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THREE LECTURES ON EFFECTIVE INTERACTION IN NUCLEI

S. A. Moszkowski

Edited By

S. D. Bloom

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THREE LECTURES ON EFFECTIVE INTERACTION IN NUCLEI

S. A. Moszkowski[†]

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[†]Present address: University of California, Los Angeles, Physics Department, Los Angeles CA

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LECTURE I: ORIGINS OF THE DENSITY DEPENDENCE OF THE EFFECTIVE INTERACTION

I. Introduction: Significance of effective interactions in nuclei

The nuclear many-body problem is interesting to treat for several reasons.

1) The nucleon-nucleon interaction is quite complicated, and not even completely determined. In fact, it is not even entirely clear to what extent this interaction can be described by a potential, since, after all, it is due to meson exchange, quark structure, etc.

2) Even if the nucleon-nucleon interaction were perfectly known, the many body problem cannot be solved exactly (except for very special potentials such as the δ -interaction or harmonic oscillator potential) and we must make approximations. One approach is to truncate the model space over which the interaction acts. This could mean, for example, that we consider only nuclear single particles relatively close to the Fermi surface. Thus we obtain an "effective" interaction.

This point is discussed in these present lectures. It is generally agreed that if the model space is truncated so as to eliminate short range correlations, then the effective interaction must contain three-body in addition to two-body terms. The three-body terms, which can be partially simulated by density dependent two-body terms, are of great importance in nuclear structure (though of course, the two-body terms, which are responsible for the nuclear binding, are even more important).

It is well known that the independent particle shell model is a good approximation for low energy nuclear properties. However, this seems to be due to a rather interesting coincidence which stems from the fact that the effective interaction between nucleons at the top of the Fermi sea in nuclear matter is very weak. Thus if the effective interaction were purely two body and therefore independent of density, it would not be able to reproduce independent particle motion without losing nuclear binding at the same time. However, introduction of effective three-body terms (or equivalently, density dependent two-body terms) makes it possible to approximately reproduce the absence of the effective interaction in the nuclear interaction and still preserve the overall nuclear binding.

In these three lectures we discuss some of these points in more detail. Section 2 reviews the lowest order treatment of the short range repulsion in the nucleon-nucleon interaction. In Section 3 we discuss an improved treatment which includes some of the effect of the nuclear medium on the interaction. The treatment can be applied to tensor correlations as well; indeed the latter seem to be even more important for nuclear matter at normal density than the short range repulsion. In Section 4 we discuss the "dispersion" effect, which gives rise to an effective three-body delta interaction, which, for some purposes, can be reproduced by a density dependent two-body interaction. We also study the properties of nuclear matter assuming a very simple effective interaction containing a two body and three body δ plus a short but finite range two-body term. One property of interest is the near vanishing of the interaction energy between nucleons on top of the Fermi sea. This is what is largely responsible for independent

particle motion in nuclei. We show explicitly that this condition can only hold if the energy due to effective three-body interactions is substantial and indeed it has to be just about the order of magnitude (~ 15 MeV per nucleon) suggested by microscopic calculations.

The equivalence between three-body delta and density dependent two-body delta interactions holds only for $N = Z$ nuclear matter. When there is a neutron excess (or spin polarization) the two interactions give quite different energies, as is discussed in Section 5. It is shown then that the empirical value of the nuclear asymmetry energy suggests that the description in terms of density dependent two-body interactions is the more realistic of the two.

The physical basis for deciding which interaction matrix elements are the most important at low energies is outlined in Section 6. It is argued on the basis of a simple energy conservation argument that the two most important quantities of interest at low energies are the average valence interaction energy F and the pairing matrix element G . If both are small (as seems to be the case) it implies that the effective interaction has zero range.

The pairing energy itself is discussed in Section 7. Its smallness in nuclear matter attests to the importance of the short range repulsion in the 1S_0 nucleon-nucleon interaction.

Section 8 contains two fantasies which describe how nuclei might behave if the interaction were a pure "exchange" interaction without any explicit momentum or density dependence. In this case the independent particle model breaks down. The smallness of the effective interaction in the nuclear

interior implies that it acts mainly at the nuclear surface. In Section 9 we discuss the surface delta interaction. The justification for this is to some extent provided by the arguments in the previous sections. We also make some remarks concerning the origin of the smallness of the effective interaction in the nuclear interior.

Section 10 contains some brief remarks on the effective interactions at high energies.

Finally, we summarize some of the salient points of our lectures in Section 11.

2. Separation Method:

Suppose we have a two-body interaction containing a strong short-range repulsion plus a longer range attraction such that the net effect, as measured, say, by the S-wave phase shift, is attractive, at least at low energies. (In fact, we restrict ourselves here to S-state interactions. At least in a rough approximation higher partial waves are not too important for nuclei.) The treatment of the many-body problem, especially of the short range repulsion, can be simplified by cutting the potential into a short range V_S and a long range part V_L . The dividing point can be chosen such that at zero energy V_S leads to zero interaction energy between two particles in free space. To first approximation we can in fact throw away V_S . The procedure of doing this can be justified as follows:

Consider the two particle Schroedinger equation at zero relative energy

$$-\frac{\hbar^2}{m} \nabla^2 f + v f = 0$$

where f is the two-body wavefunction. At some particular value of r which we call d , we have $\nabla f = 0$. This is where we make the cut. (This can always be done if the potential contains a short range repulsion followed by an attraction such that the net effect at zero energy is attractive.) Since the Schroedinger equation is linear, we can always normalize f such that $f(d) = 1$. Then the contribution of V_S to the two-body interaction energy is,

$$\Delta E_S = (\phi | V_S | f)$$

where ϕ is the unperturbed wavefunction: $\phi = 1$ for zero energy. (We normalize the wavefunction to one particle per unit volume.)

