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Physics, Computer Science & Mathematics Division

LIMITED-ANGLE IMAGING IN POSITRON CAMERAS:
THEORY AND PRACTICE

Kwok Cheong Tam
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LIMITED-ANGLE IMAGING IN POSITRON CAMERA--THEORY AND PRACTICE

Kwok-Cheong Tam

Computer Science and Applied Mathematics Department
Lawrence Berkeley Laboratory, University of California
Berkeley, California 94720

ABSTRACT

The principles of operation of planar positron camera systems made up of multiwire proportional chambers as detectors and electromagnetic delay lines for coordinate readout are discussed. Gamma converters are coupled to the wire chambers to increase detection efficiency and improve spatial resolution. The conversion efficiencies of these converters are calculated and the results compare favorably to the experimentally measured values.

Two reconstruction algorithms, Fourier deconvolution and matrix inversion, for obtaining the three-dimensional distribution of the radioisotopes from the recorded data are described. Construction of generalized tomograms capable of emphasizing the large-angle events is introduced. The effects of the angular range of data taking in reconstructions using the two algorithms are investigated. It is found that in the absence of any a priori information there are undetermined components in the reconstruction if the point response function of the positron camera does not satisfy certain criteria. However, most of the undetermined components are recovered in the case in which the transverse spacing of the object is discrete, and all of them are

recovered if the fact that the object extent is finite is utilized. An iterative scheme for recovering these undetermined components is developed, which is applicable to other transmission and emission imaging devices.

Experimental and digital methods for suppressing the noise in the data due to Compton scattering are detailed. The propagation of errors in the reconstruction algorithms is formulated, and methods to stabilize their performance in the presence of noise are developed.

Results of reconstructing a real phantom as well as computer-generated phantoms are presented.

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1. INTRODUCTION

The past decade has seen a rapid development in the field of nuclear medicine, the use of radioisotope tracers for diagnosing medical disorders. Such rapid progress is attributed to the availability of many new radiopharmaceuticals on one hand, and the improvement of radioisotope imaging devices and image reconstruction methods on the other. The general procedure is to administer a gamma-emitting radioisotope with appropriate half-life to a patient. The gamma radiation which escapes from the patient's body is detected by a gamma detector, and the data so accumulated are processed to give an estimate of the distribution or localization of the radioisotope, which is a reflection of the condition of the organ under diagnosis.

An imaging system in nuclear medicine is made up of three basic components: (1) the gamma-ray channel, which selects and directs the gamma radiation from the object to the detector; (2) the detector, which is sensitive to the gamma radiation; (3) the data processor, which processes the data acquired by the detector to produce an image of the radioisotope distribution. Depending on whether the detector is in motion or stationary while taking data, and the method of channeling the gamma rays, nuclear medicine imaging device can be classified into two categories: scanners, and gamma cameras.

Single-head scanners [1] were the earliest devices employed in radioisotope imaging: a detector equipped with a collimator scans rectilinearly over the area of interest (Fig. 1.1). At any instant of

Fig. 1.1. Schematic configuration of a scanner.

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Fig. 1.2. Schematic configuration of a Gamma camera.

data acquisition the collimator, either in the form of a single cylindrical bore, or a cluster of tapered apertures focused at a fixed point, allows the gamma rays from a single resolution element to fall on the detector while blocking out those from the rest of the organ. Recording the detector signal as a function of the scanning position permits a two-dimensional distribution of the gamma-emitting radioisotope to be mapped out. In recent years multi-head rectilinear scanners [2] have been developed and increasingly used for organ scanning, especially for the larger ones such as lung and liverspleen, and whole body scanning. Their design is fundamentally the same as that of single-head scanners, except that instead of a single scan, two or more scans are obtained from different directions.

In contrast to the scanners, gamma cameras are, in general, stationary devices, though recently some moving cameras are being developed [3]. Through image-producing collimators such as pinhole aperture or multi-channel collimator, their gamma detectors continuously receive radiation emitted from all parts of the object being diagnosed (Fig. 1.2). By recording the interaction sites of the incident gamma rays with the detector, an image of the radioisotope distribution can be obtained. There are a number of position-sensitive instruments which can be used as detectors in gamma cameras: the commercially available Anger camera [4] employs NaI(Tl) crystal viewed by a number of photomultiplier tubes (PMT) around its circumference; spatial resolution is achieved through pulse height division method. Other detectors such as image intensifier [5], spark chamber [6], multiwire proportional chamber [7],

and semiconductor detector [8] have also been used in experimental gamma-ray cameras.

The main advantage of these relatively new imaging devices, the gamma cameras, over the radioisotope scanners is their much faster rate of data acquisition. The reasons are two-fold. Firstly, the gamma-ray cameras collect data in parallel, the radiation from all parts of the object is detected at the same time, resulting in much higher sensitivity than is possible with the scanners which gather data sequentially. Secondly, data acquisition time is not limited by any mechanical motion, which is a determining factor in the case of scanners. The shorter exposure time of the cameras permits the recording of dynamic processes, such as blood flow and heart beat, as well as static distribution, whereas the scanners are restricted to the latter only. However, gamma-ray cameras are limited in their field of view, whereas the scanners are not so restricted. The problems imposed by the limited field of view will be discussed in detail in the next chapter.

One major source of errors in the data taken by the scanners and the cameras is the attenuation of the gamma rays by the quantity of tissue lying between the gamma-ray source and the detector. The amount of attenuation depends on the position of the gamma-ray source, and it is this position dependence which greatly complicates the image reconstruction problem. Furthermore, the septa of the collimators which provide for resolution in gamma scanners and cameras absorb a large fraction of the photons, limiting the imaging efficiency.

The use of positron-emitting radionuclides allows compensation for gamma-ray absorption and eliminates the need of collimators. Some proton-rich elements decay by emitting a positron and a neutrino to achieve stability. The positron produced will annihilate with an electron in the neighborhood, producing two back-to-back gamma rays. The direction and location of these annihilation gamma-ray pairs can be found by detecting them in coincidence without the need of a collimator; therefore positron imaging is capable of much higher sensitivity than that of gamma camera, resulting in lower dose to the patient. Moreover, the resolution is improved, as it is no longer limited by the collimator. Operation in coincidence also results in lower background count rate, and thus improves counting statistics. As for the absorption of the gamma rays, the sum of the distance traversed by the two annihilation gamma rays is roughly constant for a given patient thickness, so the attenuation is more or less independent of the location of the source.

More quantitatively, we assume that the two photons must travel distances x_1 and x_2 , respectively, in the object while the distance $d = x_1 + x_2$ is a slowly varying function of the annihilation point. Suppose that the effective Compton scattering attenuation coefficient for the object is given by a fixed number μ . Then the probability that both of the photons will not undergo Compton scattering is

$$P_1 P_2 = \exp(-\mu x_1) \exp(-\mu x_2) = \exp(-\mu d)$$

approximately independent of the position of the annihilation point. The probability that either photon undergoes Compton scattering is therefore proportional to $1 - \exp(-\mu d)$, also approximately independent of position.

One more advantage of positron imaging is that about one half of all the radionuclides are positron emitters, such as C^{11} , N^{13} , O^{15} , F^{18} , Ga^{68} , Cu^{64} , and Ru^{82} , etc. These biologically important radionuclides have short half-lives and high specific activity, thus it should theoretically be possible to find a positron emitter for almost any imaging problem of interest.

A positron imaging scanner utilizing the time-of-flight principle was introduced by Burnham et al., in 1967 [9]. Positron cameras using arrays of NaI crystals were constructed by Anger in 1967 [10]. There are also positron cameras using two large NaI crystals [11], using multiwire proportional chambers with liquid Xe [12]. The purpose of this thesis is to investigate the problems involved in imaging in positron cameras using multiwire proportional chambers equipped with gamma ray converters [13].

II. THEORY OF POSITRON IMAGING

In typical positron camera imaging, a β^+ active radioisotope is injected into the patient's blood stream in a carrier which is chosen to be preferentially absorbed by the medical anomaly. The radioisotope assumes a quasi-equilibrium distribution in the tissue. In undergoing β^+ decay, the nuclei of the radioisotope emit low energy positrons

about 98% of which are rapidly thermalized [14] within a short distance from their point of origin (a 1 MeV e^+ has an linear range of 5mm in tissue). The thermalized positron then combines with a valence electron of some atom in the immediate vicinity to form a positronium, usually in the singlet state; only about 1/3 of 1% of these positroniums are formed in the triplet state, which emit three photons in the subsequent annihilations [15]. Those positroniums in the singlet state annihilate with the production of 2 gammas. By the conservation of energy and momentum, these singlet photons have an energy of approximately 511 keV, and travel very nearly in opposite directions, i.e., back to back. As the momentum of the center of mass of the annihilating pair is of the order $\sim m_0 c / 137$, the 2 gammas are collinear to $\sim 1/137$ radian [16]. These annihilation gammas may subsequently escape from the patient's body and be detected by the positron camera in coincidence, with their positions of interaction with the camera recorded.

The data in a positron camera are therefore in the form of straight lines defined by the annihilation gamma pairs, with the positions

of the radioactive nuclei which have undergone β^+ decay lying somewhere along the lines. The positions of the decays can in principle be determined by the time-of-flight method, but this requires extremely high speed electronics. An alternate way is to inverse the distribution of the annihilation gamma pairs to yield the radioisotope distribution.

Thus the problem of imaging in a positron camera consists of two parts: (1) to detect the annihilation photons efficiently and record their positions accurately, and (2) to reconstruct the distribution of the radioisotope mathematically from the recorded data.

2.1. Detection of Annihilation Gammas

The requirements on the detector in positron camera are high efficiency for stopping the 511 keV photons, and good spatial resolution to define their positions of interaction. Multiwire proportional chamber (MWPC) equipped with gamma converter is a suitable combination for such purposes, besides offering large detection area at relatively low cost.

2.1.1. Multiwire Proportional Chamber

The advent of MWPC marked a great step forward in the field of particle studies. Since its development by Charpak and colleagues in 1968 [17] much work has been done in investigating the properties of the MWPC and in developing various auxiliary devices which greatly extend its scope of application. The use of MWPC in nuclear medicine imaging was first developed at Lawrence Berkeley Laboratory by Perez-Mendez and colleagues [18].

MWPC can be considered as an improved version of proportional chamber. In a MWPC, a plane grid of equally spaced wires lying between and parallel to a pair of plane cathode electrodes takes the place of the anode wire in the proportional chamber. It is equivalent to a series of quasi-independent single-wire chambers side by side, operating in the region of proportionality. The two cathode planes are made up of two planes of parallel wires orthogonal to each other.

The electric potential and field distribution in a MWPC have been calculated by Erskine [19]. For a chamber with geometric configuration as shown in Fig. 2.1, the electric potential and the electric field are given by:

$$V(x,y) \approx q \left\{ 2\pi L/s - \ln[4 \sin^2(\pi x/s) + 4 \sinh^2(\pi y/s)] \right\}$$

$$E_x(x,y) = - \frac{\partial V}{\partial x} = \frac{\pi q \sin(2\pi x/s)}{s[\sin^2(\pi x/s) + \sinh^2(\pi y/s)]}$$

$$E_y(x,y) = - \frac{\partial V}{\partial y} = \frac{\pi q \sinh(2\pi y/s)}{s[\sin^2(\pi x/s) + \sinh^2(\pi y/s)]}$$

where L is the plane spacing, s is the anode-wire spacing, and q is the charge per unit length on an anode wire. Figure 2.2 shows the general shape of the field lines and the equipotentials for such a geometry. At regions close to the anode wires ($x \ll s$, $y \ll s$), the electric field varies as $1/r$. The field near the wires is strong enough to enable electrons in that region to acquire sufficient energy between collisions to ionize gas molecules with which they collide.

When an ionizing particle passes through a MWPC, it liberates electrons from the gas molecules in the chamber through photoelectric

Fig. 2.2. Equipotentials and electric field lines in MWPC.

interaction or Compton scattering. The electrons produced drift toward the anode wire. As soon as they enter the multiplication region they produce secondary electrons by collisions which, in turn, produce more electrons by the same process. Consequently there form an avalanche of electrons which is collected by the anode wire, and a positive-ion sheath which moves toward the cathodes. Detailed consideration shows that the total number of electrons produced n (which is also the total number of positive ions) is proportional to the number of initial electrons n_0 ,

$$n = An_0.$$

The proportionality factor A is called the multiplication factor which was found by Rose and Korff [20] to be of the form

$$A = \exp 2(aNCr_0V)^{1/2} \left[\left(\frac{V}{V_s} \right)^{1/2} - 1 \right]$$

where V is the applied potential, V_s is the threshold potential (the potential at which inelastic collisions start at the surface of the wire), r_0 is the wire radius, N is the number of gas atoms per cc, a is the rate of increase of the ionization cross section with energy, and C is the chamber capacitance per unit length.

The motion of the electrons and the positive ions induces a negative voltage pulse at the avalanche anode and a positive one at the cathodes, with the major contribution to these voltage pulses coming from the positive ion drift. These signal pulses are readout by the delay line method to be described in the next section.

2.1.2. Delay Line Readout

The use of distributed LC electromagnetic delay line furnishes a suitable way of accomplishing position-to-time conversion. The signals generated on each of the two perpendicular cathode wires are capacitively coupled to a delay line. The time differences between the time of arrival of the prompt anode signal and the delayed signals from the two cathodes give the x and y coordinates of the interaction site. A distinct advantage of this readout method is the inherent ability to interpolate between signal-coupling points for the position of the centroid of the signal. The centroid of a single pulse indicates closely the center of an ionization event even though the signal includes contribution from the adjacent wires.

In its simplest form the delay line is essentially a transmission line with uniformly distributed inductance and capacitance L and C respectively. For an ideal transmission line,

$$\begin{aligned}\frac{\partial V}{\partial z} &= -L \frac{\partial I}{\partial t}, \\ \frac{\partial I}{\partial z} &= -C \frac{\partial V}{\partial t}.\end{aligned}$$

Combining these two equations we get the wave equations,

$$\begin{aligned}\frac{\partial^2 V}{\partial z^2} &= LC \frac{\partial^2 V}{\partial t^2} \\ \frac{\partial^2 I}{\partial z^2} &= LC \frac{\partial^2 I}{\partial t^2}\end{aligned}$$

which have the general solutions [21]

$$\begin{aligned}V(z,t) &= F_1(z-v_0 t) + F_2(z+v_0 t) \\ Z_0 I(z,t) &= F_1(z-v_0 t) - F_2(z+v_0 t)\end{aligned}$$

where F_1 , F_2 represent waves of arbitrary wave form travelling in opposite directions with velocity $v_0 = 1/\sqrt{LC}$, and $Z_0 = \sqrt{L/C}$ is the characteristic impedance. The time delay per unit length is, therefore, given by $\sqrt{LC}$.

The performance of the delay line deviates from that of an ideal transmission line in two aspects. Firstly, due to the finite resistance of the wire conductors and lossy dielectrics at high frequencies, the amplitude and shape of the signal are distorted. Though such frequency dependent loss mechanisms cannot be eliminated completely, they can be minimized by proper choice of construction materials. Perez-Mendez reported values for attenuation on the order of 1.5 db/sec at 3.6 MHz, with increasing attenuation beyond 5 MHz [22]. Secondly, the diminishing value of distributed inductance at higher frequencies gives rise to unequal phase velocities for different frequency components of a pulse, resulting in phase distortion. Phase distortion can be compensated by the addition of frequency dependent shunt capacity to ground.

The delayed signals from the two cathodes and the prompt anode signals are amplified and then fed into the MWPC timing discriminators to give fast timing pulses. Digitized coordinate readout can be obtained by using the anode timing pulse to initiate two digitizing scalars, which are stopped later by the delay line signals. Details of the electronics will be presented in Sec. 4.1.2.

2.1.3. Gamma Converter

The stopping power of the volume of gas mixture used in MWPC for detecting the 511 keV gammas is too low to be of any practical use. For instance, the probability for the gamma to interact with 2 cm of argon-methane (70-30) mixture is $\sim 0.03\%$. This low efficiency is due to the low atomic number of the elements in the gas mixture and its low density. If a solid plane converter is coupled to the active region of a MWPC, the gammas may interact with the plane converter, yielding conversion electrons. Those conversion electrons which could escape from the converter would be detected as in the case of a conventional MWPC. The yield of such a plane converter is limited by the maximum range of the conversion electrons. A Pb plane converter with thickness greater than the maximum range of the conversion electrons has a conversion efficiency of only about 0.26% [23]. A suitably designed gamma converter with enhanced surface area and high density for interacting with gammas can increase the detection efficiency many-fold.

Pb honeycomb converter has been developed by Perez-Mendez and colleagues [24] for such a purpose. Pb was chosen primarily because of its high atomic number. The matrix detector, with somewhat similar design and ideas, was developed at CERN [25]. Figure 2.3 shows the schematic design of the Pb converter. A positive potential is applied across its wall, and the potential at the top of the converter is negative relative to the cathode planes of the MWPC. When a gamma strikes the converter, it has some probability of interacting with the atoms in the converter, either by photoelectric absorption or by Compton scattering,

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Fig. 2.3. Schematic configuration of lead converters; showing the electric drift field lines.

producing a free electron inside the converter in either case. In case these conversion electrons escape from the converter and enter the gas volume of the converter cell, they ionize the gas molecules, producing electrons and positive ions on their paths. These electrons are drifted towards the MWPC by the drift field and detected.

