Ion-Temperature-Gradient Instability in Toroidal Plasmas

P. N. Guzdar, Liu Chen, W. M. Tang, and P. H. Rutherford

Plasma Physics Laboratory, Princeton University,
Princeton, New Jersey 08544

The stability of the ion-temperature-gradient mode in a toroidal plasma is investigated. Using the ballooning-mode formalism, an ordinary difference-differential equation is derived which includes full kinetic effects. The equation is, then, examined in various limits where it reduces to an ordinary differential equation. Analytic and computational studies show that, for $\eta_i \equiv d \ln T_i / d \ln n > 1$, toroidal effects further destabilize the mode and, hence, the corresponding growth rates exceed those obtained from the slab calculations. However, it is also found that toroidal effects give rise to higher η_i threshold compared to the slab case. Extensive numerical calculations over a wide range of parameters are carried out to delineate the regions of instability. It is concluded that the η_i instability is of relevance to present-day beam heated tokamaks.

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I. INTRODUCTION

Recent findings¹⁻⁴ regarding the stability of the universal drift mode have led to enhanced activity in the investigation of drift instabilities in sheared magnetic fields. It is well-known that there exist numerous destabilizing mechanisms associated with the electrons. Destabilization of drift waves by ion pressure gradients and impurity effects has also been considered.⁵⁻⁸ In particular, modes driven by ion temperature gradients have become the subject of special interest in light of the achievement of record high ion temperatures in the neutral-beam-heated PLT experiment. Experimental results indicate that the associated ion temperature gradient parameter, $\eta_i (\eta_i \equiv d \ln T_i / d \ln N)$, now falls in the range $\eta_i \sim 2-4$.⁹ Hence, a proper understanding of the role of η_i driven drift modes as a possible mechanism for causing anomalous transport in tokamaks becomes a relevant problem.

The discovery of the ion temperature gradient instability can be traced back to early Russian literature. Rudakov and Sagdeev⁶ found an unstable mode after carrying out a local analysis with $dT_i/dx \neq 0$ and $dN/dx = 0$ and $v_i \ll \omega/k_{\parallel} < v_e$. Pogutse⁷ reexamined the instability by assuming $dN/dx \neq 0$, and studied the effect of shear and kinetic-ion effects in the radially local limit. Later Coppi, Rosenbluth, and Sagdeev⁸ performed a systematic mode structure analysis in a slab geometry. Using adiabatic electrons and fluid ions, they obtained a Weber equation, which produced an eigenvalue condition for unstable η_i driven normal modes. However, eigenmodes derived by using the fluid-ion

approximation requires the assumption of large n_i for self-consistency. As pointed out by Kadomtsev and Pogutse¹⁰ and confirmed in more detailed numerical studies by Waltz et al.,¹¹ it is necessary to take into account kinetic-ion effects to show that the threshold for the instability occurs at $n_i \sim 1$.

The purpose of this paper is to study the instability of the n_i modes in the presence of toroidal-coupling effects. An examination of the toroidal terms arising from ion ∇B and curvature drifts indicates that, for relevant parameter range, they can be comparable to the terms appearing in the usual slab model calculations. The work outlined is as follows. In Sec. II we derive a mode structure equation in the slab geometry using fluid descriptions. Section III deals with the radially local analysis and includes a description of the basic physics of the n_i mode. In Sec. IV the eigenmode analysis for the slab case is examined and a comparison is made between the familiar electron drift mode and the n_i mode. In Sec. V starting from gyrokinetic description¹² for ions, adiabatic electron response and quasi-neutrality condition, a two-dimensional eigenmode equation in toroidal geometry is derived. Employing the ballooning-mode formalism,¹³⁻¹⁵ the eigenmode equation is, then, reduced to a one-dimensional differential-difference equation. The corresponding local analysis is considered in Sec. VI, and a detailed analytical and numerical investigation of the toroidal eigenmode equation in various limits is presented in Sec. VII. Section VIII deals with kinetic-ion effects which introduce a critical n_i for the instability. Final summary and discussions of the results and their implications are given in Sec. IX.

II. SLAB EIGENMODE EQUATION

In the following, we derive the mode structure equation for a simple slab model. Fluid descriptions are used here to elucidate the essential physics of the η_i mode. For the ions, the basic set of equations is the continuity equation

$$\partial n_i / \partial t + \nabla \cdot (n_i \mathbf{v}_i) = 0, \quad (1)$$

the equation of motion

$$m_i [\partial \mathbf{v}_i / \partial t + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i] = -e \nabla \phi - \nabla P_i / n_i + e \mathbf{v}_i \times \mathbf{B} / c - \nabla \cdot \Pi_{cf} / n_i, \quad (2)$$

the equation of energy^{10,16}

$$\partial P_i / \partial t + \nabla \cdot (P_i \mathbf{v}_i) = 0, \quad (3)$$

and

$$P_i = n_i T_i. \quad (4)$$

Here, Π_{cf} is the collision-free part of the Braginskii tensor,¹⁷ and the rest of the notation is standard. For the electrons, the dominant parallel dynamics ensure

$$n_e / N \equiv \tilde{n}_e = e \phi / T_e \equiv \tilde{\phi}. \quad (5)$$

For the slab model, $\mathbf{B} = B_0 (\mathbf{e}_z + \mathbf{e}_y x / L_s)$. Thus, $k_{||} = k_y x / L_s$