Thus

$$\Delta E_S = \int_0^d V_S f d^3r = \frac{\hbar^2}{m} \int_0^d \nabla^2 f d^3r = \frac{\hbar^2}{m} \oint \nabla f \cdot dS = 0$$

Thus the short range interaction V_S does not contribute to the two-body interaction energy of nuclear matter (in the zero-momentum limit). This conclusion still holds quite well even if the presence of the other nucleons is taken into account. The leading correction to this is the dispersion effect discussed in Section 3. This implies the assumption that the two-body wavefunction f in nuclear matter, at least for distances up to d , is the same as it is in free space, i.e., the short range repulsion leads to a "wound" in the wavefunction, i.e., a strong short range correlation. On the other hand, we also assume, in the application of the separation method, that $f = 1$ for $r > d$, i.e. that the two-body wavefunction outside the separation

distance is not affected by the interaction at all. Thus the presence of the other nucleons cancels the correlations which would occur if the particles were in free space. [In the latter case (shown as the dashed line) ψ would be the solution of the Schroedinger equation for $r > d$ as well as for $r < d$].

In this approximation then, the interaction energy ΔE is just the Born approximation result for the long range interaction

$$\Delta E = \int_d^{\infty} V_L d^3r$$

Detailed calculations show that this method gives a simple but fairly accurate description of the energy of nuclear matter at normal density. Using the separation method, we find, for typical potentials, (e.g. Reid Soft Core) $d \sim 1$ fm. The truncated potential V_L is, of course, a simple attraction. Such a potential was used by Kallio and Kolltveit and it works almost as well for some shell model calculations (e.g. in s-d shell) as more elaborate realistic or phenomenological interactions.

As we have seen, the essential point of the separation method is that a certain part of the potential, V_S , which contains both the repulsion and a part of the attraction up to a distance d , can be simply thrown away, at least for the calculation of the energy. However, strictly speaking, this separation distance d increases with momentum. Thus V_L decreases with momentum. Alternatively, if we keep d independent of k , then while V_S gives 0 interaction energy for $k = 0$, it leads to a positive interaction energy for $k > 0$. Physically, the reason for this is that the effect of the repulsive core becomes more important, relative to that of the attraction,

with increasing energy. Consider, for example, the limiting case of an infinitely hard core followed by a shallow attraction. If the radius of the attraction is large enough, it can cancel the effect of the repulsion at zero energy. However, while this combined potential gives $\Delta E_S = 0$ at 0 energy, at high energy (large compared to the depth of the attraction) it turns out that the attraction has little effect and ΔE_S is determined by the core alone.

$$\Delta E_S = \frac{4\pi\hbar^2}{m} c$$

where c is the core radius (of course in this case there exists no separation distance since the potential is repulsive).

In the lowest approximation of the separation method the short range correlations are assumed to be the same as in free space. In other words, the effect of the Pauli principle in forbidding scattering into occupied states, is neglected, at least for scattering due to the short range part of the potential. More detailed calculations, which take the Pauli principle into account, show that this leads to negligible correction to the potential energy. The main effect of the Pauli principle is to erase the phase shift due to the potential. Since V_S gives no phase shift, the Pauli principle has little effect here. It does modify V_L .

However, there is another effect, - the "dispersion effect." which can be appreciable and gives rise to a density dependence of the effective interaction. Besides these two effects there is a second order effect due to V_L , and the second order cross term between V_S and V_L , but neither of these is of crucial importance in our treatment here and we will not consider them further.

Some additional words on the separation method: It corrects the wavefunction to be essentially the scattering solution for the two-body problem for 0 to 40 MeV by using a cut-off potential which suppresses high-momentum transfer collisions. This makes it possible to use a truncated space and takes advantage of the (relative) state-independence of the short-range correlations. Of course it is the cut-off which, roughly speaking, has this state-independent feature. For instance, if we used a finite but highly repulsive potential in a truncated space, the $J = 0$ (2-body) states would be pushed up above the $J \neq 0$ states, contrary to reality. This is because the $J = 0$ state will feel the repulsion most keenly. This would not happen (or be less violent) in a very large model space.

3. Dispersion Effect:

Although the effect of V_S can be thrown away (for two particles) in free space, this is not true in the presence of other nucleons. We consider here what seems to be the most important effect due to the other nucleons - the so-called dispersion effect.

The name "dispersion" is taken from optics where it implies a change in the relation between frequency and wavelength due to the action of the refractive medium. In nuclei it means a change in the relation between single particle energy and momentum due to the momentum dependence of the single particle potential.

The dispersion effect gives rise to a density dependence of the effective interaction in nuclei. To illustrate the dispersion effect, consider the diagrammatic representation, say for V_S . We have,

$$\left| \begin{array}{c} V_S \\ \text{wavy line} \end{array} \right|$$

This represents the interaction between two interacting nucleons to first order (Born approximation) in V_S . Since V_S has a hard core, this is infinite. However, if we sum up all the higher order diagrams as well:

$$\left| \begin{array}{c} \text{---} \\ V_S \\ \text{---} \end{array} \right| + \left| \begin{array}{c} V_S \\ \text{---} \\ V_S \end{array} \right| + \left| \begin{array}{c} V_S \\ \text{---} \\ V_S \\ \text{---} \\ V_S \end{array} \right| + \dots = \left| \begin{array}{c} \text{---} \\ t_S \\ \text{---} \end{array} \right|$$

we get the reaction matrix t_S . If we proceed as in the separation method, i.e., assume that the short range potential gives no interaction energy, then the energy due to t_S vanishes:

$$\begin{array}{c} 0 \\ \left| \begin{array}{c} \text{---} \\ t_S \\ \text{---} \end{array} \right| \\ 0 \end{array}$$

We are dealing here (in the nucleus) with states of low energy (denoted by 0)

V_S alone gives $\Delta E_S = 0$ in first order (even though V_S may be infinite, as was already pointed out), and the interaction energy is due to V_L alone.

