subsection D follow Gasiorowicz's (32) lecture notes rather closely.

Let us assume that the supersymmetry transformation is generated by a unitary transformation

$$u(\bar{\epsilon}) = e^{i\bar{\epsilon}Q}$$

with the fields transforming as

$$\phi' = e^{i\bar{\epsilon}Q} \phi e^{-i\bar{\epsilon}Q}, \quad \phi = A, B, \psi. \quad (3.51)$$

This is consistent provided we assume that ϵ is a majorana spinor, anticommuting with all fermionic quantities including fermion fields. With the help of the identity $\bar{\epsilon}Q = \bar{Q}\epsilon$ (see Appendix A), we can then demonstrate that $U(\bar{\epsilon})$ is indeed unitary:

$$\begin{aligned} U^\dagger(\epsilon) &= (e^{i\bar{\epsilon}Q})^\dagger \\ &= e^{-i\bar{Q}\epsilon} \\ &= e^{-i\bar{\epsilon}Q}. \end{aligned} \quad (3.52)$$

We also obtain the infinitesimal transformations

$$\begin{aligned} \delta \phi' &= i \bar{\epsilon} [Q, \phi] + i [\bar{\epsilon}, \phi] Q, \quad \phi = A, B, \\ \delta \psi' &= i \bar{\epsilon} \{Q, \psi\} - i [\bar{\epsilon}, \psi] Q. \end{aligned} \quad (3.53)$$

If we assume ϵ anticommutes with ψ and commutes with A and B , we obtain the usual interpretation of infinitesimal transformations.

Let us examine the algebra satisfied by the spinor charge Q . From the form of Equation (3.7) we require Q to transform as a spinor under the Lorentz group, which justifies calling Q a spinor charge. Thus we require

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2} (\sigma_{\mu\nu})_{\alpha\beta} Q_\beta. \quad (3.54)$$

Since the supersymmetry transformation is a global transformation, it is independent of x . This implies

$$[P_\mu, Q_\alpha] = 0 \quad . \quad (3.55)$$

To demonstrate the closure of the algebra, we must demonstrate that the bracket of Q_α and Q_β is a linear combination of the generators. The fermionic nature of Q suggests that we define the bracket of Q_α and Q_β as the anticommutator. We will demonstrate in Subsection D that this definition is consistent with the closure of the associated group structure. The bracket $\{Q_\alpha, Q_\beta\}$ could be calculated directly using the explicit form for Q given in Subsection A. Instead, let us obtain this anticommutator by considering the action of Q upon the fields.

Let δ_1 and δ_2 be infinitesimal supersymmetry transformations. We calculate

$$\begin{aligned} \delta_2 \delta_1 A &= \delta_2 \bar{\psi} \epsilon_1 \\ &= \bar{\epsilon}_1 \delta_2 \psi \\ &= \bar{\epsilon}_1 (-i \gamma_\mu \epsilon_2 \partial^\mu A + \gamma_\mu \gamma_5 \epsilon_2 \partial^\mu B \\ &\quad + \epsilon_2 F + i \gamma_5 \epsilon_2 G) \quad . \end{aligned} \quad (3.56)$$

In the second line of (3.56) we have made use of an identity from Appendix A. We use other identities in the following. Interchanging labels in Equation (3.56) we arrive at the result

$$\begin{aligned} \delta_2 \delta_1 A &= i \bar{\epsilon}_2 \gamma_\mu \epsilon_1 \partial^\mu A + \bar{\epsilon}_1 \gamma_\mu \gamma_5 \epsilon_2 \partial^\mu B \\ &\quad + \bar{\epsilon}_2 \epsilon_1 F + i \bar{\epsilon}_2 \gamma_5 \epsilon_1 G \quad . \end{aligned} \quad (3.57)$$

Therefore

$$(\delta_2 \delta_1 - \delta_1 \delta_2) A(x) = 2 i \bar{\epsilon}_2 \gamma_\mu \epsilon_1 \partial^\mu A(x)$$

$$= -2 \bar{\epsilon}_2 \gamma_\mu \epsilon_1 [p^\mu, A(x)] \quad (3.58)$$

The same relation can be shown to hold for the fields $B(x)$ and $\psi(x)$.

We wish to write Equation (3.58) in terms of commutators. We note that

$$\begin{aligned} (\delta_2 \delta_1 - \delta_1 \delta_2) A &= -\{[\bar{\epsilon}_2 Q, [\bar{\epsilon}_1 Q, A]] - [\bar{\epsilon}_1 Q, [\bar{\epsilon}_2 Q, A]]\} \\ &= -\{[\bar{\epsilon}_2 Q, [\bar{\epsilon}_1 Q, A]] + [\bar{\epsilon}_1 Q, [A, \bar{\epsilon}_2 Q]]\} \\ &= -\{[[\bar{\epsilon}_2 Q, \bar{\epsilon}_1 Q], A]\} \end{aligned} \quad (3.59)$$

We have used the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

in deriving Equation (3.59). Since the A field can be replaced with ψ or B , comparison of Equation (3.59) with Equation (3.58) yields

$$\begin{aligned} \bar{\epsilon}_2 \gamma_\mu \epsilon_1 p^\mu &= [\bar{\epsilon}_2 Q, \bar{\epsilon}_1 Q] \\ &= [\bar{\epsilon}_2 Q, \bar{Q} \epsilon_1] \\ &= \bar{\epsilon}_{2\alpha} Q_\alpha \bar{Q}_\beta \epsilon_{1\beta} - \bar{Q}_\beta \epsilon_{1\beta} \bar{\epsilon}_{2\alpha} Q_\alpha \\ &= \bar{\epsilon}_{2\alpha} \epsilon_{1\beta} (Q_\alpha \bar{Q}_\beta + \bar{Q}_\beta Q_\alpha) \end{aligned} \quad (3.60)$$

This implies

$$\{Q_\alpha, \bar{Q}_\beta\} = 2 (\gamma_\mu)_{\alpha\beta} p^\mu \quad (3.61)$$

or, equivalently, since $\bar{Q} = Q C^{-1}$,

$$\{Q_\alpha, Q_\beta\} = 2 (\gamma_\mu C)_{\alpha\beta} p^\mu \quad (3.62)$$

This demonstrates that the Poincare algebra amended with the addition of the supersymmetry charge Q does indeed close.

It is interesting to note that, just as Dirac was able to factorize the d'Alembertian via the relations

$$\not{p}, \not{p} = \gamma p^2, [\not{p}^2, \not{p}] = 0, \quad (3.63)$$

Equations (3.62) and (3.55) accomplish a further factorization of the Dirac operator

$$\{Q_\alpha, Q_\beta\} = 2 (\not{P}C)_{\alpha\beta}, [P_\mu, Q_\alpha] = 0. \quad (3.64)$$

D. Superfields

In the previous subsection we discussed the generalized Lie algebra associated with supersymmetry. In this subsection we present a general method for constructing representations of the supersymmetry group generated by this algebra. The method is based upon the concept of superfields (18).

Let us examine the concept of the supersymmetry group somewhat closer. Any element of the group can be written as

$$g(a, \theta, \alpha) = e^{iP_\mu a^\mu} e^{iQ\bar{\theta}} e^{i/2 M_{\mu\nu} \alpha^{\mu\nu}}. \quad (3.65)$$

From the commutation rules of Subsection C it follows that

$$e^{iP_\mu a^\mu} e^{iQ\bar{\theta}} = e^{iQ\bar{\theta}} e^{iP_\mu a^\mu}, \quad (3.66)$$

but

$$e^{i/2 M_{\mu\nu} \alpha^{\mu\nu}} e^{iQ\bar{\theta}} \neq e^{iQ\bar{\theta}} e^{i/2 M_{\mu\nu} \alpha^{\mu\nu}}. \quad (3.67)$$

For the present we shall be concerned with only the group elements $g(a, \theta, 0)$, which we will denote by $g(a, \theta)$. It then follows that

$$g(a_1, \theta_1) g(a_2, \theta_2) = e^{iPa_1} e^{iQ\bar{\theta}_1} e^{iPa_2} e^{iQ\bar{\theta}_2}$$

$$= e^{iP(a_1+a_2)} e^{i\bar{Q}\theta_1} e^{i\bar{Q}\theta_2} \quad (3.68)$$

We now make use of a special case of the Baker-Hausdorff theorem

$$e^A e^B = e^{A+B + 1/2 [A,B]} \quad (3.69)$$

which is valid whenever $[A, [A, B]] = [B, [A, B]] = 0$. This yields

$$\begin{aligned} e^{i\bar{Q}\theta_1} e^{i\bar{Q}\theta_2} &= e^{i\bar{Q}(\theta_1+\theta_2) - 1/2 [\bar{Q}\theta_1, \bar{Q}\theta_2]} \\ &= e^{i\bar{Q}(\theta_1+\theta_2)} e^{\bar{\theta}_2 \gamma_\mu \theta_1 P^\mu} \\ &= e^{i\bar{Q}(\theta_1+\theta_2)} e^{-\bar{\theta}_1 \gamma_\mu \theta_2 P^\mu} \end{aligned} \quad (3.70)$$

We have used Equation (3.60) in deriving (3.70). This verifies the group closure under supersymmetry transformation. We obtain

$$g(a_1, \theta_1) g(a_2, \theta_2) = g(a_1 + a_2 + i\bar{\theta}_1 \gamma_\mu \theta_2, \theta_1 + \theta_2) \quad (3.71)$$

Note that $i\bar{\theta}_1 \gamma_\mu \theta_2$ is real since

$$[\bar{\theta}_1 \gamma_\mu \theta_2]^\dagger = \bar{\theta}_2 \gamma_\mu \theta_1 = -\bar{\theta}_1 \gamma_\mu \theta_2 \quad (3.72)$$

Equation (3.71) implies that an element of the group acts upon the parameter space according to the transformation law

$$g(a, \epsilon) \begin{cases} X_\mu \rightarrow X_\mu + a_\mu + i\bar{\epsilon} \gamma_\mu \theta \\ \theta \rightarrow \theta + \epsilon \end{cases} \quad (3.73)$$

There are some mathematical difficulties with the concept of a parameter space with the transformation properties of (3.71) (see Reference 34); however, useful results are obtained if we proceed formally with this idea.

One now assumes that the group elements can be represented by unitary operators on a Hilbert space which is defined with fields $\Phi(x, \theta)$ depending on two types of coordinates. The coordinate x is the space-time coordinate, while θ is assumed to be an anticommuting four component Majorana spinor. That is,

$$\{\theta_\alpha, \theta_\beta\} = 0 \quad (3.74)$$

Equation (3.74) implies that θ is nilpotent, i.e.,

$$\theta_\alpha \theta_\alpha = 0 \quad (3.75)$$

(No summation on indices is implied.) It then follows that the expansion of $\Phi(x, \theta)$ in powers of θ terminates, yielding

$$\begin{aligned} \Phi(x, \theta) = & A(x) + \bar{\psi}_\alpha(x) \theta_\alpha + \bar{\theta}_\beta M_{\beta\alpha}(x) \theta_\alpha \\ & + (\bar{\chi}_\alpha(x) \theta_\alpha) (\bar{\theta}_\beta \theta_\beta) + u(x) (\bar{\theta}_\alpha \theta_\alpha)^2 \end{aligned} \quad (3.76)$$

Here and in the following, summation of repeated indices is understood.

The third term can be written as

$$\begin{aligned} \bar{\theta}_\beta M_{\beta\alpha}(x) \theta_\alpha &= \theta_\beta (C^{-1} M)_{\beta\alpha} \theta_\alpha \\ &= - \theta_\alpha (C^{-1} M)_{\beta\alpha} \theta_\beta \end{aligned} \quad (3.77)$$

Therefore $C^{-1} M$ must be an antisymmetric matrix. From Appendix A, this implies that the most general form for the third term is

$$\begin{aligned} \bar{\theta}_\beta M_{\beta\alpha}(x) \theta_\alpha &= \frac{1}{2} (\bar{\theta} \theta) F(x) + \frac{i}{2} (\bar{\theta} \gamma_5 \theta) G(x) \\ &+ \frac{1}{2} \bar{\theta} \gamma_\mu \gamma_5 \theta A^\mu(x) \end{aligned} \quad (3.78)$$

Equation (3.76) can then be written as

$$\begin{aligned}\phi(x, \theta) = & A(x) + \bar{\psi}(x)\theta + \frac{1}{2} (\bar{\theta}\theta) F(x) \\ & + \frac{i}{2} (\bar{\theta}\gamma_5\theta) G(x) + \frac{1}{2} \bar{\theta}\gamma_\mu\gamma_5\theta A^\mu(x) \\ & + (\bar{\chi}(x)\theta) (\bar{\theta}\theta) + u(x) (\bar{\theta}\theta)^2.\end{aligned}\quad (3.79)$$

A scalar superfield is defined as a field which transforms under supersymmetry as

$$\begin{aligned}U(\epsilon) \phi(x, \theta) U^\dagger(\epsilon) &\equiv \phi'(x, \theta) \\ &= \phi(x_\mu + i\epsilon\gamma_\mu\theta, \theta + \epsilon)\end{aligned}\quad (3.80)$$

The coordinate $x_\mu + i\epsilon\gamma_\mu\theta$ is to be interpreted in the sense of the power series expansion of the field ϕ .

For simplicity we will also assume that $\phi(x, \theta)$ is real and is a Lorentz scalar. The fields appearing in Equation (3.79) then have the conventional transformation properties under the Lorentz group. The fields $A(x)$, $F(x)$, and $u(x)$ are scalars, $G(x)$ is a pseudoscalar, A^μ is an actual vector, and ψ and χ are Majorana spinors.

In order to calculate the variations of the fields appearing in (3.79), we expand (3.80) to first order in ϵ . From (3.79) we obtain

$$\begin{aligned}\delta\phi(x, \theta) = & \delta A(x) + \delta\bar{\psi}(x)\theta + \frac{1}{2} (\bar{\theta}\theta) \delta F(x) \\ & + \frac{i}{2} (\bar{\theta}\gamma_5\theta) \delta G(x) + \frac{1}{2} \bar{\theta}\gamma_\mu\gamma_5\theta \delta A^\mu(x) \\ & + \delta\bar{\chi}(x)\theta(\bar{\theta}\theta) + (\bar{\theta}\theta)^2 \delta u(x).\end{aligned}\quad (3.81)$$

Expanding (3.80) directly, we obtain

$$\begin{aligned}
\delta\Phi(x, \theta) = & i \bar{\epsilon} \gamma_\mu \theta [\partial^\mu A(x) + \partial^\mu \bar{\psi}(x) \theta \\
& + \frac{1}{2} \bar{\theta} \theta \partial^\mu F(x) + \frac{i}{2} (\bar{\theta} \gamma_5 \theta) \partial^\mu G(x) \\
& + \frac{1}{2} (\bar{\theta} \gamma_\nu \gamma_5 \theta) \partial^\mu A^\nu(x) \\
& + (\bar{\theta} \theta) \partial^\mu \bar{\chi}(x) \theta] \\
& + \bar{\psi}(x) \epsilon + (\bar{\theta} \epsilon) F(x) + i (\bar{\theta} \gamma_5 \epsilon) G(x) \\
& + \bar{\theta} \gamma_\mu \gamma_5 \epsilon A^\mu(x) + (\bar{\chi}(x) \epsilon) (\bar{\theta} \theta) \\
& + 2 (\bar{\chi}(x) \theta) (\bar{\theta} \epsilon) + 4 (\bar{\theta} \theta) (\bar{\theta} \epsilon) u(x) .
\end{aligned} \tag{3.82}$$

Identifying coefficients in (3.82) we find

$$\begin{aligned}
\delta A(x) &= \bar{\psi}(x) \epsilon , \\
\delta \bar{\psi}(x) &= i (\bar{\epsilon} \gamma_\mu) \partial^\mu A(x) + \bar{\epsilon} F(x) + i (\bar{\epsilon} \gamma_5) G(x) \\
&+ (\bar{\epsilon} \gamma_\mu \gamma_5) A^\mu(x) , \\
\delta F(x) &= \frac{i}{2} \partial^\mu \bar{\psi}(x) \gamma_\mu \epsilon + \bar{\chi}(x) \epsilon , \\
\delta G(x) &= - \frac{i}{2} \partial^\mu \bar{\psi}(x) \gamma_\mu \gamma_5 \epsilon - \bar{\chi}(x) \gamma_5 \epsilon , \\
\delta A^\lambda(x) &= \frac{i}{2} \partial^\mu \bar{\psi}(x) \gamma_5 \gamma^\lambda \gamma_\mu \epsilon - \bar{\chi}(x) \gamma_5 \gamma^\lambda \epsilon , \\
\delta \bar{\chi}(x) &= \frac{i}{2} \bar{\epsilon} \gamma_\mu \partial^\mu F(x) - \frac{1}{2} \bar{\epsilon} \gamma_5 \gamma_\mu \partial^\mu G(x) \\
&- \frac{i}{2} \bar{\epsilon} \gamma_\mu \gamma_\nu \gamma_5 \partial^\mu A^\nu(x) + 4 \bar{\epsilon} u(x) , \\
\delta u(x) &= \frac{i}{4} \partial^\mu \bar{\chi}(x) \gamma_\mu \epsilon .
\end{aligned} \tag{3.83}$$

This forms a representation of the supersymmetry group, but to obtain the fields of the Wess-Zumino model, we must proceed further. We note that

$$\delta(\bar{\chi} - \frac{i}{2} \partial^\mu \bar{\psi} \gamma_\mu) = 4 \bar{\epsilon} u + \frac{1}{2} \bar{\epsilon} \square A - \frac{1}{2} \bar{\epsilon} \gamma_5 \sigma_{\mu\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (3.84)$$

Defining

$$\bar{\chi} \equiv \frac{i}{2} \partial^\mu \bar{\psi} \gamma_\mu + \bar{\eta} \quad (3.85)$$

Equation (3.84) becomes

$$\delta \bar{\eta} = 4 \bar{\epsilon} u + \frac{1}{2} \bar{\epsilon} \square A - \frac{1}{2} \bar{\epsilon} \gamma_5 \sigma_{\mu\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (3.86)$$

If we define

$$A^\mu \equiv \partial^\mu B + a^\mu \quad (3.87)$$

where B is a pseudoscalar field and a^μ is an axial vector field, then

Equation (3.86) becomes

$$\delta \bar{\eta} = 4 \bar{\epsilon} u + \frac{1}{2} \bar{\epsilon} \square A - \frac{1}{2} \bar{\epsilon} \gamma_5 \sigma_{\mu\nu} (\partial^\mu a^\nu - \partial^\nu a^\mu) \quad (3.88)$$

We also note from Equation (3.83) that definition (3.85) implies

$$\delta u = \frac{i}{4} \partial^\mu \bar{\eta} \gamma_\mu \epsilon - \frac{1}{8} \square \bar{\psi} \epsilon \quad (3.89)$$

If we write

$$u = -\frac{1}{8} \square A + D \quad (3.90)$$

then it follows from Equation (3.89) and the variation given for δA in

Equation (3.48) that

$$\delta D = \frac{i}{4} \partial^\mu \bar{\eta} \gamma_\mu \epsilon \quad (3.91)$$

Substituting (3.90) into (3.88) yields

$$\delta \bar{\eta} = 4 \bar{\epsilon} D - \frac{1}{2} \bar{\epsilon} \gamma_5^{\sigma}{}_{\mu\nu} (\partial^\mu a^\nu - \partial^\nu a^\mu) \quad (3.92)$$

We now examine δA^λ given in (3.83), using the definition (3.85).