and all perturbed quantities can be expressed as $\psi(\mathbf{x}) = \psi(x) \times \exp(ik_y y - i\omega t)$. Furthermore, we choose an equilibrium pressure gradient along the negative x direction; $dP/dx < 0$. For the

frequency ordering $|\omega/\Omega_i| \ll 1$, the predominant perpendicular ion dynamics are given by the $\underline{E} \times \underline{B}$ and diamagnetic drifts; i.e.,

$$\underline{v}_{i0} = \underline{v}_E + \underline{v}_{Di} \quad , \quad (6)$$

where $\underline{v}_E = c(\nabla\phi \times \underline{e}_z)/B$ and $\underline{v}_{Di} = -c(\nabla p_i \times \underline{e}_z)/eBN$. With Corrections of $O(\omega/\Omega_i)$ (i.e., ion polarization drift) included, then we obtain

$$\nabla \cdot \underline{v}_{i1} = i\omega c \nabla_i^2 \phi / \Omega_i B - c \nabla_{Di} \cdot \nabla \nabla_i^2 \phi / BN \Omega_i \quad . \quad (7)$$

In deriving Eq. (7), we have used the Hinton-Horton¹⁸ relation and retained only the linear terms. Substituting the perpendicular ion motion into Eq. (1), we get

$$n_i/N \equiv \tilde{n}_i = \omega_{*e} \tilde{\phi} / \omega + (1 + \omega_{*pi} / \omega \tau) \rho_s^2 \nabla_i^2 \tilde{\phi} + k_{\parallel} v_{zi} / \omega \quad , \quad (8)$$

where $\omega_{*e} = -(k_y c T_e / eBN) (dN/dx)$, $\omega_{*pi} = -(k_y c / eBN T_i) (dp_i/dx)$, $\rho_s = c_s / \Omega_i$, $c_s^2 = T_e / m_i$, and $\tau = T_e / T_i$. The parallel equation of motion for the ions yields

$$v_{zi} = k_{\parallel} c_s^2 \tilde{\phi} / \omega + k_{\parallel} v_{i1}^2 / \omega = (k_{\parallel} c_s^2 / \omega) (\tilde{\phi} + \tilde{p}_i / \tau) \quad , \quad (9)$$

where $\tilde{p}_i = p_i / NT_i$. Now substituting Eq. (6) into Eq. (3), we obtain

$$\tilde{p}_i = \omega_{*pi} \tau \tilde{\phi} / \omega \quad . \quad (10)$$

In Eq. (10), we have neglected the parallel ion kinetic contributions by assuming $\omega_{*pi} \gg k_{\parallel}^2 v_{i1}^2 / \omega^2$ and the fluid-ion

approximation $k_{\parallel}^2 v_i^2 / \omega^2 \ll 1$. Equations (5), (8), (9), and (10), together with quasi-neutrality, yield the eigenmode equation in the slab geometry

$$\left[(1 + \omega_{*pi} / \omega) \rho_s^2 \nabla_{\perp}^2 - 1 + \omega_{*e} / \omega + (1 + \omega_{*pi} / \omega) k_{\parallel}^2 c_s^2 / \omega^2 \right] \tilde{\phi} = 0. \quad (11)$$

Before examining the eigenvalue problem, first we consider the local problem to identify the physical mechanisms of the instability.

III. SLAB LOCAL ANALYSIS

From Eq. (11) we can readily derive the following local dispersion relation

$$k_{\perp}^2 \rho_s^2 (1 + \omega_{*pi} / \omega) + 1 - \omega_{*e} / \omega - (k_{\parallel}^2 c_s^2 / \omega^2) (1 + \omega_{*pi} / \omega) = 0. \quad (12)$$

Equation (12) can be reduced to the dispersion relation derived by Kadomtsev and Pogutse¹⁰ by taking $k_{\perp} \rho_s \rightarrow 0$. For small $k_{\parallel}$, i.e., ω_{*e}^2 , $\omega_{*e}^2 (1 + \eta_i)^2 / \tau^2 \gg k_{\parallel}^2 c_s^2 (1 + \eta_i) / \tau$ and $k_{\perp}^2 \rho_s^2 < \tau / (1 + \eta_i)$,

we obtain

$$\omega^2 = - k_{\parallel}^2 c_s^2 (1 + \eta_i) / \tau. \quad (13)$$

Thus, Eq. (13) predicts purely growing instabilities. For larger $k_{\parallel}$, i.e., $\omega_{*e}^3 < k_{\parallel}^2 c_s^2 \omega_{*pi} < \omega_{*pi}^3$ and $k_{\parallel} c_s > (k_{\perp} \rho_s)^3 \omega_{*pi}$, we

have

$$\omega^3 \approx k_{\parallel}^2 c_s^2 \omega_{*pi} \quad , \quad (14)$$

and, again, we have instabilities though not purely growing. In both cases, the applicability of the fluid ion approximation requires $\eta_i \gg 1$.