2.1.5.1. Conversion Efficiency

To estimate the detection efficiency of the converter, the geometric form of a unit cell of the converter is simplified as shown in Fig. 2.4; L is the cell size and t is the wall thickness. We make the simplifying assumption that the electrons are generated uniformly within the converter material. This assumption is justified if the walls are quite thin and the height of the converter is relatively small. Let Q_{ij} be the average number of i th kind of conversion electrons generated in the j th unit cell of the honeycomb structure by the passage of a gamma-ray photon. By the kind of electron we mean the interaction which the gamma ray undergoes in producing the electron, either photoelectric effect or Compton scattering. The number of i th electrons, dn_{ij} , generated in the volume element at x of thickness dx is then given by

$$dn_{ij} = Q_{ij} \frac{4(L+2x)}{(2L+t)t} dx .$$

Let $P_{\text{esc}}(x, E_i)$ represent the probability that a electron of energy E_i at x will escape from the converter material to the outside. Then the probability ϵ_c that an incident gamma-ray photon gives rise to an electron that escapes to the outside is

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Fig. 2.4. Schematic of a square cross section cell array with unit cell dimension specified.

$$\epsilon_c = \sum_{i=1}^2 \sum_j \text{dni} j = \sum_{i=1}^2 \sum_j \int_0^{1/2} Q_{ij} \frac{4(1+2x)}{(21+x)t} P_{\text{esc}}(x, E_i) dx. \quad (2.1)$$

To evaluate ϵ_c we need to know $P_{\text{esc}}(x, E_i)$ and Q_{ij} . $P_{\text{esc}}(x, E_i)$ can be decomposed into 2 terms, $P_{\text{esc}}(x, E_i) = P(x, E_i) + P(t-x, E_i)$, corresponding to the probability of an electron escaping the unit cell under consideration or to an adjacent cell. Spencer [26] has given an analytic formula for $P(x, E_i)$

$$P(x, E_i) = \exp(A) \left[1 - \frac{x}{R(E_i)} \right]^p \exp \left[\frac{-A}{1 - \frac{x}{R(E_i)}} \right]$$

where $R(E_i)$ is the residual range of the electron with energy E_i . A is a parameter depending on the energy of the electron and the material of the medium, and p is a constant. The values of A can be obtained from the experimental data in Selig's work [27]. As pointed out by Jeavons [25], $P(x, E_i)$ is relatively insensitive to the value of p , and $p = -3$ is chosen to facilitate calculation. The photoelectrons have a well defined energy E_p given by

$$E_p = h\nu - E_k$$

where $h\nu$, E_k represent the energy of the gamma ray and the K shell binding energy of the atom. The Compton electrons have a continuous distribution of energy $E_c(\theta)$ depending on the scattering angle θ , $E_c(\theta) = h\nu\alpha(1 - \cos\theta)/(1 + \alpha(1 - \cos\theta))$, where $\alpha = \frac{h\nu}{m_0 c^2}$. In our calculation we assume they all have the same average energy $\bar{E}_c$ given by

$$\bar{E}_c = \frac{\int E_c(\theta) d\sigma(\theta)}{\int d\sigma(\theta)}$$

where $d\sigma(\theta)$ is the Klein-Nishima differential scattering cross section:

$$d\sigma(\theta) = r_0^2 d\Omega \left(\frac{1}{1+\alpha(1-\cos\theta)} \right)^3 \left(\frac{1+\cos^2\theta}{2} \right) \left(1 + \frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)(1+\alpha(1-\cos\theta))} \right),$$

here r_0 represents the classical electron radius.

The quantity Q_{ij} depends on the geometry and composition of the converter as well as on the position of the source. For a photon emitted at an angle Ω , the average number of i th kind of electrons generated along its path in the converter is

$$\int_0^{t_\Omega} \sigma_i \rho e^{-\sigma_T \rho s} ds = \frac{\sigma_i}{\sigma_T} (1 - e^{-\sigma_T \rho t_\Omega})$$

where t_Ω is the total path length traversed by the photon in the converter material. Thus Q_i , the average number of i th kind of electrons produced in the converter by the passage of a photon, is given by

$$Q_i = \sum_j Q_{ij} = \frac{\sigma_i}{\sigma_T} \int (1 - e^{-\sigma_T \rho t_\Omega}) d\Omega / \Omega_0$$

where the integration is extended over the total solid angle Ω_0 subtended by the converter at the source. Substituting Q_i into equation (2.1) gives

$$\epsilon_c = \frac{4 \int_{\Omega_0} (1 - e^{-\sigma_T \rho t_\Omega}) d\Omega}{\sigma_T (2L+t) t \Omega_0} \sum_{i=1}^2 \sigma_i \int_0^{t/2} (L+2x) P_{esc}(x, E_i) dx.$$

The detailed evaluation of the integral

$$\int_0^{t/2} (L+2x) P_{esc}(x, E_i) dx$$

is carried out in Appendix A. The calculated values of ϵ_c for a number of converters are presented in Figs. 2.5 and 2.6.

Conversion yield for gamma (511 keV)
converter with cell size 2mm x 2mm x 15 mm

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Fig. 2.5. Conversion efficiency of Pb-glass converters and Pb converter as a function of wall thickness.

Fig. 2.6. Conversion efficiency of Pb-glass converter as a function of wall thickness for various cell sizes.

2.1.3.2. Extraction Efficiency

The conversion electrons liberated from the converter ionize the gas molecules in the converter cell, producing ionization electrons which are extracted by the electric field applied across the converter. A profile of the drifting electric field for the Pb converter is shown in Fig. 2.3. The profile shows that some electrons are lost in the drifting process due to the termination of the field lines on the walls. The space within a cell can then be approximated by two concentric cylindrical regions (Fig. 2.7), an inner cylinder of average radius a where electric field lines are continuous along the length of the cell, and an outside cylindrical shell of outer radius b where field lines terminate on the wall. The value of a depends on b/H , where H is the width of the converter strips (see Sec. 4.1). The variation of a/b with b/H is illustrated in Fig. 2.8.

To estimate the probability that a conversion electron will trigger the MWPC, we need to calculate the track-length distribution $f(v)$ in the inner cylinder produced by a point source on the inner surface of the outer cylinder. The required probability ϵ_c is given by

$$\epsilon_c = \int_{\tau_t}^{\infty} f(v) dv$$

where τ_t is the minimum track length for detection.

Consider a cylindrical shell of infinite length, with inner and outer radii equal to a and b respectively. If we measure the track length r in units of $2a$, and let $c = b/a$, the distribution $f(r)$ of the track length is given by

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Fig.2.7. A cylindrical cell geometry used for the calculation of electric extraction field.

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Fig. 2.8. Dependence of the radius of the extraction cylinder on the ratio b/H of the cylindrical unit cell.

$$f(r) = \begin{cases} \frac{2r}{\pi} \int_0^1 \frac{(1-w^2)dw}{\sqrt{r^2(1-w^2) + (c^2-1)}\sqrt{1-r^2(1-w^2)}} & r < 1 \\ \frac{2}{\pi r^3} \int_{\phi_0}^{\pi/2} \frac{c^2 \sin^2 w - (c^2-1)dw}{\sqrt{1 - \frac{c^2 \sin^2 w - (c^2-1)}{r^2}}} & r > 1 \end{cases}$$

where $\phi_0 = \sin^{-1}(\sqrt{(c^2-1)/c})$. The derivation of these expressions is given in Appendix B. The singularity at $r=1$ is due to the fact that the differential of the track length, dr , is zero there.

For a Pb-glass converter no drifting electric field line ends on the wall. In this case $a=b$, and the above expressions reduce to

$$f(r) = \begin{cases} \frac{2}{\pi} \int_0^1 \frac{\sqrt{1-w^2}}{\sqrt{1-r^2(1-w^2)}} dw = \frac{2}{\pi} \frac{1}{r^2} [F(90^\circ, r) - E(90^\circ, r)] & r < 1 \\ \frac{2}{\pi r^3} \int_0^{\pi/2} \frac{\sin^2 w}{\sqrt{1 - \frac{\sin^2 w}{r^2}}} dw = \frac{2}{\pi r} [F(90^\circ, \frac{1}{r}) - E(90^\circ, \frac{1}{r})] & r > 1 \end{cases}$$

where F and E are the elliptical functions of the first and second kinds respectively. The distributions for some typical values of $c(=b/a)$ are plotted in Fig. 2.9.

2.1.4. Event Rate Consideration

In this section we derive the expressions for the singles count rate, the coincidence count rate, and the accidentals rate in a positron camera. From these expressions we develop the criteria for choosing the

Fig. 2.9. Track-length distributions in an infinitely long cylinder of radius a .

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appropriate count rate. First of all we define the following list of variables (see Fig. 2.10).

- N = detectable activity, i.e. the number of annihilation gamma pairs that fall within the detection cone of the positron camera in unit time
- ϵ = efficiency of each detector in detecting the 511 keV photons
- T_d = anticoincidence time or dead time (i.e. a event will be rejected if a detector detects two or more events within a time $\leq T_d$)
- T_r = resolving time (i.e. the signal detected in detector A and that in B are considered in coincidence if they occur within a time $\leq T_r$).

γ DETECTOR A

γ DETECTOR B

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Fig. 2.10. Schematic figure of planar positron camera.

2.1.4.1. Singles Rate

A typical time distribution of the detected signals in each detector is shown in Fig. 2.11. The number of signals detected in each detector per unit time is $p_1 = N_0$. The fraction p_2 of these signals that are validated is equal to the probability that no other detected signal occurs in the same converter within the time interval I in Fig. 2.11. The mean number of signals within I is $2N_0T_d$, so p_2 is equal to $\exp(-2N_0T_d)$. Thus we get

$$\begin{aligned} \text{singles rate} &= p_1 p_2 \\ &= N_0 \exp(-2N_0T_d). \end{aligned}$$

2.1.4.2. True Coincidence Rate

The number of detectable annihilation events in which both photons are detected is $p_3 = N_0^2$. The probability p_4 that no other signal is detected in either converter within the time interval I is

$$\begin{aligned} p_4 &= \sum_{n=0}^{\infty} (\text{probability of having } n \text{ annihilation events} \\ &\quad \text{within } I) \times (\text{probability that none of the } 2n \\ &\quad \text{photons is detected}) \\ &= \sum_{n=0}^{\infty} \frac{\exp(-2N_0T_d) (2N_0T_d)^n}{n!} (1 - \epsilon)^{2n}. \end{aligned}$$

Summing the series we get

$$p_4 = \exp(-2N_0T_d \epsilon (2 - \epsilon)).$$

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Fig. 2.11. Typical time distribution of the detected signals in a converter.

The true coincidence rate is thus given by

$$\begin{aligned} \text{true coincidence rate} &= p_3 p_4 \\ &= N\epsilon^2 \exp(-2NT_d\epsilon(2-\epsilon)). \end{aligned} \quad (2.2)$$

2.1.4.3. Chance Coincidence Rate

Consider an annihilation event in which only the photon incident on detector A is detected. The frequency of such events is $p_5 = N\epsilon(1-\epsilon)$. We want to find the probability p_6 that only 1 photon due to a separate annihilation event is detected in detector B within the interval I . Suppose n detectable annihilation events occur within the interval I , the probability for which to happen being equal to $\exp(-\lambda)\lambda^n/n!$ where $\lambda = 2NT_d$, the chance that only 1 photon is detected at B is

$$n\epsilon(1-\epsilon)^{2n-1}.$$

Summing over n we get

$$p_6 = \sum_{n=1}^{\infty} \frac{\exp(-\lambda)\lambda^n}{n!} n\epsilon(1-\epsilon)^{2n-1} = \epsilon(1-\epsilon) \exp(-\lambda)\lambda \exp(\lambda(1-\epsilon)^2).$$

In order for the event to be validated, the accidental signal in B has to fall within the time interval I_1 in Fig. 2.11. The probability for this to happen is T_r/T_d . Thus the chance coincidence rate is given by

$$\begin{aligned} \text{chance coincidence rate} &= p_5 p_6 T_r/T_d \\ &= 2N^2\epsilon^2(1-\epsilon)^2 \exp(-2NT_d\epsilon(2-\epsilon)) T_r. \end{aligned} \quad (2.3)$$

2.1.4.4. Ratio of True to Chance Coincidences

Combining equations (2.2) and (2.3) we get

$$\text{ratio of true to chance coincidences} = \frac{1}{2N(1-\epsilon)^2 T_R}. \quad (2.4)$$

Equation (2.4) shows that the ratio of true to chance coincidences depends on the detectable activity N , the detection efficiency of the detectors and their resolving time. For a fixed N , the ratio can be improved by increasing ϵ and decreasing T_R . The value of T_R depends on the collection time T_C of the ionization electrons in the detectors. In general T_R is chosen to be longer than T_C , which is the assumption used in the above derivation. If this is not true, then the expressions for the coincidence rate and the ratio of true to chance coincidences are modified by a factor $C(T_R/T_C)$ given by

$$C(T_R/T_C) = 2(T_R/T_C) - (T_R/T_C)^2.$$

The derivation of $C(T_R/T_C)$ is given in Appendix C.

True coincidence rates and ratios of true to chance coincidences calculated from equations (2.2) and (2.4) for a typical Pb converter and a typical PbO-glass converter respectively are plotted in Figs. 2.12 and 2.13 as a function of detectable activity. T_d is assumed to be 1.5 μsec . The values of ϵ and T_R used for the Pb-converter are 5.5% and 300 nsec respectively, whereas those for the PbO-glass converter are 10% and 150 nsec. The singles rate at the maximum of the true coincidence rate is indicated on each of the figures.

Fig. 2.12. True coincidence rate and ratio of true to chance coincidences for a Pb converter.

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Fig. 2.13. True coincidence rate and ratio of true to chance coincidences for a Pb-glass converter.

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2.2. 3-D Reconstruction in Positron Camera

In this section the general mathematics of 3-D reconstruction in positron camera are formulated.

We shall take the line joining the centers of the two detectors of the planar positron camera as the z-axis (Fig. 2.10). Due to the symmetry of the x and y axes, we shall write out only the x-axis explicitly and suppress the y-axis in most of the mathematical treatment and figures in order to facilitate presentation.

2.2.1. Solving the Integral Equation

As mentioned at the beginning of this chapter, the data in a positron camera consist of straight lines defined by the annihilation gamma pairs, with the positions of the radioactive nuclei which have undergone β^+ decay lying somewhere along the straight lines. If the number of these straight lines becomes large, their intensities at every point $\underline{r}$ in space will form a scalar field $\phi(\underline{r})$, as shown in Fig. 2.14. If the response at $\underline{r}$ due to a point source at $\underline{r}'$ is $\phi_0(\underline{r}, \underline{r}')$, then we can write

$$\phi(\underline{r}) = \int \rho(\underline{r}') \phi_0(\underline{r}, \underline{r}') d^3r' \quad (2.5)$$

where $\rho(\underline{r}')$ is the radioisotope distribution giving rise to the annihilation gamma pairs. Equation (2.5) is the basic integral equation relating the data to the object distribution.

Equation (2.5), as it stands, is not easy to deal with in general. However, if the kernel is space invariant, i.e.

$$\phi(\underline{r}) = \int \rho(\underline{r}') \phi_0(\underline{r} - \underline{r}') d^3r' \quad (2.6)$$

Fig. 2.14. Scalar field $\phi(r)$ formed by the annihilation gamma pairs.

solving for $\rho(\underline{r})$ becomes much easier, as various methods of solving integral equations with kernels of such form have been developed [28].

The space invariant point response function $\phi_0(\underline{r})$ can be constructed in a number of ways from a knowledge of the location of each annihilation gamma pair emitted from a point source and the angle θ it makes with the z-axis. Figure 2.15 shows one way to construct such a point response function. A volume element is used to measure the total length of the line segments that are contained within its volume. The average total length of these line segments defines the point response function at that position.

Another $\phi_0(\underline{r})$ which is easier for both computation and construction is shown in Fig. 2.16. Instead of a volume element, a small area element with its normal along the z-axis is used as the measuring scale. The number of lines passing through the area element is taken as a definition of the point response function. This $\phi_0(\underline{r})$ is just the conventional tomogram obtained by back projecting the data.

We can generalize the second definition of $\phi_0(\underline{r})$ to an entire family of point response functions by making use of the angle information inherent in the positron camera data. Each line passing through the area element is weighted by a factor depending on the angle θ the line makes with the z-axis, e.g. $\cos^n \theta$, $\sin^n \theta$, etc. This results in generalized tomograms in which large-angle and small-angle events are given different emphasis, and thus gives rise to the possibility of improving the quality of the reconstruction by weighting the large-angle events more. This point will be further elaborated in Sec. 3.1.2.

Fig. 2.15. Point response function defined by a volume element.

Fig. 2.16. Point response function defined by an area element with its normal along the z-axis.

To complete the discussion of the point response function, we notice that the detection apparatus does not usually subtend the full 4π steradian solid angle. As a consequence each point in the density distribution has a local cone of detection; which means any gamma pair emitted from that point cannot be recorded by the positron camera if it falls outside the local cone. The size of the local cone of detection varies with the position of the point emitter. This makes the point response function space variant.

One way to remove this dependence is to use only those gamma pairs which make an angle with the z-axis smaller than a certain limit set by the minimum local cone of detection in the density distribution. This principle is illustrated in Fig. 2.17. The local cone of detection at $\underline{r}_1$ is smaller than that at $\underline{r}_2$; but if we use only those events falling within the shaded cones which have the same size as the smaller of the two local cones at $\underline{r}_1$ and $\underline{r}_2$, the point response functions generated at these two positions would again be the same.

To solve the integral equation (2.6) directly as a matrix equation on a digital computer requires a large amount of computation and core memory. A more practical approach is to Fourier transform the equation to frequency space. As shown in [28], the transformed equation is diagonal, so the integral reduces to an algebraic equation,

$$\hat{I}(\underline{k}) = \hat{\phi}_0(\underline{k})R(\underline{k}) \quad (2.7a)$$

where

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Fig. 2.17. Making the point response function space invariant by solid angle limitation. Only those gamma pairs falling within the minimum local cone of detection (the shaded cones) are used.

$$\phi_0(\underline{k}) = \int \phi_0(\underline{r}) \exp(2\pi i \underline{k} \cdot \underline{r}) d^3 r$$

$$\phi(\underline{k}) = \int \phi(\underline{r}) \exp(2\pi i \underline{k} \cdot \underline{r}) d^3 r$$

$$R(\underline{k}) = \int \phi(\underline{r}) \exp(2\pi i \underline{k} \cdot \underline{r}) d^3 r.$$

The solution is given by

$$\phi(\underline{r}) = \int R(\underline{k}) \exp(2\pi i \underline{k} \cdot \underline{r}) d^3 k \quad (2.7b)$$

where

$$R(\underline{k}) = \begin{cases} \frac{\phi(\underline{k})}{\phi_0(\underline{k})} & \text{if } \phi_0(\underline{k}) \neq 0 \\ \text{undetermined} & \text{if } \phi_0(\underline{k}) = 0 \\ \text{(since here Eq. (2.7a) becomes } 0 = 0\text{).} \end{cases} \quad (2.7c)$$

If we Fourier transform Equation (2.6) only in the x dimension, we get the equation

$$\psi(k_x, z) = \int_{-\infty}^{\infty} \psi_0(k_x, z-z') p(k_x, z') dz' \quad (2.8a)$$

where

$$\psi_0(k_x, z) = \int_{-\infty}^{\infty} \phi_0(x, z) \exp(2\pi i k_x x) dx$$

etc.

and the solution is given by

$$\psi(x, z) = \int_{-\infty}^{\infty} p(k_x, z) \exp(-2\pi i k_x x) dk_x. \quad (2.8b)$$

Equations (2.7) and (2.8) represent two methods of solving for $\phi(\underline{r})$. The former solves for the unknown $R(k_x, k_z)$ in frequency space and then inverse transforms in the k_x and k_z dimensions back to object space, whereas the latter tackles the problem in the k_x and z space followed by

inverse transformation in the k_x dimension. Thereafter we shall refer to these methods as the deconvolution method, and the matrix method [29], respectively, since the integral equation (2.8a) becomes a matrix equation when solved numerically on a digital computer.