We obtain

$$\delta A^\lambda = i \partial^\mu \bar{\psi} \gamma_5 \gamma^\lambda \gamma_\mu \epsilon - \bar{\eta} \gamma_5 \gamma^\lambda \epsilon \quad (3.93)$$

Using definition (3.87), this becomes

$$\partial^\mu \delta B + \delta a^\mu = i \partial^\mu \bar{\psi} \gamma_5 \gamma^\lambda \gamma_\mu \epsilon - \bar{\eta} \gamma_5 \gamma^\lambda \epsilon \quad (3.94)$$

We may take

$$\delta B = i \bar{\psi} \gamma_5 \gamma^\lambda \epsilon \quad (3.95)$$

and

$$\delta a^\lambda = -\bar{\eta} \gamma_5 \gamma^\lambda \epsilon \quad (3.96)$$

Comparison of Equation (3.91), (3.92), and (3.93) shows that the variations of a^λ , $\bar{\eta}$, and D only involve the same fields. It is thus consistent to choose

$$a^\lambda = \bar{\eta} = D = 0 \quad (3.97)$$

With this choice and the definitions (3.85), (3.87), and (3.90), the variations given in (3.83) become the Wess-Zumino variations listed in (3.3).

Another interesting concept involving superfields is the possibility of mixing Lorentz and internal symmetry in a nontrivial way. Basically there are two ways of introducing an internal symmetry for a superfield $\Phi(x, \theta)$. The first, trivial, way is to introduce the internal symmetry as a direct product:

$$\phi(x, \theta) \rightarrow \phi^a(x, \theta) \quad . \quad (3.98)$$

The index a is the internal symmetry index. This is what is done in conventional field theories. The more interesting idea is to introduce the internal symmetry in the following way (32, 33):

$$\phi(x, \theta) \rightarrow \phi(x, \theta^a) \quad . \quad (3.99)$$

Then the power series expansion for ϕ becomes

$$\phi(x, \theta^a) = A(x) + \bar{\theta}^a \psi_a(x) + \dots \quad (3.100)$$

In this case the fields of different spin also have different internal spin. This method of introducing internal symmetry is compatible with the Lorentz symmetry:

$$\begin{aligned} U(\Lambda) \phi(x, \theta_a) U^\dagger(\Lambda) &= \phi(\Lambda^{-1}x, S^{-1}\theta_a) \quad , \\ U(\epsilon_a) \phi(x, \theta_a) U^\dagger(\epsilon_a) &= \phi(x_\mu + i\bar{\epsilon}_a \gamma_\mu \theta_a, \theta_a + \epsilon_a) \quad . \end{aligned} \quad (3.101)$$

Here $U(\Lambda)$ and $U(\epsilon_a)$ are the unitary operators associated with the Lorentz and supersymmetry transformations. This is essentially the only way internal symmetry and Lorentz symmetry can be mixed in a non-trivial way (34). Unfortunately, the isospin-spin correlations of (3.101) are not found experimentally.

E. Summary of the Status of Supersymmetry

Let us examine the balance sheet for supersymmetry. Some of the positive aspects are the following:

- i) The relations imposed by supersymmetry often lead to improved renormalization due to cancellations of divergences. It is hoped

that similar cancellations will aid in the renormalization of theories such as supergravity (19). The present status is that supergravity-matter coupling is nonrenormalizable at the one-loop level. Pure supergravity is renormalizable at the one- and two-loop level, although it fails at three loops. Extended supergravity avoids the one-loop catastrophe of supermatter coupling, and perhaps it will avoid the three-loop problem of pure supergravity.

ii) Although the spontaneous breakdown of supersymmetry is rare, it is possible to trigger the spontaneous breakdown of internal symmetries via supersymmetry (35).

iii) The supersymmetric Yang-Mills theories may have interesting consequences (20). For instance, the straightforward generalization of the SU_3 Yang-Mills theory of hadrons to a graded Lie algebra requires the fermion field to behave like the gluon field under the internal symmetry. This would explain the appearance of the baryons in an octet, and thus the nonexistence of quarks (31).

iv) It is possible to maintain asymptotic freedom in a spontaneously broken non-Abelian gauge theory (36).

v) Nontrivial mixing of Lorentz and internal symmetries can be achieved within supersymmetry. However, the isospin-spin correlation which is forced upon us is in disagreement with experiment (34, 35).

One must also keep in mind that supersymmetry has difficulties when compared with experiment. There are two main reasons for this. The first is that the fermions and bosons appearing within the same multiplet

probably won't be able to generate the large mass differences between fermions and mesons that are observed in nature. The second difficulty is the fact that either fermion number is not conserved or boson number is conserved.

But supersymmetry continues to be an interesting idea. The areas of supergravity (19) and supersymmetric Yang-Mills theories (20) are quite active at the time of the writing of this work. In any case, supersymmetry provides us with an interesting theoretical laboratory.

IV. THE MODEL

In this section we describe the model used for the calculation given in Section V. The Lagrangian we use is the basic Wess-Zumino model (16). It consists of a Majorana spinor field ψ , scalar field A , and a pseudo-scalar field B .

$$\begin{aligned} \mathcal{L}_S = & \frac{1}{2}[(\partial_\mu A)^2 - m^2 A^2] + \frac{1}{2}[(\partial_\mu B)^2 - m^2 B^2] + \frac{1}{2}[i\bar{\psi}\not{\partial}\psi - m\psi\psi] \\ & - \frac{1}{2}g\bar{\psi}[A - i\gamma_5 B]\psi - \frac{1}{2}mgA(A^2 + B^2) - 1/8g^2(A^2 + B^2)^2 \end{aligned} \quad (4.1)$$

The factor of $\frac{1}{2}$ in the kinetic energy term and the mass term for the spinor ψ arises from the fact that ψ is a Majorana spinor, and thus is self-conjugate. Supersymmetry requires the degeneracy of all masses and all coupling constants appearing in Equation (4.1).

This Lagrangian shares some common properties with the Yukawa model (Y model) and the neutral version of the linear σ model (37):

$$\mathcal{L}_Y = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}[(\partial_\mu \pi)^2 - \mu^2 \pi^2] + g\bar{\psi}i\gamma_5 \psi \pi \quad (4.2)$$

and

$$\begin{aligned} \mathcal{L}_\sigma = & \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}[(\partial_\mu \pi)^2 - \mu^2 \pi^2] + \frac{1}{2}[(\partial_\mu \sigma)^2 - \mu^2 \sigma^2] - g\bar{\psi}(\sigma + i\gamma_5 \pi)\psi \\ & - 4\lambda G\sigma(\sigma^2 + \pi^2) - \lambda G^2(\sigma^2 + \pi^2)^2 - 4\lambda\sigma^2 + \frac{1}{2}\mu_0^2(2G^{-1}\sigma + \sigma^2 + \pi^2) \end{aligned} \quad (4.3)$$

Here σ and π are the neutral scalar and pseudoscalar fields, respectively, and $G = g/m$.

The Feynman rules for the three theories are given in Figure 8.

The derivation of these rules for the supersymmetry model (s model) is given in Appendix B. A dashed line stands for the propagator of $B(\pi)$, a wavy line for $A(\sigma)$, and a solid line for ψ . Only diagrams (a), (b), (c), and (e) are relevant to $\mathcal{L}_Y$. Diagrams (a) - (i) apply to both the s model and the σ model. When rules of the two models are associated with different factors, the one in brackets refers to the σ model.

Before proceeding to the details of the calculation in Section V, let us summarize the results of this work and of the calculation of Reference 15.

(a) The s model. There are six possible scattering amplitudes within the s model:

$$\begin{aligned}
 \psi - \psi & , \\
 \psi - A & , \\
 \psi - B & , \\
 B - B & , \\
 B - A & , \\
 A - A & .
 \end{aligned}
 \tag{4.4}$$

Here $\psi - \psi$ denotes fermion-fermion scattering, $\psi - A$ denotes fermion-scalar scattering, $\psi - B$ denotes fermion-pseudoscalar scattering, etc.

Fermion-fermion scattering was calculated in Reference 15. In this work we consider the other amplitudes. Amplitudes containing fermions were spin averaged. For all amplitudes listed in (4.4) we found two classes of diagrams, the s diagrams and the u diagrams (see Reference 15 and Section V for definition), to dominate. The amplitudes for these two

classes were the same in all cases. We list the results:

$$\begin{aligned}
 T_u &= -ig^2 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (\bar{g}^{-2} \ln^2 s)^n \\
 &\approx \frac{-ig^2}{\sqrt{4\pi}} \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
 T_s &= -ig^2 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} [\bar{g}^{-2} (\ln s - i\pi)^2]^n \\
 &\approx \frac{-ig^2}{\sqrt{4\pi}} e^{-i2\pi\bar{g}} \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
 \end{aligned} \tag{4.6}$$

where

$$\bar{g}^{-2} = g^2/8\pi^2 \tag{4.7}$$

The scattering amplitude is

$$\begin{aligned}
 T &= T_u + T_s \\
 &\approx \frac{-ig^2}{\sqrt{4\pi}} (1 + e^{-i2\pi\bar{g}}) \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
 \end{aligned} \tag{4.8}$$

(b) The σ model. The fermion-fermion and fermion-antifermion scattering amplitudes are respectively given by T_u and T_s , i.e., by Equation (4.5) and (4.6). The fermion-boson, antifermion-boson, and boson-boson scattering amplitudes are all given by $T_u + T_s$, i.e., by Equation (4.8).

(c) The Y model. The amplitudes for fermion-fermion and fermion-antifermion scattering are given respectively by Equations (4.5) and (4.6) with g^2 and $\bar{g}^2$ replaced by $g^2/2$ and $\bar{g}^2/2$. The fermion-boson and anti-fermion-boson amplitudes are both given by Equation (4.8) with the replacement of $\bar{g}^2$ by $\bar{g}^2/2$. The boson-boson scattering amplitude is given by Equation (4.8) with g^2 and $\bar{g}^2$ replaced by $4g^2$ and $\bar{g}^2/2$.

We note that in the σ model and the Y model only the fermion-fermion and fermion-antifermion amplitudes do not have the signature factor of Equation (4.8).

V. THE CALCULATION

The details to the calculation of the scattering amplitudes listed in Equation (4.4) are presented in this section. The calculation of fermion-fermion scattering is given in Reference 15. We give only the results here. We present the calculation for pseudoscalar-pseudoscalar scattering in Subsections A - D. The pseudoscalar-fermion calculation is outlined in Subsections E - H. The modifications needed to obtain the other amplitudes are given in Subsection I.

A new feature present in the boson-boson scattering not found in the fermion-fermion scattering of Reference 15, or in fermion-boson scattering, is the fact that the leading class of diagrams (see Figure 13) are all logarithmically divergent. We have used dimensional regularization to regularize all divergent integrals. We will find in Section D that the renormalization does not affect the leading behavior.

We use the momentum labeling as given in Figure 7. The four momenta are denoted by

$$\begin{aligned}
 P_1 &= (E, -r_x, -r_y, \lambda\eta) \quad , \\
 P_2 &= (E, r_x, r_y, -\lambda\eta) \quad , \\
 P_3 &= (E, r_x, r_y, \lambda\eta) \quad , \\
 P_4 &= (E, -r_x, -r_y, -\lambda\eta) \quad .
 \end{aligned}
 \tag{5.1}$$

As usual, we define $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2 = -4\vec{r}^2$, and $u = (p_1 - p_4)^2 = 4(\lambda\eta)^2$. We consider the scattering amplitudes in the limit $s \rightarrow \infty$ with t fixed and finite.

A. The Born Terms for Pseudoscalar-Pseudoscalar Scattering

The second-order diagrams for pseudoscalar-pseudoscalar scattering are shown in Figure 9. The leading diagrams are given by Figure 9b and Figure 9d. The corresponding amplitudes are

$$T_{(9b)} = \frac{-im^2 g^2}{t - m^2}, \quad (5.2)$$

$$T_{(9d)} = -3ig^2.$$

B. Fourth-Order Pseudoscalar-Pseudoscalar Diagrams

The fourth-order diagrams include the box diagrams shown in Figure 10, as well as many radiative corrections to the Born terms. Since it will be shown below that the leading terms behave like $\ln^2 s$, the radiative corrections are nondominant, and thus will be neglected.

First let us examine the diagram given in Figure 10a.1. We shall use the Feynman parameter technique (41) to evaluate this diagram.

The amplitude for diagram 10a.1 is given by (for notation, see Figure 10)

$$T_{a.1}^{(2)} = -(-ig)^4 i^4 \int_k \frac{N_{a.1}^{(2)}}{D_{a.1}^{(2)}} \quad (5.3)$$

where

$$N_{a.1}^{(2)} = \text{Tr}[(\not{k} + m)\gamma_5(\not{p}_1 + \not{k} + m)\gamma_5(\not{p}_1 - \not{p}_3 + \not{k} + m)\gamma_5(\not{k} - \not{p}_2 + m)\gamma_5]$$

(a)

(b)

(c)

(d)

Figure 9. Born diagrams for pseudoscalar-pseudoscalar scattering

Figure 10. Fourth-order pseudoscalar-pseudoscalar diagrams

$$\begin{aligned}
&= 4\{(k^2)^2 + k^2[k \cdot (p_1 - p_3) + (k \cdot p_1) - k \cdot (p_2)] + k^2(t + s/2 - 3m^2) \\
&\quad - 2(k \cdot p_1)(k \cdot p_2) + (s/2 - 2m^2)k \cdot (p_1 - p_3) + (t/2 - m^2) \\
&\quad \times [(k \cdot p_1) - (k \cdot p_2)] - s/2 m^2 + 2m^4\} \quad (5.4)
\end{aligned}$$

and

$$\begin{aligned}
D_{a,1}^{(2)} &= [k^2 - m^2][(p_1 + k)^2 - m^2][(p_1 - p_3 + k)^2 - m^2][(k - p_2)^2 - m^2] \\
&= 3! \int_0^1 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \delta(1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2) \frac{1}{D_{a,1}^4} \quad (5.5)
\end{aligned}$$

Here

$$\begin{aligned}
D_{a,1} &= \beta_1(k^2 - m^2) + \alpha_1[(p_1 + k)^2 - m^2] + \beta_2[(p_1 - p_3 + k)^2 - m^2] \\
&\quad + \alpha_2[(p_2 - k)^2 - m^2] \\
&= (k - q_1)^2 - \Delta_1 \quad (5.6)
\end{aligned}$$

where

$$q_1 = [\beta_1(p_1 - p_3) + \alpha_2 p_2 - \alpha_1 p_1] \quad (5.7)$$

and

$$\begin{aligned}
\Delta_1 &= q_1^2 + m^2[1 - \alpha_1 - \alpha_2] - \beta_2 t \\
&= m^2[1 - \alpha_1 - \alpha_2 + (\alpha_1 + \alpha_2)^2] - \alpha_1 \alpha_2 s - \beta_1 \beta_2 t \quad (5.8)
\end{aligned}$$

The leading behavior comes from only two terms of Equation (5.4). These terms are

$$\tilde{N}_1^{(2)} = 4[s/2 k^2 - 2(k \cdot p_1)(k \cdot p_2)] \quad (5.9)$$

The contribution to $T_{a.1}^{(2)}$ from $\tilde{N}_1^{(2)}$ is given by

$$\begin{aligned} -(ig)^4 i^4 \int \frac{\tilde{N}_1^{(2)}}{D_{a.1}^{(2)}} &= -i(ig)^4 \left(\frac{-4i\pi^2}{(2\pi)^4} \right) \\ &\times \int_0^1 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \delta(1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2) \\ &\times \left[\frac{s}{\Delta_1} - s/2 \frac{q_1^2}{\Delta_1^2} - \frac{p_1 \cdot p_2}{\Delta_1} + \frac{2(p_1 \cdot q_1)(p_2 \cdot q_1)}{\Delta_1^2} \right]. \end{aligned} \quad (5.10)$$

Performing this integral in the leading logarithm approximation gives the result

$$-(ig)^4 i^4 \int \frac{\tilde{N}_1^{(2)}}{D_{a.1}^{(2)}} = -(ig)^4 \frac{i\pi^2}{(2\pi)^4} (\ln s - i\pi)^2 \quad (5.11)$$

Note that the term $4(k^2)^2$ of Equation (5.4) gives rise to a logarithmic divergence. The subtracted integral produces a term of $\ln s$. However, this logarithm is not promoted by the Feynman parameter integration, so the term is nonleading.

Since the superficial degree of divergence of $T_{a.1}^{(2)}$ is zero, the subtraction affects only terms independent of the external momenta. Therefore, the renormalization procedure does not affect the leading behavior of this diagram. This argument can be extended to all diagrams

of this class (see Subsection D).

The other terms of (5.4) give contributions of at most $\ln s$.

Therefore, Equation (5.3) becomes

$$T_{a.1}^{(2)} = \frac{-i\pi^2}{(2\pi)^4} g^4 (\ln s - i\pi)^2 \quad (5.12)$$

Diagram 10a.1 belongs to a class of diagrams we shall call the s diagrams. (See Subsection D for the definition of s diagrams.) Using obvious notation, we write

$$T_s^{(2)} = -\frac{i}{2} g_s^{2-2} (\ln s - i\pi)^2 \quad (5.13)$$

where

$$g_s^{-2} = g^2/8\pi^2 \quad (5.14)$$

Diagram 10a.2 can be handled in a similar manner. Following Tiktopolous (38), we define diagram 10a.2 to be nonplanar, while diagram 10a.1 is planar. Therefore, we expect this diagram to be non-leading. We find this to be the case, behaving at most like $\ln s$.

Diagram 10a.3 is a "crossed" planar graph. As noted in Tiktopolous (38), diagram 10a.2 is defined as nonplanar while diagram 10a.3 is taken to be planar because of the limit being considered, that is, because here we are taking $s \rightarrow \infty$ with t fixed. Therefore, diagram 10a.3 could give rise to leading behavior. We find that it indeed does.

The calculation is parallel to that for diagram 10a.1. We write

$$T_{a.3}^{(2)} = -(-ig)^4 i^4 \int_k \frac{N_{a.3}^{(2)}}{D_{a.3}^{(2)}} \quad (5.15)$$

where

$$\begin{aligned} N_{a.3}^{(2)} = & 4\{(k^2)^2 + k^2[k \cdot (p_1 - p_3) - (k \cdot p_3) - (k \cdot p_2)] + (t/2 - s/2 - m^2)k^2 \\ & + 2(k \cdot p_3)(k \cdot p_2) - (t/2 - m^2)[(k \cdot p_3) + (k \cdot p_2)] \\ & + (t/2 + s/2)m^2\} \end{aligned} \quad (5.16)$$

and

$$D_{a.3}^{(2)} = 3! \int_0^1 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \delta(1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2) \frac{1}{\tilde{D}_{a.3}^4} \quad (5.17)$$

We have defined

$$\tilde{D}_{a.3} = (k - q_2)^2 - \Delta_2 \quad (5.18)$$

where

$$q_2 = \beta_2(p_1 - p_3) + \alpha_2 p_2 + \alpha_1 p_3 \quad (5.19)$$

and

$$\Delta_2 = m^2[1 - \alpha_1 - \alpha_2 + (\alpha_1 - \alpha_2)^2] + \alpha_1 \alpha_2 s + (\alpha_1 \alpha_2 - \beta_1 \beta_2)t \quad (5.20)$$

The leading terms of (5.16) are given by

$$\tilde{N}_2^{(2)} = 4\{-\frac{s}{2}k^2 + 2(k \cdot p_3)(k \cdot p_2)\} \quad (5.21)$$

These terms give the contribution

$$\int_k \frac{\tilde{N}_1^{(2)}}{D_{a.3}^{(2)}} = \frac{i\pi^2}{(2\pi)^4} \ln^2 s \quad . \quad (5.22)$$

The other terms of (5.16) contribute at most $\ln s$. The leading behavior of (5.15) is thus given by

$$T_{a.3}^{(2)} = \frac{-i\pi^2}{(2\pi)^4} g^4 \ln^2 s \quad . \quad (5.23)$$

Diagram 10a.3 belongs to the class of diagrams we will call the u diagrams (see Subsection D for the definition). We therefore write

$$T_u^{(2)} = -\frac{i}{2} g^{2-2} \ln^2 s \quad . \quad (5.24)$$

The other diagrams shown in Figure 10 contain no fermion lines, and therefore behave like $1/s$, up to powers of $\ln s$. The leading behavior for the fourth-order diagrams is given by Equations (5.13) and (5.24).