To understand the physics of this instability, we refer to Fig. 1. An initial perturbation in the y-z plane is set up in the presence of an equilibrium pressure gradient. The shaded regions correspond to high perturbed pressure while, the unshaded encircled regions correspond to low perturbed pressure. The evolution of this initial perturbation can be best understood for the small - $k_{\parallel}$ purely growing mode. The various perturbed quantities are, then, related through Eqs. (5) and (10),

$$\omega_{*e} \tilde{\phi} / \omega = - k_{\parallel} v_{zi} / \omega \quad , \quad (15)$$

and

$$v_{zi} = k_{\parallel} c_s^2 \tilde{p}_i / \omega \tau \quad . \quad (16)$$

Equations (15) and (16) correspond, respectively, to the small $k_{\parallel}$ limits of Eqs. (8) and (9). In Eq. (10) the phase relationship between $\tilde{p}_i$ and $\tilde{\phi}$ is crucial. If we put $\omega = i\gamma$ and then determine $\tilde{\phi}$ for the given initial $\tilde{p}_i$, the phase of the y component of the electric field is determined (indicated by arrows in Fig. 1). Now, in the shaded regions of positive pressure perturbations, the $\underline{E} \times \underline{B}$ motion in the positive-x direction

moves the plasma into regions of lower equilibrium pressure and, therefore, further enhances the perturbations. Similar enhancement of perturbations occurs in the unshaded encircled regions of negative pressure perturbations. Consequently, this reinforcement of the initial perturbations leads to instability. This simple picture assumes that the parallel acoustic oscillations do not occur. The assumption which is involved is that the time scale for the instability is shorter than $(k_{\parallel} c_s)^{-1}$. This is true for large η_i , cf., Eq. (13). However, for larger $k_{\parallel}$, the restoring force along B becomes stronger and we no longer have a purely growing mode, cf., Eq. (14).

IV. SLAB EIGENMODE ANALYSIS

Now we return to Eq. (11) to solve the eigenvalue problem. The equation can be rewritten by defining the following dimensionless variable $x = x/\rho_s$, $\Omega = \omega/\omega_{*e}$, $b_{\theta} = k_y^2 \rho_s^2$ and $x_s^2 = \Omega^2 L_s^2 / L_n^2$, as

$$\{d^2/dx^2 - b_{\theta} + \tau(1-\Omega)/(1+\eta_i + \Omega\tau) + x^2/x_s^2\} \tilde{\phi}(x) = 0 \quad (17)$$

This is a standard Weber equation. The eigenvalue condition then gives the following dispersion relation

$$\begin{aligned} \Omega^2(1+b_{\theta}) + \omega[b_{\theta}(1+\eta_i)/\tau + i(2n+1)L_n/L_s - 1] \\ + i(2n+1)(1+\eta_i)L_n/L_s\tau = 0 \quad , \end{aligned} \quad (18)$$

where $n = 0, 1, \dots$. If $b_\theta(1 + \eta_i) \ll \tau$, then the two roots are

$$\Omega_1 = \frac{1}{1 + b_\theta} - i \left(\frac{2n + 1}{1 + b_\theta} \right) \left(\frac{L_n}{L_s} \right) \left(1 + \frac{1 + \eta_i}{\tau} \right), \quad (19)$$

and

$$\Omega_2 = i(2n + 1)(1 + \eta_i)L_n/L_s\tau. \quad (20)$$

The Ω_1 root corresponds to the usual Pearlstein-Berk electron drift mode.¹⁹ The damping is due to the shear-induced convective loss. The unstable root Ω_2 is the ion-temperature-gradient mode. This eigenmode growth rate can be understood in terms of the local dispersion relation, Eq. (13), by noting that the eigenmode has an effective width x_T given by the turning point

$$x_T \approx -i \Omega(L_s/L_n) [\tau/(1 + \eta_i)]^{1/2} = [(1 + \eta_i)/\tau]^{1/2}, \quad (21)$$

and, hence, an effective $k_\parallel = k_y \rho_s (1 + \eta_i)^{1/2} / L_s \tau^{1/2}$. The interesting feature to note is that, while the potential associated with the Pearlstein-Berk mode is an anti-well, the η_i mode is a well. What the shear does to the η_i mode is to localize it. This can be understood by looking at Fig. 2. Beyond a certain $k_{\parallel \max}$ there is no growth so that the localization of the mode is due to the localization in the instability region. In fact, $k_{\parallel \max} = k_y x_T \rho_s / L_s$. Finally, we note that the validity of the fluid ion approximation requires

$$|k_\parallel v_i / \omega|^2 = 2 |x_T L_n / \Omega L_s|^2 = 2 / (1 + \eta_i) \ll 1; \quad (22)$$

that is, $\eta_i \gg 1$.

V. TOROIDAL EIGENMODE EQUATION

We now give a formal derivation of the eigenmode equation for η_i modes including toroidal-coupling effects due to ion ∇B curvature drifts. For an axisymmetric toroidal plasmas with concentric, circular magnetic surfaces, the perturbed quantity $\psi(r, \theta, \xi, t)$ can be written as

$$\psi(r, \theta, \xi, t) = \tilde{\psi}(r, \theta) \exp[i(m_0 \theta - n \zeta - \omega t)] \quad , \quad (23)$$

where $|m_0| \gg |\partial \tilde{\psi} / \partial \theta| / |\tilde{\psi}|$. The θ dependence of $\tilde{\psi}$ arises because of the broken symmetry in the poloidal direction due to functional dependence of the equilibrium magnetic field on the poloidal coordinate θ ; i.e.,

$$\underline{B} = B(\underline{e}_\xi + \varepsilon \underline{e}_\theta / q) \quad , \quad (24)$$

where $q(r) = r B_T / R B_P$ is the safety factor, $\varepsilon = r / R_0$, and $B = B_0 (1 - \varepsilon \cos \theta)$.