However, the effect of V_S cannot be ignored entirely. As we have mentioned, V_S causes a wound in the wavefunctions at short distances. This implies that t_S can, in fact, scatter a pair of nucleons from states of low momenta to states of high momenta.

$$\begin{array}{c} i \\ \left| \begin{array}{c} \text{---} \\ t_S \\ \text{---} \end{array} \right| \\ 0 \end{array}$$

The amplitudes for scattering to i are proportional to the components of momenta i in t_S . This, of course, does not contribute to the ground state energy, where all nucleons must have low energy.

It might be thought that perhaps second order diagrams like

might contribute to the energy. But this turns out not to be the case. In constructing t_s we are already iterating V_s an arbitrary number of times, so that we would be double counting, were we to include the above diagram.

On the other hand, diagrams such as

where two t_s lines are separated by a V_L line, do, in fact, contribute significantly to the energy. This is the so-called "dispersion" term.

Let us look at the dispersion effect in more detail. The idea of the "dispersion" effect is as follows: Consider the defect function X , where $X = 1 - f$. Thus $X \sim 1$ in the region of the short range repulsion and falls off to zero at $r = d$. X represents the probability amplitude for a nucleon to be scattered outside the Fermi sea by virtue of the short range repulsion. We can define the Fourier transform of $X(r)$

$$X(k) = \frac{1}{(2\pi)^{3/2}} \int X(r) e^{ik \cdot r} d^3r$$

Then $X(k)$ is peaked at about $k \sim \frac{\pi}{2d} \sim 4 \text{ fm}^{-1} \text{ fm}$ if $C \sim 0.4 \text{ fm}$.

Now the probability of a nucleon being thrown out of the Fermi sea is

$$I = \int_0^\infty X^2(k) d^3k$$

which is also equal to

$$I = \int_0^\infty X^2(r) d^3r,$$

assuming a reasonable form of the X discussed in the last section. For $C = 0.4 \text{ fm}$ and $d = 1.06 \text{ fm}$ (a typical value) we find $I = 0.53$. Now nucleons far outside the Fermi sea feel very little single particle potential (V acts only in S states). Indeed, experimentally, the single particle optical potential decreases to 0 to 300 MeV. (There are some questions as to how relevant this is, however.) Now the probability P of a nucleon being scattered out of the Fermi sea, turns out to be $2 \rho I$. The factor ρ occurs because P is proportional to the number of nucleons against which a nucleon can scatter, and the factor 2 arises since, in each scattering, two nucleons get thrown out of the Fermi sea. If we assume no interaction in excited states, we find that the actual single particle potential is changed from $-U$ to $-U + 2 \rho IU$, i.e., the interaction is effectively weakened by a fraction $2 \rho I$. This effect, called "dispersion", is equivalent to a density dependent correction to the interaction. This can be seen as follows. Suppose we introduce an effective dispersion interaction given by:

$$V^D = -2IU A(r)$$

(This depends on ρ since U does). The contribution of this correction to the single particle potential is

$$U^D = \rho \int V^D d^3r = -2.1U$$

just as required. Similarly, the potential energy per particle is just half of U (in our approximation of neglecting any momentum dependence). If we neglect any momentum dependence, then in fact, U is itself proportional to ρ . In this approximation we find

$$PE^D = PE_0^2 (2r_0 I) \left(\frac{\rho}{\rho_0}\right)^2$$

For the example discussed here

$$r_0 = 0.16 \quad I = 0.53 \\ 2r_0 I = 0.17$$

Clearly the dispersion correction increases strongly with density. Using $PE^2 = -40$ MeV, we get a dispersion correction of 7 MeV at normal density. However, this estimate is oversimplified in several respects. First, if we use a more realistic short range correlation function appropriate to a soft core potential, we obtain a dispersion energy of the order of 4 MeV/A at normal density. In addition, other effects are even more important than the short range correlations. For example, the tensor force contributes in an important way. This is because the correlations for $T = 0$, for part of the time, correspond to a 3D_1 state rather than a 3S_1 state for the nucleon pair. The integral I now represents the probability of nucleons being in a 3D_1 state. This can happen without hindrance in free space but in nuclear matter this process is inhibited by the Pauli principle. Roughly speaking,

the dispersion correction can be obtained by an expression similar to the one above. We have

$$\frac{PE^D}{A} = \frac{1}{2} \left(\frac{\rho}{\rho_0} \right) U^D = - \frac{1}{2} \frac{\rho^2}{\rho_0} I(U) = \rho_0 I(U) \left(\frac{\rho}{\rho_0} \right)^2$$

Thus

$$PE^D \propto \rho^2$$

Let

$$K = \rho_0 I$$

Then, the contribution of dispersion to the energy per nucleon at normal density is

$$\left. \frac{PE^D}{A} \right|_{\rho_0} = K |U|$$

As a rough estimate, we take $|U|$ to be the same for all correlation processes to be considered here. But K is roughly twice as large for the $T=0$ tensor correlation as for the short range repulsion. Thus the former contributes about $2 \times 4 = 8$ MeV to the energy per nucleon.

While the dispersion term is approximately proportional to ρ^2 at low densities, this is not the case for large densities. It is of interest to consider also the latter as a limiting case. Here the Pauli principle cuts out all higher order contributions to the energy. Furthermore the first order interaction energy due to the tensor forces vanishes because of its symmetry properties. Thus, in the limit of $\rho \rightarrow \infty$, the dispersion energy approaches a constant rather than varying as ρ^2 , insofar as the tensor contribution is concerned.