We note that the dominant fourth-order diagrams support the conjecture made by Chang and Ma (23), based on their QED calculation in the infinite momentum frame, that amplitudes are dominated by the highest spin exchanges in the crossed channel. This is similar to the rule of Regge pole theory that an amplitude is dominated by the highest J -plane trajectory allowed in the t channel.

C. Sixth-Order Pseudoscalar-Pseudoscalar Diagrams

We find the sixth-order box diagrams of Figure 11 to give the leading behavior. Let us examine the diagram shown in Figure 11a.1.

Figure 11. Leading sixth-order pseudoscalar-pseudoscalar diagrams:
 (a) s diagrams and (b) u diagrams

The corresponding integral is given as

$$T_{a.1}^{(3)} = ig^6 \int_k \int_\ell \frac{N_{a.1}^{(3)}}{D_{a.1}^{(3)}} \quad (5.25)$$

This integral is logarithmically divergent. The subtracted integral of the divergent part of $T_{a.1}^{(3)}$ gives rise to a power of $\ln s$. However, as in the case for the fourth-order diagram, the Feynman parameter integration does not promote this logarithm. The leading behavior thus comes from the convergent part of $T_{a.1}^{(3)}$.

The numerator in Equation (5.24) is given by

$$\begin{aligned} N_{a.1}^{(3)} &= \text{Tr}\{[\ell + m]\gamma_5[\not{p}_2 + \ell + m]\gamma_5[\not{p}_2 - \not{p}_4 + \ell + m]\gamma_5 \\ &\quad \times [\not{p}_2 - \not{p}_4 + k + m]\gamma_5[k - \not{p}_1 + m]\gamma_5[k + m]\gamma_5\} \\ &= -4\{-t[(\ell \cdot p_2)(k \cdot p_1) - (\ell \cdot p_1)(k \cdot p_2) + (\ell \cdot k)(s/2 - m^2)] \\ &\quad + (k^2 - m^2)[-(\ell \cdot p_2)t/2 + (\ell \cdot p_1)t/2 - \ell \cdot (p_2 - p_4)(s/2 - m^2) \\ &\quad + t(\ell \cdot p_2) + (\ell \cdot p_2)k \cdot (p_2 - p_4) - \ell \cdot (p_2 - p_4)(k \cdot p_2) \\ &\quad + t/2(\ell \cdot k) + 2(p_2 \cdot \ell)(p_1 \cdot \ell) + 2(p_2 \cdot \ell)\ell \cdot (p_2 - p_4) \\ &\quad + 2(p_2 \cdot \ell)(k \cdot \ell) - t/2 m^2 - 2(p_2 \cdot \ell)m^2] + (\ell^2 - m^2)[t/2(p_1 \cdot k) \\ &\quad - t/2(p_2 \cdot k) - (s/2 - m^2)k \cdot (p_2 - p_4) + 2(k \cdot p_1)(k \cdot p_2) \\ &\quad - t(k \cdot p_1) - \ell \cdot (p_2 - p_4)(k \cdot p_1) + (\ell \cdot p_1)k \cdot (p_2 - p_4) + t/2(\ell \cdot k) \\ &\quad - 2(k \cdot p_1)k \cdot (p_2 - p_4) - 2(k \cdot p_1)(k \cdot \ell) - t/2 m^2 + 2(p_1 \cdot k)m^2] \end{aligned}$$

$$\begin{aligned}
& - (k^2 - m^2)(\ell^2 - m^2)[(s/2 - m^2) + (p_2 \cdot k) - (p_1 \cdot \ell) - (k \cdot \ell)] \\
& - \ell \cdot (p_2 - p_4) - k \cdot (p_2 - p_4) + m^2] - 2(k \cdot p_1)[(\ell \cdot p_2)k \cdot (p_2 - p_4) \\
& - \ell \cdot (p_2 - p_4)(k \cdot p_2) + t/2(\ell \cdot k)] - 2(\ell \cdot p_2)[(k \cdot p_1)\ell \cdot (p_2 - p_4) \\
& - k \cdot (p_2 - p_4)(\ell \cdot p_1) - t/2(\ell \cdot k)] - 4(p_2 \cdot \ell)(p_1 \cdot k)(k \cdot \ell) \\
& + m^2[t(s/2 - m^2) + t(p_1 \cdot k) - t(p_2 \cdot \ell) + 4(p_2 \cdot \ell)(p_1 \cdot k)] \}
\end{aligned} \tag{5.26}$$

We Feynman parameterize the denominator of (5.25) (22, p. 31, 41), obtaining

$$D_{a.1}^{(3)} = 6! \int_0^1 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 (1 - \Sigma\alpha_j - \Sigma\beta_j) \left(\frac{1}{\tilde{D}_1} \right)^7, \tag{5.27}$$

where

$$\begin{aligned}
\tilde{D}_1^{(3)} &= h^T A h - 2h^T B p - \hat{\sigma}, \\
h &= \begin{pmatrix} k \\ \ell \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_4 - p_2 \end{pmatrix}, \\
A &= \begin{pmatrix} \eta_1 & -\beta_3 \\ -\beta_3 & \eta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_1 & 0 & \alpha_4 \\ 0 & -\beta_2 & \alpha_3 \end{pmatrix}, \\
\eta_1 &= \alpha_1 + \alpha_4 + \beta_1 + \beta_3, \quad \eta_2 = \alpha_2 + \alpha_3 + \beta_2 + \beta_3, \\
\hat{\sigma} &= m^2[1 - \beta_1 - \beta_2] - (\alpha_3 + \alpha_4)t - i\varepsilon, \\
C \equiv \det A &= \eta_1 \eta_2 - \beta_3^2.
\end{aligned} \tag{5.28}$$

We make the variable change

$$h = Rh + A^{-1}Bp \quad (5.29)$$

where

$$h' = \begin{pmatrix} h'_1 \\ h'_2 \end{pmatrix}$$

and

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

Then

$$\bar{D}_1^{(3)} = h' R^T A R h' - (Bp)^T A^{-1} (Bp) - \hat{\sigma} \quad (5.30)$$

We choose R to diagonalize A , so that

$$R^T A R = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad C = \lambda_1 \lambda_2 \quad (5.31)$$

It can be shown that

$$\begin{aligned} C\Delta &\equiv C[(Bp)^T A^{-1} (Bp) - \hat{\sigma}] \\ &= [-\beta_1 \beta_2 \beta_3 s - ft + C\sigma] \quad (5.32) \end{aligned}$$

where

$$f = \alpha_1 \alpha_4 \eta_2 + \alpha_2 \alpha_3 \eta_1 + \beta_3 (\alpha_2 \alpha_4 + \alpha_1 \alpha_3) \quad (5.33)$$

and

$$\sigma = m^2 [1 - \beta_1 - \beta_2] \quad (5.34)$$

The exact form for R is not needed, since the elements R_{11} , R_{12} , etc., always appear in the combinations

$$R_{11}R_{22} - R_{12}R_{21} = 1$$

$$R_{11}R_{21}\lambda_2 + R_{12}R_{22}\lambda_1 = \beta_3$$

$$R_{11}^2\lambda_2 + R_{12}^2\lambda_1 = \eta_2$$

$$R_{21}^2\lambda_2 + R_{22}^2\lambda_1 = \eta_1$$

We scale the integration variables:

$$k'' = \frac{1}{\sqrt{\lambda_1}} k', \quad \ell'' = \frac{1}{\sqrt{\lambda_2}} \ell' \quad (5.35)$$

The numerator given in Equation (5.26) becomes quite complicated after the variable changes given in (5.29) and (5.35). We list the effect for only the leading terms:

$$\begin{aligned} -s/2 k^2 \ell^2 = & -s/2 C^{-2} \{ [3/2 \beta_3^2 + C] k''^2 \ell''^2 + C \eta_2 G_2^2 k''^2 + C \eta_1 G_1^2 \ell''^2 \\ & + 4 C \beta_3 (k'' \cdot G_1) (k'' \cdot G_2) + C^2 G_1^2 G_2^2 \} \quad (5.36) \end{aligned}$$

$$\begin{aligned} -4(p_1 \cdot k)(p_2 \cdot \ell)(k \cdot \ell) = & -4C^{-2} \{ [\beta_3^2 + 1/6 C] k''^2 (k'' \cdot p_1)(k'' \cdot p_2) \} \\ & + C^{-1} \{ \beta_3 k''^2 (G_1 \cdot p_1)(G_2 \cdot p_2) + \eta_2 (G_2 \cdot p_2)(k'' \cdot p_1) \} \end{aligned}$$

$$\begin{aligned}
& \times (k'' \cdot G_2) + \beta_3 (G_1 \cdot p_1) (k'' \cdot p_2) (k'' \cdot G_2) \\
& + \beta_3 (G_2 \cdot p_2) (k'' \cdot p_1) (k'' \cdot G_1) + \eta_1 (G_1 \cdot p_1) (k'' \cdot p_1) (k'' \cdot G_1) \\
& + \beta_3 (G_1 \cdot G_2) (k'' \cdot p_1) (k'' \cdot p_2) \} + (G_1 \cdot G_2) (G_1 \cdot p_1) (G_2 \cdot p_2) \} , \\
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
2 \ell^2 (k \cdot p_1) (k \cdot p_2) &= 2 c^{-1} \{ [1 + 3/2 \beta_3^2 c^{-1}] \ell'^2 (k'' \cdot p_1) (k'' \cdot p_2) \\
&+ \eta_1 (G_1 \cdot p_1) (G_1 \cdot p_2) k''^2 + 2 \beta_3 (k'' \cdot G_2) (k'' \cdot p_1) \\
&\times (G_1 \cdot p_2) + 2 \beta_3 (k'' \cdot G_2) (k'' \cdot p_2) (G_1 \cdot p_1) \\
&+ \eta_2 G_2^2 (k'' \cdot p_1) (k'' \cdot p_2) + c G_2^2 (G_1 \cdot p_1) (G_1 \cdot p_2) \} , \\
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
2 k^2 (p_2 \cdot \ell) (p_1 \cdot \ell) &= 2 c^{-1} \{ [1 + 3/2 \beta_3^2 c^{-1}] k''^2 (\ell'' \cdot p_1) (\ell'' \cdot p_2) \\
&+ \eta_2 (G_2 \cdot p_1) (G_2 \cdot p_2) k''^2 + 2 \beta_3 (k'' \cdot G_1) (k'' \cdot p_1) \\
&\times (G_2 \cdot p_2) + 2 \beta_3 (k'' \cdot G_1) (k'' \cdot p_2) (G_2 \cdot p_1) \\
&+ \eta_1 G_1^2 (k'' \cdot p_1) (k'' \cdot p_2) + c G_1^2 (G_2 \cdot p_1) (G_2 \cdot p_2) \} . \\
\end{aligned} \tag{5.39}$$

Here

$$G_1 = c^{-1} [\eta_2 \beta_1 p_1 - \beta_2 \beta_3 p_2 + (\eta_2 \alpha_4 + \alpha_3 \beta_4) (p_4 - p_2)] ,$$

$$G_2 = c^{-1} [\beta_1 \beta_3 p_1 - \eta_1 \beta_2 p_2 + (\alpha_4 \beta_3 + \eta_1 \alpha_3) (p_4 - p_2)] .$$

Several of the terms in Equations (5.36) - (5.39) give rise to leading behavior. However, several cancellations occur, leaving the final result

$$T_{a.1}^{(3)} = \frac{1}{2} \left[-ig^2 \frac{1}{2!3!} g^{-4} (\ln s - i\pi)^4 \right] \quad (5.40)$$

This complex cancellation will be seen to occur to all orders in Subsection D.

The analysis for diagram 11a.2 is similar. We obtain the identical result as given in (5.40):

$$T_{a.2}^{(3)} = \frac{1}{2} \left[-ig^2 \frac{1}{2!3!} g^{-4} (\ln s - i\pi)^4 \right] \quad (5.41)$$

Diagrams 11a.1 and 11a.2 belong to the class of s diagrams, to be defined in Subsection D. We therefore obtain

$$\begin{aligned} T_s^{(3)} &= T_{a.1}^{(3)} + T_{a.2}^{(3)} \\ &= -ig^2 \frac{1}{2!3!} g^{-4} (\ln s - i\pi)^4 \quad (5.42) \end{aligned}$$

The calculation for diagrams 11b.1 and 11b.2 follow the same form as that for diagram 11a.1. They belong to the class of u diagrams as defined in Subsection D. We obtain the result

$$\begin{aligned} T_u^{(3)} &= T_{b.1}^{(3)} + T_{b.2}^{(3)} \\ &= -ig^2 \frac{1}{2!3!} g^{-4} \ln^4 s \quad (5.43) \end{aligned}$$

The other sixth-order diagrams include nonplanar diagrams, diagrams involving quartic vertices, radiative corrections to the lower order

diagrams, etc. We discuss some of these briefly.

The calculation of the diagrams in Figure 12a and 12b follows the same procedure as above. They behave at most like $\ln^3 s$. The diagrams in Figure 12c do not involve fermion lines, and thus behave like $1/s$, up to powers of $\ln s$.

The nonplanar diagrams can be neglected, following the general analysis of Tiktopolous (38). This is valid as long as cancellations do not occur to prevent the attainment of the maximum powers of $\ln s$. The maximum powers of $\ln s$ are obtained in this case, justifying the neglect of nonplanar diagrams.

Another class of diagrams is illustrated in Figure 13. We find that diagrams 13a - 13c behave like $\ln s$, while diagrams 13d and 13e behave like $1/s$, up to powers of $\ln s$.

We note that the dominant sixth-order diagrams support the conjecture made by Chang and Ma, as mentioned in Subsection B.

D. Pseudoscalar-Pseudoscalar Scattering to All Orders

In the last two subsections we found that the ladder diagrams with fermions exchanged in the t -channel were the dominant diagrams. We shall assume that these ladders, as illustrated in Figure 14, dominate in all orders. There are several reasons for assuming this: the results of the fourth- and sixth-order calculations, the similarity to the ladders of $(\psi^3)_6$, and the Chang-Ma conjecture. In this subsection we calculate the diagrams shown in Figure 14 to all orders in the forward direction. Diagrams like the one shown in Figure 14a we will call u diagrams, while those in Figure 14b we will call s diagrams.

Figure 12. Sixth-order pseudoscalar-pseudoscalar diagrams

Figure 13. Sixth-order pseudoscalar-pseudoscalar diagrams

Figure 14. N-loop pseudoscalar-pseudoscalar ladder diagrams:
 (a) u diagrams and (b) s diagrams

To aid in comparison with Reference 15, we will calculate the $2(n+1)$ order u diagrams shown in Figure 14b. They all have the same denominator,

$$D_n = \prod_{j=1}^n \Delta_j^2 \prod_{\ell=1}^{n+1} [(k_\ell - k_{\ell+1})^2 - m^2] [(p_2 - k_1)^2 - m^2] \\ \times [(p_1 - k_n)^2 - m^2] \quad , \quad (5.44)$$

where

$$\Delta_j = -(k_j^2 - m^2) \quad . \quad (5.45)$$

The numerator of each individual diagram is given by

$$\text{Tr}[\Gamma_1(k_1 + m)\Gamma_2(k_2 + m) \dots \Gamma_n(k_n + m)\Gamma_{n+1}(p_1 + k_n + m) \\ \times \Gamma_{n+1}(k_n + m) \dots \Gamma_1(k_1 + p_2 + m)] \quad , \quad (5.46)$$

where

$$\Gamma_j = (i) \quad \text{or} \quad (\gamma_5) \quad .$$

If we denote the sum of all contributions as N_n , then

$$T_u^{(n+1)} = -g^{2(n+1)} i^{3n+1} \int_{k_1} \dots \int_{k_n} \frac{N_n}{D_n} \quad . \quad (5.47)$$

In Appendix D we show that, in the leading logarithm approximation,

$$N_n = (-2)^{n+1} [A_n(p_1 + k_n) \cdot (p_2 + k_1) + \sum_{j=1}^n B_n^j(p_1 + k_n) \cdot k_j] \quad (5.48)$$

where

$$A_n = \prod_{j=1}^n \Delta_j \quad ,$$

$$B_n^j = \left(\prod_{\ell=j+1}^n \Delta_\ell \right) \sum_{i=1}^j \sum_{1 \leq \lambda_1 < \lambda_2 \dots < \lambda_i = j} L_0^{\lambda_1} L_{\lambda_1}^{\lambda_2} \dots L_{\lambda_{i-1}}^{\lambda_j},$$

$$L_\ell^j = 2k_\ell \cdot k_j \prod_{i=\ell+1}^{j-1} \Delta_i,$$

$$L_0^1 = 2[(p_2 + k_1) \cdot k_1],$$

$$L_0^j = 2[(p_2 + k_1) \cdot k_j] A_{j-1}. \quad (5.49)$$

In (5.48), we define

$$\prod_{\ell=j+1}^j \Delta_\ell \equiv 1.$$

The relations in (5.49) are identical to the corresponding relations (5.7) of Reference 15 except for the replacement of p_a with $(p_2 + k_1)$ and p_b with $(p_1 + k_n)$.

By power counting, one finds that the integral in (5.47) is logarithmically divergent. Performing a subtraction on the divergent part of $T_u^{(n+1)}$ produces a $\ln s$ term. However, we note in Appendix E that, as in the case for the fourth- and sixth-order diagrams, this logarithm is not promoted by the Feynman parameter integrations, and thus remains nonleading. The diagrams of Figure 13 contribute to the coupling constant renormalization. Since the integral of (5.47) has degree of divergence zero (logarithmically divergent), the renormalization will affect only terms independent of the external momentum. Thus the leading energy behavior we find for these diagrams should be

unaffected by the renormalization.

We find it convenient to rewrite Equation (5.48) as

$$N_n = (-2)^{n+1} [A_n(p_1 \cdot p_2) + A_n(p_2 \cdot k_n) + A_n(p_1 \cdot k_1) + A_n(k_1 \cdot k_n) \\ + \sum_{j=1}^n B_n^j \{(p_1 \cdot k_j) + (k_n \cdot k_j)\}] \quad (5.50)$$

The first term of (5.50) is identical to the corresponding term in Equation (5.7) of Reference 15. It can be evaluated using Mellin transform techniques, yielding a contribution to $T_u^{(n+1)}$ of

$$-ig^2 \frac{g^{2n}}{n!(n+1)!} \ln^{2n} s \quad (5.51)$$

The analysis of the other terms in Equation (5.50) is quite complicated. The result is that they are all nonleading. Therefore the leading behavior is given entirely by (5.51). We outline the general argument below. More details appear in Appendix E.