2.2.2. The Optical Transfer Function and the Undetermined Frequency Components

In order to reconstruct the object uniquely using the deconvolution method from Equation (2.7c), all the frequency components of the optical transfer function $\phi_0(\underline{k})$ must be non-zero. For the matrix method, the corresponding requirement is that the integral operator (2.8a) does not have zero as eigenvalues [30]. Neither of these two conditions holds when the ranges of integration involved cover all space, and there is some plane passing through the origin of the point response function and that plane contains only one non-zero point. To show this, we first calculate the optical transfer function $\phi_0(\underline{k})$. Due to the property of the positron camera event to be shown later (Equation (2.9), we make the following coordinate transformation

$$r(x,z) \rightarrow r(\theta,z)$$

$$\text{where } \theta = \tan^{-1}\left(\frac{x}{z}\right)$$

$$z = z.$$

As mentioned in the last section, the general point response function at (x,z) is defined as the number of straight lines emitted from the point source which pass through a line segment of unit length at (x,z) oriented perpendicular to the z -axis, with each line weighted by a factor $F(\theta)$ depending on the angle θ the line makes with the z -axis.

$F(\theta)$ is positive inside the cone and zero outside; the case of $F(\theta) = 1$ inside the cone corresponds to the conventional tomograms obtained from back-projecting the events. For $z \neq 0$, consider a line segment dx pointing along the x -axis centered at (θ, z) relative to the point source (Fig. 2.18A)

$$\begin{aligned}\phi_0(\theta, z) &= F(\theta) \frac{\text{angle subtended by } dx \text{ at origin}}{\pi} \frac{1}{dx} \\ &= \frac{F(\theta)}{\pi} \frac{\cos^2 \theta}{|z|}.\end{aligned}$$

$$\text{Thus } \phi_0(\theta, z)|z| = \frac{F(\theta)\cos^2 \theta}{\pi} \quad (2.9)$$

Equation (2.9) shows that $\phi_0(\theta, z)|z|$ is a function of θ only. This expression is valid also at $z = 0$. The optical transfer function is

$$h_0(k_x, k_z) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \phi_0(x, z) \exp(2\pi i(k_x x + k_z z)). \quad (2.10)$$

Performing this integral (Appendix D) we get

$$h_0(k_x, k_z) = \begin{cases} \frac{F(z)}{\pi} \int_{-\theta_0}^{\theta_0} F(\theta) d\theta & \text{if } k_x = 0 \\ \frac{1}{\pi|k_x|} F(\theta_1) \cos^2 \theta_1 & \text{if } k_x \neq 0, \\ \text{where } \tan \theta_1 = -\frac{k_z}{k_x} \end{cases}$$

Figure 2.18B shows schematically the shape of the two-dimensional optical transfer function. Three-dimensional point response functions and optical transfer functions are treated in Appendix E, and the result (E.3) for the pyramidal $\phi_0(\underline{r})$ is illustrated in Fig. 2.19. Figures 2.18B and 2.19B show that there are regions in frequency space where

A. $\phi_0(\underline{r})$

B. $\phi_0(\underline{k})$

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Fig. 2.18. A two-dimensional point response function $\phi_0(\underline{r})$ and its optical transfer function $\phi_0(\underline{k})$.

A. $\phi_0(\underline{r})$

B. $\phi_0(\underline{k})$

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Fig. 2.19. A pyramidal three-dimensional point response function $\phi_0(\underline{r})$ and its optical transfer function $\phi_0(\underline{k})$.

$\phi_0(\underline{k})$ is zero, and thus $R(\underline{k})$ cannot be recovered by the deconvolution method from Equation (2.7c). Hereafter we shall refer to the region where $\phi_0(\underline{k}) \neq 0$ as the "allowed cone", and that where $\phi_0(\underline{k}) = 0$ as the "missing cone", respectively. Note that Equations (D.2) and (E.2) show that the ϕ_0 of those point response functions satisfying Equations (2.9) and (E.1), respectively, are non-negative.

The existence of the missing cone can also be shown in a simple way using the projection theorem [55], which states that the one-dimensional Fourier transform of the projection $P(r, \theta)$ in r of a two-dimensional function gives the components of the two-dimensional Fourier transform of the function on a line at angle $(\theta + \pi/2)$. Now it can be shown easily from Equation (2.9) that the total integrated intensity of $\phi_0(x, z)$ on any line which intercepts the detection cone completely is a constant independent of the position of the line (see the expression of $\Psi_0(0, z)$ in Sec. 2.2.3.2(a)). This means that the projection $P(r, \theta)$ of $\phi_0(x, z)$ is a constant function in r for $\theta (= \tan^{-1} x/z)$ in the range $(\theta_0, -\theta_0)$, and therefore its Fourier transform is a delta function at the origin.

However, if the point response function in Fig. 2.19A is rotated around the x-axis, a cylindrically-shaped point response function will be generated. Making use of the linearity of the Fourier transformation and the non-negativity of ϕ_0 , the corresponding optical transfer function can be obtained by rotating the function in Fig. 2.19B around the k_x -axis. The resulting function is non-zero everywhere in the frequency space.

This example illustrates the following general principle:

"the necessary and sufficient condition for the optical transfer function in a positron camera to be non-zero everywhere is that every plane passing through the origin (the location of the point emitter) of the point response function contain at least a line of non-zero values." (2.11)

The proof is given in Appendix F.

In the case of the matrix method, we claim that the integral operator (2.8a) does have zero as its eigenvalue for every value of k_x if condition (2.11) is not satisfied. First we show that $\{\exp(-2\pi i k_z z)\}$ is the set of eigenfunctions of the integral operator. Setting $p(k_x, z) = \exp(-2\pi i k_z z)$, the integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_0(k_x, z-z') \exp(-2\pi i k_z z') dz' \\ = \exp(-2\pi i k_z z) \phi_0(k_x, k_z). \end{aligned}$$

Thus $\{\exp(-2\pi i k_z z)\}$ is the set of eigenfunctions for the integral operator with eigenvalues $\phi_0(k_x, k_z)$. However, we have already shown that ϕ_0 has zero components if some cross-section of ϕ_0 through the origin contains one non-zero point only. Since the eigenfunctions form a complete set, the general solution of the integral equation (2.8a) is

$$p(k_x, z) = \int_{-\infty}^{\infty} C(k_x, k_z) \exp(-2\pi i k_z z) dk_z$$

where the coefficients $C(k_x, k_z)$ are given by

$$C(k_x, k_z) = \begin{cases} \frac{\int_{-\infty}^{\infty} \phi(k_x, z) \exp(2\pi i k_z z) dz}{\phi_0(k_x, k_z)} = \frac{\phi(k_x, k_z)}{\phi_0(k_x, k_z)}, & \text{if } \phi_0(k_x, k_z) \neq 0 \\ \text{undetermined,} & \text{if } \phi_0(k_x, k_z) = 0. \end{cases}$$

The solution $p(x, z)$ in object space is obtained by inverse transforming $p(k_x, z)$ in the k_x dimension, and it can be seen that the result is identical to that obtained by the deconvolution method, as expected.

It follows that $p(x, z)$ cannot be determined uniquely using either the deconvolution method or matrix method represented by Equations (2.7) and (2.8), respectively, if the point response function does not satisfy (2.11) and no further information is available.

2.2.3. Use of A Priori Information

In this section we discuss the effects of some of the a priori constraints on the undetermined components of $R(\underline{k})$.

In all physically interesting situations we will be dealing with a bounded, finite object in which the appropriate density $\rho(r)$ is to be determined. Furthermore, due to the limitation in resolution of the measuring device and the fact that computation is usually done on a digital computer, this object distribution will be analyzed on a grid of points with finite spacings $\Delta x, \Delta z$. In order to understand the effect of each constraint separately, we will consider the following cases:

2.2.3.1. Finite Spacing

As mentioned in Sec. 2.2.2, the vanishing of $t_0(k_x, k_z)$ in the missing cone is due to the property of $t_0(x, z)$ as expressed in equation (2.9). If the assumption is made that the distribution of $\rho(x, z)$ is not continuous but discrete, $t_0(x, z)$ can be sampled accordingly. Equation (2.9) is no longer valid for the sampled $t_0(x, z)$, and hence part or all of the zeroness in the missing cone is removed. The detailed analysis is done in Appendix G. The results show that the number of undetermined components of $R(k)$ is greatly reduced if $t_0(x, z)$ is sampled in the z dimension, and is completely removed if sampled in the x dimension. For the matrix method, the number of eigenfunctions with zero as eigenvalue is also reduced in a corresponding manner.

2.2.3.2. Finite Extent of Object

By making use of the fact that the extent of the object is finite, all of the undetermined coefficients in the solution can be recovered. This is a consequence of two theorems: (1) the Fourier transform of a finite object is an entire function, and (2) an entire function can be

continued throughout the whole complex plane from a knowledge of the function on any finite continuous line segment [31]. We consider the effects of the finiteness of the object in two steps.

(a) Finiteness in z dimension ($z_1 < z < z_2$)

The Fourier transform $R(k_x, k_z)$ of $p(r)$ is entire in k_z . So, for any $k_{x_1} \neq 0$, the function $R(k_{x_1}, k_z)$ can be calculated for all values of k_z from a knowledge of the function on the line segment $\{(k_{x_1}, k_z) \mid |k_z| \leq |k_{x_1}| \tan \theta_0\}$ (Fig. 2.20). This cannot be done at $k_{x_1} = 0$, where the line segment degenerates into a point.

This process of analytic continuation in k_z can be carried out for any non-zero angle θ_0 . The limiting case is when $\theta_0 = 0$, where for every k_{x_1} the line segment $\{(k_{x_1}, k_z) \mid |k_z| \leq |k_{x_1}| \tan \theta_0\}$ degenerates into a point. For the general case $\theta_0 \neq 0$, the only information that cannot be recovered is $R(0, k_z)$ with $k_z \neq 0$. Since $p(0, z)$ and $R(0, k_z)$ are a Fourier transform pair, and

$$p(0, z) = \int_{-\infty}^{\infty} \rho(x, z) dx$$

represents the total slice density on the z-plane, the indeterminacy of $R(0, k_z)$ where $k_z \neq 0$ means that the total slice density on each z-plane cannot be determined uniquely. However, the total object density is known, since

$$R(0, 0) = \iint \rho(x, z) dx dz$$

is known.

As for the matrix method, the integral operator (2.8a) in this case becomes positive definite when the range of integration in z is

Fig. 2.20. Analytic continuation of $R(k_x, k_z)$ in k_z dimension.

finite. This can be seen as follows.

$$\mathcal{P}_0(k_x, z-z') = \int_{-\theta_0}^{\theta_0} \frac{F(\theta)}{\pi} \exp(2\pi i k_x \tan \theta (z-z')) d\theta.$$

For any function $f(z)$ piece-wise continuous and spatially bounded in $z_1 \leq z \leq z_2$,

$$\begin{aligned} & \int_{z_1}^{z_2} dz' \int_{z_1}^{z_2} dz \mathcal{P}_0(k_x, z-z') f(z) f^*(z') \\ &= \int_{z_1}^{z_2} dz' \int_{z_1}^{z_2} dz \int_{-\theta_0}^{\theta_0} d\theta \frac{F(\theta)}{\pi} \exp(2\pi i k_x \tan \theta (z-z')) f(z) f^*(z') \\ &= \frac{1}{\pi} \int_{-\theta_0}^{\theta_0} F(\theta) \left| \int_{z_1}^{z_2} \exp((2\pi i k_x \tan \theta) z) f(z) dz \right|^2 d\theta. \end{aligned} \quad (2.12)$$

Since $F(\theta) \geq 0$ for all $-\theta_0 \leq \theta \leq \theta_0$, the vanishing of expression (2.12) would require that

$$\begin{aligned} I(k_x \tan \theta) &= \int_{z_2}^{z_1} \exp((2\pi i k_x \tan \theta) z) f(z) dz = 0 \\ &\text{for all } -|k_x \tan \theta_0| \leq k_x \tan \theta \leq |k_x \tan \theta_0|. \end{aligned}$$

Now the integral $I(k_x \tan \theta)$ is an entire function [32]. If it is zero within the interval $[-k_x \tan \theta_0, k_x \tan \theta_0]$, it must be zero everywhere. Thus $f(z)$ must be zero everywhere. This argument proves that the integral operator

$$g(z) = \int_{z_1}^{z_2} \mathcal{P}_0(k_x, z-z') f(z') dz'$$

is positive definite, and by [30], the eigenfunctions associated with it form a complete set in the class of functions square integrable in (z_1, z_2) ,

and thus unique solutions exist for the integral operator.

This argument breaks down if $k_x \tan \theta_0 = 0$. Therefore, there is no unique solution for $p(k_x, z)$ if $k_x = 0$ or $\theta_0 = 0$. For the general case $\theta_0 \neq 0$, the only missing information is $p(0, z)$. As in the case of the deconvolution method, this implies that the total slice density on each z -plane cannot be determined. The total object density, however, can still be determined. This can be seen as follows:

$$\begin{aligned} \Psi_0(0, z) &= \int_{-\infty}^{\infty} \phi_0(x, z) dx \\ &= \int_{-\theta_0}^{\theta_0} \phi_0(0, z) |z| \sec^2 \theta d\theta \\ &= \int_{-\theta_0}^{\theta_0} \frac{F(\theta)}{\pi} d\theta \quad (\text{from Equation (2.9)}) \\ &= A(\theta_0) \quad (\text{say}). \end{aligned}$$

Substituting into (2.8a) with $k_x = 0$ we get

$$\begin{aligned} \Psi_0(0, z) &= \int_{z_1}^{z_2} A(\theta_0) p(0, z') dz' \\ &= A(\theta_0) \int_{-\infty}^{\infty} dx' \int_{z_1}^{z_2} dz' \rho(x', z'). \end{aligned}$$

$$\begin{aligned} \text{Hence, total object density} &= \int_{-\infty}^{\infty} dx' \int_{z_1}^{z_2} dz' \rho(x', z') \\ &= \frac{\Psi_0(0, z)}{A(\theta_0)}. \end{aligned}$$

(b) Finiteness in both x and z dimensions: ($x_1 \leq x \leq x_2$, $z_1 \leq z \leq z_2$)

In this case, $R(k_x, k_z)$ is entire in both k_x and k_z , so for the deconvolution method, $R(k_x, k_z)$ can be continued throughout the k -space from a knowledge of the function within the cone defined by any two

intersecting line segments. Again, the procedure cannot be used when $\theta_0 = 0$, since then the only region in which $R(\underline{k})$ is known is the line $k_z = 0$.

The above analysis shows that all the information of any density distribution of finite extent is contained in its Fourier components within any open cone in frequency space with its apex at the origin. This result is in agreement with the fact the such a distribution is uniquely determined by any infinite set of projections [33, 34].

For the matrix method, the fact that $p(k_x, z)$ is entire in k_x for all $z_1 \leq z \leq z_2$ can be employed to fill in the undetermined components at $k_x = 0$ on each z -plane from values at $k_x \neq 0$ on the same z -plane. If $\theta_0 = 0$, the components at $k_x \neq 0$ are not known themselves, not to mention the continuation to $k_x = 0$.

To continue $R(\underline{k})$ from the allowed cone, the most direct approach would be to calculate the successive derivatives of $R(\underline{k})$ at some point k_0 in the allowed cone to form a Taylor series expansion of $R(\underline{k})$ which converges everywhere. In practice such a series has to be truncated, so the error for the values of the series calculated at region far away from k_0 would be very serious. Besides, to accurately determine the derivatives of a function numerically is by no means simple.

A practical way to continue an analytic function is by means of the prolate spheroidal function expansion [35, 36]. The function $R(\underline{k})$ to be continued is expanded in a series of prolate spheroidal functions $\psi_i(\underline{k})$

$$R(\underline{k}) = \sum_i a_i \psi_i(\underline{k})$$

the coefficients a_1 being determined from the known values of $R(\underline{k})$ in the allowed cone.

Besides the prolate spheroidal functions, Fourier series can also be used in expanding the function to fill in the missing cone. This method was employed by Harris [31] to continue a one-dimensional spectrum, and by Inouye [37] to reconstruct a two-dimensional image.

A rather obvious way to extend the known portion of the spectrum $R(\underline{k})$ to the missing cone is by means of the iterative scheme shown in Fig. 2.21. The spectrum $S^{(0)}(\underline{k})$ obtained from deconvolution, with the undetermined components set to zero, is Fourier transformed to the object space. There the values outside the known extent of the object are set to zero, and then inverse transformed to the frequency space, giving $R^{(1)}(\underline{k})$. The components of $R^{(1)}(\underline{k})$ inside the allowed cone are reset to the original values given by $S^{(0)}(\underline{k})$, and the cycle repeats yielding $R^{(n)}(\underline{k})$ after n iterations.