There is also an $n - \Lambda$ isobar effect which acts only in $T = 1$ states. This operates as follows: for part of the time, one of the nucleons in a nn pair may be in a $\Lambda(S = 3/2)$ virtual state. Thus a 1S_0 nn pair may be, for part of the time, in a 5D_0 state. (That is the only state besides 1S_0 which conserves angular momentum and parity). Otherwise this effect operates in a manner quite similar to the tensor force (except that it acts in $T = 1$ states) and its contribution to the energy of nuclear matter has been estimated to be approximately half that of the tensor force, i.e. ~ 4 MeV. However, it is very important for the density dependence of the $T = 1$ interaction. Thus, the density dependent components of the interaction contribute a total of about $4 + 8 + 4 = 16$ MeV per particle to the energy of nuclear matter. On the other hand, the total potential energy of nuclear matter must be -38 MeV, (so that, combined with 22 MeV kinetic energy, we get the known volume energy of -16 MeV). To obtain this value of -38 MeV, we see that the two body (density independent) and dispersion contributions should be approximately $PE^{(2)} = -54$ MeV and $PE^D = 16$ MeV, respectively.

4. Density dependence and the effective three body interaction

We have pointed out that the dispersion effect due to the short range repulsion can be roughly simulated by a density dependent interaction. But since three particles are involved here, in this effect, the dispersion effect acts like a three-body interaction, rather than a conventional

two-body interaction. This difference has important consequences as we now show. For example, suppose we also represent the density independent part of the interaction by a δ -interaction. Then we can write for the entire interaction

$$V = -\alpha \delta(r_{12}) + \gamma \delta(r_{123})$$

If we write this as an effective two-body interaction (by summing over particle 3), we obtain

$$V^{(2)} = -\alpha \delta(r_{12}) + \gamma \delta(r_{12})$$

Next, let us calculate the potential energy per unit volume of nuclear matter. This is (neglecting all exchange effects)

$$P = -\frac{\alpha \rho^2}{2} + \frac{\gamma \rho^3}{6}$$

The factor 6 corresponds to the number of combinations of nucleons in each triplet.

Next, let us calculate the ratio of the strengths of the effective two-body interaction to the potential energy. For the two-body interaction we obtain

$$R^{(2)} = \frac{2}{\rho}$$

while for the three-body interaction

$$R^{(3)} = \frac{6}{\rho^2}$$

three times as large. Thus a three-body force yields a larger effective two-body force than a true two-body force does, even if they give equal potential energies. It is easily shown that if the interaction is represented by the form

$$(-\alpha + \gamma\rho)\delta(r)$$

and the parameters α and γ are adjusted to fit nuclear matter saturation properties

$$\begin{aligned} W &= -16 \text{ MeV per particle} \\ \text{at } \rho &= 0.16 \text{ nucl/fm}^3 \end{aligned}$$

then the contribution of two-body and three-body potential energies are

$$\begin{aligned} PE^{(2)} &= -61.5 \text{ MeV/A} \\ PE^{(3)} &= 23.4 \text{ MeV/A} \end{aligned}$$

The ratio

$$\frac{PE^{(3)}}{PE^{(2)}} = 0.38 = 2k$$

$$k = \rho I, \text{ defined in Section 3.}$$

The more precise estimate made in the previous section gives

$$\frac{PE^{(3)}}{PE^{(2)}} = \frac{16}{54} = 0.30 = 2k$$

As we have seen, the effect of dispersion is to reduce the potential energy by a fraction $2k$.

$$PE_{L+D} = (1 - 2k) PE_L$$

However, the effect of dispersion on the effective two-body interaction is three times as large,

$$v_{L+D}^{(2)} = (1 - 6k) v_L^{(2)}.$$

With either of our values of K , ($K = 0.15$ or 0.19) we see that $v_{L+D}^{(2)}$ nearly vanishes at normal nuclear density.

Thus, we can write approximately

$$v_{L+D}^{(2)} = \left(1 - \frac{\rho}{\rho_0}\right) v_L^{(2)}$$

Clearly then, the effective two-body interaction is weaker in the interior than on the surface.

Now we have also argued that the spin dependence vanishes in the interior. Thus it seems that both $v^{(2)}, {}^1S_0$ and $v^{(2)}, {}^3S_1$ can be regarded as vanishing in the interior. Consequently, $v^{(2)}$ must be a surface interaction. Of course, while the two-body interaction, which is proportional to $1-6k$, seems to approximately vanish in the interior, the potential energy, which is $1-2k$, definitely does not vanish there. The apparent paradox is resolved if we realize that while our original interaction, including hard core, is a pure two-body interaction, the effective interaction, in which the hard core has been removed, contains both two-body and three-body components. While it can be represented as a pure two-body interaction for some purposes, it behaves differently in other ways, i.e. the ratio between two-body strength and total potential energy. These points are summarized in Table 1.

We wish to remark here that for a more general density dependent delta interaction of the form,

$$-\alpha(\rho)\delta(r)$$

it turns out that the renormalization factor in going from PE to V_L is given by

$$\frac{1}{2} = \frac{1}{2\alpha(\rho_0)} \frac{d^2}{d\rho^2} (\rho^2 \alpha(\rho)) \rho_0$$

If $\alpha \propto \rho^n$

then $\frac{1}{2} = (n+1)(n+2)/2$

which reduces to our previous value of 3 for a linear density dependence.

5. Isospin Asymmetry Effects

The above argument regarding the density dependent interaction is incomplete in one important respect: it neglects the symmetry properties of the wavefunction.

To illustrate this point, suppose the interaction acts only in spatially even states (a delta interaction is a special case of this). Now two nucleons with the same charge, i.e. t_z and spin component s_z cannot be in the same spatial state, thus they cannot interact. However, the antisymmetrization does not affect the interaction between two nucleons for which t_z and/or s_z differ. Now in nuclear matter, ($A \rightarrow \infty$) there are four possible values of (t_z, s_z) . Any given nucleon cannot interact with another one of the same (t_z, s_z) i.e. 1/4 of the nucleons. Thus the antisymmetrization multiplies the potential by the factor 3/4. However, now

Table 1. Table of the different contributions of $v^{(2)} \delta(r_{12})$ and $v^{(3)} \delta(r_{123})$ to the three quantities of interest here: a) The effective two-body interaction $v_{\text{eff}}^{(2)}$, b) The single-particle potential v_{sp} , and c) The average potential energy per particle of nuclear matter, PE/A . Only self-conjugate nuclei ($N = Z$) are considered and charge exchange has been neglected. Note in the text we considered the specific cases $v_{12}^{(2)} = -\alpha$ and $v_{123}^{(3)} = \gamma$. The total interactions for $v_{\text{eff}}^{(2)}$, $v(r)$, and W would be in this case: $A + B = -\alpha + \gamma\rho$, $-\alpha + \gamma\rho/2$, and $-\alpha + \gamma\rho/3$, respectively.