We need to consider two types of terms (see Equation (5.16) of Reference 15):

$$J_{n,\ell}^{(1)} = g_{\mu_0 \nu_0} g_{\mu_1 \nu_1} \dots g_{\mu_\ell \nu_\ell} p_2^{\mu_0} \int_{k_1} \dots \int_{k_n} k_{\lambda_1}^{\nu_0 \mu_1} k_{\lambda_1}^{\nu_1 \mu_2} \dots k_{\lambda_\ell}^{\nu_{\ell-1} \mu_\ell} k_n^{\nu_\ell} \\ \times [(k_{\lambda_1}^2 - m^2 + i\epsilon)(k_{\lambda_2}^2 - m^2 + i\epsilon) \dots (k_{\lambda_\ell}^2 - m^2 + i\epsilon) \\ \times \{ \prod_{\ell=1}^n (k_\ell^2 - m^2 + i\epsilon) \} \{ \prod_{j=1}^{n-1} [(k_j - k_{j+1})^2 - m^2 + i\epsilon] \} \\ \times \{(p_2 + k_1)^2 - m^2 + i\epsilon\} \{(p_1 + k_n)^2 - m^2 + i\epsilon\}]^{-1} \quad (5.52a)$$

and

$$\begin{aligned}
 J_{n,\ell}^{(2)} = & g_{\mu_0 \nu_0} g_{\mu_1 \nu_1} \dots g_{\mu_\ell \nu_\ell} \int_{k_1} \dots \int_{k_n} k_1^{\mu_0 \nu_0} k_{\lambda_1}^{\mu_1 \nu_1} k_{\lambda_2}^{\mu_2 \nu_2} \dots k_{\lambda_\ell}^{\mu_\ell \nu_\ell} \\
 & \times [(k_{\lambda_1}^2 - m^2 + i\epsilon)(k_{\lambda_2}^2 - m^2 + i\epsilon) \dots (k_{\lambda_\ell}^2 - m^2 + i\epsilon)] \\
 & \times \left\{ \prod_{e=1}^n (k_e^2 - m^2 + i\epsilon) \right\} \left\{ \prod_{j=1}^{n-1} (k_j - k_{j+1})^2 - m^2 + i\epsilon \right\} \\
 & \times \{(p_2 + k_1)^2 - m^2 + i\epsilon\} \{(p_1 + k_n)^2 - m^2 + i\epsilon\}^{-1} \quad , \quad (5.52b)
 \end{aligned}$$

where $1 \leq \lambda_1 < \lambda_2 \dots < \lambda_\ell < n$ for (5.52a) and $1 < \lambda_1 < \lambda_2 \dots < \lambda_\ell < n$ for (5.52b). All other terms have been examined in Reference 15. Each of the other terms was shown to be nonleading but only after a complicated cancellation of possible leading terms (called p terms in Reference 15).

It is useful to parameterize the propagators in exponential form:

$$\begin{aligned}
 (k^2 - m^2 + i\epsilon)^{-1} &= -i \int_0^\infty d\beta e^{i\beta(k^2 - m^2 + i\epsilon)} \quad , \\
 k_\mu (k^2 - m^2 + i\epsilon)^{-1} &= -i \int_0^\infty d\beta (2i\beta)^{-1} \frac{\partial}{\partial a^\mu} e^{i\beta[(k+a)^2 - m^2 + i\epsilon]} \Big|_{a=0} \quad , \\
 k_\mu k_\nu (k^2 - m^2 + i\epsilon)^{-2} &= \frac{i}{2} \int_0^\infty d\beta [(2i\beta)^{-1} \frac{\partial^2}{\partial a^\mu \partial a^\nu} - g_{\mu\nu}] e^{i\beta[(k+a)^2 - m^2 + i\epsilon]} \Big|_{a=0} \quad . \\
 & \quad (5.53)
 \end{aligned}$$

Following Blaha (39), we introduce n extraneous momenta (see Figure 15). We can then use the relations given in (5.53) to write

Figure 15. N-loop pseudoscalar-fermion u diagram with spurious momenta a_j and Feynman parameters α_j and β_j

Equation (5.52a) as

$$\begin{aligned}
 J_{n,\ell}^{(1)} &\sim g_{\mu_0\nu_1} g_{\mu_1\nu_1} \dots g_{\mu_\ell\nu_\ell} p_2^{\mu_0} \int_0^\infty d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n (2i\beta_1)^{-1} \mathfrak{D}_n^{\nu_\ell} \\
 &\times \left\{ \prod_{e=1}^{\ell} [g^{\nu_e-1\mu_e} - (2i\beta_{\lambda_e})^{-1} \mathfrak{D}_{\lambda_e}^{\nu_e-1} \mathfrak{D}_{\lambda_e}^{\mu_e}] \right. \\
 &\times \int_{k_1}^{\ell} \dots \int_{k_n} \exp \{ i \sum_{j=1}^n [(k_j + a_j)^2 - m^2 + i\epsilon] \beta_j \\
 &+ i \sum_{j=1}^{n-1} [(k_j - k_{j+1})^2 - m^2 + i\epsilon] \alpha_j + i[(p_2 + k_1)^2 - m^2 + i\epsilon] \alpha_0 \\
 &\left. + i[(p_1 + k_n)^2 - m^2 + i\epsilon] \alpha_n \right\} \Big|_{a_1=a_2=\dots=a_n=0} \\
 &\equiv \hat{J}_{n,\ell}^{(1)}, \tag{5.54}
 \end{aligned}$$

where

$$\mathfrak{D}_{\lambda}^{\mu} = \frac{\partial}{\partial a_{\lambda\mu}} \tag{5.55}$$

In (5.54) we have neglected irrelevant constants.

The momentum integrations can be performed. This yields

$$\begin{aligned}
 \hat{J}_{n,\ell}^{(1)} &= g_{\mu_0\nu_0} g_{\mu_1\nu_1} \dots g_{\mu_\ell\nu_\ell} p_2^{\mu_0} \int_0^\infty d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n (2i\beta_n)^{-1} \mathfrak{D}_n^{\nu_\ell} \\
 &\times \left\{ \prod_{e=1}^{\ell} [g^{\nu_e-1\mu_e} - (2i\beta_{\lambda_e})^{-1} \mathfrak{D}_{\lambda_e}^{\nu_e-1} \mathfrak{D}_{\lambda_e}^{\mu_e}] \frac{e^{iD(a)/C}}{C^2} \right\} \Big|_{a=0}, \tag{5.56}
 \end{aligned}$$

where

$$\begin{aligned} c \equiv c_n^0 &= d_{1,n} + (\alpha_0 + \beta_1) c_n^1 + \sum_{\ell=1}^{n-1} d_{1,\ell} \beta_{\ell+1} c_n^{\ell+1} \\ &= d_{0,n-1} + (\alpha_n + \beta_n) c_{n-1}^0 + \sum_{\ell=1}^{n-1} d_{\ell,n-1} \beta_{\ell} c_{\ell-1}^0, \end{aligned} \quad (5.57)$$

and

$$\begin{aligned} D(a) &= -d_{0,n} s + \sum_{\ell=1}^n [(p_2 + a_{\ell})^2 G_{\ell} + (p_1 + a_{\ell})^2 H_{\ell}] + \sum_{\substack{j,\ell=1 \\ j < \ell}}^n (a_j - a_{\ell})^2 I_{j\ell} \\ &\quad - (m^2 - i\epsilon) \left[c \left(\sum_{i=0}^n \alpha_i + \sum_{i=1}^n \beta_i \right) - 4 d_{0,n} \right]. \end{aligned} \quad (5.58)$$

We have defined

$$\begin{aligned} d_{j,\ell} &= \prod_{i=j}^{\ell} \alpha_i, \quad d_{j,j} = \alpha_j, \\ c_{\ell}^j &= d_{j,\ell-1} + (\alpha_{\ell} + \beta_{\ell}) c_{\ell-1}^j + \sum_{i=j+1}^{\ell-1} d_{1,\ell-1} \beta_i c_{i-1}^j \\ &= d_{j+1,\ell} + (\alpha_j + \beta_{j+1}) c_{\ell}^{j+1} + \sum_{i=j+1}^{\ell-1} d_{j+1,i} \beta_{i+1} c_{\ell}^{i+1}, \quad j < \ell, \\ c_j^j &= 1, \end{aligned} \quad (5.59)$$

and

$$\begin{aligned} G_j &= d_{0,j-1} \beta_j c_n^j, \\ H_j &= d_{j,n} \beta_j c_{j-1}^j, \end{aligned}$$

$$I_{j\ell} = d_{j,\ell-1} \beta_j \beta_\ell c_{j-1}^0 c_n^\ell \quad (5.60)$$

The expression (5.58) for $D(a)$ can be obtained by the graphical cutting rules (22, pp. 31-36) generalized to graphs containing $2n+6$ external lines.

Following a similar procedure, we obtain

$$J_{n,\ell}^{(2)} \sim g_{\mu_0 \nu_0} g_{\mu_1 \nu_1} \dots g_{\mu_\ell \nu_\ell} \int_0^\infty d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n (2i\beta_1)^{-1} (2i\beta_n)^{-1} \\ \times \prod_{e=1}^{\ell} [g^{\nu_{e-1} \nu_e} - (2i\beta_{\lambda_e})^{-1} \delta_{\lambda_e}^{\nu_{e-1} \nu_e}] \frac{e^{iD(a)/C}}{C^\omega} \Big|_{a=0} \quad (5.61)$$

By power counting, we see that (5.52b) is divergent. Expression (5.61) is a result of dimensional regularization, where 2ω is the spatial dimension in which all momentum integrations have been performed.

We show in Appendix E that the leading terms in the integrals of (5.56) and (5.61) are of the form

$$\frac{(p_1 \cdot p_2)}{C^2} f(\alpha, \beta) e^{iD/C} (d_{0,n} p_1 \cdot p_2)^j, \quad j = 0, 1, 2, \dots, \ell \quad (5.62)$$

where

$$f(0, \beta) = 0 \quad (5.63)$$

Since $f(0, \beta) = 0$, we find that expressions (5.56) and (5.61) are non-leading. The leading behavior of (5.48) is therefore given by (5.51):

$$T_u^{(n+1)} = -ig^2 \frac{1}{n!(n+1)!} (g^{-2} \ln^2 s)^n \quad (5.64)$$

This is identical to the spin-average result, Equation (5.32), of Reference 15. The results obtained in Subsections B and C are special cases of (5.64) for $n = 1$ and 2 , i.e., for fourth-order and sixth-order (see Equations (5.23) and (5.42)). It does not give the result found for the Born terms of Subsection A. However, in performing the summation to all orders, we will see that Equation (5.64) can also be used for $n = 0$ to give the leading behavior.

The high energy behavior for the s diagrams can be obtained in the same manner. All changes can be accounted for by replacing p_1 with $-p_1$. For the denominator, this amounts to replacing the factor $[(p_1 - k_n)^2 - m^2]$ with $[(p_1 + k_n)^2 - m^2]$. This results in changing s to $-s$ in $D(a)$ and D given in Equations (5.58) and (5.44). Again we find that the term $A_n(p_1 \cdot p_2)$ (see Equation (5.48)) gives the leading behavior. This also gives the effect $s \rightarrow -s$ in the numerator. We finally obtain

$$T_s^{(n+1)} = -ig^2 \frac{1}{n!(n+1)!} [g^2 (\ln s - i\pi)^2]^n \quad (5.65)$$

Again this is identical to the spin-average result, Equation (5.33), of Reference 15.

With the expressions given in (5.64) and (5.65) we are able to sum to all orders the leading logarithms of the two classes of diagrams:

$$\begin{aligned} T_u &= \sum_{n=0}^{\infty} T_u^{(n+1)} \\ &= -ig^2 \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} (g^2 \ln s)^n \end{aligned}$$

$$\begin{aligned}
&= -ig^2 \frac{I_1(2\bar{g} \ln s)}{\bar{g} \ln s} \\
&\approx \frac{-ig^2}{\sqrt{4\pi}} \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
\end{aligned} \tag{5.66}$$

and

$$\begin{aligned}
T_s &= -ig^2 \frac{I_1(2\bar{g}(\ln s - i\pi))}{(\bar{g} \ln s)} \\
&\approx \frac{-ig^2}{\sqrt{4\pi}} e^{-i2\pi\bar{g}} \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
\end{aligned} \tag{5.67}$$

For the s model, the amplitude is given by

$$\begin{aligned}
T &= T_s + T_u \\
&\approx \frac{-ig^2}{\sqrt{4\pi}} (1 + e^{-i2\pi\bar{g}}) \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}
\end{aligned} \tag{5.68}$$

This corresponds to a fixed cut in the complex angular momentum plane.

It is identical to the result obtained in Reference 15 for the spin averaged fermion-fermion scattering amplitude. It is also similar to the results of $(\varphi^3)_6$ and the truss bridge diagrams of $\varphi^3 + \varphi^4$ (12, 40):

$$T \approx \frac{-ig^2}{\sqrt{4\pi}} (1 + e^{-i\pi(2\bar{g}-1)}) \frac{s^{2\bar{g}-1}}{(\bar{g} \ln s)^{3/2}}$$

In this case the branch cut begins at $J = (2\bar{g} - 1)$ while in the s model the branch cut begins at $J = 2\bar{g}$.

The amplitude for pseudoscalar-pseudoscalar scattering in the σ

model is also given by (5.68). The amplitude for the Y model can be obtained from (5.68) with the replacement of g^2 and $\bar{g}^2$ by $4g^2$ and $\bar{g}^2/2$.

In deriving (5.66) and (5.67), we have used the expressions (5.64) and (5.65) for the Born terms as well. The value of (5.64) and (5.65) for $n = 0$ is just $-ig^2$, which is different from the expressions given in (5.2). However, all these terms behave like $g^2 f(t)$. When compared to the all-orders sum given in (5.68) or in (5.66) and (5.67) which behave like $g^2 s^{2\bar{g}}$, these terms are negligible. Therefore, the use of expressions (5.64) and (5.65) for the Born terms does not affect the leading behavior of the all-orders sum given in (5.68).

E. The Born Terms for Pseudoscalar-Fermion Scattering

The Born diagrams for pseudoscalar-fermion scattering are shown in Figure 16. All diagrams contribute to the leading behavior. We obtain

$$\begin{aligned}
 T_{15.a} &\sim -ig^2 \delta_{\lambda_1 \lambda_3}, \\
 T_{15.b} &\sim -ig^2 \delta_{\lambda_1 \lambda_3}, \\
 T_{15.c} &\sim \frac{-2ig_m^2 \delta_{\lambda_1 \lambda_3}}{t - m^2}.
 \end{aligned} \tag{5.69}$$

F. Fourth-Order Pseudoscalar-Fermion Diagrams

Figure 17 illustrates the fourth-order diagrams for pseudoscalar-fermion scattering. Besides the diagrams shown in Figure 17, there are many radiative corrections to the Born terms. These radiative corrections

(a)

(b)

(c)

Figure 16. Born diagrams for pseudoscalar-fermion scattering

Figure 17. Fourth-order pseudoscalar-fermion diagrams

are nonleading and will not be discussed further.

In analogy with the pseudoscalar-pseudoscalar scattering, we call the diagrams of Figure 16a.1 and 16a.2 s diagrams and those of Figure 16a.3 and 16a.4 u diagrams. They are the leading diagrams.

We discuss the diagram of Figure 16a.1. The corresponding amplitude is given by

$$T_{16a.1}^{(2)} = g^4 \int_k \frac{N_{16a.1}^{(2)}}{D_{a.1}^{(2)}} \quad (5.70)$$

The denominator $D_{a.1}^{(2)}$ is given by Equation (5.5). The numerator is given by

$$\begin{aligned} \tilde{N}_{16a.1}^{(2)} = & \bar{u}(p_3) [-\not{p}_1 \not{p}_2 \not{k} + k^2 \not{k} + 2(k \cdot p_2) \not{k} - k^2 \not{p}_2 - k^2 \not{m} + \not{p}_1 \not{p}_2 \not{m} - k \not{p}_2 \not{m} \\ & - \not{k} \not{m} + m^3] u(p_1) \end{aligned} \quad (5.71)$$

The leading behavior comes from the term

$$\begin{aligned} N_{16a.1}^{(2)} &= -\bar{u}(p_3) \not{p}_2 u(p_1) \\ &\approx -s \delta_{\lambda_1 \lambda_3} \end{aligned} \quad (5.72)$$

The calculation follows the same procedure as in Subsection B. We arrive at the result

$$I_{16a.1} = \frac{-i\pi^2}{(2\pi)^4} g^4 \frac{(\ln s - i\pi)^2}{2} \quad (5.73)$$

The analysis of the diagram in Figure 16a.2 is entirely analogous.

We obtain the identical result:

$$T_{17a.2} = \frac{-i\pi^2}{(2\pi)^4} g^4 \frac{(\ln s - i\pi)^2}{2} \quad (5.74)$$

Combining Equations (5.73) and (5.74), we obtain

$$T_s^{(2)} = -\frac{i}{2} g^2 \frac{-2}{g^2} (\ln s - i\pi)^2 \quad (5.75)$$

The u diagrams can be handled in the same manner as the s diagrams. The denominator is changed by the replacement of $[(p_2 + k_1)^2 - m^2]$ by $[(p_2 - k_1)^2 - m^2]$. This is equivalent to the replacement of s by $-s$. The numerator undergoes the change of replacing p_2 with $-p_2$. This also is equivalent to changing s into $-s$. We obtain the results

$$T_{17a.3}^{(2)} = T_{17a.4}^{(2)} = \frac{-i\pi^2}{(2\pi)^4} g^4 \frac{\ln^2 s}{2} \quad (5.76)$$

Thus

$$T_u^{(2)} = -\frac{i}{2} g^2 \frac{-2}{g^2} \ln^2 s \quad (5.77)$$

Some of the other fourth-order diagrams are shown in Figure 17b - 17d. They are all nonleading. We list only the results. The diagrams of Figure 17b and 17d behave at most like a constant. The diagrams of Figure 17c behave like $1/s$, up to powers of $\ln s$, while their planar counterparts behave at most like $\ln s$. The leading behavior for the fourth-order diagrams is thus given by Equations (5.75) and (5.77).

G. Sixth-Order Pseudoscalar-Fermion Diagrams

The planar sixth-order diagrams are illustrated in Figures 18 and 19. We expect the diagrams of Figure 18 to give the leading behavior. The calculation of the leading diagrams is quite tedious. However, if one performs the calculation as the forward spin average, it becomes tractable. In this case the general analysis of the following subsection applies, and we obtain

$$T_s^{(3)} = -ig^2 \frac{1}{2!3!} [g^2 (\ln s - i\pi)^2]^2 \quad (5.78)$$

and

$$T_u^{(3)} = -ig^2 \frac{1}{2!3!} [g^2 \ln^2 s]^2 \quad (5.79)$$

H. Pseudoscalar-Fermion Scattering to All Orders

We assume that the ladder diagrams such as that shown in Figure 20 give the leading contribution in any given order. We shall work with the forward spin averaged amplitude. The calculation is parallel to that for pseudoscalar-pseudoscalar scattering given in Subsection D. Let us denote the sum of all contributions from diagrams such as that shown in Figure 20b by

$$T_u^{(n+1)} = -g^{2(n+1)} i^{3n+1} \int_{k_1} \dots \int_{k_n} \frac{\hat{N}_n}{D_n} \quad (5.80)$$

The denominator D_n is the same as for pseudoscalar-pseudoscalar scattering and is given by Equation (5.43). The expression for the numerator is

Figure 18. Leading sixth-order pseudoscalar-fermion s diagrams

Figure 19. Nonleading sixth-order pseudoscalar-fermion planar diagrams

Figure 20. N-loop pseudoscalar-fermion ladder diagrams: (a) s diagrams and (b) u diagrams

derived in Appendix D. The result is

$$\hat{N}_n = (-2)^{n+1} [A_n(p_1 \cdot p_2) + A_n(p_2 \cdot k_1) + \sum_{j=1}^n \hat{B}_n^j (p_1 + k_n) \cdot k_j] \quad (5.81)$$

This equivalent to the replacement of $(k_1 + p_2)$ with p_2 in expressions (5.47) and (5.48). As in Subsection D, we find the term $A_n(p_1 \cdot p_2)$ to give the leading behavior. The other terms have been analyzed in Appendix E. They have already occurred in the calculation of Subsection D. They were shown to be nonleading.