We note that this iterative scheme has been used by Gerchberg [38] and Papoulis [39] to improve the resolution of one-dimensional signals in band-limited systems. The convergence of the scheme in one dimension has been proved by both of these authors. The proof given in Gerchberg's paper made use of the property that a real analytic function in one-dimensional space either vanishes everywhere or only has isolated zeros. This proof can be generalized to higher dimensions by using the more general property that real analytic functions on n -dimensional space cannot vanish on an infinite set of $n-1$ dimensional planes through the origin without vanishing identically.

Fig. 2.21. Iterative scheme to extend the spectrum $R(k)$ beyond the allowed cone.

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A more quantitative proof of the convergence of the iterative scheme is now given. In actual reconstruction, we will be dealing with frequency components below a certain maximum frequency determined by such factors as the spatial resolution of the imaging system, the available computer core memory, etc. In Fig. 2.22, R_a represents the region in frequency space where $R(\underline{k})$ is known, and R_b is the extent of the object. Define the operators A and B operating on functions f defined in frequency space as follows:

$$\begin{aligned} A f &= \chi_A f \\ B f &= F^{-1} \chi_B F f \end{aligned}$$

where F and F^{-1} represent Fourier transformation and its inverse, and χ_A , χ_B are respectively the characteristic functions of R_a and R_b , defined as:

$$\begin{aligned} \chi_A(\underline{k}) &= \begin{cases} 1 & \underline{k} \in R_a \\ 0 & \underline{k} \notin R_a \end{cases} \\ \chi_B(\underline{x}) &= \begin{cases} 1 & \underline{x} \in R_b \\ 0 & \underline{x} \notin R_b \end{cases} \end{aligned}$$

With these operators we can formulate the iteration procedure as follows. If $R(\underline{k})$ represents the Fourier spectrum of the object $p(\underline{r})$, and $S^{(0)}(\underline{k})$ the Fourier spectrum of $p(\underline{r})$ obtained from deconvolution, then

$$S^{(0)}(\underline{k}) = A R(\underline{k}).$$

Fourier transforming $S^{(0)}(\underline{k})$ to the object space, setting the values outside the known extent of the object to zero, and then inverse transforming to the frequency space, we get the first iterated spectrum $R^{(1)}(\underline{k})$:

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Fig. 2.22. Schematic representations of the allowed cone and the object extent.

$$R^{(1)}(\underline{k}) = B S^{(0)}(\underline{k}) = B A R(\underline{k}).$$

Resetting the components of $R^{(1)}(\underline{k})$ inside the allowed cone to the original values given by $S^{(0)}(\underline{k})$ we get $S^{(1)}(\underline{k})$:

$$\begin{aligned} S^{(1)}(\underline{k}) &= S^{(0)}(\underline{k}) + (I - A) R^{(1)}(\underline{k}) \\ &= A R(\underline{k}) + (I - A) R^{(1)}(\underline{k}) \end{aligned}$$

Iterating we obtain the spectra $R^{(2)}(\underline{k}), S^{(2)}(\underline{k})$ -----, $R^{(n)}(\underline{k})$,

$$\begin{aligned} R^{(2)}(\underline{k}) &= B S^{(1)}(\underline{k}) \\ S^{(2)}(\underline{k}) &= A R(\underline{k}) + (I - A) R^{(2)}(\underline{k}) \\ \vdots \\ S^{(n-1)}(\underline{k}) &= A R(\underline{k}) + (I - A) R^{(n-1)}(\underline{k}) \\ R^{(n)}(\underline{k}) &= B S^{(n-1)}(\underline{k}) = B A R(\underline{k}) + (I - BA) R^{(n-1)}(\underline{k}). \end{aligned} \quad (2.13)$$

Subtracting $R(\underline{k})$ from both sides of equation (2.13)

$$\begin{aligned} R^{(n)}(\underline{k}) - R(\underline{k}) &= B A R(\underline{k}) - R(\underline{k}) + (I - B A) R^{(n-1)}(\underline{k}) \\ &= (I - BA) (R^{(n-1)}(\underline{k}) - R(\underline{k})) \\ &\vdots \\ &= (I - BA)^{n-1} (R^{(1)}(\underline{k}) - R(\underline{k})) \\ &= (I - BA)^n (R^{(0)}(\underline{k}) - R(\underline{k})) \end{aligned}$$

where we have defined $R^{(0)}(\underline{k}) = 0$. Thus we have

$$R^{(n)}(\underline{k}) = R(\underline{k}) - (I - BA)^n R(\underline{k}). \quad (2.14)$$

Note that throughout the iterations, all the spectra $R^{(i)}(\underline{k})$ operated on by BA satisfy

$$B R^{(i)}(\underline{k}) = R^{(i)}(\underline{k}). \quad (2.15)$$

In Appendix II it is shown that the operator BA operating on the functions satisfying equation (2.15) is a positive definite operator. Thus the set of eigenfunctions $\{\psi_i\}$ of BA forms a complete set of functions which are orthonormal in the entire k-space, and orthogonal in the region R_a [40],

$$\int_{\text{k-space}} \psi_i(\underline{k}) \psi_j^*(\underline{k}) d^3k = \delta_{ij}$$

$$\int_{R_a} \psi_i(\underline{k}) \psi_j^*(\underline{k}) d^3k = \lambda_i \delta_{ij}.$$

Thus any function f in k-space which satisfies (2.15) can be expanded in a series of ψ_i which converges to f everywhere; if $Bf \neq f$, the expansion is still valid in the region R_a , but it does not necessarily converge outside R_a . All the eigenvalues of BA lie between 0 and 1, i.e., $0 < \lambda_i < 1$.

Decompose $R(\underline{k})$ into a linear combination of ψ_i

$$R(\underline{k}) = \sum_{i=0}^{\infty} a_i \psi_i(\underline{k}).$$

Then equation (2.14) becomes

$$R^{(n)}(\underline{k}) = \sum_{i=0}^{\infty} a_i (1 - (1 - \lambda_i)^n) \psi_i(\underline{k}).$$

The truncation error in terminating the iteration after n steps is thus given by

$$\begin{aligned} E_t^{(n)}(\underline{k}) &= R^{(n)}(\underline{k}) - R(\underline{k}) \\ &= - \sum_{i=0}^{\infty} a_i (1 - \lambda_i)^n \psi_i(\underline{k}) \end{aligned} \quad (2.16)$$

which is identical to that obtained by Papoulis [39] for the case of one-dimensional signals. This error tends to zero as $n \rightarrow \infty$.

III. DATA TREATMENTS AND ERROR ANALYSIS

In this chapter we shall discuss the effects of noise in the imaging problem. Data pre-treatments to reduce the influence of scattering will be dealt with first, followed by analyzing the propagation of errors in reconstructions, and developing methods to stabilize the performance of the reconstruction algorithms.

3.1. Compton Scattering

Despite the compensation effect mentioned in Chapter I, Compton scattering is still a major source of errors in positron imaging. Scattering, which takes place either inside the object being imaged or in the detector itself, modifies the point response function $\phi_0(\underline{r})$, making it space variant and thus violating the condition for equation (2.6) to hold.

Scattered events can be rejected effectively by using pulse height selection in the case of NaI crystals for energies below 300 keV, and MWPC where the gamma ray is below 60 keV [41]. For MWPC detectors equipped with honeycomb converters, the spectrum of conversion electrons is continuous and hence rejection of such events is not feasible by pulse height selection. In this section we shall describe two ways to reduce the influence of scattering in imaging.

3.1.1. Delay Line Sum Pulse Rejection

Those events in which a gamma gives rise to more than one signal in the same detector can be identified by the method of reading the

signal from each end of the delay line. The time interval between arrival of the pulse on the anode wires and that of the delayed signal at each end of the delay line is measured. If the gamma generates a pulse at only one location of the detector assembly, the sum of the intervals to each end should equal the length of the delay line plus some fixed delay in the processing electronics, a constant to within the accuracy of the timing measurements. Figure 3.1 shows the time sum distribution for an Fe^{55} source whose 5.9 keV photon, being stopped entirely within the chamber, did not interact with the converter. As expected for a single interaction type event, the distribution is symmetrical and narrow with a 5 nsec FWHM. In the case of multiple signals, however, the sum of the 2 intervals will be some value less than the length of the line, and thus these events can be distinguished and rejected. One additional advantage of reading from both ends is that in the case where the accuracy is limited by timing errors, averaging the positions obtained from each end of the line improves the resolution by a factor of $\sqrt{2}$.

To illustrate this effect we measured the time sum distribution for a point source of Ge^{68} , a positron emitter. A small NaI detector was used in coincidence with the prompt signal from the anode plane to provide a well collimated beam of 511 keV photons (from the geometry the size of the beam at the converter was approximately 3mm). In Fig. 3.2 an x-projection of the detector point response function for the positron source is plotted with accidental background subtracted. In all cases discussed here accidental background is removed by applying identical

Fig. 3.1. Time sum distribution for an Fe^{55} source.

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Fig. 3.2. X-projection of detector point response function for the positron source.

cuts to the delayed coincidence data and subtracting the resultant distributions. The FWHM of the distribution is 7 mm, larger than one would expect from the geometry discussed previously. There are also a number of events outside the main peak.

Figure 3.3 shows the time sum distribution for this data. When compared with the distribution for Fe^{55} , it is seen that the distribution is wider (FWHM = 20 nsec), skewed to the low side and there are more events on the high side of the peak than one would expect. It has been determined that the events on the right side are due to saturation of the linear electronics by signals whose amplitude out of the amplifier is greater than 3V. Such large pulses, which also give incorrect spatial information, can be removed by applying a cut above 168 ns, as indicated in Fig. 3.3. Figure 3.4 shows the detector point response function after this cut has been applied; the FWHM is reduced to 6.5 mm.

On the left hand side of the time sum peak there appear to be a significant number of cases where a multiple interaction has occurred. Figures 3.5 and 3.6 show the detector point response functions for events with a time sum between 141 and 168 nsec, and 150 and 168 nsec respectively. These cuts should remove many of these multiple events. As can be seen the number of off-target events (outside the main peak) is reduced and the detector point response function is sharpened.

Table 3.1 summarizes the results giving the number of events contained within different intervals about the mean for various sum cuts. As can be seen improvement in the fraction of events contained within a given interval can be obtained with some reduction in the total number of events within the interval.

Fig. 3.3. Time sum distribution for the positron source. The cut on the high side of the peak is shown.

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Fig. 3.4. X-projection of detector point response function for those events with time sums less than 168 ns.

Fig. 3.5. X-projection of detector point response function for those events with sums between 141 and 168 ns.

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Fig. 3.6. X-projection of detector point response function for those events with sums between 150 and 168 ns.

Table 3.1. The number of events contained in intervals about the mean of x-projection of the detector point response function for various time sum cuts. The fraction of those events passing the cut, which fall within the interval, is also given.

Interval	No Sum Cuts	Sum <168 ns	141<Sum<168 ns	150<Sum <168ns
5 mm	14003	12495	11481	8464
	49%	50%	65%	72%
10 mm	21604	19105	15057	10655
	75%	77%	86%	91%
15 mm	24867	21942	16472	11346
	86%	88%	94%	97%

A Monte Carlo program was written to determine the probability of secondary interactions from photoelectric escape photons and Compton scattering, and their contributions to the spatial resolution. The 511 keV photon was assumed to enter the converter perpendicularly. The material in the converter was assumed to be distributed uniformly throughout its volume. The point at which the 511 keV photon converted was sampled and an average detection probability was assigned. The conversion electron was considered detected if it escaped from the wall of the honeycomb into the gas. The Compton scattered photon or the 88 keV escape photon for Pb was followed until it converted or escaped from the converter. The program successfully predicted the detection efficiency of the converter to within 20% of the experimental value. However, it showed that the probability of having a multiple event due to scattering is only 4%, which is not enough to explain the effect observed. This

indicates that there are other mechanisms, besides scattering, causing the multiple events. Nevertheless, reading out from both ends of the delay line provides a means of correcting the problem.

3.1.2. Digital Filtering

Another manifestation of the scattered events is in the angular distribution of the events. If there were no scattering, the annihilation gammas detected should be distributed isotropically, i.e., the number of gammas lying within the solid angle $d\Omega$ is proportional to $d\Omega = d(\cos\theta)d\phi$, where θ, ϕ are respectively the polar angle and the azimuth measured with respect to the axis of the positron camera. Thus the distribution of the photons as a function of $\cos\theta$ should be a flat line, independent of $\cos\theta$. In the presence of scattering, however, the distribution would be skewed towards the small angle region. This can be understood by considering Fig. 3.7. A pair of annihilation gammas is shown emitted within the detection cone of the positron camera. Now suppose one of them, say the upper one, is scattered by the medium while the other is not. As can be seen from the relative size of the two detection cones shown in the figure, this gamma has higher probability of being scattered anticlockwisely within the detection cone of the camera. If this gamma strikes the upper detector at A' , the angle inferred from $A'B$ would be smaller than that of AB . This argument also applies to scattering in the converter.

To verify this analysis, a Monte Carlo simulation was performed to study the angular distributions of the events detected in a planar positron camera from a point source with and without scattering. The

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Fig. 3.7. Effect of scattering on the angle of the detected annihilation gamma pair.

results are displayed in Fig. 3.8. The two solid histograms show the angular distributions for a point source in air and that for a point source located at the center of a sphere of water medium of radius 10 cm. The total number of events generated is equal in the two cases. For the point source in water, the portion of events that were unscattered is shown as the broken histogram. No scattering was assumed in the positron camera in both cases.

We have also studied the angular distributions of the events taken from the MWPC-gamma converter positron camera (see Sec. 4.1). The two histograms in Fig. 3.9 show the angular distribution of annihilation gammas from a point Cu source in air and one embedded in a bucket of water respectively. Again, it can be seen that those events scattered by water are more populated at small angles. The fact that the distribution for the source in air is also peaked in the small angle region, though to a much smaller extent, is attributed to scattering in the detectors.

Knowing that the scattered events occur more at small angles, data can be filtered to put more emphasis on the large-angle events. Such filtering can be achieved through the use of those angular factors $F(\theta)$ in equation (2.9) which peak at large θ , such as $\cos^{-n}\theta$, $\sin^n\theta$, etc., where n is a positive integer.

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Fig. 3.8. Angular distributions of the simulated detected events from 2 point sources: one in air and the other in a sphere of water.

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Fig. 3.9. Angular distributions of the real detected events from 2 point sources: one in air and the other in a bucket of water. The two histograms have been normalized at the right-most bin.

5.2. Error Propagation and Stabilization in Reconstructions

As shown in Sec. 2.2, complete 3-D reconstructions can be achieved through deconvolution followed by iterations, or through matrix inversion. In this section we formulate the propagation of errors in each of these two approaches, and develop methods to stabilize their performance in the presence of noise.

5.2.1. Deconvolution Followed by Iterations

The deconvolution method solves for $R(\underline{k})$, the Fourier components of $r(\underline{r})$. If noise $\mathcal{E}(\underline{k})$ is present in the data $i(\underline{k})$, the propagated error $E_d(\underline{k})$ in the reconstruction will be given by

$$E_d(\underline{k}) = \frac{\mathcal{E}(\underline{k})}{i_0(\underline{k})}, \quad i_0(\underline{k}) \neq 0. \quad (5.1)$$

Equation (5.1) shows that the error in the data is multiplied by the factor $1/i_0(\underline{k})$ in deconvolution. In the region where $i_0(\underline{k})$ is very small, the error will be greatly magnified. This is the case in the missing cone as well as in the high $|k_x|$ region, since equation (D.2) shows that for fixed $|k_z|/|k_x|$, $i_0(\underline{k}) \propto 1/|k_x|$.

In the iterative scheme (2.15), $R(\underline{k})$ is set to zero in the missing cone before iterating. This procedure removes the instabilities there. A way to deal with the instabilities in the high $|k_x|$ region has been described by Phillips [42]. However, we shall recast his treatment, since the convolution integral permits a particularly elegant realization of the technique.

Suppose the scalar field $\phi(\underline{r})$ is known only up to some error $\epsilon(\underline{r})$. Then, the equation to be solved is,

$$\phi(\underline{r}) + \epsilon(\underline{r}) = \int d^3r' \phi_0(\underline{r}-\underline{r}') \rho(\underline{r}') \quad (3.2)$$

where the error $\epsilon(\underline{r})$ is an arbitrary function except for some condition on its magnitude, such as $|\epsilon(\underline{r})| \leq M$. Equation (3.2) may be solved for $\rho(\underline{r})$ by taking the Fourier transform,

$$\rho(\underline{r}) = \int d^3r' [\epsilon(\underline{r}') + \phi(\underline{r}')] \int d^3k \frac{\exp[-2\pi i \underline{k} \cdot (\underline{r}-\underline{r}')] }{\phi_0(\underline{k})}.$$

The functional derivative of $\rho(\underline{r})$ with respect to $\epsilon(\underline{r}')$ is a function of $\underline{r}-\underline{r}'$,

$$\frac{\delta \rho(\underline{r})}{\delta \epsilon(\underline{r}')} = \int d^3k \frac{\exp[-2\pi i \underline{k} \cdot (\underline{r}-\underline{r}')] }{\phi_0(\underline{k})} = \alpha(\underline{r}-\underline{r}'). \quad (3.3)$$

The error tends to be random from point to point, and it generates instabilities in the solution $\rho(\underline{r})$ which are manifested as sharp fluctuations. Thus, a reasonable condition on $\rho(\underline{r})$ is a requirement for smoothness, in which a solution is sought such that

$$\int d^3r [\nabla^2 \rho(\underline{r})]^2 = \text{minimum}. \quad (3.4)$$

Suppose the total error is some fixed number e where

$$e^2 = \int d^3r [\epsilon(\underline{r})]^2. \quad (3.5)$$

Then the smoothness condition may be re-expressed by introducing a Lagrange multiplier $1/\gamma$ and minimizing the following expression with respect to variations in $\epsilon(\underline{r})$:

$$\int d^3r [V_{\text{eff}}(\underline{r})]^2 + \frac{1}{2} \int d^3r \psi(r)^2.$$

It is clear that ψ must be non-negative, if there is to be a meaningful solution. The functional derivative of this expression with respect to $\psi(r')$ gives an expression for the smoothness condition on the solution $\psi(r)$,

$$0 = -\frac{1}{2} \int d^3r [\psi^2(r)]^2 + \frac{1}{2} \frac{\delta}{\delta \psi(r')} + \lambda \psi(r')$$

subject to the constraint, equation (5.5).