The ion gyrokinetic equation can be written as

$$\begin{aligned} (\omega - i v_{||} \underline{e}_{||} \cdot \nabla - i v_{Di} \cdot \nabla) \hat{h}_i &= (\omega + \omega_{*Ti}) (e F_{Mi} / T_i) < \hat{\phi}(r' + v_\perp \sin \phi' / \Omega_i, \theta) \\ &\times \exp(i k_\theta v_\perp \cos \phi' / \Omega_i) >_\phi \quad , \quad (25) \end{aligned}$$

where $v_{Di} = \underline{e}_{||} \times \nabla B (v_\perp^2 / 2 + v_{||}^2) / \Omega_i B$ is the sum of the ∇B and curvature drift, F_{Mi} is the Maxwellian distribution, $k_\theta = m_0 / r_0$,

$\omega_{*Ti} = \omega_{*i} \times [1 + \eta_i (x^2 - 3/2)]$, and $x^2 = (v_\perp^2 + v_{||}^2) / v_i^2$.

Meanwhile, we have

$$r' = r - v_{\perp} \sin \phi / \Omega_i ,$$

$$\theta' = \theta - v_{\perp} \cos \phi / \Omega_i ,$$

$$\text{and} \quad \phi' = \phi \quad (26)$$

representing the guiding-center coordinates, $\langle A \rangle_{\phi} = \int_0^{2\pi} d\phi' A(\phi') / 2\pi$, $h_i = f_i + F_{Mi} e\phi / T_i$, and $h_i(r', \theta', \xi') = \hat{h}_i(r', \theta')$ $\exp [i(m_0 \theta' - n\xi' - \omega t)]$. Since we have to evaluate perturbed ion density, a second phase average appears when $\hat{h}_i$ is replaced by h_i and the velocity space integration involving the operation $\int_0^{2\pi} d\phi$ has to be done. In the small ion-Larmor-radius limit, the gyrophase averaging yields

$$\begin{aligned} & \langle \hat{\phi}(r', v_{\perp} \sin \phi' / \Omega_i \theta) \exp(ik_{\theta} v_{\perp} \cos \phi' / \Omega) \rangle_{\phi} \\ & \quad \exp(-ik_{\perp} v_{\perp} \cos \phi / \Omega_i) \rangle_{\phi} \\ & \approx [1 - (k_{\theta} v_{\perp} / \Omega_i)^2 / 2 + (1/2)(v_{\perp} / \Omega_i)^2 \partial^2 / \partial r^2] \hat{\phi}(r, \theta) \\ & \equiv (1 - b_{\perp i} v_{\perp}^2 / v_i^2) \hat{\phi}(r, \theta) . \end{aligned} \quad (27)$$

Following standard procedures,¹² we then derive the perturbed ion density to be

$$\begin{aligned} n_i / N = & -\{1 + [1 + \omega_{*i}(1 - 3\eta_i/2)/\omega] (I_1 - I_2 b_{\perp i}) \\ & + (\eta_i \omega_{*i} / \omega) (I_3 - I_4 b_{\perp i})\} e\hat{\phi} / T_i , \end{aligned} \quad (28)$$

where

$$\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \frac{2\xi_i}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty \frac{\begin{Bmatrix} x_\perp \\ x_\parallel \end{Bmatrix}^3 dx_\perp dx_\parallel \exp[-(x_\perp^2 + x_\parallel^2)]}{x_\parallel - \xi_i - \hat{\xi}_{Di}(x_\perp^2/2 + x_\parallel^2)} , \quad (29)$$

and

$$\begin{Bmatrix} I_3 \\ I_4 \end{Bmatrix} = \frac{2\xi_j}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty \frac{\begin{Bmatrix} x_\perp \\ x_\parallel \end{Bmatrix}^3 x_\perp^2 dx_\perp dx_\parallel \exp[-(x_\perp^2 + x_\parallel^2)]}{x_\parallel - \xi_i - \hat{\xi}_{Di}(x_\perp^2/2 + x_\parallel^2)} . \quad (30)$$

Here, $x_\parallel = v_\parallel/v_i$, $x_\perp = v_\perp/v_i$, $x = v/v_i$, $\hat{\xi}_{Di} = 2\omega_{di}C/k_\parallel v_i \equiv \xi_{Di}C$,
 $\omega_{di} = k_y c T_i / e B R_0$, and

$$\hat{C}\phi(r, \theta) = [\cos \theta + (ir/m_0) \sin \theta \partial/\partial r] \hat{\phi}(r, \theta) . \quad (31)$$

Again, using adiabatic electron response and quasi-neutrality, we get the required two-dimensional eigenmode equation

$$\{1/\tau + [1 + (1 - 3\eta_i/2)/\Omega\tau] I_1 + \eta_i I_3/\Omega\tau - b_{\perp i} \\ [1 + (1 - 3\eta_i/2)/\Omega\tau] I_2 - b_{\perp i} \eta_i I_4/\Omega\tau\} \hat{\phi} = 0 . \quad (32)$$

A convenient radial coordinate is $s = (r - r_0)\Delta r_s$, where r_0 is a reference mode-rational surface defined by $m_0 = nq(r_0)$ and