Since the leading term is identical to that given in Subsection D, we obtain the identical result:

$$T_u^{(n+1)} = -ig^2 \frac{1}{n!(n+1)!} (\bar{g}^2 \ln^2 s)^n \quad (5.82)$$

The analysis for the s diagrams is parallel to that for the u diagrams. We obtain the same result as in Subsection D:

$$T_s^{(n+1)} = -ig^2 \frac{1}{n!(n+1)!} [\bar{g}^2 (\ln s - i\pi)^2]^n \quad (5.83)$$

Therefore, we arrive at the same results as in pseudoscalar-pseudoscalar scattering for the all-order amplitudes:

$$T_u = \frac{-ig^2 I_1(2\bar{g} \ln s)}{\bar{g} \ln s} \approx \frac{-ig^2}{\sqrt{4\pi}} \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}, \quad (5.84)$$

$$T_s \approx \frac{-ig^2}{\sqrt{4\pi}} \frac{e^{-i2\pi\bar{g}} s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}}, \quad (5.85)$$

$$T = T_s + T_u$$

$$\approx \frac{-ig^2}{\sqrt{4\pi}} (1 + e^{-i2\pi\bar{g}}) \frac{s^{2\bar{g}}}{(\bar{g} \ln s)^{3/2}} \quad (5.86)$$

The result for the σ model is also given by (5.86). The amplitudes for pseudoscalar-fermion and pseudoscalar-antifermion scattering are both given by (5.86) with the replacement of $\bar{g}^2$ by $\bar{g}^2/2$.

1. Summary

In this work we have explicitly examined pseudoscalar-pseudoscalar and pseudoscalar-fermion scattering. Fermion-fermion scattering was calculated in Reference 15. Amplitudes containing external fermion lines were spin averaged. The other amplitudes listed in (4.4) differ from the amplitudes mentioned above only by changing two external pseudoscalar lines to scalar lines. As noted in Appendix D, this affects only the mass terms. Therefore, these amplitudes have the same leading behavior, and we are justified in speaking of fermion-boson and boson-boson amplitudes.

In the s model we found all amplitudes to be identical. They are given by (5.86).

The fermion-boson, antifermion-boson, and boson-boson amplitudes in the σ model are also given by (5.86). The fermion-fermion and fermion-antifermion amplitudes are respectively given by (5.84) and (5.85). They lack the signature factor.

In the Y model the boson-boson scattering amplitude is given by (5.86) with g^2 and $\bar{g}^2$ replaced by $4g^2$ and $\bar{g}^2/2$. The fermion-boson and

antifermion-boson amplitudes is given by (5.86) with $\bar{g}^2$ replaced by $\bar{g}^2/2$. The amplitudes for fermion-fermion and fermion-antifermion scattering are given respectively by (5.84) and (5.85) with g^2 and $\bar{g}^2$ replaced by $g^2/2$ and $\bar{g}^2/2$. They also lack the signature factor.

VI. CONCLUSION

In this work we review the supersymmetry transformation and the construction of a field theory satisfying this symmetry. The Feynman rules are derived and renormalizability is studied to the lowest order of perturbation theory.

We have studied boson-boson and boson-fermion scattering in the high energy limit within a class of field theories containing fermions and spinless bosons with the aim of investigating the role supersymmetry plays in the theory. Fermion-fermion scattering was studied in an earlier work (15). In all cases we found the amplitudes to be dominated by the diagrams with fermions exchanged in the t channel. This is similar to the rule in Regge theory that an amplitude is dominated by the highest J -plane trajectory allowed in the t channel.

The Wess-Zumino model of supersymmetry used in this calculation is clearly renormalizable since the dimension of all interaction terms is less than or equal to four. The supersymmetry places restrictions between the coupling constants and between the masses. As demonstrated in Section III, these restrictions greatly improve the renormalization, causing cancellations of divergences to occur between different diagrams. All remaining divergences are removed by the introduction of a single wavefunction renormalization constant.

In contrast, the symmetry restrictions in spontaneously broken non-Abelian gauge theories are essential for their renormalizability. The symmetry also plays a strong role in their high energy behavior. All integrations over transverse momentum are convergent, due to cancellations

between diagrams. All $\ln s$ factors come from the integrations over longitudinal momentum. This causes the theory to Reggeize. The vector-meson self-couplings play an important role in these cancellations.

Despite the cancellations in the renormalization of supersymmetry, the s model has the same basic features of the high energy behavior as the Y and σ models. Mesons play only a passive role, providing momentum transfer for the fermions exchanged in the t channel. Meson-meson couplings are unimportant. There is no $P_{\perp}$ damping and the maximum energy dependence exhibited by the individual Feynman diagrams is attained.

It is interesting to note that the spin-averaged amplitudes for fermion-fermion, fermion-boson, and boson-boson scattering are all equal in the high energy limit within the s model. In the σ model the spin-averaged fermion-boson and boson-boson amplitudes are equal, while, as noted in Subsection 1 of Section V, the fermion-fermion scattering lacks the signature factor. This is due to the fact the fermion wavefunction is no longer self-conjugate. The Y model is slightly more complicated. The boson-boson amplitude is given by four times the spin-averaged fermion-boson amplitude. The spin-averaged fermion-fermion amplitude is equal to $\frac{1}{2}$ times the spin-averaged fermion-boson amplitude, minus the signature factor. These factors arise because there is only one type of meson which can be exchanged. Again the fermion-fermion amplitude lacks the signature factor because the fermion wavefunction is not self-conjugate. All amplitudes of the three theories possess a similar fixed square root cut.

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$$J_{\mu\nu}^{(3)} \equiv \int d^n k \frac{k_\mu k_\nu}{(k^2 - m^2)^3}$$

$$= (-1)^3 i (\sqrt{\pi})^n \left\{ - \frac{\Gamma(2 - n/2)}{2\Gamma(3)} \frac{g_{\mu\nu}}{(m^2)^{2-n/2}} \right\}$$

and

$$J^{(2)} \equiv \int d^n k \frac{1}{(k^2 - m^2)^2}$$

$$= (-1)^2 i (\sqrt{\pi})^n \left\{ \frac{\Gamma(2 - n/2)}{\Gamma(2)} \frac{1}{(m^2)^{2-n/2}} \right\}$$

Comparison of $J_{\mu\nu}^{(3)}$ and $J^{(2)}$ yields Equation (3.26).

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IX. APPENDIX A

In this appendix we give our notational conventions. We also list some results for Majorana spinors that are used in the text.

We define our metric according to

$$\begin{aligned} g_{00} &= 1, \quad g_{11} = g_{22} = g_{33} = -1, \\ g_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu. \end{aligned} \tag{A.1}$$

The Dirac matrices we use satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{A.2}$$

We define

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.3}$$

so that

$$\gamma_5^2 = 1. \tag{A.4}$$

We take as charge conjugation matrix

$$C = i\gamma^0\gamma^2. \tag{A.5}$$

Our charge conjugation matrix C satisfies the conditions

$$\begin{aligned} CC^\dagger &= C^\dagger C = 1, \\ C^T &= -C, \\ C^{-1}\gamma^\mu C &= -(\gamma^\mu)^T. \end{aligned} \tag{A.6}$$

We also define

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \quad . \quad (A.7)$$

Integrals over four-dimensional volumes in momentum space are denoted by

$$\int_k \equiv \int \frac{d^4 k}{(2\pi)^4} \quad . \quad (A.8)$$

The amplitude T used in the text is defined by

$$S = 1 + T(2\pi)^4 \delta(p_f - p_i) \quad . \quad (A.9)$$

The coupling constant, g , we employ in the discussion of the Wess-Zumino model in Sections III, IV, and V differs from that defined in Reference 16 by a factor $\frac{1}{2}$.

A Majorana spinor ψ is defined by

$$\psi = \psi^C = (\bar{\psi}C)^T \quad , \quad (A.10)$$

where C is the charge conjugation matrix. Because of (A.10), ψ and $\bar{\psi}$ are not independent fields:

$$\bar{\psi}_\alpha = \psi_\beta C_{\beta\alpha}^{-1} \quad . \quad (A.11)$$

This is the reason a factor of $\frac{1}{2}$ is needed in front of the kinetic energy and mass terms of $\mathcal{L}_S$, Equation (4.1).

The spinor wavefunctions we employ follow the convention of Bjorken and Drell (41), except we use a different normalization:

$$\sum_s u(p, s) \bar{u}(p, s) = \not{p} + m \quad (A.12)$$

and

$$\sum_s v(p, s) \bar{v}(p, s) = \not{p} - m \quad . \quad (A.13)$$

The explicit forms of the spinor wavefunctions, with helicity $\pm\frac{1}{2}$, are

$$\begin{aligned} u(p, s) &= (E + m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi_s \end{pmatrix} , \\ v(p, s) &= (E + m)^{\frac{1}{2}} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \xi_s \\ \xi_s \end{pmatrix} , \end{aligned} \quad (A.14)$$

where χ_s is a nonrelativistic spinor wavefunction satisfying

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \chi_s = s \chi_s \quad , \quad s = \pm 1 \quad (A.15)$$

and

$$\xi_s = -i\sigma_2 (\chi_s^\dagger)^T \quad . \quad (A.16)$$

If we choose the reaction plane to coincide with the xz plane so that $r = (0, r, 0, 0)$, then in the limit $r/E \rightarrow 0$

$$\begin{aligned} \chi_\pm^{(1)} &\approx \eta_\pm \mp \frac{r}{\sqrt{s}} \eta_\mp \quad , \quad \chi_\pm^{(3)} \approx \eta_\pm \pm \frac{r}{\sqrt{s}} \eta_\mp \quad , \\ \chi_\pm^{(2)} &\approx \eta_\mp \pm \frac{r}{\sqrt{s}} \eta_\pm \quad , \quad \chi_\pm^{(4)} = \eta_\mp \mp \frac{r}{\sqrt{s}} \eta_\pm \quad , \\ \xi_\pm^{(2)} &= \mp \chi_\mp^{(2)} \quad , \quad \xi_\pm^{(4)} = \mp \chi_\mp^{(4)} \end{aligned} \quad (A.17)$$

and

$$\eta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{A.18})$$

where $\chi_{\pm}^{(1)}$, etc., apply to particle 1, etc.

We now list some identities satisfied by two Majorana spinors, ψ_1 and ψ_2 . The proof of these relations appears in Gasiorowicz (32).

$$\begin{aligned} \bar{\psi}_1 \psi_2 &= \bar{\psi}_2 \psi_1, \\ \bar{\psi}_1 \gamma_\mu \psi_2 &= -\bar{\psi}_2 \gamma_\mu \psi_1, \\ \bar{\psi}_1 \gamma_5 \psi_2 &= \bar{\psi}_2 \gamma_5 \psi_1, \\ \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 &= \bar{\psi}_2 \gamma_\mu \gamma_5 \psi_1, \\ \bar{\psi}_1 \gamma_\mu \gamma_\nu \psi_2 &= -\bar{\psi}_2 \gamma_\mu \gamma_\nu \psi_1, \\ \bar{\psi}_1 \sigma_{\mu\nu} \psi_2 &= -\bar{\psi}_2 \sigma_{\mu\nu} \psi_1. \end{aligned} \quad (\text{A.19})$$

Using the Fierz transformation, Gasiorowicz also demonstrates that

$$\begin{aligned} (\bar{\psi}_1 \gamma_5 \psi_1) (\bar{\psi}_2 \gamma_5 \psi_1) &= -(\bar{\psi}_1 \psi_1) (\bar{\psi}_2 \psi_1), \\ (\bar{\psi}_1 \gamma_\mu \gamma_5 \psi_1) (\bar{\psi}_2 \gamma^\mu \gamma_5 \psi_1) &= 4(\bar{\psi}_1 \psi_1) (\bar{\psi}_2 \psi_1). \end{aligned} \quad (\text{A.20})$$

X. APPENDIX B

In this appendix we derive the Feynman rules of Figure 8 in Section IV. We will do this using functional methods and reduction techniques. Reduction techniques simply mean that one may obtain the S-matrix element by taking the appropriate n-point function, removing the propagators from the external legs, and placing the external momenta on the mass shell.

Functional techniques allow us to calculate the n-point functions as functional derivatives of the generating functional. In the case of the Lagrangian given in Equation (4.1), the generating functional is given by

$$Z(J) \sim \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \left\{ \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_J] \right\} \quad (\text{B.1})$$

where

$$\mathcal{L}_0 = \frac{1}{2} [(\partial_\mu A)^2 - m^2 A^2] + \frac{1}{2} [(\partial_\mu B)^2 - m^2 B^2] + \frac{1}{2} [i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi] \quad , \quad (\text{B.2})$$

$$\mathcal{L}_1 = -\frac{1}{2}g\bar{\psi}(A - i\gamma_5 B)\psi - \frac{1}{2}mgA(A^2 + B^2) - 1/8 g^2(A^2 + B^2)^2 \quad , \quad (\text{B.3})$$

and

$$\mathcal{L}_J = J_A A + J_B B + \bar{\eta}\psi \quad . \quad (\text{B.4})$$

We can write $\mathcal{L}_1$ in terms of functional derivatives. Then (B.1) becomes

$$Z(J) \sim \int \mathcal{D}A \mathcal{D}B \mathcal{D}\psi \exp i \left\{ \int d^4x \mathcal{L}_1 \left(\frac{\delta}{\delta J_A}, \frac{\delta}{\delta J_B}, \frac{\delta}{\delta \bar{\eta}} \right) \right. \\ \left. \times \int d^4y [\mathcal{L}_0 + \mathcal{L}_J] \right\} \quad , \quad (\text{B.5})$$

where

$$\begin{aligned}
 \mathfrak{L}_1 \left(\frac{\delta}{\delta J_A}, \frac{\delta}{\delta J_B}, \frac{\delta}{\delta \bar{\eta}} \right) &= -\frac{1}{2} g \left(\frac{1}{i} \right)^3 \left[-C_{\beta\alpha}^{-1} \right] \frac{\delta}{\delta \bar{\eta}_\alpha} \left[\frac{\delta}{\delta J_A} \delta_{\beta\sigma} - (i\gamma_5)_{\beta\sigma} \frac{\delta}{\delta J_B} \right] \\
 &\quad \times \frac{\delta}{\delta \bar{\eta}_\sigma} - \frac{1}{2} m g \left(\frac{1}{i} \right)^3 \frac{\delta}{\delta J_A} \left[\frac{\delta^2}{\delta J_A^2} + \frac{\delta^2}{\delta J_B^2} \right] - 1/8 g^2 \left(\frac{1}{i} \right)^4 \\
 &\quad \times \left[\frac{\delta^4}{\delta J_A^4} + 2 \frac{\delta^4}{\delta J_A^2 \delta J_B^2} + \frac{\delta^4}{\delta J_B^4} \right] . \quad (B.6)
 \end{aligned}$$

Summation on repeated indices is implied.

The integration can now be performed, using the formula

$$\int d\xi e^{i\int (\frac{1}{2}\xi A \xi + j\xi)} = (\det A)^{-\frac{1}{2}} e^{-\frac{1}{2} j A^{-1} j} . \quad (B.7)$$

Det A is an irrelevant constant, so (B.5) becomes

$$Z(J) \sim e^{i \int \mathfrak{L}_1 \left(\frac{\delta}{\delta J_A}, \frac{\delta}{\delta J_B}, \frac{\delta}{\delta \bar{\eta}} \right)} e^{-\frac{1}{2} \int \left[J_A \Delta J_A + J_B \Delta J_B + \bar{\eta} S_F^C \bar{\eta}^T \right]} \quad (B.8)$$

where

$$\begin{aligned}
 \Delta(x-y) &= \int_p \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} , \\
 S_F^C(x-y) &= S_F(x-y) C^T , \quad (B.9)
 \end{aligned}$$

with

$$S_F(x-y) = \int_p \frac{e^{-ip(x-y)}}{p - m + i\epsilon} .$$

Note that

$$\Delta(x - y) = \Delta(y - x) \quad , \quad (B.10)$$

$$S_F^C(x - y) = -[S_F^C(y - x)]^T \quad . \quad (B.11)$$

The propagators $\Delta(x - y)$ and $S_F(x - y)$ provide the rules for Figures 8a, 8b, and 8c.

For simplicity, let us first examine the diagram in Figure 8f of Section III:

This is a term in the expansion of

$$\langle T (A(x)A(y)A(z)) \rangle = \left(\frac{1}{i} \right)^3 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_A(y)} \frac{\delta}{\delta J_A(z)} Z(J) \quad . \quad (B.12)$$

It corresponds to the term $-\frac{1}{2}mg A^3$ in $\mathcal{L}_1$. We expand the two exponentials of $Z(J)$ given in (B.8), keeping only the term which can contribute to this diagram. Let us call this term A_3 . Then

$$\begin{aligned} A_3 = & \left(\frac{1}{i} \right)^3 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_A(y)} \frac{\delta}{\delta J_A(z)} (i) (-\frac{1}{2}mg) \left(\frac{1}{i} \right)^3 \int \frac{\delta^3}{\delta J_A^3} (-\frac{1}{2})^3 \frac{1}{3!} \\ & \times \int J_A \Delta J_A \int J_A \Delta J_A \int J_A \Delta J_A \quad . \end{aligned} \quad (B.13)$$

To obtain diagram 8f, the first three derivatives must act on each of the last three integrals. There are $3!$ ways to do this. The other derivatives can act on the remaining currents in $3!$ ways. We get another factor of $(2)^3$ from the fact that

$$\frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_A(w)} \int J_A \Delta J_A = 2 \Delta(x - w) \quad . \quad (B.14)$$

This follows from Equation (B.10). We therefore obtain

$$\begin{aligned} A_3 &= \left(\frac{1}{i}\right)^3 (i)(-\frac{1}{2}mg) \left(\frac{1}{i}\right)^3 (-\frac{1}{2})^3 (2)^3 (3!)(3!) \frac{1}{3!} \\ &= -3 \text{ img} \quad . \end{aligned} \quad (B.15)$$

This is the Feynman rule listed in Figure 8.

Next let us look at the vertex shown in Figure 8g:

This corresponds to the term $-\frac{1}{2}mg^2 AB^2$ in $\mathcal{L}_1$. This vertex appears in the expansion of

$$\langle T (A(x)B(y)B(z)) \rangle = \left(\frac{1}{i}\right)^3 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_B(y)} \frac{\delta}{\delta J_B(z)} Z(J) \quad . \quad (B.16)$$

Expanding $Z(J)$ as given by (B.8), we obtain

$$\begin{aligned}
AB_2 = & \left(\frac{1}{i}\right)^3 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_B(y)} \frac{\delta}{\delta J_B(z)} i(-\frac{1}{2}mg) \left(\frac{1}{i}\right)^3 \int \frac{\delta}{\delta J_A} \frac{\delta^2}{\delta J_B^2} \\
& \times (-\frac{1}{2})^3 \frac{1}{3!} 3 \int J_A \Delta J_A \int J_B \Delta J_B \int J_B \Delta J_B .
\end{aligned} \tag{B.17}$$

We have kept only the term contributing to the vertex shown in Figure 8g, and have called this term AB_2 . To obtain this vertex, $\delta/\delta J_B(y)$ and $\delta/\delta J_B(z)$ must act on each of the last two integrals, giving a factor $2!$. The derivative $\delta^2/\delta J_B^2$ gives another factor of $2!$. We obtain a factor of (2) since, e.g.,

$$\frac{\delta}{\delta J_B(y)} \frac{\delta}{\delta J_B(w)} \int J_B \Delta J_B = 2 \Delta(y - w) \tag{B.18}$$

and another factor of 2 because of Equation (B.14). Therefore, we obtain

$$\begin{aligned}
AB_2 = & \left(\frac{1}{i}\right)^3 i(-\frac{1}{2}mg) \left(\frac{1}{i}\right)^3 (-\frac{1}{2})^3 \frac{1}{3!} 3(2)^3 (2)^2 \\
= & -img .
\end{aligned} \tag{B.19}$$

This agrees with the rule given in Figure 8.