Integration by parts and use of equation (5.3) yield an expression for the error which is proportional to ψ ,

$$\psi(r) = -\frac{1}{2} \int d^3r' \psi(r') \psi^4(r') \psi(r'-r) \quad (5.6)$$

where

$$\psi^4(r'-r) = (2\pi)^4 \int d^3k \frac{k^4}{\epsilon_0(k)} \exp[-2\pi i \underline{k} \cdot (r'-r)].$$

Substituting equation (5.6) into equation (5.2) gives a new convolution equation,

$$\int d^3r' [\epsilon_0(r-r) + \psi^4(r'-r)] \psi(r') = \psi(r).$$

The solution for the density distribution in frequency space is simply

$$R(k) = \frac{\epsilon(k)}{\epsilon_0(k) + \frac{1}{2} \frac{k^4}{\epsilon_0(k)}} \quad (5.7)$$

while the error is expressed in terms of the Lagrange multiplier

$$\psi(r) = -\frac{1}{2} (2\pi)^4 \int d^3k \exp(-2\pi i \underline{k} \cdot \underline{r}) \frac{k^4 R(k)}{\epsilon_0(k)},$$

The ideal case in which there is no error in the determination of the scalar field corresponds to setting $\gamma = 0$. Then equation (3.7) reduces to the canonical solution of the convolution equation given in equation (2.7c).

The modification made in equation (3.7) can be viewed as the action of a low spatial frequency pass filter. The additional term in equation (3.7)

$$\frac{\gamma(2\pi)^4 k^4}{\phi_0(\underline{k})}$$

is negligible at low frequencies compared to $\phi_0(\underline{k})$, but increases rapidly in magnitude with frequency as both k^4 increases and $\phi_0(\underline{k})$ decreases. Thus the information at low frequencies is undistorted whereas the noise at high frequencies is suppressed.

A natural generalization of the filter is

$$\frac{\gamma(2\pi)^m k^m}{\phi_0(\underline{k})} \quad (3.8)$$

where m is a positive integer governing the sharpness of the filter and m corresponds to different powers of $V^2 \rho(\underline{r})$ in the smoothness condition (3.4). By adjusting m an optimum filter can be constructed to suit the noise characteristics of individual imaging system.

A convenient way for specifying γ is to note the surface $S_{1/2}$ in k -space where the two terms in the denominator of equation (3.7) become equal. At these frequencies the original Fourier components of the object are attenuated by a factor of $1/2$. The surface $S_{1/2}$ should be chosen not too close to the origin so that reconstructions are not oversmoothed beyond the desirable resolution.

As for the iterative scheme (2.13), the rate of convergence, i.e., the rate $E_t^{(n)}$ goes to zero, depends on the distribution of $\{\lambda_j\}$ and $\{a_j\}$. The distribution of $\{\lambda_j\}$ is determined by the regions R_a and R_b . In general the region R_b , which represents the extent occupied by the object, is fixed, whereas the region R_a can be changed by varying the angle subtended by the imaging device.

Figure 3.10 shows a plot of the eigenvalues for various opening angles of R_a while R_b is chosen to be a 9×9 square sub-lattice in a 21×21 reconstruction lattice. It can be seen that the spectrum shifts towards zero as the angle decreases. The implication is that the convergence as expressed by equation (2.16) will become worse when the angle of R_a is reduced.

To show this effect we applied the iterative algorithm to restore the missing-cone components for a 2-D phantom. The reconstruction area is a 128×32 lattice, with equal lattice spacings in the $x(i)$ and $z(k)$ directions. The phantom has a square boundary with perpendicular diagonals which are both 11 lattice spacings long in the x and z directions, respectively. The Fourier components of the phantom outside the allowed cone were first set to zero, and then the iterative scheme was employed to recover them. The solid curve in Fig. 3.11 shows the root mean square error σ of the results after 20 iterations as a function of the half-angle of R_a . Here σ is defined as

$$\sigma = \sqrt{\frac{\sum_{i,j,k} (\text{reconstruction}(i,j,k) - \text{phantom}(i,j,k))^2}{\text{total number of picture elements}}} \quad (3.9)$$

Fig. 3.10. Eigenvalues of BA for a two-dimensional problem for various semi-vertical angles of the allowed cone.

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Fig. 3.11. Root mean square error in reconstruction a 2-D phantom as a function of the semi-vertical angle of the allowed cone for various statistics.

Figure 3.12 shows the corresponding results for a three-dimensional phantom. The shape of the point response function ϕ_0 was in the form of a square pyramid with semi-vertical angle θ_0 (Fig. 2.19A). These two results show clearly the dependence of the truncation error $E_t^{(n)}(\underline{k})$ on the size of the opening angle of the allowed cone.

The above results can be viewed as reconstructions from perfect data generated by the phantoms, using deconvolution + iterations. For comparison, deconvolution + iterations were performed on a number of sets of finite statistical positron annihilation events generated by the 2-D phantom. The values of σ for these results are plotted as the broken curves in Fig. 3.11. It can be seen that the truncation error is the main source of error at small angles of R_a , whereas the statistical error dominates at large angles. The minimum in σ which occurs for finite statistical reconstructions is due to the competition of two effects: the improvement in the behavior of the eigenvalues $\{\lambda_i\}$ on the one hand, and the increase in the error magnitude of $\phi(\underline{r})$ on the other, as the angle increases while keeping the number of annihilation events fixed.

Besides the truncation error $E_t^{(n)}$, the measured error $\Delta S(\underline{k})$ in the frequency components in the allowed cone R_a also propagates in the iterations. Following Papoulis [39], we expand $\Delta S(\underline{k})$ in a series of eigenfunctions of BA in the region R_a

$$\Delta S(\underline{k}) = \sum_{i=0}^{\infty} c_i \psi_i(\underline{k}) \quad \underline{k} \in R_a$$

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Fig. 3.12. Root mean square error after 20 iterations as a function of the semi-vertical angle of the allowed cone for a three-dimensional phantom.

$$\text{with} \quad |\Delta S|^2 = \int_{R_a} |\Delta S(\underline{k})|^2 d^3 \underline{k} = \sum_i c_i^2 \lambda_i.$$

Since the iteration is a linear process, the result after n steps of iterations on the measured frequency components, $S(\underline{k}) + \Delta S(\underline{k})$, is given by $R^{(n)}(\underline{k}) + \Delta R^{(n)}(\underline{k})$ where

$$\Delta R^{(n)}(\underline{k}) = \sum_{i=0}^{n-1} c_i (1 - (1-\lambda_i)^n) \phi_i(\underline{k}).$$

An upper bound of the magnitude of the propagated error $\Delta R^{(n)}(\underline{k})$ can be estimated as follows:

$$\begin{aligned} |\Delta R^{(n)}|^2 &= \sum_{i=0}^{n-1} c_i^2 (1 - (1-\lambda_i)^n)^2 \\ &= \sum_{i=0}^{n-1} c_i^2 \lambda_i^2 (1 + (1-\lambda_i) + \dots + (1-\lambda_i)^{n-1}). \end{aligned}$$

Since all the λ_i 's lie in (0,1), we have

$$1 + (1-\lambda_i) + \dots + (1-\lambda_i)^{n-1} < n.$$

Thus

$$|\Delta R^{(n)}|^2 < n^2 \sum_{i=0}^{n-1} c_i^2 \lambda_i = n^2 |\Delta S|^2.$$

Therefore

$$|\Delta R^{(n)}| < n |\Delta S|.$$

3.2.2. Matrix Inversion

The matrix method solves the integral equation (2.8a) in the z dimension for every spatial frequency k_x . The unique solution is given by

$$p(k_X, z) = \sum_1 \frac{(g_1^*, \mathcal{P})}{\alpha_1} g_1(k_X, z)$$

where α_1 , g_1 are the eigenvalues and eigenfunctions of the integral operator (2.8a), and

$$(g_1^*, \mathcal{P}) = \int_{-z_1}^{z_2} g_1^*(k_X, z) \mathcal{P}(k_X, z) dz.$$

If the data $\mathcal{P}(k_X, z)$ contain error $\delta \mathcal{P}(k_X, z)$, then the propagated error in inversion E_m will be given by

$$E_m(k_X, z) = \sum_1 \frac{(g_1^*, \delta \mathcal{P})}{\alpha_1} g_1(k_X, z). \quad (3.10)$$

Again, the expression (3.10) shows that the major errors in the reconstruction come from the small eigenvalues.

Now $\mathcal{P}_0$ is of the form (see Sec. 2.2.3.2)

$$\mathcal{P}_0(k_X, z-z') = \int_{-\theta_0}^{\theta_0} \frac{F(\theta)}{\pi} \exp\{2\pi i k_X \tan\theta (z-z')\} d\theta.$$

For the particular case $F(\theta) = \cos^{-2}\theta$ inside the detection cone, the expression for $\mathcal{P}_0$ simplifies to

$$\begin{aligned} \mathcal{P}_0(k_X, z-z') &= \frac{1}{\pi} \int_{-\theta_0}^{\theta_0} \exp\{2\pi i k_X \tan\theta (z-z')\} d(\tan\theta) \\ &= \frac{\sin[2\pi k_X \tan\theta_0 (z-z')]}{\pi^2 k_X^2 (z-z')} \end{aligned}$$

The eigenvalue equation of this kernel is

$$\lambda_1 g_1(k_X, z) = \int_{z_1}^{z_2} g_1(z') \frac{\sin[2\pi k_X \tan\theta_0 (z-z')]}{\pi^2 k_X^2 (z-z')} dz'. \quad (3.11)$$

On rearranging equation (3.11) becomes

$$(\pi k_{x,i+1}) g_1(k_x, z) = \int_{z_1}^{z_2} \frac{\sin(2\pi k_x \tan \theta_0 (z-z'))}{(z-z')} g_1(k_x, z') dz'$$

which is the zeroth order prolate spheroidal eigenvalue equation. Thus $\{\pi k_{x,i+1}\}$ is a set of zeroth order prolate spheroidal eigenvalues. As pointed out in [35,43], the distribution of those eigenvalue depends on the quantity $c = 2\pi k_x \tan \theta_0 (z_2 - z_1)$, as shown in Fig. 3.13, and for a fixed c the eigenvalues fall off to zero rapidly with increasing i once i exceeds $(2/\pi)c$. This means that noise multiplication would be especially serious whenever k_x , $\tan \theta_0$, or $(z_2 - z_1)$ becomes small.

One way to stabilize the method, for fixed $\tan \theta_0$ and $(z_2 - z_1)$, is to discard the results at small $|k_x|$ where the errors dominate, and, by making use of the finite extent of the object in the x dimension, fill in those values using the results obtained from the higher $|k_x|$ values through the iterative scheme shown in Fig. 3.14.

Another way to stabilize the matrix method is by means of the smoothing procedure proposed by Phillips [42] and Twomey [44]. Instead of solving the ill-conditioned matrix equation

$$Y = AX$$

which is the digital version of equation (2.8a), another matrix equation with a modified kernel

$$Y = (A + \gamma B)X$$

is solved. Here the matrix B is obtained from A in the following manner:

45.74.0112

Fig. 5.15. Eigenvalues of the zeroth order prolate spheroidal equation.

Fig. 3.14. Iterative scheme to stabilize the matrix inversion algorithm.

$$b_{k,z} = a_{k-2,z} - 4a_{k-1,z} + 6a_{k,z} - 4a_{k+1,z} + a_{k+2,z}$$

and γ is a parameter dependent on the noise level in the data. This procedure will remove the instabilities at the high spatial frequencies k_z for each of the operators (2.8a) characterized by different k_x .

3.2.5. Comparison between Deconvolution + Iterations and Matrix Inversion

After analyzing the basic properties of the deconvolution + iterations and the matrix methods, a comparison of their relative merit is now in order. If the data contain no error or only a negligible amount of error, the main error in the result of deconvolution + iterations will come from the truncation error $E_t^{(n)}$ in iterating, as the deconvolution error E_d will be insignificant in this case. For the matrix method, the inversion error E_m will also be negligible. The only unknown solution at $k_x = 0$ can be filled in by continuation from other non-zero k_x values, and the error introduced in continuing the solution to one point will be very small compared to the error $E_t^{(n)}$ in continuing the solution outside R_a for general half-angle α . Thus in the case of very small amounts of noise, the matrix inversion has an advantage over the deconvolution + iterations approach, unless some accelerated scheme can be devised to reduce $E_t^{(n)}$.

To predict their relative performance in the presence of significant amounts of noise, it suffices to compare the condition number of the deconvolution operation with that of the matrix inversion. For deconvolution, the condition number r_d is given by

$$\kappa_d = \frac{(\phi_0(k))_{\max}}{(\phi_0(k))_{\min}}.$$

And for each $k_x \neq 0$, the condition number $\kappa_m(k_x)$ of the integral operator (2.8a) is given by

$$\kappa_m(k_x) = \frac{\alpha_{\max}}{\alpha_{\min}}.$$

Now equation (D.2) in Appendix D shows that $\phi_0(k)$ is in the form

$$\phi_0(k_x, k_z) = \frac{F(\theta_i) \cos^2 \theta_i}{\pi |k_x|}$$

where $\tan \theta_i = -k_z/k_x$, and $F(\theta)$ is the angular factor used in constructing $\phi_0(r)$. For each $k_x \neq 0$, the condition number is thus given by

$$\kappa_d(k_x) = \frac{(F(\theta) \cos^2 \theta)_{\max}}{(F(\theta) \cos^2 \theta)_{\min}}.$$

In general the maximum and minimum values of $F(\theta) \cos^2 \theta$ do not differ by several orders of magnitude; in fact, for the generally used angular factors $F(\theta) = 1$ and $F(\theta) = \cos^{-2} \theta$, $\kappa_d(k_x)$ equals $\sec^2 \theta_0$ and 1 respectively. On the other hand, $\alpha_{\max}$ and $\alpha_{\min}$ can differ by a factor of order hundreds of thousands; in fact, α_i asymptotically approaches zero as the index of the eigenvalue increases, as shown in Appendix I. The decrease of α_i with the index is especially fast at small values of k_x and θ_0 . Thus matrix inversion is expected to be more unstable to noise than deconvolution + iterations.

This comparison is still valid even if the iterative scheme in Fig. 3.14 is employed to stabilize the matrix method. The reason is the following. In iterating, all the solutions from inversion with $|k_x|$ below some $k_0 (> 0)$ are discarded, and they are filled in using the one with $|k_x| \geq k_0$. But as the solution from inversion for every k_x contains both reliable and unreliable components corresponding to the large and small eigenvalues of the integral operator (2.8a) characterized by that k_x , the solutions used to start the iterations for the matrix method always contain some unreliable components. In contrast, the deconvolution results used to start iteration do not have unreliable components.

IV. IMPLEMENTATION

4.1. MWPC-Gamma Converter Positron Camera System

The MWPC-gamma converter positron camera system is shown in Fig. 4.1. The camera structure can be roughly divided into 3 parts, as illustrated in the block diagram in Fig. 4.2. The first group consists of MWPCs, gamma converters and delay lines for detecting the 511 keV annihilation gammas and localizing their positions of interaction. Next is the electronics system for signal processing. It includes low noise amplifiers, timing discriminators, a data selector unit, and an interface unit. The last group of hardware, which processes and stores the data, consists of a Digital Equipment Co. (DEC) PDP 11/20 computer with an extended memory of 28K, and peripherals including a display storage scope, two data storage disc units, a fast paper tape reader/punch, and a Decwriter keyboard terminal.

The sequence of data acquisition is as follows: On detecting a pair of annihilation gammas, 3 amplified signals from each detector (1 prompt anode signal and 2 delay line signals) are fed to multichannel timing discriminators. The 6 discriminator outputs are transmitted to the data selector logic unit which performs decision logic functions such as valid coincident event selection and invalid event inhibit/reset. At the same time the anode signals initiate 4 digitizing scalers which are stopped later by the delay line signals. If the event satisfies some data selection requirements (to be described in Sec. 4.1.2), an interrupt signal is transferred via the buffer unit to the memory of the computer. Otherwise, an inhibit/reset signal is issued by the data

Fig. 7.28b

Fig. 4.2. Schematic configuration of the MMPC positron camera.

XBL 796-10275

selector to all units. Some important components are described in detail in the following sections.

4.1.1. The Detection Assembly

Four MWPCs each with $48\text{ cm} \times 48\text{ cm}$ sensitive area were used to form a pair of detectors, each of which was made up of 2 MWPCs and 2 Pb converters housed in an air-proof Al box. The cathode wires in each chamber were $50\text{ }\mu\text{m}$ in diameter and spaced 2 mm apart. They were terminated through 220 K_{Ω} resistors to a common bus. The anode wires were $20\text{ }\mu\text{m}$ in diameter and spaced 3 mm apart. The chamber frames were made of Nema G-10 fiberglass; the two central frames and the outer frames were each 4 mm thick. On these frames were epoxied the coupling strip PC boards. The delay lines were mounted on the coupling strips with 0.1 mm thick mylar between them and the strips for insulation. The detailed construction of the chamber is shown in Fig. 4.3.

The delay lines used were the phased compensated electromagnetic delay lines for wire chamber readout developed by Grove, Perez-Mendez, et al. [22,45]. The basic design, illustrated in Fig. 4.4, consists of a helical winding of #32 gauge copper Formvar wire on a plastic core with longitudinal copper bands on one side. A mylar strip with etched metallic bands of copper is cemented onto one of the flat sides of the delay line for phase compensation. The delay to rise time ratio is 28:1 and the total delay is $1.1\text{ }\mu\text{sec}$.

The gamma converters were made of Pb shaped in the form of honeycomb, as shown in Fig. 4.5. The cell size is 3.5 mm and cell wall is $75\text{ }\mu\text{m}$. The height of the converter is 15 mm . The detailed procedures

XBL 7311-1476

Fig. 4.3. Multiwire proportional chamber construction, the location of gamma converters is indicated.

