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ANALYTICAL DESCRIPTION OF THE BASIN AND THE TRANSIENTS OF A POINT
ATTRACTOR OF THE HÉNON MAPPING

Theo P. Valkering
Center for Theoretical Physics
Twente University of Technology
7500 AE Enschede
The Netherlands

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Abstract

Explicit expressions for closed curves C_n about the attractor that enclose a part of its basin are obtained. If $n \rightarrow \infty$ the interior of C_n covers the complete basin. A sequence of invertible polynomial transformations is given, that converges rapidly towards the normal transformation, that transforms the nonlinear motion in the basin to a linear one. Using these transformations we obtain some practical tools to describe one aspect of the transient behavior in the basin.

1. Results

Consider a quadratic invertible mapping of the real plane, $x \rightarrow x'$, with

$$x' = Hx, \quad Hx = Px + Q(x) \quad (1)$$

with constant positive Jacobian b , smaller than unity. $Q(x)$ is a homogeneous quadratic expression in the components x_1 and x_2 . The matrix P is assumed to have two different complex eigenvalues, such that the origin is a spiral attractor.

We obtain a sequence of polynomial expressions, representing closed curves C_n about the origin, which enclose a part of its basin of attraction. For each enclosed domain we shall obtain a polynomial Lyapunov function $L_n(x)$, which controls the rate of convergence,

$$L_n(Hx) \leq \theta_n(L_n(x)) L_n(x), \quad \theta_n(L_n) = b^{1/2} + b^{n/2} q L_n, \quad (2)$$

where q is some positive constant representing the 'strength' of the nonlinear term (cf. §2). At sufficiently large n , each x in the basin is enclosed by a contour C_n . Figure 1 shows some contours for the Hénon mapping ([1]).

To describe the transient behavior we introduce a functional $n_\epsilon(x)$ that represents the number of steps necessary to map x into a well defined ϵ -neighborhood of the origin: below a Lyapunov function $L(x)$ is defined, for which

$$L(Hx) = b^{1/2} L(x) \quad (3)$$

for each x in the basin. Eq (3) shows that a level line $L(x) = L_0$ is mapped onto the level line $L(x) = b^{1/2} L_0$. Thus the number of steps necessary to map x into a small region given by $L(x) \leq \epsilon$, depends on L only and is

$$n_\epsilon(x) = \text{INT}[(\ln L(x) - \ln \epsilon) / \ln b^{-1/2}], \quad (4)$$

i.e. the integer part of the expression in brackets.

The irregular shape of the area enclosed by a level line (cf. Fig. 2) indicates irregular transient behavior. One aspect of this behavior is formulated more precisely in terms of the gradient of L : in a (sufficiently) small neighborhood of some x_0 , the average value of $n_\epsilon(x)$ equals $n_\epsilon(x_0)$ and one readily shows [2] that its root mean square is proportional to the length of $(\nabla L/L)_{x=x_0}$. Both for the level lines and the gradient we have explicit approximate expressions.

2. Method: transformation to normal form.

To obtain the results above a sequence of invertible coordinate transformations $x_n(u)$, with inverse $u_n(x)$

$$x_n(u) = H^{-n}(P^n u), \quad u_n(x) = P^{-n}H^n(x), \quad (5)$$

is introduced. It can be proved [2] that the sequence $\{x_n(u)\}$ converges to an invertible function $x(u)$, that is defined for $u \in \mathbb{R}^2$ and whose range is the basin of attraction of the origin. Furthermore, $x(u)$ transforms the mapping $x' = H(x)$, restricted to this basin, to the linear mapping $u' = Pu$. The existence of this function and its inverse $u(x)$, is guaranteed by an easy extension of a theorem of Poincaré [3].

With these transformations the functionals L_n and L are defined

$$L_n(x) = (u_n(x), A u_n(x)), \quad L(x) = (u(x), A u(x)), \quad (6)$$

where A is a real symmetric positive matrix such that $(Pu, APu) = b(u, Au)$. Such an A exists since P has complex eigenvalues. One readily proves (2) and (3), with q being the smallest positive number such that $(Q(x), AQ(x)) \leq q^2(x, A_0 x)^2$.

Furthermore observe that a level line $L_n(x) = L_0$ is the image in the x plane of an ellipse $(u, Au) = L_0$ in the u plane by the mapping $x_n(u)$. Such an ellipse is parametrized by $u = L_0^{1/2}(a_1 \cos \phi + a_2 \sin \phi)$, a_1 and a_2 being eigenvectors of A of appropriate length. Consequently we have explicit expressions for the level lines of L_n . Since $L_n(x)$ has only one stationary point, which is a minimum at the origin, it is clear from (2) that curves C_n for which $\theta_n = 1$ enclose a domain in the basin of attraction. In [2] it will be proved that a level line $L_n(x) = L_0$ converges to a level line $L(x) = L_0$, and that $\nabla L_n(x)$ along a level line converges to $\nabla L(x)$.

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Figure 1

Contours C_2 , C_4 and C_{10} for the Hénon mapping (1) with

$$P = \begin{pmatrix} 2c & -b \\ 1 & c \end{pmatrix} \text{ and } Q(x) = \begin{pmatrix} 2x_1^2 \\ 0 \end{pmatrix} \text{ (cf. [4])}, b=0.7 \text{ and } c=0.$$

+ denotes the attractor and x is a saddle fixed point. The curve through x is the stable manifold W^s of the saddle. The basin of attraction is certainly within the region enclosed by this stable manifold. There can be other attracting orbits within this region however ([5]).

Figure 2

Level lines of L for $b=0.9$ and $c=0$. Curve 3 is mapped by H in 5 steps onto curve 2, etc.

Observe that the shape of the enclosed area is more complicated for larger values of L .

Figure 3

$||\nabla L||/L$ along the level lines of Figure 2. Observe that the maximal value and the complexity of the graphs increase with L .