Let's examine the following vertex of Figure 8h

corresponding to the term $-1/8 g^2 B^4$ of $\mathcal{L}_1$. As before, we expand the

following 4-point function

$$\langle T (B(x)B(y)B(z)B(w)) \rangle = \left(\frac{1}{i}\right)^4 \frac{\delta}{\delta J_B(x)} \frac{\delta}{\delta J_B(y)} \frac{\delta}{\delta J_B(z)} \frac{\delta}{\delta J_B(w)} Z(J) \quad (B.20)$$

In the expansion of (B.20), we keep only the term contributing to the above diagram. Let us call this term B_4 . We obtain

$$B_4 = \left(\frac{1}{i}\right)^4 \frac{\delta}{\delta J_B(x)} \frac{\delta}{\delta J_B(y)} \frac{\delta}{\delta J_B(z)} \frac{\delta}{\delta J_B(w)} (i) (-1/8 g^2) \left(\frac{1}{i}\right)^4 \int \frac{\delta^4}{\delta J_B^4} (-\frac{1}{2})^4 \frac{1}{4!} \\ \times \int J_B \Delta J_B \int J_B \Delta J_B \int J_B \Delta J_B \int J_B \Delta J_B \quad .$$

Now is it simply a question of combinatorics, keeping only the terms which contribute to the above diagram. Proceeding as above, we obtain

$$B_4 = \left(\frac{1}{i}\right)^4 \frac{4! \cdot 4!}{4!} (i) (-1/8 g^2) \left(\frac{1}{i}\right)^4 (-\frac{1}{2})^4 (2)^4 \\ = -3 i g^2 \quad (B.21)$$

The analysis of the other diagram of Figure 8h is entirely the same. We obtain the same result, given by Equation (B.21). This agrees with Figure 8.

To obtain the rule for the diagram appearing in Figure 8i

corresponding to the term $-1/8 g^2 (2A^2 B^2)$ of $\mathcal{L}_1$, we expand the 4-point

function

$$\langle T \left(A(x)A(y)B(z)B(w) \right) \rangle = \left(\frac{1}{i} \right)^4 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_A(y)} \frac{\delta}{\delta J_B(z)} \frac{\delta}{\delta J_B(w)} Z(J) \quad .(B.22)$$

Let us call the desired term in the expansion of (B.22) $A_2 B_2$. We obtain

$$\begin{aligned} A_2 B_2 = & \left(\frac{1}{i} \right)^4 \frac{\delta}{\delta J_A(x)} \frac{\delta}{\delta J_A(y)} \frac{\delta}{\delta J_B(z)} \frac{\delta}{\delta J_B(w)} i(-1/8 g^2) \left(\frac{1}{i} \right)^4 (2) \int \frac{\delta^2}{\delta J_A^2} \frac{\delta^2}{\delta J_B^2} \\ & \times \left(-\frac{1}{2} \right)^4 \frac{1}{4!} (6) \int J_A \Delta J_A \int J_A \Delta J_A \int J_B \Delta J_B \int J_B \Delta J_B \quad . \end{aligned} \quad (B.23)$$

Keeping only the terms which correspond to diagram 8i, the combinatorics yields

$$\begin{aligned} A_2 B_2 = & (i) (-1/8 g^2) \frac{(2)^4 (2)^2 (2)^2}{4!} \left(-\frac{1}{2} \right)^4 6(2) \\ = & -ig^2 \quad , \end{aligned} \quad (B.24)$$

in agreement with Figure 8.

Lastly, let us calculate the rule corresponding to Figure 8d.

corresponding to the term $-g/2 \bar{\psi} A \psi$ of $\mathcal{L}_1$. The result for Figure 8e can be obtained from this result due to symmetry.

As before, we will expand the vertex function

$$\langle T (\psi_\alpha(x) \bar{\psi}_\beta(y) A(z)) \rangle = \left(\frac{1}{i}\right)^3 (-C_{\beta\sigma}^{-1}) \frac{\delta}{\delta \bar{\eta}_\sigma(y)} \frac{\delta}{\delta \bar{\eta}_\beta(x)} \frac{\delta}{\delta J_A(z)} Z(J) \quad (B.25)$$

In order to obtain all signs correctly, let us explicitly exhibit the reduction technique. To be specific, let us consider the amplitude for particle A coming in, with an outgoing antifermion and fermion. This amplitude is given by

$$\begin{aligned} T(p_1 s_1, p_2 s_2, p_3) &= \left(\frac{1}{i}\right)^3 \int d^4 x d^4 y d^4 z \bar{u}(p_1 s_1)_\gamma (\overrightarrow{i \not{x}} - m)_{\gamma\alpha} \\ &\quad \times \langle T (\psi_\alpha(x) \bar{\psi}_\beta(y) A(z)) \rangle (-\overleftarrow{\not{z}} - m^2) (\overleftarrow{-i \not{y}} - m)_{\beta\tau} \\ &\quad \times v_\tau(p_2 s_2) e^{i(p_1 x + p_2 y - p_3 z)} \end{aligned} \quad (B.26)$$

We expand (B.25), keeping the term, called $A\psi_2$, corresponding to diagram 8d:

$$\begin{aligned} A\psi_2 &= \left(\frac{1}{i}\right)^6 i(-\frac{1}{2}g)(-\frac{1}{2})^3 \frac{1}{2} [C_{\beta\sigma}^{-1}] [C_{\gamma\varphi}^{-1}] \frac{\delta}{\delta \bar{\eta}_\sigma(y)} \frac{\delta}{\delta \bar{\eta}_\alpha(x)} \frac{\delta}{\delta J_A(z)} \int \frac{\delta}{\delta J_A} \frac{\delta}{\delta \bar{\eta}_\gamma} \frac{\delta}{\delta \bar{\eta}_\phi} \\ &\quad \times \int J_A \Delta J_A \int \bar{\eta} S_F^C \bar{\eta}^T \int \bar{\eta} S_F^C \bar{\eta}^T \end{aligned} \quad (B.27)$$

Taking the functional derivatives, and keeping only the terms contributing to the vertex of Figure 8d, we obtain

$$\begin{aligned} A\psi_2 &= (-i)(-\frac{1}{2}g)(\frac{1}{2})^3 (\frac{1}{2}) (C_{\beta\sigma}^{-1}) (C_{\gamma\varphi}^{-1}) \int d^4 u [2 \Delta(z - u)] \cdot 2 \left\{ - [S_F^C(y - u)_{\sigma\gamma} \right. \\ &\quad - S_F^C(u - y)_{\gamma\sigma}] [S_F^C(x - u)_{\alpha\varphi} - S_F^C(u - x)_{\varphi\alpha}] + [S_F^C(y - u)_{\sigma\varphi} \\ &\quad \left. - S_F^C(u - y)_{\varphi\sigma}] [S_F^C(x - u)_{\alpha\gamma} - S_F^C(u - x)_{\gamma\alpha}] \right\} \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{ig}{2} \right) \left(\frac{1}{2} \right)^3 \left(\frac{1}{2} \right) (C_{\beta\sigma}^{-1}) (C_{\gamma\varphi}^{-1}) \int d^4u [2 \Delta(z - u)]^2 \left\{ 4[S_F^C(u - y)_{\gamma\sigma}] \right. \\
&\quad \times [S_F^C(x - u)_{\alpha\varphi}] - 4[S_F^C(u - y)_{\varphi\sigma}] [S_F^C(x - u)_{\alpha\gamma}] \left. \right\} . \quad (B.28)
\end{aligned}$$

We have used Equation (B.11) in deriving (B.28). Using the relation $(C^{-1})^T = -(C^{-1})$ from Appendix A, we arrive at

$$\begin{aligned}
A_{\psi_2} &= \left(-\frac{ig}{2} \right) \left(\frac{1}{2} \right)^3 \left(\frac{1}{2} \right) \int d^4u [2 \Delta(z - u)] (2)^3 \left\{ [S_F^C(x - u)C^{-1} S_F^C(u - y)C^{-1}]_{\alpha\beta} \right. \\
&\quad \left. + [S_F^C(x - u)C^{-1} S_F^C(u - y)C^{-1}]_{\alpha\beta} \right\} . \quad (B.29)
\end{aligned}$$

$$= -ig \int d^4u \Delta(z - u) [S_F(x - u)S_F(u - y)]_{\alpha\beta} . \quad (B.30)$$

Noting that

$$i \not{x} - m S_F(x - u) = i \delta(x - u) ,$$

and

$$-i \not{y} - m S_F(u - y) = i \delta(u - y) ,$$

Equation (B.26) becomes

$$T(p_1 s_1, p_2 s_2, p_3) = \bar{u}(p_1 s_1) v(p_2 s_2) (-ig) (2\pi)^4 \delta(p_1 + p_2 - p_3) , \quad (B.31)$$

valid to order g^2 . The Feynman rule we obtain from (B.31) is that given in Figure 8d.

As mentioned above, the Feynman rule for the diagram of Figure 8e can immediately be obtained from that for Figure 8d. It agrees with the rule listed in Figure 8.

We have verified these rules with the calculation of higher order diagrams. We obtain the additional rule that every fermion loop is associated with a factor (-1) . Also, if two diagrams contain closed fermion loops differing only in the direction of the charge flow, we keep only one of them.

XI. APPENDIX C

In this appendix we derive a procedure for evaluating traces of terms often appearing in calculations involving fermions. We employ the convention for γ matrices such that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where the metric $g^{\mu\nu}$ is given by $(+, -, -, -)$. We follow essentially the methods outlined in Caianiello (42). For the purposes of this work we do not need all the results listed below. We list them for completeness, as well as to show the full power of the methods.

Let

$$\hat{p}_i = p^{(i)} + m^{(i)} \quad (C.1)$$

where $p_4^{(i)} = m^{(i)}$.

Let

$$R = \hat{p}_1 \hat{p}_2 \hat{p}_3 \dots \hat{p}_n \quad (C.2)$$

We prefer to rewrite this as

$$R = \hat{p}_1 \gamma^5 \cdot \gamma^5 \hat{p}_2 \cdot \hat{p}_3 \gamma^5 \cdot \gamma^5 \hat{p}_4 \dots \hat{p}_{2m-1} \gamma^5 \cdot \gamma^5 \hat{p}_{2m} \quad \text{for } n = 2m, \quad (C.3)$$

$$R = \hat{p}_1 \gamma^5 \cdot \gamma^5 \hat{p}_2 \cdot \hat{p}_3 \gamma^5 \cdot \gamma^5 \hat{p}_4 \dots \hat{p}_{2m-1} \gamma^5 \cdot \gamma^5 \quad \text{for } n = 2m-1.$$

Here $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

We introduce the dual representation

$$\Gamma^\mu \equiv i\gamma^\mu\gamma^5, \quad \Gamma^4 \equiv i\gamma^5 \quad (C.4)$$

It is then easy to verify that

$$\begin{aligned} \{\Gamma^\mu, \Gamma^\nu\} &= 2g^{\mu\nu}, \\ \Gamma^4\Gamma^4 &= -1, \\ \{\Gamma^4, \Gamma^\mu\} &= 0. \end{aligned} \quad (C.5)$$

Letting primed indices run from 0 to 4, we can summarize this as

$$\{\Gamma^{\mu'}, \Gamma^{\nu'}\} = 2G^{\mu'\nu'} \quad (C.6)$$

where the metric $G^{\mu'\nu'}$ is given by $(+, -, -, -, -)$.

Let us now define

$$\varrho_i \equiv \hat{P}_i(i\gamma^5) = p_\mu^{(i)}(i\gamma^\mu\gamma^5) + m^{(i)}(i\gamma^5) \quad (C.7)$$

We can then write

$$\varrho_i = q_{\mu'}^{(i)}\Gamma^{\mu'} \quad (C.8)$$

where

$$q_{\mu'}^{(i)} \equiv (p_{\mu'}^{(i)}, m^{(i)}) \quad (C.9)$$

We find it also convenient to define

$$\bar{\varrho}_i \equiv (i\gamma^5)\hat{P}_i \quad (C.10)$$

and

$$\varrho_0 \equiv (i\gamma^5) \quad , \quad \bar{\varrho}_0 \equiv \varrho_0 \quad . \quad (C.11)$$

Then

$$\bar{\varrho}_{\mu'}^{(i)} = (-P_{\mu'}^{(i)}, m^{(i)}) = -(P_{\mu}^{(i)}, -m^{(i)}) \quad (C.12)$$

and

$$\bar{\varrho}_{\mu'} = \varrho_{\mu'}^{(0)} = (0, 0, 0, 0, 1) \quad . \quad (C.13)$$

We immediately obtain

$$\gamma^5 \varrho_i \gamma^5 = \bar{\varrho}_i \quad . \quad (C.14)$$

We also obtain the following results

$$\varrho_i \cdot \varrho_j = P_i \cdot P_j - m^{(i)} m^{(j)} \quad , \quad (C.15)$$

$$\bar{\varrho}_i \cdot \bar{\varrho}_j = \varrho_i \cdot \varrho_j \quad , \quad (C.16)$$

and

$$\varrho_i \cdot \bar{\varrho}_j = \bar{\varrho}_i \cdot \varrho_j = -(P_i \cdot P_j + m^{(i)} m^{(j)}) \quad . \quad (C.17)$$

We are now able to rewrite (C.3) as

$$R = (-1)^m \varrho_1 \bar{\varrho}_2 \varrho_3 \bar{\varrho}_4 \dots \varrho_{2m-1} \bar{\varrho}_{2m} \quad \text{for } n = 2m \quad , \quad (C.18)$$

$$R = (-1)^m \varrho_1 \bar{\varrho}_2 \varrho_3 \bar{\varrho}_4 \dots \varrho_{2m-1} \bar{\varrho}_0 \quad \text{for } n = 2m-1 \quad .$$

The power of this notation is seen when we take the trace of R .

The otherwise troublesome m 's are handled automatically by the formalism.

Let us take the trace of (C.16) (and thus of (C.3)). For illustrative purposes, we will consider only the case $n = 2m$. The application to the case $n = 2m-1$ is straightforward.

We use the fact that the trace of gamma matrices can be written as a pfaffian. We use the notation outlined in Reference 42. Let

$$\begin{aligned} (i \ j) &\equiv Q_i \cdot Q_j \\ (i \ \bar{j}) &\equiv Q_i \cdot \bar{Q}_j \end{aligned} \quad , \quad (C.19)$$

etc. Then we can write the trace of R as

$$\begin{aligned} \frac{1}{4} \text{Tr}[R] &= \frac{1}{4} (-1)^m \text{Tr}[Q_1 \bar{Q}_2 Q_3 \bar{Q}_4 \dots Q_{2m-1} \bar{Q}_{2m}] \\ &= (-1)^m (1 \ \bar{2} \ 3 \ \bar{4} \dots 2m-1 \ \overline{2m}) \end{aligned} \quad (C.20)$$

where $(1 \ \bar{2} \ 3 \ \bar{4} \dots 2m-1 \ \overline{2m})$ is the pfaffian (see Reference 42).

For instance

$$\begin{aligned} \frac{1}{4} \text{Tr}[Q_1 \bar{Q}_2 Q_3 \bar{Q}_4] &= (1 \ \bar{2} \ 3 \ \bar{4}) \\ &= (1\bar{2})(3\bar{4}) - (13)(\bar{2}\bar{4}) + (1\bar{4})(\bar{2}3) \\ &= (Q_1 \cdot \bar{Q}_2)(Q_3 \cdot \bar{Q}_4) - (Q_1 \cdot Q_3)(\bar{Q}_2 \cdot \bar{Q}_4) + (Q_1 \cdot \bar{Q}_4)(\bar{Q}_2 \cdot Q_3). \end{aligned} \quad (C.21)$$

If we define

$$\bar{R} \equiv (-1)^m \bar{Q}_1 Q_2 \bar{Q}_3 Q_4 \dots \bar{Q}_{2m-1} Q_{2m} \quad , \quad (C.22)$$

that is, we replace Q_i with $\bar{Q}_i$ and $\bar{Q}_j$ with Q_j , then we note from (C.14) and from $(\gamma^5)^2 = 1$ that

$$\text{Tr}[R] = \text{Tr}[\bar{R}] \quad . \quad (\text{C.23})$$

XII. APPENDIX D

A. Pseudoscalar-Pseudoscalar Scattering

In order to calculate all the ladder diagrams of order $2(n+1)$, we need to evaluate the numerator of each individual diagram. To do this, we use the methods developed in Appendix C.

First let us evaluate the expression for the numerator associated with the diagram shown in Figure 21.

In this diagram all the rungs are pseudoscalars. The numerator, neglecting factors of i from the propagators, is given by

$$M_n = \text{Tr}[(k_n + m)\gamma_5(k_{n-1} + m)\gamma_5 \dots (k_1 + m)\gamma_5(k_1 - p_2 + m)\gamma_5 \\ \times (k_1 + m)\gamma_5(k_2 + m)\gamma_5 \dots (k_n + m)\gamma_5(p_1 + k_n + m)\gamma_5] \quad (D.1)$$

Let us define

$$l_1 = k_n, \quad l_2 = k_{n-1}, \quad \dots, \quad l_n = k_1, \\ l_{n+1} = (k_1 + p_2), \quad l_{n+2} = k_1, \quad l_{n+3} = k_2, \quad \dots, \quad l_{2n+1} = k_n, \\ l_{2n+2} = (k_n - p_1) \quad (D.2)$$

We can then write (D.1) as

$$M_n = \text{Tr}[(l_1 + m)\gamma_5(l_2 + m)\gamma_5(l_3 + m)\gamma_5 \dots (l_{2n+2} + m)\gamma_5] \quad (D.3)$$

Now define

$$L_i = l^{(i)} + m \quad (D.4)$$

Figure 21. A ladder diagram for pseudoscalar-pseudoscalar scattering

and, as in Appendix C, let

$$\varrho_i = L_i(i\gamma_5) \quad (D.5)$$

Then

$$M_n = (-1)^{n+1} \text{Tr}[\varrho_1 \varrho_2 \varrho_3 \dots \varrho_{2n+2}] \quad (D.6)$$

Using the notation of Appendix C, this is just

$$M_n = 4(-1)^{n+1} (1 \ 2 \ 3 \ \dots \ 2n+2) \quad (D.7)$$

(See Figure 22)

We note that

$$Q_1 = Q_{2n+1}, \quad Q_2 = Q_{2n}, \quad Q_3 = Q_{2n-1}, \quad \dots, \quad Q_n = Q_{n+2} \quad (D.8)$$

With this in mind, we find it convenient to rewrite (D.7) as

$$\begin{aligned} \frac{1}{4} M_n &= (-1)^{2n+1} (n+1 \ 2n+2 \ 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \\ &= - (n+1 \ 2n+2 \ 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \end{aligned} \quad (D.9)$$

For example, for $n = 3$, we have

$$\frac{1}{4} M_n = -(4 \ 8 \ 1 \ 7 \ 2 \ 6 \ 3 \ 5) \quad (D.10)$$

We have thus grouped equal momenta together. The extra factor of $(-1)^n$ in (D.9) comes from the sign of the permutation to bring the pfaffian to this form.

We define

$$A_n = (-1)^n (1 \ 2n+2-1 \ 2 \ 2n+2-2 \ 3 \ 2n+2-3 \ \dots \ n \ 2n+2-n) \quad (D.11)$$

Figure 22. This is a relabeling of Figure 21 (see text)

and

$$B_n^j = 2(-1)^{n+1} (1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n+1 \ n+j+1 \ \dots \ n \ 2n+2-n). \quad (D.12)$$

That is, B_n^j is defined by replacing the element $n+1-j$ in A_n with $n+1$ and multiplying by an overall factor (-2) . This will be illustrated later.