$\Delta r_s = 1/k_\theta \hat{s}$ ($\hat{s} = rq'/q$) is the separation between adjacent meridional surfaces. Furthermore, we can Fourier decompose $\hat{\phi}$ as

$$\hat{\phi}(r, \theta) = \sum_j \hat{\phi}_j(s) \exp(ij\theta) \quad (33)$$

The $\hat{\phi}_j$'s are then poloidal "sidebands" coupled together by the toroidal effects and $|j| \ll |m_0|$ is assumed here. In this coordinate representation, we have

$$b_{\perp i} = -b_{\theta i} [\hat{s}^2 d^2/ds^2 - 1] \quad (34)$$

with $b_{\theta i} = k_{\theta i}^2 \rho_i^2 / 2$, and

$$|k_{\parallel}| = |s - j|/qR \quad (35)$$

If we further assume that $|\omega_{di}/\omega| < 1$, then the integrals in Eqs. (29) and (30), can be simplified to

$$I_1 = \xi_i Z(\xi_i) + [Z'/2 + Q_1] \xi_i \xi_{Di} C \quad (36)$$

$$I_2 = \xi_i Z(\xi_i) + [Z' + Q_1] \xi_i \xi_{Di} C \quad (37)$$

$$I_3 = \xi_i Z(\xi_i) + \xi_i^2 (1 + \xi_i Z_i) + [Z' + 3Q_1/2 + Q_2] \xi_i \xi_{Di} C \quad (38)$$

and

$$I_4 = 2\xi_i Z(\xi_i) + \xi_i^2 (1 + \xi_i Z_i) \xi_i \xi_{Di} C ; \quad (39)$$

where

$$Q_1 = 1 + 2\xi_i Z_i + \xi_i^2 Z' ,$$

$$Q_2 = 1/2 + 3\xi_i^2 Q_1 - 2\xi_i^3 (Z_i + \xi_i Z') ,$$

and Z is the plasma dispersion function.

In summary, the two-dimensional eigenmode equation can be written as

$$(b_{\theta i} \hat{s}^2 d^2/ds^2 + L_1) \hat{\phi}_j = (L_2/2) [\hat{\phi}_{j+1} + \hat{\phi}_{j-1} + \hat{s}(d/ds) (\hat{\phi}_{j-1} - \hat{\phi}_{j+1})] , \quad (40)$$

where $L_1 = G_1 \tau - b_{\theta i}$, $G_1 = (1 + 1/\tau + N_1 D_1 + N_2 D_2)/A_1$,
 $N_1 = 1 + (1 - 3\eta_i/2)$, $N_2 = \eta_i/\Omega\tau$, $D_1 = \xi_i Z_i$, $D_2 = \xi_i^2 (1 + \xi_i Z_i)$
 $+ \xi_i Z_i$, $A_1 = N_1 D_1 + N_2 (D_1 + D_2)$, $L_2 = 2\varepsilon_n G_2/\Omega$, $G_2 =$
 $[(B_1/A_1)(1 + 1/\tau) + N_1 D_3 + N_2 D_4]/A_1$, $D_3 = B_1 D_1/A_1 - D_2 + 2\xi_i^2$
 $(1 + D_1)$, $D_4 = (B_1/A_1 - 1)[D_1 + \xi_i^2(1 + D_1)] + \xi_i^2[1 + \xi_i^2(1 + D_1)]$,
 $B_1 = N_1 D_5 + N_2 D_6$, $D_5 = -1 - 2\xi_i^2(1 + \xi_i Z_i)$, and $D_6 = -5/2 - \xi_i^2$
 $(3 + 2D_1) - 2\xi_i^4(1 + D_1)$. The terms on the right-hand-side of

Eq. (40) arise due to the toroidal effects and are responsible for the couplings between the neighboring side bands. In fact, we have an infinite set of coupled differential equations. Since $|m_0| \sim |n| \gg 1$, we can employ the large- n ballooning-mode formalism¹³⁻¹⁵ and introduce the variable $z = s - j$ such that $k_{\parallel} = z/qR$ and $\hat{\phi}_{j \pm 1}(s) = \hat{\phi}(s, j \pm 1) \equiv \phi(z \mp 1)$. In the zeroth order, Eq. (40) then becomes a one-dimensional difference-differential equation

$$[b_{\theta i} \hat{s}^2 \frac{d^2}{dz^2} + L_1] \phi(z) = \frac{L_2}{2} \{ \phi(z-1) + \phi(z+1) + \hat{s} \frac{d}{dz} [\phi(z+1) - \phi(z-1)] \} . \quad (41)$$

To make Eq. (41) more tractable, we discuss several limits in which Eq. (41) can be reduced to a second-order differential equation. First, we consider the Taylor's strong-coupling approximation,²⁰ which is

$$\begin{aligned} \phi(z-1) + \phi(z+1) + \hat{s} d[\phi(z+1) - \phi(z-1)]/dz \\ \approx 2[1 + (1/2 - \hat{s}) d^2/dz^2] \phi(z) . \end{aligned} \quad (42)$$

Equation (41) then becomes

$$([b_{\theta i} \hat{s}^2 - L_2(1/2 - \hat{s})] d^2/dz^2 + L_1 - L_2) \phi(z) = 0 . \quad (43)$$

This equation contains the toroidal effects in the strong-coupling approximation together with full kinetic-ion effects. The strong-coupling approximation requires that the eigenmodes extend over a few mode rational surfaces (since z is normalized to Δr_s). If, in addition, we made the fluid-ion approximation, Eq. (43) further reduces to a Weber equation

$$\{[b_0 \hat{s}^2 + \epsilon_n p_2 (2\hat{s} - 1)/\Omega] d^2/dz^2 + p_1 \tau - b_0 - 2\epsilon_n p_2/\Omega\} \phi = 0, \quad (44)$$

with $p_1 = (1/\Omega - 1)/A\tau + [(1 + \eta_i/A\Omega\tau)(1 + 1/\tau + \eta_i/\Omega\tau) - \eta_i/\Omega\tau]/\xi_i^2 A$, $p_2 = 1 + 3(1 + \eta_i/A\Omega\tau)(1 - 1/\Omega)/2A\tau$, and $A = 1 + (1 + \eta_i)/\Omega\tau$. We note here that Eq. (44) is analytically soluble and the solutions will be discussed later. At this point, we consider the possibility of abandoning Taylor's strong-coupling approximation. So we proceed in a different way. We first make the fluid-ion approximation and retain terms up to ξ_i^{-2} . Equation (41) then reduces to

$$(b_0 \hat{s}^2 d^2/dz^2 + p_1 \tau - b_0) \phi(z) = (2\epsilon_n p_2/\Omega) \{\phi(z-1) + \phi(z+1) + \hat{s} d[\phi(z+1) - \phi(z-1)]/dz\}. \quad (45)$$