	A	B	
Contribution from $\rightarrow$	$v^{(2)} \delta(r_{12})$	$v^{(3)} \delta(r_{123})$	$B/\rho A$
$v_{\text{eff}}^{(2)}$	$v_{12}^{(2)}$	$\rho \langle v_{123}^{(3)} \rangle_1$	1
$v_{\text{sp}} \equiv v(r)$	$\rho \langle v_{12}^{(2)} \rangle_1$	$(1/2) \rho^2 \langle v_{123}^{(3)} \rangle_{12}$	1/2
$\text{PE}/A \equiv W - \text{KE}/A$	$(1/2) \rho \langle v_{12}^{(2)} \rangle_{12}$	$(1/6) \rho^2 \langle v_{123}^{(3)} \rangle_{123}$	1/3

	(A), $\delta(\mu_{12})$	(B), $\delta(\mu_{123})$	$P(B)/A$
$V_{eff}^{(2)}$			1
V_{SP}			$1/2$
PE/A			$2/6 = 1/3$

Figure 1. Diagrammatic representations of interactions of Table I.

suppose we turn all the protons into neutrons, keeping the density the same. If it were not for antisymmetrization, the potential energy (which depends only on the density) would not change. But consideration of antisymmetrization reduces the result by 1/2. This is because a neutron cannot interact with another one of the same s_z , i.e. 1/2 of the neutrons (rather than 1/4 as in the nuclear matter case). Thus when we go from nuclear matter to a neutron gas at the same density and take antisymmetry into account, we find that the potential energy is multiplied by 2/3. All the above results hold also if the interaction is explicitly density dependent, i.e. $\rho\delta(\vec{r})$.

Let us now consider the potential energy due to three-body interactions, such as discussed in the last section. We again suppose an even state interaction, of which a three-body interaction is a special case. This time the interaction can occur only between triplets of nucleons, no two of which have the same $(t_z s_z)$. For nuclear matter, a fraction 3/8 of triplets are of this form. But for a neutron gas it is impossible to form three neutrons all with different spins - since there are only two allowed spin components. Then even if we can simulate a three-body interaction by a density dependent δ -interaction for $N = Z$ nuclear matter, this correspondence breaks down for $N \neq Z$. More specifically, the factor by which the potential energy should be multiplied due to antisymmetry turns out to be

$$f_2 = \frac{3}{4} \left(1 - \frac{1}{3} I^2\right)$$

where,

$$I \equiv \frac{N-Z}{A}$$

for any two-body even-state interaction. We have already derived the values of f_2 in the limiting case $I = 0$ (nuclear matter) and $I = 1$ (neutron gas). On the other hand, for three-body even-state interactions the factor is

$$f_3 = \frac{3}{8} (1 - I^2)$$

Thus to reproduce the three-body dispersion due to an S-state interaction with a density dependent two-body δ -function for all I , we should use an effective interaction proportional to

$$\frac{1 - I^2}{1 - \frac{1}{3} I^2} \delta(r)$$

The two different assumptions: density dependent two-body delta interaction or three-body delta interactions, lead to different values of the asymmetry energy of nuclear matter W_A , where $W(I) = W_0 + W_A I^2$. Empirically, the value of W_A deduced from nuclear binding energies seems to be about 30 MeV.

Now an elementary calculation shows that the kinetic energy contributes an amount

$$W_A^{KE} = \frac{1}{3} T_F \sim 12 \text{ MeV}$$

Thus the empirical value of the potential energy contribution to W_A is

$$W_A^{PE} \sim 18 \text{ MeV}$$

This is the number we should try to account for. Let us assume that the average interaction which is responsible for the potential energy of nuclear matter is a density dependent delta interaction

$$V = -\alpha \left(1 - \frac{1}{3} \frac{\rho}{\rho_0}\right) \delta(r)$$

The factor $1/3$ is put in so that the effective interaction between valence nucleons, (in which the δ -dependent term must be multiplied by 3 (see previous section) vanishes in normal nuclear matter. Then the potential energy of nuclear matter due to the density independent term would be

$$U = \frac{3}{2} PE = -57 \text{ MeV}$$

and

$$PE = U - \frac{1}{3} U = -38 \text{ MeV}$$

This is true for both possible assumptions regarding the density dependence. On the other hand, the isospin dependence of the two terms is different. Using results obtained above we see that for the density dependent two-body interaction

$$PE(I) = U(1 - \frac{1}{3} I^2) - \frac{1}{3} U(1 - \frac{1}{3} I^2)$$

i.e. the two components have the same I dependence and the potential asymmetry energy is

$$W_A^{PE} = -\frac{2}{9} U = 13 \text{ MeV}$$

fairly close to the empirical value. On the other hand, if the density dependence is due to a three-body delta interaction, then

$$PE(I) = U(1 - \frac{1}{3} I^2) - \frac{1}{3} U(1 - I^2)$$

and the asymmetry effects cancel

$$W_A^{PE} = 0$$

in flagrant disagreement with experiment. Thus within the framework of a delta function interaction, the empirical asymmetry energy favors a density dependent two-body interaction rather than a three-body δ -interaction.

Finally, what would be the physical origin of the difference between these two types of interaction?

Very roughly this difference can be accounted for as follows. The effective three-body delta interaction results if all the short range correlations occur only in even states; on the other hand, if the short range correlations are state independent, then antisymmetry corrections do not appear and we obtain a single density dependent two-body interaction. (Of course, as discussed above, for an even state long range interaction we still get a reduction of 1/3 in the potential energy in going from nuclear matter to the neutron gas.)