We will show that (D.7) can then be written as

$$\frac{1}{2}M_n = (-1)^{n+1} [((k_1 - p_2) \cdot (k_n + p_1) - m^2)A_n + \sum_{j=1}^n [(k_n + p_1) \cdot k_j - m^2 B_n^j]]. \quad (D.13)$$

This is similar in form to the result obtained in Reference 15. We will be able to ignore the mass terms, which should be nonleading. The major difference between this form and that obtained in Reference 15 is the replacement of p_1 with $(p_1 + k_n)$ and the replacement of p_2 with $(p_2 - k_1)$.

We will find the following relations to hold

$$A_n = \prod_{j=1}^n \Delta_j \quad (D.14a)$$

$$B_n^j = \prod_{\ell=j+1}^n \Delta_\ell \sum_{i=1}^{\ell} \sum_{1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_i = j} L_0^{\lambda_1} L_{\lambda_1}^{\lambda_2} \dots L_{\lambda_{i-1}}^{\lambda_i} \quad (D.14b)$$

where

$$\Delta_j = -(k_j^2 - m^2) \quad (D.15)$$

$$k_0 = k_1 - p_2 \quad (D.16)$$

$$L_j^i = 2[(k_i \cdot k_j) - m^2] \prod_{\ell=j+1}^{i-1} \Delta_\ell \quad (D.17)$$

and we define

$$\prod_{i=\ell+1}^{\ell} = 1 \quad . \quad (D.18)$$

It follows that

$$\begin{aligned} L_0^n &= 2[(k_1 - p_2) \cdot k_n - m^2] A_{n-1} \quad , \quad n > 1 \quad , \\ L_0^1 &= 2[(k_1 - p_2) \cdot k_1 - m^2] \quad . \end{aligned} \quad (D.19)$$

We see that, apart from irrelevant mass terms, the only change from Reference 15 arises in the L_0^j terms occurring in B_n^j . This change is the replacement of p_2 with $k_1 - p_2$ (when all signs are taken into account, it is the replacement of p_2 with $p_2 - k_1$).

Let us illustrate these statements with some simple examples. Consider first the diagram shown in Figure 23a.

From (D.9)

$$\frac{1}{4} M_1 = -(2 \ 4 \ 1 \ 3) \quad . \quad (D.20)$$

Using the properties of pfaffians (see Reference 42), we can expand M_1 as

$$\frac{1}{4} M_1 = -[(2 \ 4)(1 \ 3) + \overset{0}{(2 \ 4)} \overset{0}{(1 \ 3)}] \quad (D.21)$$

where we define

$$\begin{aligned} \overset{0}{(i \ j)} &= 0 \\ \overset{0}{(i \ j)} &= \overset{0}{(i \ j)} = (k \ j) \quad . \end{aligned} \quad (D.22)$$

(a)

(b)

Figure 23. Diagram (b) is a relabeling of diagram (a) (see text)

The second term of (D.21) can be expanded as

$$\begin{aligned} \overset{0}{2} \overset{0}{4} 1 3 &= -[(1 4)(2 3) + (3 4)(1 2)] \\ &= -2[(1 4)(2 3)] \end{aligned} \quad (\text{D.23})$$

where we have used the fact that $Q_1 = Q_3$.

Thus we can write (D.21) as

$$\frac{1}{4}M_1 = -[(2 4)(1 3) - 2(1 4)(2 3)] \quad (\text{D.24})$$

From (D.11) and (D.12) we have

$$A_1 = -(1 3) \quad , \quad (\text{D.25})$$

$$B_1^1 = 2(2 3) \quad . \quad (\text{D.26})$$

Substituting these into (D.24) and using the values for (2 4) and (1 4), we obtain

$$\frac{1}{4}M_1 = [[(k_1 - p_2) \cdot (k_1 + p_1) - m^2]A_1 + [(k_1 + p_1) \cdot k_1 - m^2]B_1^1] \quad . \quad (\text{D.27})$$

This is in agreement with (D.13).

If we now evaluate (D.25) and (D.26), we obtain

$$\begin{aligned} A_1 &= -(k_1^2 - m^2) \\ B_1^1 &= 2[(k_1 - p_2) \cdot k_1] - m^2 \end{aligned} \quad (\text{D.28})$$

This is in agreement with (D.14).

We now proceed to the two loop case. (See Figure 24).

From (D.9),

Figure 24. Diagram (b) is a relabeling of diagram (a) (see text)

$$\frac{1}{4}M_2 = -(3 \ 6 \ 1 \ 5 \ 2 \ 4) \quad (D.29)$$

This can be expanded as

$$\frac{1}{4}M_2 = -[(3 \ 6)(1 \ 5 \ 2 \ 4) + \overset{0}{3} \overset{0}{6} 1 \ 5 \ 2 \ 4] \quad (D.30)$$

We can expand the second term as

$$\overset{0}{3} \overset{0}{6} 1 \ 5 \ 2 \ 4 = -[(1 \ 6)(3 \ 5 \ 2 \ 4) + (5 \ 6)(1 \ 3 \ 2 \ 4) + (2 \ 6)(1 \ 5 \ 3 \ 4) + (4 \ 6)(1 \ 5 \ 2 \ 3)] \quad (D.31)$$

The first two terms of (D.31) are equal. Since $Q_1 = Q_5$, it is clear that

$$(1 \ 6) = (5 \ 6) \quad (D.32)$$

We can also see that

$$(3 \ 5 \ 2 \ 4) = (1 \ 3 \ 2 \ 4) \quad (D.33)$$

by rewriting the left hand side as

$$(3 \ 5 \ 2 \ 4) = -(5 \ 3 \ 2 \ 4) \quad (D.34)$$

We can compare this to $(1 \ 3 \ 2 \ 4)$ by noting that $Q_1 = Q_5$, that 1 is less than everything to its right while 5 is greater than everything to its right, and by using the fact that

$$\begin{aligned} (i \ j) &= -(j \ i) \\ &= Q_i \cdot Q_j \end{aligned} \quad \text{for } i < j \quad (D.35)$$

Thus the first two terms of (D.31) are equal.

We can also show that the next two terms are equal. Again, it is

clear that

$$(2\ 6) = (4\ 6) \quad . \quad (D.36)$$

We now make use of the fact that

$$\begin{pmatrix} i & j \\ 1 & m \end{pmatrix} = 0 \quad \text{when } Q_i = Q_j, \quad i < 1, m; j > 1, m, \quad (D.37)$$

where we write

$$\begin{pmatrix} i & j \\ 1 & m \end{pmatrix} = (i\ 1)(j\ m) - (i\ m)(j\ 1) \quad . \quad (D.38)$$

This is just the determinant of the matrix $\begin{pmatrix} i & j \\ 1 & m \end{pmatrix}$ written in the notation due to Cayley (see Reference 42).

Thus we are able to expand (see Reference 42)

$$\begin{aligned} (1\ 5\ 3\ 4) &= (1\ 5)(3\ 4) - \begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix} \\ &= (1\ 5)(3\ 4) \quad . \end{aligned} \quad (D.39)$$

(One could also interpret this as a trace, and use the fact

$$Q_1 = Q_5 \Rightarrow Q_1 Q_5 = Q_1^2 \quad .)$$

The second term is zero from (D.37). Similarly,

$$(1\ 5\ 2\ 3) = (1\ 5)(2\ 3) \quad . \quad (D.40)$$

Using (D.35) plus the fact that $Q_2 = Q_4$, we arrive at the equality of the second two terms of (D.31). Thus we are able to write (D.30) as

$$\frac{1}{4}M_2 = -[(3\ 6)(1\ 5\ 2\ 4) - 2(1\ 6)(3\ 5\ 2\ 4) - 2(2\ 6)(1\ 5\ 3\ 4)] \quad . \quad (D.41)$$

Using (D.11) and (D.12) and substituting for (1 6), (2 6), and (3 6), we

obtain the validity of (D.13) for $n = 2$.

Now let us verify that the relations given in (D.14) hold for $n = 2$. Using (D.37) and expanding as in (D.39), we immediately obtain

$$(1 \ 5 \ 2 \ 4) = (1 \ 5)(2 \ 4) \quad (D.42)$$

Substituting for $(1 \ 5)$ and $(2 \ 4)$, we verify that

$$A_2 = (k_2^2 - m^2)(k_1^2 - m^2) \quad (D.43)$$

in agreement with (D.14a). We note for later purposes that (D.42) is of the form

$$A_2 = \Delta_2 A_1 \quad (D.44)$$

where we have taken into account the relabeling

$$\Delta_1 = (2 \ 4) \quad (D.45)$$

In order to develop a recurrence relation for the proof of (D.14), we find it convenient to expand the second term of (D.30) in a different manner.

$$\begin{aligned} \overset{\circ}{3} \overset{\circ}{6} 1 \ 5 \ 2 \ 4 &= (1 \ 5 \ \overset{\circ}{3} \ \overset{\circ}{6} \ 2 \ 4) \\ &= (1 \ 5)(\overset{\circ}{3} \ \overset{\circ}{6} \ 2 \ 4) - \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix} (2 \ 4) - \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix} (3 \ 6) \\ &\quad + \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} (3 \ 4) + \begin{pmatrix} 1 & 5 \\ 4 & 6 \end{pmatrix} (2 \ 3) \\ &= (1 \ 5)(\overset{\circ}{3} \ \overset{\circ}{6} \ 2 \ 4) - \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix} (2 \ 4) + 2 \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} (3 \ 4) \end{aligned} \quad (D.46)$$

In the second line of (D.46), the third term is zero from (D.37) and the

last two terms can be shown to be equal.

Now $\overset{0}{3} \overset{0}{6} 2 5$ is essentially just a relabeling of (D.23), yielding

$$\begin{aligned} \overset{0}{3} \overset{0}{6} 2 4 &= -2(2 6)(3 4) \\ &= -[(p_1 + k_2) \cdot k_1 - m^2] B_1^1 \end{aligned} \quad (D.47)$$

So the first term of (D.46) becomes, upon substitution for (1 5),

$$(1 5) \overset{0}{3} \overset{0}{6} 2 4 = [(p_1 + k_2) \cdot k_1 - m^2] \Delta_2 B_1^1 \quad (D.48)$$

The second term is

$$\begin{aligned} - \begin{pmatrix} 1 & 5 \\ 3 & 6 \end{pmatrix} (2 4) &= -2(1 3)(5 6)(2 4) \\ &= -2[(k_1 - p_2) \cdot k_2 - m^2][(p_1 + k_2) \cdot k_1 - m^2][k_1^2 - m^2] \\ &= 2[(k_1 - p_2) \cdot k_2 - m^2][(p_1 + k_2) \cdot k_1 - m^2] \Delta_1 \\ &= [(p_1 + k_2) \cdot k_1 - m^2] L_0^2 \end{aligned} \quad (D.49)$$

The last term is

$$\begin{aligned} 2 \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} (3 4) &= \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} B_1^1 \\ &= 2(1 2)(5 6) B_1^1 \\ &= 2[(p_1 + k_2) \cdot k_2 - m^2][(k_2 \cdot k_1) - m^2] B_1^1 \end{aligned} \quad (D.50)$$

Collecting (D.48), (D.49), and (D.50), we obtain

$$\begin{aligned}
{}^0_3 {}^0_6 1 5 2 4) &= [[(p_1 + k_2) \cdot k_2 - m^2] \Delta_2 B_1^1 + [(p_1 + k_2) \cdot k_1 - m^2] L_0^2 \\
&+ 2[(p_1 + k_2) \cdot k_2 - m^2][(k_2 \cdot k_1) - m^2] B_1^1] . \quad (D.51)
\end{aligned}$$

This is the same recurrence relation for B_2^j as given in Reference 15, so we conclude that (D.14b) is valid for $n = 2$. One can also verify this directly by expanding (D.41).

Now let us consider the three loop case (see Figure 25).

From (D.9)

$$\pm M_3 = -(4 \ 8 \ 1 \ 7 \ 2 \ 6 \ 3 \ 5) \quad (D.52)$$

This can be expanded as

$$\pm M_3 = -[(4 \ 8)(1 \ 7 \ 2 \ 6 \ 3 \ 5) + ({}^0_4 {}^0_8 1 \ 7 \ 2 \ 6 \ 3 \ 5)] \quad (D.53)$$

We can expand the second term as in the two loop case (see Equation (D.41)), yielding

$$\begin{aligned}
\pm M_3 &= -[(4 \ 8)(1 \ 7 \ 2 \ 6 \ 3 \ 5) - 2(1 \ 8)(4 \ 7 \ 2 \ 6 \ 3 \ 5) \\
&- 2(2 \ 8)(1 \ 7 \ 4 \ 6 \ 3 \ 5) - 2(3 \ 8)(1 \ 7 \ 2 \ 6 \ 4 \ 5)] \quad (D.54)
\end{aligned}$$

Substituting for (1 8), (2 8), (3 8), and (4 8), and using definitions (D.11) and (D.12), we can write

$$\begin{aligned}
\pm M_3 &= [[(k_1 - p_2) \cdot (k_3 + p_1) - m^2] A_3 + [(k_3 + p_1) \cdot k_3 - m^2] B_3^3 \\
&+ [(k_3 + p_1) \cdot k_2 - m^2] B_3^2 + [(k_3 + p_1) \cdot k_1 - m^2] B_3^1] \quad (D.55)
\end{aligned}$$

This is in agreement with the general form (D.13).

Figure 25. Diagram (b) is a relabeling of diagram (a) (see text).

To compare with (D.14a), A_3 can be expanded as

$$\begin{aligned}
 A_3 &= -(1 \ 7 \ 2 \ 6 \ 3 \ 5) \\
 &= -[(1 \ 7)(2 \ 6 \ 3 \ 5) - \begin{pmatrix} 1 & 7 \\ 2 & 6 \end{pmatrix} (3 \ 5) - \begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix} (2 \ 6) - \begin{pmatrix} 1 & 7 \\ 6 & 3 \end{pmatrix} (2 \ 5) \\
 &\quad - \dots] \\
 &= -(1 \ 7)(2 \ 6 \ 3 \ 5) .
 \end{aligned} \tag{D.56}$$

Only the first term in the expansion of A_3 is nonzero due to (D.37). We recognize (D.56) as

$$A_3 = \Delta_3 A_2 . \tag{D.57}$$

This verifies (D.14a) for $n = 3$.

For comparison with (D.14b) we expand the second term of (D.53) in a different manner. As in the two loop case, we can write

$$\begin{aligned}
 \overset{\circ}{4} \ \overset{\circ}{8} \ 1 \ 7 \ 2 \ 6 \ 3 \ 5 &= (1 \ 7) \overset{\circ}{4} \ \overset{\circ}{8} \ 2 \ 6 \ 3 \ 5 \\
 &= (1 \ 7) \overset{\circ}{4} \ \overset{\circ}{8} \ 2 \ 6 \ 3 \ 5 - \begin{pmatrix} 1 & 7 \\ 4 & 8 \end{pmatrix} (2 \ 6 \ 3 \ 5) \\
 &\quad + 2 \begin{pmatrix} 1 & 7 \\ 2 & 8 \end{pmatrix} (4 \ 6 \ 3 \ 5) + 2 \begin{pmatrix} 1 & 7 \\ 3 & 8 \end{pmatrix} (2 \ 6 \ 4 \ 5) .
 \end{aligned} \tag{D.58}$$

When we take into account relabeling, we can compare these terms with the two loop expansion and find

$$(1 \ 7) \overset{\circ}{4} \ \overset{\circ}{8} \ 2 \ 6 \ 3 \ 5 = -\Delta_3 \sum_{j=1}^2 [(p_1 + k_3) \cdot k_j - m^2] B_2^j , \tag{D.59}$$

$$\begin{aligned}
- \begin{pmatrix} 1 & 7 \\ 4 & 8 \end{pmatrix} (2 \ 6 \ 3 \ 5) &= -2(1 \ 4)(7 \ 8)(2 \ 6)(3 \ 5) \\
&= -2[(k_1 - p_2) \cdot k_3 - m^2][(p_1 + k_3) \cdot k_3 - m^2]A_2 \quad (D.60) \\
&= -[(p_1 + k_3) \cdot k_3 - m^2]L_0^3,
\end{aligned}$$

and

$$\begin{aligned}
2 \begin{pmatrix} 1 & 7 \\ 2 & 8 \end{pmatrix} (4 \ 6 \ 3 \ 5) + 2 \begin{pmatrix} 1 & 7 \\ 3 & 8 \end{pmatrix} (2 \ 6 \ 4 \ 5) &= 4(1 \ 2)(7 \ 8)(4 \ 6 \ 3 \ 5) \\
&\quad + 4(1 \ 3)(7 \ 8)(2 \ 6 \ 4 \ 5) \quad (D.61) \\
&= -[(p_1 + k_3) \cdot k_3 - m^2] \\
&\quad \times [2[(k_3 \cdot k_2) - m^2]B_2^2 + 2[(k_3 \cdot k_1) - m^2]B_2^1].
\end{aligned}$$

Collecting (D.59), (D.60), and (D.61), we obtain

$$\begin{aligned}
-\begin{pmatrix} 0 & 0 \\ 4 & 8 \end{pmatrix} (1 \ 7 \ 2 \ 6 \ 3 \ 5) &= \Delta_3 \sum_{j=1}^2 [(p_1 + k_3) \cdot k_j - m^2]B_2^j + [(p_1 + k_3) \cdot k_3 - m^2]L_0^3 \\
&\quad + [(p_1 + k_3) \cdot k_3 - m^2]\{2[(k_3 \cdot k_2) - m^2]B_2^2 \\
&\quad + 2[(k_3 \cdot k_1) - m^2]B_2^1\} \quad (D.62)
\end{aligned}$$

This is just the recurrence given in Reference 15. Substituting for

L_0^3 , B_2^1 , and B_2^2 verifies (D.14b) for $n = 3$.

With the previous examples in mind, we find that there are two useful expansions for (D.9). The first expansion verifies (D.13) while the second verifies (D.14). The validity of the expansions can be proven by induction. The general form is given by

$$\begin{aligned}
\frac{1}{2}M_n &= -(n+1 \ 2n+2 \ 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \\
&= -[(n+1 \ 2n+2)(1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \\
&\quad + \overset{0}{(n+1 \ 2n+2 \ 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n)}] \quad (D.63) \\
&= (-1)^{n+1} [(n+1 \ 2n+2)(-1)^n(1 \ 2n+2-1 \ \dots \ n \ 2n+2-2) \\
&\quad + (-1)^n \overset{0}{(n+1 \ 2n+2 \ 1 \ 2n+2-1 \ \dots \ n \ 2n+2-n)}] .
\end{aligned}$$

Comparing with (D.11), we see that the first term is just

$$(n+1 \ 2n+2)(-1)^n(1 \ 2n+2-1 \ \dots \ n \ 2n+2-2) = [(k_1 - p_2) \cdot (k_n + p_1) - m^2]A_n . \quad (D.64)$$

Using (D.37) one can prove by induction that the following recurrence relation holds for A_n :

$$A_n = \Delta_n A_{n-1} . \quad (D.65)$$

Using the methods illustrated in the previous examples, one can prove by induction that the second term of (D.63) can be expanded in the following form.

$$\begin{aligned}
(-1)^n \overset{0}{(n+1 \ 2n+2 \ 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n)} &= \\
&(-1)^{n+1} 2[(1 \ 2n+2)(n+1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \\
&+ (2 \ 2n+2)(1 \ 2n+2-1 \ n+1 \ 2n+2-2 \ \dots \ n \ 2n+2-n) + \dots \\
&+ (n \ 2n+2)(1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n+1 \ 2n+2-n)] . \quad (D.66)
\end{aligned}$$

By comparison with (D.12), this is just

$$\sum_{j=1}^n [(k_n + p_1) \cdot k_j - m^2] B_n^j \quad . \quad (D.67)$$

Combining (D.65) with (D.67) yields (D.13).