Defining

$$\phi(z) = \int_{-\infty}^{\infty} \phi(\eta) \exp(i\eta z) d\eta, \quad (46)$$

we now Fourier transform Eq. (45). Since we have terms like $\xi_i^2 = z_i^2 v_i^2 / q^2 R^2 \omega^2$ we get a second order differential equation for $\phi(\eta)$; i.e.,

$$\{d^2/d\eta^2 + \beta_i^2 [b_\theta (1 + \hat{s}^2 \eta^2) + (1 - 1/\Omega)/A + (2\epsilon_n P_2/\Omega)$$

$$(\cos \eta + \hat{s} \eta \sin \eta)]\} \phi(\eta) = 0$$

with $\beta_i^2 = b_\theta (q\Omega/\epsilon_n)^2 / [1 + (1 - 1/\Omega)(1 + \eta_i/A\Omega\tau)/A\tau]$. Hence,

if the fluid-ion approximation is valid, we can do much better than just using the Taylor's strong-coupling approximation. We can now ascertain the validity of strong coupling by first establishing what the strong-coupling approximation is in the transformed variable, η . If we assume $\cos \eta \approx 1 - \eta^2/2$ and $\sin \eta \approx \eta$, Eq. (47) becomes

$$\{d^2/d\eta^2 + \beta_i^2 [b_\theta (1 + \hat{s}^2 \eta^2) + 2\epsilon_n P_2/\Omega + (\epsilon_n P_2/\Omega)$$

$$(2\hat{s} - 1) \eta^2]\} \phi(\eta) = 0. \quad (48)$$

We note that Eq. (48) is the Fourier transform of Eq. (44).

Hence, the strong-coupling approximation in the transformed variable demands that the width of the mode, η_w , is small; i.e., $\eta_w < \pi$.

Before we proceed the investigation of the eigenvalue problem in the toroidal case, we first examine the simple case of the local analysis.

VI. TOROIDAL LOCAL ANALYSIS

To understand the role played by toroidal-coupling effects, we look at the local dispersion relation in the limit $b_\theta \rightarrow 0$, $k_\parallel \rightarrow 0$ and near $\delta = 0$. Equation (47) then becomes, for $\eta_i \gg 1$,

$$1 - i/\Omega + (1 + \eta_i/\Omega\tau)(2\epsilon_n/\Omega) = 0 \quad . \quad (49)$$

For $|\epsilon_n \eta_i| > 1$, Eq. (49) yields an unstable root

$$\Omega \approx i(\epsilon_n \eta_i/\tau)^{1/2} \quad . \quad (50)$$

This is basically a Rayleigh-Taylor type instability where the role of the effective gravity is played by the bad curvature. This source of instability can therefore quite significantly modify the stability of the η_i modes arising from ion inertial effects ($k_\parallel c_s$); especially, if the mode is localized in the bad curvature region. This motivates the study of the stability of the η_i mode using the ballooning mode formalism.

VII. TOROIDAL EIGENMODE ANALYSIS

This section is devoted to the nonlocal analyses in the toroidal geometry. By solving the Weber equation, Eq. (48), analytically and Eq. (47) numerically, we are able to establish the range of validity for Taylor's strong-coupling approximation. Having done that, we can then ascertain the kinetic-ion effects by using Eq. (40), which is derived using Taylor's strong-coupling approximation but with kinetic-ion responses retained.

Equation (48) can be written as

$$[d^2/d\eta^2 + \beta_i^2(\lambda + \Gamma\eta^2)]\phi(\eta) = 0, \quad (51)$$

where $\lambda = b_\theta + (1 - 1/\Omega)/A + 2\epsilon_n p_2/\Omega$ and $\Gamma = b_\theta \hat{s}^2 + (2\epsilon_n p_2/\Omega) \times (1 - \hat{s}/2)$. The eigenvalue condition $(2n + 1)^2 \Gamma + \beta_i^2 \lambda^2 = 0$ gives us the following dispersion relation for the lowest $n = 0$ eigenstate

$$\begin{aligned} & \{b_\theta \hat{s}^2 \Omega A'^2 + \epsilon_n (2\hat{s} - 1) [(3/2)(A' + \eta_i)(\Omega - 1) + A'^2]\} \\ & [A'^2 + (\Omega - 1)(A' + \eta_i)] + (q^2 \Omega b_\theta / \epsilon_n^2) \{b_\theta A'^2 \Omega \\ & + 2\epsilon_n [(3/2)(A' + \eta_i)(\Omega - 1) + A'^2] \\ & + \tau \Omega (\Omega - 1)\}^2 = 0, \quad (52) \end{aligned}$$

with $A' = \Omega\tau + 1 + \eta_i$.