- ① Plastic Core
- ② Floating Metal Strips On Mylar Base: Strips = 1.8mm Wide;
Gaps = 0.3mm Wide; Mylar = 25 Microns Thick
- ③ Winding = #30 Formvar Wire
- ④ 8 Copper Strips On Mylar Base: Strips = 1.8mm Wide;
Gaps = 0.3mm Wide; Mylar = 25 Microns Thick

XBL 714-679

Fig. 4.4. MWPC delay line construction.

XBB 7410-7153

Fig. 4.5. A section of the layered, honeycomb shaped gamma converter. Graded voltages are applied through the bus-wires to individual cells.

for constructing these Pb converters were given in [46].

The sensitivity of camera system was measured to be 900 counts/min/ μ Ci. The spatial resolution in FWHM was 8-10 mm for a Na^{22} point source.

4.1.2. The Electronics System

A modified version of a charge-sensitive amplifier (MCSA), shown in Fig. 4.6, was specially designed to be suitable for amplifying signals from either the anode wires or the delay lines. The principle of "electronic cooling" [47] has been utilized in the circuit to reduce noise. The MCSA has an effective bandwidth of 10 MHz and an adjustable gain from 50 to 2000.

Pulse shaping is performed by a RC differentiator at the input to the operational amplifier (μL733). Further differentiation can be accomplished by adding a RC network at the output of the amplifiers. With an output differentiator of time constant 250 nsec, the rise time and fall time of the anode signal from the MCSA are 100 nsec and 400 nsec respectively. The delay line signals, however, have already been differentiated through the capacitive coupling to the wire planes before input to the amplifiers, resulting in a rise time of 100 nsec and fall time 250 nsec. Therefore no differentiation is necessary at the output of a delay line amplifier.

The spatial resolution is determined by the time interval between the arrival of the prompt anode signal and the delayed cathode signal, and hence depends critically on the timing accuracy of the timing discriminator. The timing discriminator is of the differentiating zero-cross type which operates on the principle that the occurrence-time of the peak

is common to a family of pulses, regardless of their amplitude [48].

The contributions to the timing inaccuracies include time slew which arises from the difference in pulse amplitude, and time jitter which is caused by the noise [49]. For the timing discriminator we used, the time slew is 6 nsec over a dynamic range of 20 in signal amplitude, and the time jitter associated with 100 mV signal and 20 mV noise is 8 nsec; but improves to 1 nsec as the input signal is increased to 1 V.

The Data Selector unit is made up of a network of flip-flops and gates to which the outputs from the timing discriminators are applied. Figure 4.7 shows its logic diagram. An event will be validated if it satisfies the following conditions:

Prompt anode signals

1. Only one of the two MWPCs is triggered in each of the upper and lower detector boxes (2 chambers were housed in one detector box).
2. When one of the two chambers in a detector box is triggered, the coincident signal from one of the chambers in the opposite detector box must arrive within the coincidence resolving time.

Delayed cathode signals

3. If conditions (1) and (2) in the prompt signals are met, there must exist 4 delayed cathode signals associated with the chambers which were triggered.

An event is rejected by the initiation of an inhibit/reset signal if,

Fig. 4.7. Logic diagram of the data selector.

4. Any one of the conditions (1), (2), (3) is not satisfied.
5. A second prompt signal should occur on either one of the four chambers before a valid coincident signal is initiated.
6. More than one signal should occur on any one of the delay lines within delay line time (1.5 μ sec) before a valid signal is initiated.

Whenever any one of the four MWPCs is triggered, further signals from that particular chamber would be gated off for 1.5 μ sec while the other detectors are still operative. This would prevent accidental coincidence and minimize the detector system dead time.

4.2. Reconstruction Results

We have tested the results developed in Sections 2.2 and 3.2 by performing studies on some computer-generated phantoms and a real phantom. As the computations had to be done on digital computers, all the mathematical quantities were digitized on a finite-extent lattice with lattice spacings Δx , Δy , Δz , the choice of which is governed by the spatial resolution of the imaging device and the available core memory capacity of the computer. In digitizing we have restricted ourselves to frequency components below the maximum frequencies given by

$$k_x(\text{max}) = \frac{1}{2\Delta x}, \text{ etc.}$$

The components at frequencies above this value were very small in magnitude as a consequence of the finite resolution, and were set to zero for computational purposes. The constraint that the density distribution is

non-negative was also utilized in the reconstructions.

The use of lattice of finite extent introduces errors into the results of the Fourier transforms: the finite lattice spacing gives rise to "aliasing", while the finite extent causes "leakage" [50]. Aliasing, the distortion of the desired Fourier transform due to sampling, is minimized by making the lattice spacings Δx , Δy , Δz small enough, while leakage, the distortion due to truncation, can be reduced by windowing [51]. In this work Gaussian window functions appeared to give the best results, but the type of window did not seem to be critical. Windowing is not required when taking the inverse transform of $R(\underline{k})$, as $R(\underline{k})$ is a periodic function with period equal to the truncation interval in which case there is no leakage.

All the Fourier transforms were performed using Fast Fourier Transform algorithms: if each dimension of the array being transformed was a power of 2, the subroutine FOUR2 by K. F. Subhani and F. Chu (private communication) was employed, otherwise the subroutines FFT and REALR by Singleton [52] for computing mixed radix Fourier transform were used. The subroutine REALR has been modified to three dimensions by the author.

4.2.1. Computer-Simulated Data

We first examined the capability of the iterative scheme shown in Fig. 2.21 to recover missing-cone components. Figures 3.11-3.12 give only the relative magnitude of the truncation error in iterating as a function of the angular size of the allowed cone. To get a qualitative feeling of how well the iterations work at small allowed-cone angles,

we applied the algorithm to restore the missing-cone components of a 2-D point source located inside the square boundary of the 2-D phantom of Fig. 3.11. The assumed allowed cone subtends a semi-vertical angle of $\tan^{-1}(0.5)$ along the k_x -axis.

Figure 4.8A shows the shape of the point source at the center of the square with the Fourier components in the missing cone set to zero. Only the middle 32×32 picture elements are shown. Two kinds of distortions are seen. Firstly, the point source is considerably widened. Secondly, decaying oscillating ridges appear on the edges of the detection cone corresponding to the assumed missing cone and centered at the point source.

Figure 4.8B shows the same point source after 30 iterations. The ridges are lowered in height. Also, the point source is significantly narrowed.

Figure 4.9 illustrates the corresponding results for a point source located on the boundary of the square: in this case at the corner with the lowest k -index. The improvement after iterations is much more impressive in this case than that with the point source at the center of the square. The greater improvement is due to the fact that for the point source on the boundary, at least two of the four decaying ridges lie outside the square and thus are repeatedly reset to zero in iterating; whereas for the point source at the center, only those low-amplitude lobes of the ridges far away from the point source are reset to zero, producing smaller effect in restoring the missing-cone components in comparison.

YSL 7910-12148

Fig. 4.8. Recovering the missing-cone components of a 2-D point source located at the center of a square which acts as the finite object extent in the iterations. The semi-vertical angle of the allowed cone is $\tan^{-1}(0.5)$.

(A) The point source with the missing-cone components set to zero.

(B) The point source after 30 iterations.

XBL 7910-12147

Fig. 4.9. Recovering the missing-cone components of a 2-D point source located on the boundary of a square which acts as the finite object extent in the iterations. The semi-vertical angle of the allowed cone is $\tan^{-1}(0.5)$.

(A) The point source with the missing-cone components set to zero.

(B) The point source after 30 iterations.

We then tested the effect of the size of the camera angle on deconvolution using the computer-generated 3-D phantom shown in Fig. 4.10, which was a spherical skull with a tumor located off-center inside.

The skull was of inner radius 9 cm and thickness 2 cm, while the tumor had radius 1.5 cm. The concentration ratio was

$$\text{tumor:skull:background} = 10:5:1.$$

We performed reconstructions using the deconvolution method in three positron camera configurations with three different point response functions. The first point response function subtended a viewing angle (the solid angle of the detection cone in Fig. 2.19A) of one-third of 4π along the z-axis; the second one subtended two-thirds of 4π ; one-third along the z-axis and the other third along the y-axis; and the last one had complete 4π viewing angle. The corresponded to camera configurations having one, two, and three pairs of detectors, respectively, with the axes joining each pair perpendicular to each other. The last configuration, though not very realistic practically, was included for making comparisons, because its point response function obviously contained no zero Fourier components. According to condition (2.11) in Sec. 2.2, the first point response function contained zero Fourier components, whereas the second one did not. The phantom generated a total number of 1.2 million events in each case. Reconstructions were done on a $48 \times 48 \times 48$ lattice, with lattice spacings $1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm}$. Each event was weighted by $F(\theta) = \cos^{-3}\theta$ in constructing $\phi(\underline{r})$. $R(\underline{k})$ was set to zero wherever $\phi_0(\underline{k})$ is zero or close to zero. The value of γ was set to zero in all the 3 deconvolutions.

XBL 797-10701

Fig. 4.10. A computer-generated phantom simulating a brain tumor. The skull is 2 cm thick and has an inner radius of 9 cm. The tumor has a radius of 1.5 cm. The concentration is 5:1 for the skull and 10:1 for the tumor. Each picture element is $1 \times 1 \text{ cm}^2$. The planes are 3 cm apart in the z direction.

The deconvolved images obtained in each of the camera configurations are shown in Figs. 4.11-4.13. For the sake of comparison, the image obtained by back-projecting the data in the two-sided camera configuration is illustrated in Fig. 4.14. Figures 4.15-4.18 show the profile of the various reconstruction results along the lattice line with lateral indices $i = j = 25$, which runs parallel to the z -axis passing through the tumor; the corresponding profile of the phantom is shown as the dotted histogram in each of these figures. Table 4.1 summarizes the values of σ of these results as defined in equation (3.9).

It can be seen that quality of the deconvolved image in the four-sided camera is essentially the same as that in the camera with 4π viewing angle, showing clearly that the Fourier components of the four-sided camera point response function are also non-zero everywhere, as stated above. On the other hand, the deconvolved image in the two-sided camera is considerably worse than the other two, as can be expected from the presence of zero components in its Fourier transform.

Next the iterative scheme was applied to recover the missing-cone components of the deconvolved image in the first camera configuration (two detectors: $1/3 \times 4\pi$ viewing angle). The values of σ as a function of the number of iterations are shown in Fig. 4.19. It can be seen that the scheme converged at about 10 iterations. The image after 10 iterations is illustrated in Fig. 4.20, and its profile along the lattice line with lateral indices $i = j = 25$ is shown in Fig. 4.21. After iterations σ is reduced to a level much closer to, though still higher than, that obtained using the point response function with 4π viewing angle,

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Fig. 4.11. Deconvolved image in the two-sided camera having a point response function with viewing angle one-third of 4π along the z-axis. Each event is weighted by $\cos^{-3}\theta$ in constructing ϕ . The value of γ is 0.

XBL 797-10702

Fig. 4.12. Deconvolved image in the four-sided camera having a point response function with viewing angle one-third of 4π along the z-axis and one-third of 4π along the y-axis. Each event is weighted by $\cos^{-3}\theta$ in constructing ϕ . The value of γ is 0.

XBL 797-10706

Fig. 4.13. Deconvolved image in the six-sided camera having a point response function with complete 4π viewing angle. Each event is weighted by $\cos^{-3}\theta$ in constructing ϕ . The value of γ is 0.

XBL 797-10705

Fig. 4.14. Back-projection image in the two-sided camera.

Fig. 4.15. Profile of the back-projection image in Fig. 4.14 along the line $i = j = 25$.

Fig. 4.16. Profile of the deconvolved image in Fig. 4.11 along the line $i = j = 25$.

Fig. 4.17. Profile of the deconvolved image in Fig. 4.12 along the line $i = j = 25$.

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Fig. 4.18. Profile of deconvolved image in Fig. 4.13 along the line $i = j = 25$.

Table 4.1. Root mean square error of the images.

Number of detectors in the positron camera	2	4	6	
Reconstruction	Back - projection	Deconvolu- tion	Deconvolu- tion	Deconvolu- tion
σ	1.652	1.000	0.582	0.569

XBL 798-2382

Fig. 4.19. The values of the root mean square deviation of the deconvolved image in the two-sided camera as a function of the number of iterations.

XBL 797-10703

Fig. 4.20. The deconvolved image in Fig. 4.11 after 10 iterations using the scheme shown in Fig. 2.21.

XBL 7910-12144

Fig. 4.21. Profile of the image in Fig. 4.20
along the line $i = j = 25$.

there being a factor of three in the reduction of the difference in σ . The residual difference is attributed to the truncation error in terminating the iterations and the propagation of errors in iterating, as discussed in detail in Sec. 3.2.1.

4.2.2. Real Data

The deconvolution-and-iterations reconstruction algorithm was applied to the real data generated by a cylindrical head phantom in the MWPC-Pb converter positron camera. The head contained 2 simulated tumors: one had activity concentration of 5:1, and the other 10:1. The head phantom was immersed in a container filled with Ga^{68} (concentration 5:1) simulating the peripheral activity around the 'skull'. The 'skull' was in the form of a cylindrical shell which was 1.5 cm thick and 16.5 cm high, with an inner radius of 7.5 cm. The two tumors were also cylindrical in shape, with 1 cm radius and 4 cm height.

For this phantom we have used a lattice of dimensions $64 \times 64 \times 30$, and the lattice spacings were $.5 \times .5 \times 2 \text{ cm}^3$. The point response function ϕ_0 was in the form of a square pyramid with semi-vertical angle $\theta_0 = \tan^{-1} (0.5)$. 437229 annihilation events were used in the reconstructions.

Figure 4.22 shows the results obtained by simple back-projection with solid angle limitation. The images of the tumors are almost completely buried under the off-plane activities and can hardly be seen. Figure 4.23 was obtained from Fig. 4.22 by subtracting from the content of each pixel 55% of the peak activity. This helps to bring out the contrast, and the tumors faintly appear near the central planes.

Fig. 4.22. Back-projection image of a real phantom which consists of a cylindrical shell to simulate a skull, and 2 cylindrical tumors. The shell is 1.5 cm thick, 16.5 cm high, and has an inner radius of 7.5 cm. Each tumor is 4 cm high and has a radius of 1 cm. The concentration ratio is 5:1 for the skull and one tumor, and 10:1 for the other. Each picture element is $0.5 \times 0.5 \text{ cm}^2$. The planes are 2 cm apart in the z direction.

XBB 799-11798

Fig. 4.22.

Fig. 4.23. Back-projection of the data from the real phantom with 55% background subtraction.

Figure 4.24 shows the results obtained by deconvolution + iterations using the angular factor $F(\theta) = \cos^{-12}\theta$ in constructing the scalar fields. The point response function that was used was obtained by fitting a smooth analytic function to the experimentally measured values. Compton scattering has the effect of making the point response function non-zero in the missing cone. Though these missing-cone components are relatively small in magnitude and thus are less reliable, they nevertheless furnish some useful information. For this reason the missing-cone components of $R(\underline{k})$ were not set to zero before iterating. 4 iterations were performed. The parameters used in the frequency filter were $m = 10$ and $\gamma = 35.9$. The two tumors and the skull region are clearly visible, and well separated.

As a comparison, the result of deconvolution + iterations using the angular factor $F(\theta) = \cos^4\theta$ in the scalar fields is shown in Fig. 4.25. The parameters used in the frequency filter were also $m = 10$ and $\gamma = 35.9$. The two tumors and the skull also appear distinctly, but the two tumors are not so well separated as those in Fig. 4.24. Their difference verifies the conclusion in Sec. 3.1.2 that putting more weight on the large-angle events would improve the signal-to-noise ratio in data affected by scattering.

A

B

C

Fig. 4.24. Reconstruction of the real phantom by Fourier deconvolution + 4 iterations.

Angular factor: $F(\theta) = \cos^{-12}\theta$.

Filtering parameters: $m = 10$, $\gamma = 35.9$.

XBB 799-11800

A

B

C

Fig. 4.25. Reconstruction of the real phantom by Fourier deconvolution + 4 iterations.
Angular factor: $F(\theta) = \cos^4\theta$.
Filtering parameters: $m = 10$, $\gamma = 35.9$.

V. DISCUSSIONS AND CONCLUSIONS

5.1. Discussions

Compared to cameras using detectors such as NaI, the present MWPC-Pb converter position camera has excellent spatial resolution and the advantage of low cost, but suffers in the area of inadequate detection efficiency. Roughly speaking, the conversion efficiency of the gamma converter is determined by the total amount of surface area available for interacting with the gammas, and thus can be increased either by decreasing the cell size of the converter, as illustrated in Fig. 2.6, or by increasing its height. Both decrease in cell size and increase in height are made possible through the use of PbO-glass converter [53].

Due to its more uniform electric drift field, the drift time spread in electron collection T_c in a PbO-glass converter is much shorter than in a Pb converter, being 110 nsec for a 15 mm high PbO-glass converter compared to 330 nsec for a Pb converter of the same height. The reduction in T_c makes it possible to use converters of increased height. The increased uniformity of the drift field also means higher extraction efficiency ϵ_e , as implied in Fig. 2.9, and thus converters with smaller cell size can be used. Fabrication of PbO-glass converters with cell size ~ 1 mm is readily achievable by fusing together PbO-glass tubings followed by slicing. Recent measurements on a PbO-glass converter of cell size 1.4 mm and height 2 cm yielded 8% detection efficiency [54]. This translates into a nine-fold increase in the coincidence count rate over the present camera system equipped with Pb converters.

Equation (2.4) shows that the reduction by a factor of 3 in T_c , and thus T_r , carries with it a corresponding gain in the true-to-chance-coincidences ratio. Alternately, for a fixed ratio of true to chance coincidences, the PbO-glass converter can afford going to much higher count rate, and thus shortening the exposure time. This is evident from the curves shown in Figs. 2.12 - 2.13.

The use of PbO-glass also gives better spatial resolution, due to the reduced T_c and smaller cell size. On a test chamber, a PbO-glass converter of cell size 2 mm and height 15 mm gave a spatial resolution of 3 mm for a Na^{22} point source, compared to 5 mm for a Pb converter of the same height with cell size 2.5 mm.