Expanding (D.66) in a different manner yields

$$\begin{aligned} & (-1)^n \binom{0}{n+1} \binom{0}{2n+2} 1 \ 2n+2-1 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n \\ &= (-1)^n (1 \ 2n+2-1 \ n+1 \ 2n+2 \ 2 \ 2n+2-2 \ \dots \ n \ 2n+2-n) \\ &= [-(-1)^n \binom{1}{n+1} \binom{2n-1}{2n+2} (2 \ 2n+2-2 \ 3 \ 2n+2-3 \ \dots \ n \ 2n+2-n) \\ &\quad + 2(-1)^n \binom{1}{2} \binom{2n-1}{2n+2} (n+1 \ 2n+2-2 \ 3 \ 2n+2-3 \ \dots \ n \ 2n+2-n) \\ &\quad + 2(-1)^n \binom{1}{3} \binom{2n-1}{2n+2} (2 \ 2n+2-2 \ n+1 \ 2n+2-3 \ \dots \ n \ 2n+2-n) + \dots \\ &\quad + 2(-1)^n \binom{1}{n} \binom{2n-1}{2n+2} (2 \ 2n+2-2 \ 3 \ 2n+2-3 \ \dots \ n+1 \ 2n+2-n)] \quad . \end{aligned} \quad (D.68)$$

Identifying terms in (D.68), we can write the following recurrence relation

$$B_n^j = \Delta_n B_{n-1}^j \quad \text{for } j < n, \quad (D.69)$$

$$B_n^n = L_0^n + \sum_{j=1}^{n-1} 2(k_n \cdot k_j) B_{n-1}^j \quad .$$

These relations, along with (D.65), are the same as those obtained in Reference 15, and thus lead to the same solution. Our functions L_j^i are defined slightly differently, but upon neglect of irrelevant mass terms, the definitions become equivalent.

In order to complete the analysis of pseudoscalar-pseudoscalar

scattering let us examine what change occurs upon the replacement of one of the pseudoscalar rungs, say, the j th rung, with a scalar rung. This is a change in the coupling from $(-g\gamma_5)$ to $(-ig)$. In (D.1) this corresponds to the replacement of the appropriate γ_5 's with i 's. From (C.14) this amounts to changing (D.7) to

$$M_n = 4(-1)^n (1 \ 2 \ 3 \ \dots \ \bar{j} \ \overline{j+1} \ \dots \ \overline{2n+2-j} \ 2n+2-j+1 \ \dots \ 2n+2) \quad . \quad (D.70)$$

That is, we replace all elements between j and $2n+2-j$, inclusive, with barred elements. The extra factor of (-1) comes from the i^2 . From (D.12) this is equivalent to replacing m with $-m$ in all the barred elements and multiplying by an overall (-1) (we obtain a factor (-1) for each rung we pass going up the ladder from j , a (-1) for each rung going down the ladder to $2n+2-j$, and a (-1) going across the top. This results in an overall minus sign.) Differences due to mass terms can be neglected, since they should be nonleading. This brings (D.70) back to the form (D.7), up to mass terms.

This argument can be repeated for the replacement of an arbitrary number of pseudoscalar rungs. Thus, to leading order in $\log s$, all combinations contribute equally. There are $(n-1)$ rungs which can be replaced with scalar lines. This leads to 2^{n-1} combinations. Adding all diagrams thus leads to a factor of 2^{n-1} . We thus obtain the following result for the sum over all combinations, correct to leading order in $\log s$.

$$\begin{aligned}
N_n &= (2^{n-1}) M_n \\
&= (-2)^{n+1} [(k_1 - p_2) \cdot (k_n + p_1) A_n + \sum_{j=1}^n (p_1 + k_n) \cdot k_j B_n^j] \quad .
\end{aligned}
\tag{D.71}$$

This is in agreement with Reference 15.

The numerator for the u diagrams evaluated in subsection D of section V can be obtained from Equation (D.71) by replacing $-p_2$ with p_2 .

B. Pseudoscalar-Fermion Scattering

We now consider the numerator for pseudoscalar-fermion scattering. Let us examine the $2(n+1)$ order diagram shown in Figure 26, taking the forward spin average.

Here all rungs are pseudoscalars. The numerator is given by

$$\begin{aligned}
M_n &= \frac{1}{2} \text{Tr}[(k_n + m) \gamma_5 (k_{n-1} + m) \gamma_5 \dots (k_1 + m) \gamma_5 (k_1 - p_2 + m) \gamma_5 \\
&\quad \times (k_1 + m) \gamma_5 \dots (k_n + m) \gamma_5 (p_1 + m)] \quad .
\end{aligned}
\tag{D.72}$$

We see that this is the same form as (D.1) except for the factor of $\frac{1}{2}$ due to the spin average and for the replacement of $(p_1 + k_n + m)$ with $(p_1 + m)$.

We are able to carry out the analysis of the preceding section with only minor changes. In this case we have n rungs which can be either pseudoscalars or scalars. They all contribute with the same sign. The extra rung gives another factor of 2, which cancels the $\frac{1}{2}$ due to the spin average. We thus obtain the same result as in the preceding section, except for the change of everywhere replacing $(p_1 + k_n)$ with p_1 .

Figure 26. A ladder diagram for pseudoscalar-fermion scattering

C. Other Amplitudes

The numerators for all the other amplitudes mentioned in the Summary of Section V are all given by Equation (D.71). These amplitudes differ from the ones discussed in Sections I and II above by the replacement of two external pseudoscalar lines with scalar lines. This causes a change only in the mass terms, which are nonleading. For instance, in order to obtain scalar-pseudoscalar scattering, one simply changes the external lines of Figure 20 labeled with momentum p_2 to scalar lines. One also changes the corresponding couplings from $-g\gamma_5$ to ig . But, as argued above in Section I, this causes no change to the leading terms. Thus the pseudoscalar-pseudoscalar, pseudoscalar-scalar, scalar-scalar, fermion-fermion, fermion-pseudoscalar, and fermion-scalar amplitudes for the s diagrams (respectively, the u diagrams) in any given order have the same numerator in the leading logarithm approximation (and when external fermion lines are forward spin averaged). This numerator is given by Equation (D.71). Therefore, one can speak of boson-boson, boson-fermion, and fermion-fermion scattering in the high energy limit without ambiguity.

XIII. APPENDIX E

In this appendix we present some of the details for the proof of the statement that Equations (5.56) and (5.61) do not contain any leading terms. This amounts to the proof of statements (5.62) and (5.63). We base our approach on the methods of Reference 15.

We will need the following definitions:

$$\begin{aligned}
 (j)_\mu &\equiv d_{0,j-1} c_n^j p_{2\mu} + d_{j,n} c_{j-1}^0 p_{1\mu} , \\
 (jj) &\equiv \frac{1}{\beta_j} [\alpha_{j-1} c_n^j (c_{j-1}^0 - \alpha_{j-1} c_{j-2}^0) + \alpha_j c_{j-1}^0 (c_n^j - \alpha_j c_n^{j+1})] , \\
 (j\ell) &\equiv -d_{j,\ell-1} c_{j-1}^0 c_n^\ell , \quad j < \ell ,
 \end{aligned} \tag{E.1}$$

where $d_{j,\ell}$ and c_ℓ^j are given in (5.58).

We note the following formulas:

$$\begin{aligned}
 \frac{1}{\beta_\ell} \frac{i}{c} \frac{\partial D(a)}{\partial a_\ell^\mu} \bigg|_{a_j=0} &= \frac{2i}{c} (j\ell)_\mu , \\
 \frac{1}{\beta_\ell \beta_j} \frac{i}{c} \frac{\partial^2 D(a)}{\partial a_\ell^\mu \partial a_j^\nu} \bigg|_{a_i=0} &= \frac{2i}{c} (j\ell) g_{\mu\nu} .
 \end{aligned} \tag{E.2}$$

Keeping terms with lowest power in α_j 's and highest power in $(p_1 \cdot p_2)$, we have for $j_1 < j_2$

$$\begin{aligned}
 (j_1 \cdot j_2) &\equiv (j_1)_\mu (j_2)^\mu \approx d_{j_1,j_2-1}^{-1} c_n^{j_1} c_{j_2-1}^0 [d_{0,n}(p_1 \cdot p_2)] , \\
 (j_1 \cdot j_1) &\equiv (j_1)_\mu (j_1)^\mu \approx c_{j_1-1}^0 c_n^{j_1} [2d_{0,n}(p_1 \cdot p_2)] ,
 \end{aligned}$$

$$\begin{aligned}
(p_2 \cdot j) &\equiv (p_2)^\mu(j)_\mu \approx d_{j,n} c_{j-1}^0 (p_1 \cdot p_2) \quad , \\
(p_1 \cdot j) &\equiv (p_1)^\mu(j)_\mu \approx d_{0,j-1} c_n^j (p_1 \cdot p_2) \quad , \\
(p_2 \cdot j_1)(p_1 \cdot j_2) &\approx d_{j_1,j_2-1} c_{j_1-1}^0 c_n^{j_2} [d_{0,n} (p_1 \cdot p_2)^2] \quad , \\
(p_2 \cdot j_2)(p_1 \cdot j_1) &\approx d_{j_1,j_2-1}^{-1} c_{j_2-1}^0 c_n^{j_1} [d_{0,n} (p_1 \cdot p_2)^2] \quad , \\
(p_2 \cdot j_1)(p_1 \cdot j_1) &\approx c_{j_1-1}^0 c_n^{j_1} [d_{0,n} (p_1 \cdot p_2)^2] \quad .
\end{aligned} \tag{E.3}$$

In order to illustrate the general argument, let us examine a simple case of (5.55) and of (5.61).

For an example of (5.55), consider the term $A_n(p_2 \cdot k_n)$ in Equation (5.48). In this case (5.55) becomes

$$\begin{aligned}
\tilde{j}_{n,2}^{(1)} &= g_{\mu_0 \nu_0} p_2^{\mu_0} \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n (2i\beta_n)^{-1} d_n^{\nu_0} \frac{e^{iD(a)/C}}{c^2} \Big|_{a=0} \\
&= g_{\mu_0 \nu_0} p_2^{\mu_0} \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \frac{(n)^{\nu_0}}{c} \frac{e^{iD/C}}{c^2} \\
&= \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \frac{(p_2 \cdot n)}{c} \frac{e^{iD/C}}{c^2} \\
&\sim \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \frac{\alpha_n c_{n-1}^0}{c} \left(\frac{s}{2}\right) \frac{e^{iD/C}}{c^2} \quad .
\end{aligned} \tag{E.4}$$

We have used expressions (E.1), (E.2), and (E.3) in deriving (E.4).

If we let $f(\alpha, \beta) = \alpha_n c_{n-1}^0 / C$, we obtain the results stated in (5.62)

and (5.63). The expression in (E.4) obviously gives less leading behavior than the term

$$\int \frac{A_n(p_1 \cdot p_2)}{D_n} \sim s \int \frac{e^{iD/C}}{c^2} \quad (E.5)$$

We illustrate the technique of handling (5.61) with the examination of the term $A_n(k_1 \cdot k_n)$. In this case (5.61) becomes

$$\begin{aligned} J_{n,l}^{(2)} &\sim g_{\mu_0 \nu_0} \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n (2i\beta_1)^{-1} (2i\beta_n)^{-1} \vartheta_1^{\mu_0} \vartheta_n^{\nu_0} \frac{e^{iD(a)/C}}{c^\omega} \Big|_{a=0} \\ &\sim g_{\mu_0 \nu_0} \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \left[\left(\frac{2i}{c} \right) (1n) g^{\mu_0 \nu_0} - (1)^{\mu_0} (n)^{\nu_0} \right] \frac{e^{iD/C}}{c^\omega} \\ &\sim \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \left[\frac{2i}{c} (1n)(4) - \frac{(1 \cdot n)}{c^2} \right] \frac{e^{iD/C}}{c^\omega} \quad (E.6) \end{aligned}$$

The first term of (E.6) is divergent in the limit $\omega \rightarrow 2$ and requires a subtraction. From (E.1) we obtain

$$(1n) = -d_{1,n-1} \quad (E.7)$$

which is of degree $n - 1$.

Let us call the first term of (E.6) I . Scaling the variables as in (2.37) and (2.38), we arrive at the result

$$I \sim \int_0^1 d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \delta(1 - \Sigma\alpha - \Sigma\beta) \int_0^\infty d\rho \rho^{n(2-\omega)-1} e^{i\rho D/C} \quad (E.8)$$

We have used the fact that D , C , $c_{\lambda-1}^0$, c_n^λ , and d_{λ_1, λ_2} are respectively

homogeneous functions of degree $n + 1$, n , $\lambda - 1$, $n - \lambda$, and $\lambda_2 - \lambda_1 + 1$.

Using Equation (2.39), the ρ integral can be performed, giving the result

$$I \sim \int_0^1 d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \delta(1 - \Sigma\alpha - \Sigma\beta) \frac{d_{1,n-1}}{c^{\omega+1}} \left(\frac{C}{D}\right)^{n(2-\omega)} \times \Gamma(n[2 - \omega]) \quad (E.9)$$

The pole at $\omega = 2$ is explicitly exhibited in the gamma function. Performing the subtraction yields

$$I \sim \int d\alpha_0 \dots d\alpha_n d\beta_1 \dots d\beta_n \delta(1 - \Sigma\alpha - \Sigma\beta) \frac{d_{1,n-1}}{c^3} [\ln D/C + A_0] \quad (E.10)$$

where A_0 is an arbitrary constant to be fixed by the renormalization.

The term $\ln D/C$ is not promoted by the Feynman integrations. Therefore, this term behaves at most like $\ln s$, and is nonleading.

Let us call the integral of the second term of (E.6) I_2 . Performing the scaling as before, we obtain

$$I_2 \sim \int_0^1 \frac{d_{1,n-1}^{-1} c_n^1 c_{n-1}^U}{c^2} \frac{[d_{0,n}(p_1 \cdot p_2)]}{c^2} \left(\frac{C}{D}\right) \quad (E.11)$$

We have taken the limit $\omega \rightarrow 2$ since the integral is convergent. After factoring the term $(p_1 \cdot p_2)$, the remaining polynomial vanishes for $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Therefore, this term is nonleading.

We now proceed to the general case of Equation (5.56). In Appendix C of Reference 15, it was shown that the leading terms of Equation (5.18) of that paper could be written as

$$g(\alpha, \beta) \frac{(p_1 \cdot p_2)}{c^2} e^{iD/c} [d_{0,n}(p_1 \cdot p_2)]^j, \quad j = 0, 1, \dots, \ell. \quad (E.12)$$

In the case we consider here, the proof of (5.62) is entirely analogous. There are only slight modifications. We treat each modification separately.

i) The replacement of the factor $(p_1 \cdot p_2)$ with $(p_2 \cdot \lambda_n)/c$. In this case we replace (E.13) with

$$\frac{(p_2 \cdot \lambda_n)}{c} g(\alpha, \beta) e^{iD/c} [d_{0,n}(p_1 \cdot p_2)]^j.$$

But

$$\frac{(p_2 \cdot \lambda_n)}{c} \sim \frac{1}{c} d_{\lambda_n, n} c_{n-1}^0 (p_1 \cdot p_2).$$

Thus if we define

$$f(\alpha, \beta) = \frac{g(\alpha, \beta)}{c} d_{\lambda_n, n} c_{n-1}^0, \quad (E.13)$$

we obtain Equations (5.62) and (5.63).

ii) The replacement of a factor $(p_1 \cdot \lambda_k)$ with $(\lambda_k \cdot \lambda_n)$ with $(\lambda_k < \lambda_n)$. But

$$(p_1 \cdot \lambda_k) \sim d_{0, \lambda_k - 1} c_n^{\lambda_k} (p_1 \cdot p_2),$$

$$(\lambda_k \cdot \lambda_n) \sim d_{0, \lambda_k - 1} d_{\lambda_n, n} c_n^{\lambda_k} c_{\lambda_n - 1}^0 (p_1 \cdot p_2).$$

This is equivalent to the change

$$f(\alpha, \beta) = \frac{d_{\lambda_k, n} c_{n-1}^0}{c} g(\alpha, \beta) \quad (E.14)$$

This verifies Equations (5.62) and (5.63) for this case.

iii) Lastly, it is possible to obtain a factor $(\lambda_k \lambda_n)$. This arises from a term

$$\frac{1}{\beta_{\lambda_k} \beta_{\lambda_n}} \frac{i}{c} \frac{\partial^2 D(a)}{\partial a_{\lambda_k}^\mu \partial a_{\lambda_n}^\nu} \Big|_{a=0} = \frac{2i}{c} (\lambda_k \lambda_n) g_{\mu\nu}$$

This can be interpreted as the change

$$\lambda_k^\mu p_1^\nu \rightarrow \frac{(\lambda_k \lambda_n)}{c} g^{\mu\nu} \quad (E.15)$$

in

$$\frac{g(\alpha, \beta) (p_1 \cdot p_2)}{c^2} e^{iD/c} [d_{0,n} (p_1 \cdot p_2)]^j = p_2^\sigma p_1^\nu \lambda_k^\mu h(\alpha, \beta, p_1, p_2)_{\mu\nu\sigma} \quad (E.16)$$

From (E.1) the change (E.15) is equivalent to

$$\begin{aligned} & [d_{0, \lambda_k-1} c_n^{\lambda_k} p_2^\mu + d_{\lambda_k, n} c_{\lambda_k-1}^0 p_1^\mu] p_1^\nu \\ & \rightarrow - \left[\frac{d_{\lambda_k, \lambda_n-1} c_{\lambda_k-1}^0 c_n^{\lambda_n}}{c} \right] g^{\mu\nu} \quad (E.17) \end{aligned}$$

One can interpret this as a modification in (E.16) of

$$\begin{aligned}
& \frac{g(\alpha, \beta)}{c^2} (p_1 \cdot p_2) e^{iD/C} [d_{0,n}(p_1 \cdot p_2)]^j \rightarrow \left[\frac{d_{\lambda, \lambda_n-1} c_{\lambda_k-1}^0 c_n^{\lambda_n}}{c} \right] \\
& \times \frac{g'(\alpha, \beta)}{c^2} (p_1 \cdot p_2) e^{iD/C} [d_{0,n}(p_1 \cdot p_2)]^j, \quad j > 0, \\
& \rightarrow \left[\frac{d_{\lambda_k, \lambda_n-1} c_{\lambda_k-1}^0 c_n^{\lambda_n}}{c} \right] g(\alpha, \beta) e^{iD/C}, \quad j = 0. \quad (E.18)
\end{aligned}$$

The function $g'(\alpha, \beta)$ is different from that given in Equation (E.16). But the vanishing of $f(0, \beta)$ has been explicitly shown. This verifies Equations (5.62) and (5.63).

The general case of Equations (5.61) can be handled similarly. The divergent terms are treated exactly as in the special case $A_n(k_1 \cdot k_n)$ considered earlier. The subtracted integral gives rise to a term $\ln(D/C)$ which is not promoted by the Feynman parameter integrations. The convergent terms give rise to only one case not already treated. This is the replacement of $(p_1 \cdot p_2)$ in Equation (E.12) with

$$\frac{(1 \cdot n)}{c^2} \sim \frac{c_n^1 c_{n-1}^0}{c^2} [\alpha_0 \alpha_n (p_1 \cdot p_2)] \quad (E.19)$$

This is obviously less leading than the term $(p_1 \cdot p_2)$ since the coefficient vanishes for $\alpha_0 = \alpha_n = 0$.