In the limit $\tau > 1$, $\eta_i > 1$ and $|b_\theta \hat{s}^2 \Omega A'^2| > |\epsilon_n (2\hat{s} - 1)| \times [(3/2)(A' + \eta_i)(\Omega - 1) + A'^2]$, the dispersion relation yields

$$\Omega \approx \frac{1}{2(1+b_\theta)} \left\{ 1 - 2\epsilon_n - \frac{b_\theta(1+\eta_i)}{\tau} - i\hat{s} \frac{\epsilon_n}{q} + \left[\left(1 - 2\epsilon_n - \frac{b_\theta(1+\eta_i)}{\tau} - i\hat{s} \frac{\epsilon_n}{q} \right)^2 - \frac{4(1+\eta_i)}{\tau} \left(2\epsilon_n + i\hat{s} \frac{\epsilon_n}{q} \right) \right]^{1/2} \right\} . \quad (53)$$

Furthermore, for $\epsilon_n(1 + \eta_i) < 1$ and $b_\theta(1 + \eta_i)/\tau < 1$, we have

$$\Omega_1 \approx (1 + b_\theta)^{-1} \{ 1 - 2\epsilon_n [1 + (1 + \eta_i)/\tau] - b_\theta(1 + \eta_i)/\tau - i\hat{s}\epsilon_n [1 + (1 + \eta_i)/\tau]/q \} , \quad (54)$$

which is the shear damped electron drift branch. We also obtain the other unstable ion branch

$$\Omega_2 \approx (1 + \eta_i)(i\hat{s}\epsilon_n/q + 2\epsilon_n)/\tau , \quad (55)$$

which corresponds to the η_i mode. In the absence of toroidal effects, the growth rate is the same as that obtained in the slab case with L_s^{-1} replaced by $\hat{s}/Rq$. For $b_\theta(1 + \eta_i)/\tau > 1$ and $\epsilon_n(1 + \eta_i) < 1$, however, we have

$$\Omega_2 \approx -\epsilon_n [is/q + 2q] / [(1 + \eta_i) \tau b_\theta] ; \quad (56)$$

i.e., the eigenmode is damped. The interesting case is $\epsilon_n(1 + \eta_i) > 1$ and $b_\theta(1 + \eta_i)/\tau < 1$; for which, we have

$$\Omega \approx 1 - 2\epsilon_n - b_\theta(1 + \eta_i)/\tau - is\epsilon_n/q + i[(1 + \eta_i)\epsilon_n \times (2 + is/q)/\tau]^{1/2} ; \quad (57)$$

which gives an instability with the growth from toroidal effects being the dominant term. We remark that for the case considered here the validity of strong-coupling approximation requires

$$(2\epsilon_n \tau)^{1/2} < \pi^2 b_\theta (1 + \eta_i)^{1/2} . \quad (58)$$

For more general parameter regimes, we solve the dispersion relation Eq. (52) numerically and compare the growth rates with those obtained by directly integrating Eq. (47). In Fig. 3, we compare also the toroidal case with the slab case. For the chosen set of parameters, the toroidal effects are definitely destabilizing. In Figs. 4(a), 4(b), and 4(c), we show the wave functions. Figures 4(a) and 4(b) correspond to the limit in which the strong-coupling approximation is valid. The mode is highly localized in the η coordinate and extended in the z coordinate. On the other hand, Fig. 4(c) corresponds to parameters for which the Taylor's approximation is expected to break down and, indeed, the mode is well extended in the η coordinate.

We next address ourselves to the validity criterion of the fluid-ion approximation, $|k_{\parallel} v_i / \omega|^2 \ll 1$. In the η coordinate, the fluid-ion validity criterion becomes

$$2 |(\epsilon_n^2 / q^2 \Omega^2 b_{\theta} \tau) d^2 \phi / d\eta^2| \ll |\phi| \quad (59)$$

Using Eq. (47) and defining $G(\eta) = 2\epsilon_n^2 \beta_i^2 Q(\eta) / q^2 \Omega^2 b_{\theta} \tau$, where $Q(\eta) = b_{\theta} (1 + \hat{s}^2 \eta^2) + (1 - 1/\Omega)/A + 2\epsilon_n P_2(\cos \eta + \hat{s} \eta \sin \eta) / \Omega$, define a parameter λ ; such that

$$\lambda = \frac{\int G(\eta) |\phi|^2 d\eta}{\int |\phi|^2 d\eta} \quad (60)$$

λ , thus, gives a measure of the fluid-ion approximation weighted over the entire wave function. In Fig. 5, we have plotted λ against b_{θ} and find that for the chosen set of parameters the fluid-ion approximation becomes consistently bad as we go into the region of weak coupling.

VIII. EFFECTS OF KINETIC IONS

In this section, we investigate the kinetic-ion effects by solving Eq. (44) numerically. Figure 6 shows the modified growth rates with the inclusion of kinetic-ion effects. Again, the toroidal case is more unstable than the slab case. Another interesting feature that emerges is that the introduction of

the kinetic-ion effect does not affect the strong-coupling approximation as can be seen by comparing the width of the wave functions in Figs. 7 and 4(b).