The continuation method developed in Sec. 2.2.3.2 is not restricted to positron imaging only. In transmission imaging using parallel beams of x-rays and some other applications, the measured data are in the form of projections $P(r, \theta)$ of the object at certain angles θ . By the projection theorem [55], the one-dimensional Fourier transform of $P(r, \theta)$ in r gives the components of the two-dimensional Fourier transform of $\rho(x, z)$ on a line at angle $(\theta + \pi/2)$. If the projections are taken in a limited range of angle, we will get the Fourier components of $\rho(x, z)$ within a cone, giving rise to the situation shown in Fig. 2.20. The process of analytic continuation (2.13) can therefore be applied to obtain the rest of the Fourier components of $\rho(x, z)$.

The iterative scheme (2.13) works very well at large allowed-cone angle, but deteriorates rapidly when the angle is reduced as the spectrum of the eigenvalues shifts towards zero. This poses a practical

limitation on the minimum size of the angle of the allowed cone that can be used in limited-angle imaging. The minimum practical allowed-cone angle is primarily governed by the accuracy of the digital computer employed for computation, which sets a limit on the minimum magnitude of the eigenvalues that can be computed, and hence determines the number of eigenfunction components of the object that can be recovered in the iterations.

One way to improve on the situation is to use direct inversion to solve for the missing-cone frequency components instead of using iterations. Sabri and Steenaart developed the extrapolation matrix [56] for such a purpose. This approach requires inverting a matrix of order $m \times m$, where m is the number of frequency components in the allowed cone and missing cone combined. As m usually ranges from a few thousand upwards in two-dimensional and even more in three-dimensional imaging, extrapolation matrix is not very practical in such applications.

Another approach is to devise schemes to accelerate the iterations, possibly through the use of other constraints on the object or its Fourier components, such as boundedness in object density, continuity, etc. Positivity [57, 58] has already been utilized in our iterations: all the negative values in the object density were set to zero after each Fourier transform. This procedure produced a 20% reduction in σ for the 2-D point sources in Figs. 4.8 - 4.9. For ordinary phantoms with extended regions of positive density, the reduction in σ was much smaller, about 0.5%, due to swamping of most of the negative-undershoot artifact of each point source by the positive density values of other point sources.

In implementing the deconvolution + iterations algorithm, two Fourier transforms are required for the deconvolution process, and two more for each iteration. For an $N_x \times N_y \times N_z$ array, deconvolution + n iterations requires approximately $2(n+1)N_x N_y N_z \log_4 (N_x N_y N_z)$ complex multiplications. In comparison, the matrix inversion algorithm needs $N_x N_y N_z (2 \log_4 (N_x N_y N_z) + N_z)$ complex multiplications. For a CDC 7600 computer with a cycle time of 27.5 nsec, Fast Fourier transform of an $32 \times 32 \times 32$ array takes 0.2 sec, and 5 sec for an $64 \times 64 \times 30$ array (different Fast Fourier Transform algorithms were used for the two arrays). If computing has to be done on a small machine with limited high speed storage, data could be stored in slower memory devices such as tape or drum. Separate transforms are computed and then combined [59, 60, 61].

5.2. Conclusions

We have shown that MWPC equipped with gamma converters is capable of imaging the distribution of β^+ -active radioisotopes in three dimensions. The Fourier deconvolution + iterations algorithm provides a practical method for reconstructing the object from the positron camera data. The use of a suitable angular factor $F(\theta)$ in constructing the data scalar field ϕ improves the signal-to-noise ratio through digital filtering of the scattered events. In principle, one can reconstruct the object completely no matter how small the solid angle the positron camera subtends, though the noise in the data and the numerical implementation pose a practical limit to the quality of the reconstruction. The method of analytic continuation by iterations is applicable to other limited-angle imaging problems.

APPENDIX A

$$\text{EVALUATION OF } I = \int_0^{t/2} (L + 2x) P_{\text{esc}}(x, E_1) dx$$

We consider the following 2 cases:

Case 1 $R(E_1) < t/2$

$$t - x > 2 R(E_1) - x > R(E_1)$$

$$\therefore P(t - x, E_1) = 0.$$

In this case we only have to evaluate

$$\begin{aligned} & \int_0^{t/2} (L + 2x) P(x, E_1) dx \\ &= \int_0^{R(E_1)} (L + 2x) P(x, E_1) dx. \end{aligned}$$

Putting $y = \frac{1}{1 + \frac{x}{R(E_1)}}$ and integrating we get

$$I = \frac{R(E_1)}{A^2} [2R(E_1) + L(A + 1)].$$

Case 2 $R(E_1) \geq t/2$

$$\begin{aligned} I &= \int_0^{t/2} (L + 2x) P_{\text{esc}}(x, E_1) \\ &= \int_0^{t/2} (L + 2x) P(x, E_1) dx + \int_0^{t/2} (L + 2x) P(t - x, E_1) dx. \end{aligned}$$

Changing variables and collecting terms, we have

$$\begin{aligned}
 I &= I_1 \int_0^t P(x, E_1) dx + 2 \left\{ \int_0^{t/2} x P(x, E_1) dx + \int_{t/2}^t x P(x, E_1) dx \right\} \\
 &\quad + 2t \int_{t/2}^t P(x, E_1) dx \\
 &= I_1 + I_2 + I_3
 \end{aligned}$$

$$\begin{aligned}
 I_1 &= I \int_0^t P(x, E_1) dx \\
 &= I \exp(A) \int_0^t \exp\left(\frac{-Ax}{1 - \frac{x}{R(E_1)}}\right) \left(1 - \frac{x}{R(E_1)}\right)^{-5} dx \\
 &= \frac{I \cdot R(E_1)}{A} \exp(A) \left(f(1) - \begin{cases} f(y_{t,1}) \\ f(0) \end{cases} \right) \quad \begin{array}{l} \text{if } t \leq R(E_1) \\ \text{if } t > R(E_1) \end{array}
 \end{aligned}$$

where

$$f(y) = \exp(-Ay) \left(y + \frac{1}{A}\right), \quad y_{t,1} = \left(1 - \frac{t}{R(E_1)}\right)^{-1}.$$

To find I_2 , we first evaluate the integral

$$\begin{aligned}
 &\int_0^t x P(x, E_1) dx \\
 &= \exp(A) \int_0^t x \exp\left(\frac{-Ax}{1 - \frac{x}{R(E_1)}}\right) \left(1 - \frac{x}{R(E_1)}\right)^{-5} dx.
 \end{aligned}$$

Putting $y = \left(1 - \frac{x}{R(E_1)}\right)^{-1}$, we get

$$\int_0^t x P(x, E_1) dx = \frac{\exp(A) \cdot R(E_1)^2}{A} \exp(-Ay) \left(1 - y + \frac{1}{A}\right).$$

Substituting into I_2 we obtain

$$I_2 = \frac{2 \exp(A) R^2(E_1)}{A} \left(2g(y_{t/2,i}) - g(1) - \begin{cases} g(y_{t,i}) \\ 0 \end{cases} \right) \begin{cases} \text{if } t \leq R(E_1) \\ \text{if } t > R(E_1) \end{cases}$$

where

$$g(y) = \exp(-Ay) \left(1 - y - \frac{1}{A} \right).$$

Similarly we have

$$I_3 = \frac{2t}{A} \left(f(y_{t/2,i}) - \begin{cases} f(y_{t,i}) \\ 0 \end{cases} \right) \begin{cases} \text{if } t \leq R(E_1) \\ \text{if } t > R(E_1) \end{cases}.$$

The results are summarized as follows:

$$\begin{aligned} t \leq R(E_1), \quad I = \frac{\exp(A) R(E_1)}{A} \left\{ L[f(1) - f(y_{t,i})] \right. \\ \left. + 2R(E_1) [2g(y_{t/2,i}) - g(y_{t,i}) - g(1)] \right. \\ \left. + 2t[f(y_{t/2,i}) - f(y_{t,i})] \right\} \end{aligned}$$

$$\begin{aligned} R(E_1) < t \leq 2R(E_1), \quad I = \frac{\exp(A) R(E_1)}{A} \left\{ L f(1) + 2R(E_1) [2g(y_{t/2,i}) - g(1)] \right. \\ \left. + 2t f(y_{t/2,i}) \right\} \end{aligned}$$

$$2R(E_1) < t, \quad I = \frac{R(E_1)}{A^2} [2R(E_1) + L(A+1)].$$

APPENDIX B

TRACK-LENGTH DISTRIBUTION INSIDE AN INFINITELY LONG CYLINDER WITH SOURCE POINT OUTSIDE THE CYLINDER

Consider the family of lines generated from the source point (located at the origin) and intersecting the cylinder (see Fig. B.1). Let the line characterized by the polar angle and the azimuth (θ, ϕ) intersect the surface of the cylinder at distances R_1 and R_2 from the source point. R_1 and R_2 both satisfy the equation

$$(R \sin\theta \cos\phi)^2 + (R \sin\theta \sin\phi + b)^2 = a^2.$$

Solving the quadratic equation gives

$$R_{1,2} = \frac{2b \sin\theta \sin\phi \pm \sqrt{4b^2 \sin^2\theta \sin^2\phi - 4 \sin^2\theta (b^2 - a^2)}}{2 \sin^2\theta}.$$

The track length inside the cylinder ΔR is

$$\begin{aligned} \Delta R = |R_1 - R_2| &= \frac{2\sqrt{b^2 \sin^2\phi - (b^2 - a^2)}}{|\sin\phi|} \\ \frac{\Delta R}{2a} &= \frac{\sqrt{(\frac{b}{a})^2 \sin^2\phi - ((\frac{b}{a})^2 - 1)}}{|\sin\phi|}. \end{aligned}$$

Define $r = \frac{\Delta R}{2a}$, $c = \frac{b}{a}$, $x = \cos\phi$. The above expression then becomes

$$r = \frac{\sqrt{c^2 \sin^2\phi - (c^2 - 1)}}{\sqrt{1 - x^2}}.$$

Due to the symmetry of the geometry, the domains of ϕ and x can be taken to be $(\phi_0, \pi/2)$ and $(0,1)$ respectively, where $\phi_0 = \sin^{-1} \frac{\sqrt{c^2 - 1}}{c}$.

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Fig. B.1. Geometry for calculating the track-length distribution inside an infinitely long cylinder.

This distribution functions of ϕ and x are given by

$$f_{\phi}(\phi) = \begin{cases} 0 & 0 \leq \phi < \phi_0 \\ \frac{1}{\pi/2 - \phi_0} & \phi_0 \leq \phi \leq \frac{\pi}{2} \end{cases} \quad (\text{B.1})$$

$$f_x(x) = 1. \quad (\text{B.2})$$

We consider the following two cases:

Case 1: $r < 1$

Following [62], we define the following

$$r = \frac{\sqrt{c^2 \sin^2 \phi - (c^2 - 1)}}{\sqrt{1 - x^2}} \quad (\text{B.3})$$

$$w = x. \quad (\text{B.4})$$

Solving (B.3) and (B.4) yields

$$\phi = \sin^{-1} \frac{\sqrt{r^2(1-w^2) + (c^2 - 1)}}{c} \quad (\text{B.5})$$

$$x = w.$$

From (B.5) we get

$$\cos \phi = \frac{\sqrt{1 - r^2(1 - w^2)}}{c}.$$

The Jacobian is then given by

$$J \left(\begin{matrix} r, w \\ \phi, x \end{matrix} \right) = \frac{c^2 \sin \phi \cos \phi}{\sqrt{1 - x^2} \sqrt{c^2 \sin^2 \phi - (c^2 - 1)}} \frac{\sqrt{r^2(1 - w^2) + (c^2 - 1)} \sqrt{1 - r^2(1 - w^2)}}{r(1 - w^2)}. \quad (\text{B.6})$$

The distribution of (r, w) is obtained from (B.1), (B.2) and (B.6):

$$\begin{aligned} f_{rw}(r, w) &= \frac{f_{\phi, x}(\phi, x)}{J\left(\frac{r, w}{\phi, x}\right)} \\ &= \frac{f_{\phi}(\phi) f_x(x)}{J\left(\frac{r, w}{\phi, x}\right)} \\ &= \frac{2r}{\pi} \frac{(1-w^2)}{\sqrt{r^2(1-w^2) + (c^2-1)}\sqrt{1-r^2(1-w^2)}}. \end{aligned}$$

The track-length distribution is thus given by

$$f(r) = \frac{2r}{\pi} \int_0^1 \frac{(1-w^2)dw}{\sqrt{r^2(1-w^2) + (c^2-1)}\sqrt{1-r^2(1-w^2)}}.$$

Case 2: $r > 1$

In this case we put

$$r = \frac{\sqrt{c^2 \sin^2 \phi - (c^2-1)}}{\sqrt{1-x^2}}$$

$$w = \phi.$$

Following exactly the same procedure used in case 1, we get

$$f(r) = \frac{2}{\pi r^3} \int_0^{\pi/2} \frac{c^2 \sin^2 w - (c^2-1)}{\sqrt{1 - \frac{c^2 \sin^2 w - (c^2-1)}{r^2}}} dw.$$

APPENDIX C.

DERIVATION OF THE FACTOR $C(T_r/T_c)$ IN THE TRUE COINCIDENCE RATE WHEN $T_r \leq T_c$.

If $T_r \leq T_c$, true coincidence rate and hence the ratio of true to chance coincidences as given by equations (2.2) and (2.4) are modified by a factor $C(T_r/T_c)$, which is the probability that the two electrons e_1 and e_2 generated in detector A and detector B respectively in an annihilation event will be in coincidence within the time T_r .

Referring to Fig. C.1, this quantity is given by

$$\begin{aligned} C(T_r/T_c) &= \text{probability that } |t_1 - t_2| \leq T_r \\ &= P \left\{ |t_1 - t_2| \leq T_r \right\}. \end{aligned}$$

We consider the following two cases:

Case 1. $T_r < T_c/2$

Partition the time interval T_c for detector A into the 3 intervals I, II and III, as shown in Fig. C.2.

If e_1 occurs within the interval I,

$$P \left\{ e_1 \text{ occurring within } dt_1 \text{ at } t_1 \right\} = \frac{dt_1}{T_r}$$

$$P \left\{ |t_1 - t_2| \leq T_r \mid e_1 \text{ occurring within } dt_1 \text{ at } t_1 \right\} = \frac{t_1 + T_r}{T_c},$$

$$\begin{aligned}
 \therefore C_I(T_R/T_C) &= \int_0^{T_R} P \left\{ |t_1 - t_2| \leq T_R \mid e_1 \text{ occurs within } dt_1 \text{ at } t_1 \right\} \\
 &\quad \times P \left\{ e_1 \text{ occurs within } dt_1 \text{ at } t_1 \right\} \\
 &= \int_0^{T_R} \frac{(t+T_R)}{T_C} \frac{dt_1}{T_R} \\
 &= \frac{3}{2} \frac{T_R}{T_C} .
 \end{aligned}$$

By symmetry,

$$C_{III}(T_R/T_C) = C_I(T_R/T_C) = \frac{3}{2} \frac{T_R}{T_C} .$$

If e_1 occurs within the interval II,

$$C_{II}(T_R/T_C) = \frac{2T_R}{T_C} .$$

Combining the contributions from I, II and III we get

$$\begin{aligned}
 C(T_R/T_C) &= P \left\{ e_1 \text{ occurs within I} \right\} \times C_I(T_R/T_C) \\
 &\quad + P \left\{ e_1 \text{ occurs within II} \right\} \times C_{II}(T_R/T_C) \\
 &\quad + P \left\{ e_1 \text{ occurs within III} \right\} \times C_{III}(T_R/T_C) \\
 &= \frac{T_R}{T_C} \frac{3}{2} \frac{T_R}{T_C} + \frac{T_C - 2 T_R}{T_C} \cdot \frac{2 T_R}{T_C} + \frac{T_R}{T_C} \frac{3}{2} \frac{T_R}{T_C} \\
 &= 2 (T_R/T_C) - (T_R/T_C)^2 .
 \end{aligned}$$

Fig. C.1. Occurrence of detected signals.

Fig. C.2. Partition for $T_r < T_c/2$.

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Fig. C.3. Partition for $T_r \geq T_c/2$.