LECTURE 11: Surface peaking of the effective interaction and why the shell model works.

We will first discuss $N=Z$ nuclear matter with a simple density dependent interaction. Then we will briefly discuss pairing. Finally we will make some remarks regarding the question of why the shell model works.

6. Average interaction energy of valence nucleons.

Let us now extend the δ -function model used in the last sections slightly so as to allow the two-body interaction to have a finite range. This form is known as the Skyrme Interaction. But we assume it acts only in even states. Thus we find that the effective two-body interaction can be written as

$$v^{(2)} = (-\alpha + \beta k^2 + \gamma \rho) \delta(r)$$

where we have taken advantage of the fact that the finite range character is simulated by the momentum-dependent term. Note that β is proportional to the mean square radius of the interaction. The potential energy is determined by summing the interaction,

$$v^{(PE)} = [-\alpha + \beta k^2 + (\gamma \rho/3)] \delta(r)$$

over all the nucleon pairs. The potential energy (per nucleon) is given by

$$\frac{PE}{A} = -\frac{3}{8} \alpha \rho + \frac{3}{8} \times \frac{3}{10} \beta k_F^2 + \frac{3}{8} \frac{\gamma \rho^2}{3}$$

The factor 3/8 is a product of the 1/2 (correcting for counting pairs twice), and 3/4 (to correct for exchange in $N=Z$ nuclear matter). The average value of k^2 for nucleons in nuclear matter is $3/10 k_F^2$.

The average two-body interaction energy between valence nucleons $\langle \vec{k}_i \vec{k}_i' | V^{(2)} | \vec{k}_i \vec{k}_i' \rangle$ (with $k_i = k_i' = k_F$), denoted here by F_0 , is

$$F_0 = -\epsilon + \frac{1}{2} k_F^2 + ,$$

The average k^2 for valence nucleons averaged over all angles is $1/2 k_F^2$.

Using the saturation condition for nuclear matter,

$$\left. \frac{dE}{d\epsilon} \right|_{\epsilon=0} = 0$$

we can show that,

$$F_0 = \frac{1}{5} \left(-\frac{4}{5} T_F + 2W^{(3)} \right)$$

where T_F = Fermi kinetic energy. Note also that,

$$W^{(3)} = 3 \text{ body energy} = 1/8 \mu^2.$$

Now since

$$T_F = 36 \text{ MeV}$$

and

$$W^{(3)} = 15 \text{ MeV}$$

we find $F_0 \approx 0$.

7. Pairing

Pairing is a very important aspect of the nuclear shell model. It means that in the ground state, each pair of identical nucleons will tend to couple their angular momenta to a resultant $J_0 = 0$. This pairing energy Δ is related to the odd-even mass differences and is roughly given by,

$$\Delta_{\text{expt}} \sim 12 A^{-1/2} \text{ MeV}$$

i.e., 1 MeV for medium heavy nuclei. The pairing energy in nuclear matter can only be roughly deduced from the observed pairing energies, but it is probably less than 0.5 MeV.

Δ can be roughly related to the two-body interaction. A simple-minded theory gives for Δ in nuclear matter (see Jakeman and Moszkowski, Phys. Rev. 141, 933 (1966)),

$$\Delta_{\text{NM}} = 2 T_F \exp[-\pi/(2\delta_S(k_F))]$$

$\delta_S(k_F)$ is the S-wave phase shift in free space at $k = k_F$. Δ is a non-analytic function of the strength of the interaction. Using known 1S_0 phase shifts we have

$$\delta_S(k_F) = 15^\circ$$

(quite small because of the action of the short range repulsion) and $\Delta_{\text{NM}} = 0.17 \text{ MeV}$, also quite small. At the nuclear surface, ρ , and thus k_F , is smaller. Thus $\delta_S(k_F)$ is larger and so is Δ . In fact, in this way, one can roughly account for the variation of Δ with A using the simple-minded theory referred to above.

Let us now try to relate the pairing effect to the effective interaction discussed here. Instead of $S_S(k_F)$, which refers to free nucleon-nucleon scattering, we should be using the pairing matrix element G in nuclear matter. Now G = average matrix element for scattering of a pair of nucleons in conjugate states m, m^* into m', m'^* . In nuclear matter, we have plane waves. Thus we have scattering of a pair $\vec{k}, -\vec{k}$ into $\vec{k}', -\vec{k}'$, where

$$k = k' = k_F.$$

Thus

$$G = \langle \vec{k}', -\vec{k}' | V^{(2)} | \vec{k}, -\vec{k} \rangle$$

averaged over all angles. Note that the pairing matrix element G is not to be confused with the nuclear reaction matrix (often referred to as the G matrix). G can involve large momentum transfer (unlike F_0), but the center of mass momentum of the nucleons must vanish. For the interaction discussed above, this is

$$\begin{aligned} G &= -\alpha + \beta k_F^2 + \gamma \rho_0 \\ &= F_0 + \frac{1}{2} \beta k_F^2. \end{aligned}$$

As we have seen, $F_0 \sim 0$. Thus, since β is positive, due to the finite range of the interaction, we find $G > 0$, which means that the pairing in nuclear matter is repulsive; i.e., no pairing. Of course, at the nuclear surface, the density is lower, and G may be attractive. Thus the pairing effect occurs mainly at the nuclear surface.

Although in quantum mechanics it is permissible to violate energy conservation for a short time, we expect that energy conserving scatterings should be the ones of the greatest importance for low energy nuclear structure. These are precisely a) the forward scatterings (average interaction energy) described by F , and b) the pairing interaction G discussed here. For both F and G the energy is conserved, but for F only the relative momentum is also conserved.