The next question is the determination of a threshold in η_i . It is well known that for the slab case $\eta_{\text{critical}} \approx 1$. In Fig. 8, we plot the growth rates as a function of η_i and, to our surprise, we find that the toroidal threshold is higher than the slab case. We ask ourselves if it is possible that as $\eta_i \rightarrow 1$ the kinetic-ion effects become very dominant for the toroidal case and therefore introduce a strong stabilization. To answer this, we would like to know the behavior of $(k_{\parallel} v_i / \omega)^2 = 2(z_w \epsilon_n / q\Omega)^2 / b_{\theta} \tau$ as $\eta_i \rightarrow 1$. We find that neither Ω_r , nor z_w , the width of the wave function, changes significantly as $\eta_i \rightarrow 1$. Thus, kinetic-ion effects do not seem to have led to the dramatic raising of the threshold. However, we do find (Fig. 9) is that, in the fluid ion limit, the growth rate decreases as η_i is reduced and, in fact, becomes less than that of the slab case. So that for $\eta_i \lesssim 1$ the toroidal effects seem to be stabilizing and, hence, the threshold η_i can be expected to be higher. Another result that emerges is that, if we look for critical η_i as a function of τ (Fig. 10), we find, for $\tau = 1/2$, the threshold becomes even higher. This is due to the kinetic-ion effects which become more important as τ becomes less than unity. In fact, we find that the minimum critical η_i occurs for $\tau = 1$.

IX. SUMMARY AND DISCUSSIONS

The stability of the ion-temperature-gradient mode has been examined in this paper. Starting with the local and slab model theories to understand the basic physics of the mode and, hence, the localization of the mode in the shear case, we have studied the eigenmode problem associated with this mode in a toroidal plasma. The ballooning mode formalism reduces the 2D problem to that represented by a 1D difference-differential equation. This equation is further reduced to a second-order ordinary differential equation by either using the fluid-ion approximation and Fourier transforming the equation or by making the Taylor's strong-coupling approximation. In the latter case, the full kinetic-ion effect can also be retained. The toroidal effects are found to be destabilizing for $\eta_i > 1$. However, around $\eta_i \sim 1$ the toroidal effect becomes stabilizing so that with full kinetic-ion effects the η_i threshold turns out to be higher than that found from the slab model calculations. Also, the critical η_i is found to depend on τ and has a minimum at $\tau \approx 1$.

The present results are of relevance to present-day beam-heated tokamaks where one finds $\eta_i \sim 2-4$. However, at this point it is not very clear whether the observed fluctuations in PLT are due to the η_i modes. For one thing, the threshold condition for the onset of the fluctuations is not very conclusive, i.e., whether the onset is due to a critical beam power (which can give rise to beam-driven instabilities) or a critical T_i , or η_i . Further experiments may help isolate the

triggering mechanisms and thereby help in identifying the source of the instability. Furthermore, a number of instability mechanisms could be responsible as has been suggested by Rewoldt, Marchand, Tang²¹; although the η_i mode may tend to dominate in these range of parameters. Hence, we feel that η_i mode may be a very favorable candidate to account for the enhanced fluctuations observed in P.I.T.

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FIGURE CAPTIONS

FIG. 1. Physical Model for ion-temperature-gradient instability in a slab geometry.

FIG. 2. Plots of γ/ω_{*e} , ω_r/ω_{*e} versus $k_{\parallel}c_s/\omega_{*e}$ from local dispersion relation, Eq. (12), with $k_{\perp}^2\rho_s^2 = 0$ and $\eta_i = 3$.

FIG. 3. Plots of γ/ω_* versus b_{θ} with the fluid-ion approximation for $\eta_i = 3$, $\epsilon_n = 0.1$, $q = \tau = \hat{s} = 1$.

FIG. 4. Plots of eigenfunction (a) $\hat{\phi}(\eta)$ for $b_{\theta} = 0.25$, (b) $\phi(z)$ for $b_{\theta} = 0.25$, and (c) $\hat{\phi}(\eta)$ for $b_{\theta} = 0.01$. The rest of the parameters are $\eta_i = 3$, $\epsilon_n = 0.1$, $q = \tau = \hat{s} = 1$.

FIG. 5. Plot of the validity parameter for the fluid-ion approximation, λ , versus b_{θ} for $\eta_i = 3$, $\epsilon_n = 0.1$, and $q = \tau = \hat{s} = 1$.

FIG. 6. γ/ω_* versus b_{θ} with kinetic-ion responses for $\eta_i = 3$, $\epsilon_n = 0.1$ and $q = \tau = \hat{s} = 1$. The toroidal case employs the strong-coupling approximation.

FIG. 7. Plot of the eigenfunction $\phi(z)$ for $\eta_i = 3$, $\epsilon_n = 0.1$, $q = \tau = \hat{s} = 1$ and $b_{\theta} = 0.25$ with kinetic ions.

FIG. 8. γ/ω_* versus η_i with kinetic-ion responses. The parameters are $\epsilon_n = 0.1$, $q = \tau = \hat{s} = 1$, $b_\theta = 0.2$ for the slab case and $b_\theta = 0.1$ for the toroidal case.

FIG. 9. Same as Fig. 8 except the fluid-ion approximation is used.

FIG. 10. The critical η_i versus τ for $\hat{s} = q = 1$, $\epsilon_n = 0.1$, $b_\theta = 0.2$ for the slab case, and $b_\theta = 0.1$ for the toroidal case.

Fig. 1. (PPPL-792336)

Fig. 2. (PPPL-792333)

Fig. 3. (PPPL-792334)

Fig. 4. (PPPL-792332)

Fig. 5. (PPPL-792340)

Fig. 6. (PPPL-792337)

Fig. 7. (PPPL-792341)

Fig. 8. (PPPL-792338)

Fig. 9. (PPPL-792335)

Fig. 10. (FPPL-792339)