Case 2. $T_r \geq T_c/2$

Partition the time interval T_c for detector A into the 3 intervals I, II and III, as shown in Fig. C.3. Similar to case 1, we have

$$C_I (T_r/T_c) = C_{III} (T_r/T_c) = \frac{3}{2} \frac{T_r}{T_c}.$$

For the interval II,

$$C_{II} (T_r/T_c) = 1.$$

Combining the contributions from I, II, and III we have

$$\begin{aligned} C (T_r/T_c) &= \frac{T_c - T_r}{T_c} \frac{3 T_r}{2 T_c} + \frac{2 T_r - T_c}{T_c} + \frac{T_c - T_r}{T_c} \frac{3 T_r}{T_c} \\ &= 2 (T_r/T_c) - (T_r/T_c)^2. \end{aligned}$$

APPENDIX D

EVALUATION OF THE TWO-DIMENSIONAL OPTICAL TRANSFER FUNCTION

$$\begin{aligned}\phi_0(k_x, k_z) &= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \phi_0(x, z) \exp(2\pi i(k_x x + k_z z)) \\ &= \int_{-\infty}^{\infty} dz \int_{-\pi}^{\pi} d\theta \phi_0(\theta, z) \exp(2\pi i z(k_x \tan\theta + k_z)) |z| \sec^2\theta \\ &= \int_{-\theta_0}^{\theta_0} d\theta \frac{F(\theta)}{\pi} \int_{-\infty}^{\infty} dz \exp(2\pi i z(k_x \tan\theta + k_z)).\end{aligned}$$

Now

$$\int_{-\infty}^{\infty} \exp(2\pi i z(k_x \tan\theta + k_z)) dz = \delta(k_x \tan\theta + k_z) \quad (D.1)$$

therefore

$$\begin{aligned}\phi_0(k_x, k_z) &= \int_{-\theta_0}^{\theta_0} d\theta \frac{F(\theta)}{\pi} \delta(k_x \tan\theta + k_z) d\theta \\ &= \begin{cases} \frac{\delta(k_z)}{\pi} \int_{-\theta_0}^{\theta_0} F(\theta) d\theta & \text{if } k_x = 0 \\ \frac{F(\theta_1) \cos^2\theta_1}{\pi |k_x|} & \text{if } k_x \neq 0 \end{cases} \quad (D.2)\end{aligned}$$

$$\text{where } \tan\theta_1 = -\frac{k_z}{k_x}.$$

APPENDIX E

EVALUATION OF THE THREE-DIMENSIONAL OPTICAL TRANSFER FUNCTION

In three dimensions, equation (2.9) takes the form

$$\phi_0(\underline{s}, z) |z|^2 = H(\underline{s}/z) \quad (E.1)$$

where $\underline{s} = (x, y)$, and the angular function H is positive inside the data cone and zero outside. Defining $\underline{t} = \underline{s}/z$, $\underline{w} = (k_x, k_y)$, and performing the Fourier transformation, we get

$$\begin{aligned} \phi_0(\underline{w}, k_z) &= \int dz \iint d^2 \underline{t} \phi_0(\underline{s}, z) \exp(2\pi i (\underline{w} \cdot \underline{s} + k_z z)) \\ &= \iint H(\underline{t}) \delta(\underline{w} \cdot \underline{t} + k_z) d^2 \underline{t}. \end{aligned}$$

The vector $\underline{t} = (t_1, t_2)$ can be chosen so that t_1 lies along $\underline{w}$, giving $\underline{w} \cdot \underline{t} + k_z = |\underline{w}| t_1 + k_z$. Then we get

$$\phi_0(\underline{w}, k_z) = \begin{cases} \delta(k_z) \iint H(\underline{t}) d^2 \underline{t} & \text{if } \underline{w} = 0 \\ \int H(-k_z/|\underline{w}| - t_2) dt_2 & \text{if } \underline{w} \neq 0. \end{cases} \quad (E.2)$$

In the particular case where ϕ_0 is in form of a square pyramid, i.e.,

$$\begin{aligned} \phi_0(x, y, z) &> 0 \text{ whenever } 0 \leq |x| \leq |z \tan \theta_0| \\ &\text{and } 0 \leq |y| \leq |z \tan \theta_0| \end{aligned}$$

application of (E.2) shows that

$$\begin{aligned} \phi_0(k_x, k_y, k_z) &> 0 \text{ if } (|k_x| + |k_y|) > |k_z|/\tan \theta_0 \text{ when } |k_x|, |k_y| > 0 \\ &\text{or } (|k_x| + |k_y|) \geq |k_z|/\tan \theta_0 \text{ when } |k_x| = 0 \text{ or } |k_y| = 0 \\ &= 0 \text{ otherwise.} \end{aligned} \quad (E.3)$$

APPENDIX F

NECESSARY AND SUFFICIENT CONDITION FOR AN OPTICAL TRANSFER FUNCTION TO BE NON-ZERO EVERYWHERE

In this Appendix we prove the statement (2.11), namely, the necessary and sufficient condition for an optical transfer function to be non-zero everywhere is that every plane passing through the origin of the corresponding point response function contain at least a line of non-zero values.

Necessity

If there is a plane A through the origin of ϕ_0 on which only the origin is non-zero, then we take the normal to A at the origin to be the z-axis. Equation (E.2) shows that ϕ_0 is zero on the k_z -axis except at the origin.

Sufficiency

Given a point response function satisfying condition (2.11), we want to show that the value of ϕ_0 at any arbitrary point P in the frequency space is non-zero. First rotate the coordinate system in frequency space so that P lies on the k_x -axis. The plane $x = 0$ contains at least one line on which ϕ_0 is non-zero. Take any such line to be the z-axis. Then decompose ϕ_0 into two parts in the following manner:

$$\begin{aligned}\phi_0^{(1)}(x, y, z) &= \begin{cases} 0 & \text{if } z = 0 \\ \phi_0(x, y, z) & \text{if } z \neq 0 \end{cases} \\ \phi_0^{(2)}(x, y, z) &= \begin{cases} \phi_0(x, y, z) & \text{if } z = 0 \\ 0 & \text{if } z \neq 0. \end{cases}\end{aligned}$$

Clearly $\phi_0 = \phi_0^{(1)} + \phi_0^{(2)}$. Since $\phi_0^{(1)}$ satisfies (E.1), equation (E.2) shows that $\phi_0^{(1)} > 0$, because $H(\underline{t}) > 0$ at $\underline{t} = 0$. Now it remains to show that $\phi_0^{(2)}$ is non-negative at P. $\phi_0^{(2)}$ can be written as

$$\phi_0^{(2)}(s, \theta, z) = \frac{F(\theta) \delta(z)}{s}$$

where we have made the coordinate transformation $r(x, y) \rightarrow r(s, \theta)$.

Transforming $k(k_x, k_y)$ to polar coordinate $k(w, \alpha)$, and taking the Fourier transform of $\phi_0^{(2)}$, we get

$$\begin{aligned}\phi_0^{(2)}(w, \alpha, k_z) &= \iiint \frac{F(\theta) \delta(z)}{s} \exp(2\pi i (sw \cos(\theta - \alpha) + k_z z)) ds d\theta dz \\ &= \frac{F(\alpha + \pi/2)}{2\pi |w|}\end{aligned}$$

which is clearly non-negative.

APPENDIX G

EFFECTS OF SAMPLING $\phi_0(x, z)$ IN THE x AND z DIMENSIONS.

Let the object distribution $\rho(r)$ be sampled in the z dimension by the sampling function:

$$s_{\Delta z}(z) = \sum_{n=-\infty}^{\infty} \delta(z - n\Delta z).$$

The point response function $\phi_0(x, z)$ can thus be sampled in the same way. By the convolution theorem, the optical transfer function $\phi_0^{(z)}(k_x, k_z)$ of the sampled $\phi_0(x, z)$ is given by

$$\phi_0^{(z)}(k_x, k_z) = \phi_0(k_x, k_z) * S_{\Delta k_z}(k_z) \quad (G.1)$$

where $\phi_0(k_x, k_z)$ is the original optical transfer function, $S_{\Delta k_z}(k_z)$ is the Fourier transform of $s_{\Delta z}(z)$, and $*$ denotes convolution. Now reference [63] shows that $S_{\Delta k_z}(k_z)$ is also a sampling function in the k_z dimension given by

$$S_{\Delta k_z}(k_z) = \Delta k_z \sum_{n=-\infty}^{\infty} \delta(k_z - n\Delta k_z), \quad \Delta k_z = \frac{1}{\Delta z}. \quad (G.2)$$

From equations (G.1) and (G.2) we see that $\phi_0^{(z)}(k_x, k_z)$ is the superposition of repetitions of the original optical transfer function in the k_z dimension at intervals of Δk_z . Thus $\phi_0^{(z)}(k_x, k_z)$ is non-zero in the region $2|k_x| \tan \theta_0 \geq \Delta k_z$; as for the region $2|k_x| \tan \theta_0 < \Delta k_z$,

the number of zero Fourier components is reduced by a factor of 2.

If sampling of $\phi_0(x, z)$ is done in the x dimension instead, similar considerations show that the corresponding optical transfer function $t_0^{(x)}(k_x, k_z)$ will be non-zero everywhere.

APPENDIX II

PROOF FOR THE POSITIVE DEFINITENESS OF BA

We want to prove that BA is a positive definite operator for all functions ψ satisfying $B\psi \neq 0$.

Lemma 1. For all ψ such that $A\psi \neq 0$, B is a positive definite operator.

Proof: $B\psi \neq 0$

$$\Rightarrow F^{-1}(\chi_B F \psi_A) = 0$$

$$\Rightarrow \psi_B F \psi_A = 0$$

$$\Rightarrow F \psi_A = 0 \quad (\because F \psi_A \text{ is an entire function})$$

$$\Rightarrow \psi_A = 0$$

$$\Rightarrow \psi = 0.$$

Lemma 2. For all ψ such that $B\psi \neq 0$, A is a positive definite operator.

Proof: $A\psi \neq 0$

$$\Rightarrow \psi_A F^{-1} \chi_B F \psi = 0$$

$$\Rightarrow F^{-1} \chi_B F \psi = 0 \quad (\because F^{-1} \chi_B F \psi \text{ is an entire function})$$

$$\Rightarrow \psi = 0.$$

Combining Lemma 1 and Lemma 2 we get the desired result.

APPENDIX I

LOWER BOUND OF THE EIGENVALUES OF THE MATRIX INTEGRAL OPERATOR

We first simplify the kernel $\mathcal{V}_0$

$$\begin{aligned}\mathcal{V}_0(k_x, z) &= \int_{-\theta_0}^{\theta_0} \frac{F(\vartheta)}{\pi} \exp((2\pi i k_x \tan \vartheta) z) d\vartheta \\ &= \int_{-\tan \theta_0}^{\tan \theta_0} \frac{g(t)}{\pi} \exp(2\pi i k_x t z) dt\end{aligned}$$

where $t = \tan \vartheta$, and $g(t) = F(\tan^{-1}(t)) \cos^2(\tan^{-1}(t))$. Let $s = k_x t$ and $s_0 = k_x \tan \theta_0$, then we have

$$\mathcal{V}_0(k_x, z) = \int_{-s_0}^{s_0} \frac{G(s)}{k_x \pi} \exp(2\pi i s z) ds \quad (I.1)$$

where $G(s) = g(\frac{s}{k_x})$. Equation (I.1) shows that we can treat z and s as reciprocal Fourier transform variables. We define a function $F(s)$ to be z -limited if its Fourier transform $f(z)$ is zero outside (z_1, z_2) . For such a function $F(s)$ the ratio of the energy of $F(s) \sqrt{\frac{G(s)}{k_x \pi}}$ in $(-s_0, s_0)$ to the total energy of $F(s)$ is given by

$$\frac{\int_{-s_0}^{s_0} |F(s)| \sqrt{\frac{G(s)}{k_x \pi}}^2 ds}{\int_{-\infty}^{\infty} |F(s)|^2 ds} = \frac{\int_{z_1}^{z_2} dz f(z) \int_{z_1}^{z_2} dz' f^*(z') \int_{-s_0}^{s_0} ds \exp(-2\pi i s(z-z')) \frac{G(s)}{k_x \pi}}{\int_{z_1}^{z_2} |f(z)|^2 dz} \quad (I.2)$$

By well known theorem in variational calculus, the lower bound of this ratio for all z -limited functions is equal to the infimum $\alpha_{\min}$ of the eigenvalues $\{\alpha_i\}$ of the integral operator.

$$\alpha_i g_i(z) = \int_{z_1}^{z_2} \left(\int_{-s_0}^{s_0} \frac{G(s)}{k_x \pi} \exp(-2\pi i s(z-z')) ds \right) g_i(z') dz'.$$

We want to prove that $\alpha_{\min} = 0$. Assume $\alpha_{\min} > 0$. Let G_0 be the maximum value of $\frac{G(s)}{k_x \pi}$ in $(-s_0, s_0)$. For the special case $\frac{G(s)}{k_x \pi} = 1$, the eigenvalues λ_i of the integral equation

$$\lambda_i f_i(z) = \int_{z_1}^{z_2} \left(\int_{-s_0}^{s_0} \exp(-2\pi i s(z-z')) ds \right) f_i(z') dz'$$

go to zero as the index $i \rightarrow \infty$ [35], so there is an integer m such that $G_0 \lambda_m < \alpha_{\min}$. Substituting the z -limited function $f_m(z)$ and its Fourier transform $F_m(s)$ into the expression (I.2) we get

$$\begin{aligned} 0 &< \frac{\int_{-s_0}^{s_0} |F_m(s)|^2 \frac{G(s)}{k_x \pi} ds}{\int_{-\infty}^{\infty} |F_m(s)|^2 ds} \leq \frac{G_0 \int_{-s_0}^{s_0} |F_m(s)|^2 ds}{\int_{-\infty}^{\infty} |F_m(s)|^2 ds} \\ &= \lambda_m G_0 < \alpha_{\min} \end{aligned}$$

which contradicts the definition of $\alpha_{\min}$. Thus $\alpha_{\min}$ is equal to zero.

- [13] Lim, C.B., Chu, D., Kaufman, L., Perez-Mendez, V., Hattner, R., and Price, D. C., IEEE Trans. Nucl. Sci., NS-22 (1975) 388.
- [14] Heitler, W., 1944, Quantum Theory of Radiation (London: Oxford University Press) p. 230 ff.
- [15] Ore, A., Powell, J. L., Phys. Rev., 75 (1949) 1697.
- [16] DeBenedetti, S., Cowan, C. E., Konneker, W. R., and Primakoff, H., Phys. Rev., 77 (1950) 205.
- [17] Charpak, G., Bouclier, R., Bressani, T., Favier, J., and Zupancic, C., Nucl. Instr. and Meth., 62 (1968) 262.
- [18] Kaufman, L., Perez-Mendez, V., Shames, D., and Stoker, G., IEEE Trans. Nucl. Sci., NS-19 (1972) 169.
- [19] Erskine, G. A., CERN report, Nucl. Instr. and Meth., 105 (1972) 565.
- [20] Rose, M. E., and Korff, S. A., Phys. Rev., 59 (1941) 850.
- [21] Bleaney, B. J., and Bleaney, B., 1965, Electricity and Magnetism (London: Oxford University Press) p. 304.
- [22] Grove, R., Ko, I., Leskovar, B., Perez-Mendez, V., Nucl. Instr. and Meth., 99 (1972) 381.
- [23] Bradt, H., Helv. Phys. Acta., 19 (1946) 77.
- [24] Lim, C. B., Chu, D., Kaufman, L., Perez-Mendez, V., and Sperinde, J., IEEE Trans. Nucl. Sci., NS-21 (1975) 85.
- [25] Jeavons, A. P., Charpak, G., Stubbs, R. J., IEEE Trans. Nucl. Sci., NS-22 (1975) 297.
- [26] Spencer, L. V., Phys. Rev., 98 (1955) 1597.
- [27] Seliger, H. H., Phys. Rev., 88 (1952) 408.

- [28] Morse, P. M., and Feshbach, H., 1953, Methods of Theoretical Physics (New York: McGraw-Hill) Part I: chapter 8.
- [29] Chang, L. T., Macdonald, B., and Perez-Mendez, V., IEEE Trans. Nucl. Sci., NS-23 (1976) 568.
- [30] Riesz, F., and Sz.-Nagy, B., 1955, Functional Analysis (New York: Frederick Ungar) chapter 6.
- [31] Harris, J. L., J. Opt. Soc. Am., 54 (1964) 931-936.
- [32] Whittaker, E. T., and Watson, G. N., 1952, A Course of Modern Analysis (England: Cambridge University Press) p. 67.
- [33] Solomon, D. C., J. Math. Anal. Appl., 56 (1976) 61-83.
- [34] Smith, K. T., Solomon, D. C., and Wagner, S. L., Bull. Amer. Math. Soc., 83 (1977) 1227-1270.
- [35] Slepian, D., and Pollak, H. O., Bell Syst. Tech. J., 40 (1961) 43.
- [36] Barnes, C. W., J. Opt. Soc. Am., 56 (1966) 575.
- [37] Inouye, T., IEEE Trans. Nucl. Sci., NS-26 (1979) 2666.
- [38] Gerchberg, R. W., Optica Acta, 21 (1974) 709-720.
- [39] Papoulis, A., IEEE Trans. Circuits Syst., CAS-22 (1975) 735-742.
- [40] Slepian, D., Bell Syst. Tech. J., 43 (1964) 3009.
- [41] Bateman, J. E., Nucl. Instr. and Meth., 140 (1977) 211.
- [42] Phillips, D. L., J. Ass. Comp. Mach., 9 (1962) 84.
- [43] Slepian, D., and Sonnenblick, E., Bell Syst. Tech. J., 44 (1965) 1745.
- [44] Twomey, S., J. Ass. Comp. Mach., 10 (1963) 97.
- [45] Grove, R., Perez-Mendez, V., Sperinde, J., Nucl. Instr. and Meth., 106 (1973) 407.

- [46] Chu, D., Ph.D Thesis, University of California, Berkeley, Calif. (1976).
- [47] Radeka, V., IEEE Trans. Nucl. Sci., NS-21 (1973) 51.
- [48] Bhandari, N., Perez-Mendez, V., and Kirsten, F., LBL-report (1976).
- [49] Perez-Mendez, V., Greenstein, M., and Ortendahl, D., IEEE Trans. Nucl. Sci., NS-24 (1977) 209.
- [50] Brigham, E. O., 1974, The Fast Fourier Transform (Englewood Cliffs, N. J. : Prentice Hall).
- [51] Blackman, R. B., and Tukey, J. W., 1958, The Measurement of Power Spectra from the Point of View of Communications Engineering (New York: Dover) p. 98.
- [52] Singleton, R. C., IEEE Trans. Audio and Electroacoustics, AU-17 (1969) 93.
- [53] Chu, D., Tam, K. C., Perez-Mendez, V., Lim, C. B., Lambert, D., Kaplan, S. N., IEEE Trans. Nucl. Sci., NS-23 (1976) 634.
- [54] Lum, G. K., Green, M. I., Perez-Mendez, V., and Tam, K. C., IEEE Trans. Nucl. Sci., to be published.
- [55] Crowther, R. A., DeRosier, D. J., and Klug, A., Proc. Roy. Soc. Lond. A, 317 (1970) 319-340.
- [56] Sabri, M. S., and Steennaart, W., IEEE Trans. Circuits Syst., CAS-25 (1978) 74.
- [57] Frieden, B. R., J. Opt. Soc. Am., 62 (1972) 511.
- [58] Frieden, B. R., and Burke, J. J., J. Opt. Soc. Am., 62 (1972) 1202.

- [59] Brenner, N. M., IEEE Trans. Audio and Electroacoustics,
AU-17 (1969) 128.
- [60] Buijs, H. L., Applied Optics, 8 (1969) 211.
- [61] Singleton, R. C., IEEE Trans. Audio and Electroacoustics,
AU-15 (1967) 91.
- [62] Papoulis, A., 1965, Probability, Random Variables, and
Stochastic Process (New York: McGraw-Hill) p. 201.
- [63] Papoulis, A., 1962, The Fourier Integral and Its Applications
(New York: McGraw-Hill) p.44.