8. Why the nuclear shell model works and how it could break down if the interaction were not density dependent.

Suppose now that the effective interactions were not dependent on density. This could happen, for example, if the interactions are attractive in even states but equally repulsive in odd states ($V_{\text{odd}} = -V_{\text{even}}$). In this case we can get nuclear saturation even without any repulsive core or tensor forces. Indeed, in the early days of nuclear physics the nuclear forces were thought to be more or less of this type. In the case we find, using a Skyrme interaction,

$$F_0 = G = -\alpha + \beta k_F^2.$$

In this case, F_0 and G are equal. (The previous expression for F_0 holds only for even state interactions.) But since $W^{(3)} = 0$ we find, using the relations given before

$$F_0 = -\frac{4}{5} \frac{T_F}{\rho_0}$$

which leads to

$$\delta_s(k_F) = \frac{3}{10} \frac{\pi}{2}$$

and the following pairing energy of nuclear matter

$$\Delta_{NM} = 2T_F e^{-10/3} \sim 3 \text{ MeV}$$

attractive but much larger than the experimental value.

We obtain an even larger result if renormalization is taken into account and also for finite nuclei. In any case, Δ is larger than the average spacing between single particle levels. Thus we would indeed find that for even-even nuclei the ground state $J_0 = 0$, but for odd A nuclei $J_0 \neq J$ (single particle), i.e., the single particle shell model would lose its validity. Another way of saying this is that the Fermi surface would be unstable. The pairing correlations would trigger all kinds of instabilities, such as breaking into alpha particles (where all interactions are even state, i.e., attractive). Fortunately, thanks to the density dependence, all this does not happen and so the shell model does survive.

We can look at this in another way in terms of the single particle potential. The validity of the independent particle model really means that the actual (instantaneous) potential acting on a nucleon in the nuclear interaction does not fluctuate strongly about its average (constant) value. Since the force is proportional to the gradient of the potential, this force is small in the interior. At the nuclear surface there is, of course, a significant average force which tends to keep the nucleons confined within the nucleus. All this is the case for the kind of

interaction considered at the beginning of this lecture, i.e. with $F_0 \sim 0$. On the other hand, for the "exchange" interaction discussed at the beginning of this section, the average potential may be reasonable, but its fluctuations must be large, even in the nuclear interior. Thus the independent particle model would break down.

LECTURE III: Essential features of the effective interactions for nuclear structure calibration.

9. The surface-delta interaction.

We have shown that the average interaction F_0 between valence nucleons as well as the pairing interaction G are both small, but probably somewhat repulsive, in nuclear matter. Thus they will not have much effect there. (If anything, nucleons will simply stay away from the region of the repulsion.) But at the nuclear surface where ρ and k_F are lower, both are attractive. Furthermore, we can argue that at least the pairing interaction G is, in fact, isotropic, with respect to the angle of the momentum, i.e., it acts like a delta function in the angle (in coordinate space). Thus we can assume that the effective interaction has zero range and that it vanishes in the interior. This is the physical argument for the surface delta interaction.

Now, in a finite nucleus, the valence nucleons all have approximately the same energy and the same radial dependence $\exp[-cE^{1/2}r]$ at large distances, i.e., at the surface. Detailed calculations show that the amplitudes at the surface are also not very different for different valence states. Of course, different valence wavefunctions are quite different in the interior, and even have different numbers of nodes there, but this is not relevant if the effective interaction acts only at the surface. Thus we can set all radial integrals equal. This assumption leads to the surface delta interaction,

$$V_{SDI} = 4\pi F_0 \delta(\Omega)$$

where Ω is the angle between the nucleons. The surface delta interaction is quite easy to treat, all matrix elements are simply Clebsch-Gordan coefficients. Further, with such an interaction, it has proven possible to fit a large variety of nuclear structure data. The required strength F_0 turns out to be about $-25A^{-1}$ MeV, if a model space of about one oscillator shell on each side of the Fermi surface is used. This guarantees that the wavefunction has full angular but no radial degrees of freedom. Approximately this value of F_0 can also be derived theoretically by using an interaction of the Skyrme form discussed in an earlier section. (Also see the discussion below.)

Let us compare the surface delta interaction with a conventional δ -interaction acting over the entire nuclear volume

$$g\delta(\vec{r}) = g \frac{\delta(r)}{M^2} \delta(\Omega)$$

For the latter, the radial integrals differ widely. For particles in the same state j , i.e. j^n configuration, all the radial wavefunctions are identical, and the SDI matrix elements are equivalent to those of the δ -function, except for a proportionality constant. However, for mixed configuration, i.e. nucleons in several shells, the matrix elements of SDI and δ -functions are different.

Multipole expansion of SDI

The SDI generates nuclear deformations for nucleons far from closed shells. To see this, consider the expansion of the SDI into multipoles. (These are particle hole interactions and describe the coupling strengths of nucleons to deformations.) We start with the expansion of the δ -function into Legendre polynomials,

$$4\pi \delta(\Omega) = \sum_L (2L+1) P_L(\cos \theta)$$

For $\theta \approx 0$ all $P_L \approx 1$ and terms add coherently. For $\theta \neq 0$, we get cancellation. Now for any $V(\theta)$ depending only on angle (we assume all radial integrals are equal), we can write:

$$V(\theta) = \sum_L (2L+1) F_L P_L(\cos \theta)$$

But for the SDI

$$V_{SDI} = 4\pi F\delta(\Omega) = F\sum (2L+1) P_L(\cos \theta)$$

Thus for the SDI we have

$$F_L = F_0$$

i.e. all multipole coefficients are equal.

Interestingly enough, an analysis of the spectra and electromagnetic transition rates in heavy deformed nuclei, shows that

$$F_2, F_3, F_4 \text{ are all } \sim \frac{25}{A} \text{ MeV}$$

The near equality of these quantities favor zero range. On the other hand, for a finite range interaction we might expect that

$$F_L \sim F_0 f_1\left(\frac{La}{R}\right)$$

where a = range of interaction, R = nuclear radius, and $f_1(x)$ decreases with x . For a simple model we find

$$f_1(x) \sim e^{-x^2}$$

The near equality of G_0 and F_0 is consistent with the expectations for an SDI. On the other hand for a finite range interaction we obtain,

$$G_0 \sim F_0 f_2(k_F a)$$

where $f_2(x)$ might be of the same order as $f_1(x)$

We can also look at pairing in higher J states using two nucleon transfer reaction in Pb. There it is found that G_2, G_4, G_6 are all fairly close to G_0 . This again favors the SDI.

To summarize, for the SDI we have

$$F_0 = F_2 = \dots = G_0 = G_2$$

All multipole coefficients as well as pairing interaction coefficients are equal.

Estimate of F_0

We can make a rough theoretical estimate of the magnitude of F_0 . We have the effective interaction discussed in Sec. 4.

$$V_{\text{eff}} = (-\alpha + \gamma \rho(R)) \delta(r)$$

where ρ is now a function of R . Then the magnitude of F_0 is just the strength of V_{eff} , averaged over the nucleus, divided by the nuclear volume, which is

$$\text{Vol} = A/\rho_0$$

Thus

$$F_0 = -\frac{\alpha \rho_0}{A} + \frac{\gamma \rho_0}{A} \rho_{AV}$$

Here we have to take an average of ρ over the nucleus. Now, as we have shown, for an interaction of this form, the two body and three-body energies are

$$W(2) = -\frac{3}{8} 4\rho_0 = -61.5 \text{ MeV}$$

$$W(3) = \frac{1}{8} \rho_0^2 = 23.4 \text{ MeV}$$

Thus we find

$$F_0 = -\frac{164}{A} \left(1 - \frac{1.14 \rho_{AV}}{\rho_0}\right) \text{ MeV}$$

Note that according to our assumptions, F_0 is weakly repulsive, not absent, inside nuclear matter.

Let us next make a rough estimate of ρ_{AV} . For a typical nucleus, roughly half is interior, where $\rho = \rho_0$, and half is surface, where we guess $\rho \sim 1/2 \rho_0$.

Thus

$$\rho_{AV} = \frac{3}{4} \rho_0$$

and

$$F_0 = \frac{164}{A} [1 - 1.14 \times 0.75] \text{ MeV}$$

The second term nearly cancels the first and we obtain

$$F_0 = -\frac{24}{A} \text{ MeV}$$

in surprisingly good agreement with the empirical results.

10. Some remarks concerning the effective interaction at high energies.

As we have shown, at low energies the effective interaction is roughly given by

$$V^{\text{eff}} = V_L - 2 I U \delta(r)$$

Here V_L is the "long range" part of the interaction discussed in Lecture 1.

I is the integral involving the wound in the wavefunction

$$I = \int (\phi - \psi^2) d^3r$$

and U is the average single particle potential for particles in the Fermi sea. As the energy increases, the following things happen.

1. V_L , properly defined, approaches the free particle t matrix denoted by t_F .
2. Both I and U become smaller with increasing energy. The details of this depend on the particular model used, but at high energies the impulse approximation, according to which $V^{\text{eff}} \rightarrow t_F$, is expected to be valid. This probably happens in the energy range 100 to 1000 MeV.

11. Summary

Here are what I consider the most important points of these three lectures.

1. The effective nucleon-nucleon interactions are strongly density dependent. Regardless of the cause of this density dependence (repulsive core, tensor, or n - Δ) it can be approximately represented as a three-body delta interaction. (There are, however, some problems associated with $N \neq Z$ asymmetry.)

2. The three-body interaction can be represented as an effective two body interaction. However, the latter is different depending on what quantities we wish to calculate. Thus if the interaction is

$$V = -V_0 + \gamma \delta(r_{123})$$

where V_0 is a conventional two-body interaction, then the density dependent two-body interaction between valence nucleons is (neglecting exchange terms)

$$V^{(2)} = -V_0 + v_{\text{val}}(r_{12})$$

but the two-body interaction which should be used for the calculation of the total potential energy is

$$V^{\text{PE}} = -V_0 + \frac{1}{3} v_{\text{val}}(r_{12})$$

3. The ratio of 3 which appears here is very important. Thus the interaction in the Fermi sea $V^{\text{F.S.}}$ leads to nuclear binding, yet the valence interaction is very weak. In fact, with our parameters we find that the angle average of V^{val} , denoted by F_0 , is close to zero.

4. The average pairing matrix element G_0 also happens to be very small in nuclear matter.

5. From 2, 3, and 4 we conclude that the valence interaction is small or absent in nuclear matter. Thus nucleons move nearly independently in nuclear matter.

6. From (2-4) we can also conclude that there is no momentum dependence of V^{val} at any density, at least up to nuclear matter density.

7. From 5 and 6 it follows that the valence interaction can be written something like

$$V = -\alpha \left(1 - \frac{\rho}{\rho_0}\right) \delta(\vec{r})$$

i.e. a surface peaked delta interaction.

8. Since all valence wavefunctions at the nuclear surface have essentially the same radial dependence, the $\delta(\vec{r})$ reduces to a $\delta(\Omega)$, function in the angle Ω between the nucleons.

9. The $\delta(\Omega)$ interaction, called the surface delta interaction (SDI), gives the same multipole interaction and pairing strength for all multipoles. Both are given by,

$$G_L \sim F_L \sim -\frac{25}{A} \text{ MeV}$$

for all L.

10. The Surface Delta Interaction, which contains only one free parameter (or two if we allow for isospin) explains the gross features of low lying spectra. It is not too much worse than realistic (but also more complicated) interactions.

11. The value of the SDI strength can be deduced using the interaction deduced in Sect. (9) (the section on the estimate of F_0) assuming that the nucleus is approximately half surface. It can also be independently derived from empirical data on nuclear spectra.

12. The density dependence of the effective interaction is expected to decrease with increasing energy so that at high energy, the effective interaction is the same as between particles in free space, i.e. the impulse approximation.