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**Equilibrium and Ballooning Mode
Stability of an Axisymmetric
Tensor Pressure Tokamak**

W. A. Cooper
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D. B. Nelson
T. Kammash

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EQUILIBRIUM AND BALLOONING MODE STABILITY OF AN
AXISYMMETRIC TENSOR PRESSURE TOKAMAK

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CONTENTS

ABSTRACT	v
1. INTRODUCTION	1
2. AXISYMMETRIC TOROIDAL TENSOR PRESSURE EQUILIBRIA	5
2.1 BASIC EQUILIBRIUM EQUATIONS	5
2.2 FORCE BALANCE RELATION AND POLOIDAL BETA	7
2.3 THE CURRENT DENSITIES	11
2.4 PRESSURE MOMENTS FROM A BEAM DISTRIBUTION FUNCTION	13
3. HIGH n -MODE STABILITY OF A TENSOR PRESSURE PLASMA	17
3.1 THE ENERGY PRINCIPLE	17
3.2 THE ENERGY PRINCIPLE IN AXISYMMETRIC TORI	24
3.3 HIGH n PERTURBATIONS AND THE KINK TERM	27
3.4 MINIMIZATION OF THE FIELD-COMPRESSION TERM	30
3.5 THE EULER EQUATION	31
3.6 THE CONNOR, HASTIE, TAYLOR TRANSFORMATION	38
4. CONCLUSIONS	43
REFERENCES	45

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ABSTRACT

A force balance relation, a representation for the poloidal beta (β_p), and expressions for the current densities are derived from the MHD equilibrium relations for an axisymmetric tensor pressure tokamak. Perpendicular and parallel beam pressure components are evaluated from a distribution function that models high energy neutral particle injection. A double adiabatic energy principle is derived from that of Kruskal and Oberman,¹ with correction terms added. The energy principle is then applied to an arbitrary cross-section axisymmetric tokamak to examine ballooning instabilities of large toroidal mode number. The resulting Euler equation is remarkably similar to that of ideal MHD. Although the field-bending term is virtually unaltered, the driving term is modified because the pressures are no longer constant on a flux surface. Either a necessary or a sufficient marginal stability criterion for a guiding center plasma can be derived from this equation whenever an additional stabilizing element unique to the double adiabatic theory is either kept or neglected, respectively.

1. INTRODUCTION

The heating of a tokamak to reactor-like temperatures using neutral beam injection is expected to produce an anisotropy in the pressure if the beams are preferentially directed either parallel or perpendicular to the magnetic field. The macroscopic behavior of a plasma can be altered by the pressure anisotropy, currents, and flows induced by the beams. Here we will concentrate on the first of these three effects.

Injection experiments carried out so far on ORMAK, TFR, DITE, T11, and PLT² indicate that the behavior of the injected hot ions is classical in nature, implying that the beam particles tend to slow down faster than they scatter in pitch angle. The resulting distribution function can display a strong energy anisotropy, the appropriate moments of which yield pressures along the field lines that are different from those across them. The experimental observation of an energy anisotropy in the distribution function of low density, high temperature discharges was complicated by the presence of the beams, which tended to swamp the signal in the parallel charge exchange analyzer.³ In that respect the question of beam-induced pressure anisotropies remains unanswered.

Neutral beam injection is not the only path by which a tensor pressure can be developed and maintained. The perpendicular charge exchange analyzer has shown an enhancement of the tail of the ion distribution function during lower hybrid heating experiments in such machines as PETULA and WEGA.⁴ Ion cyclotron heating can be expected to produce a similar tail, and theoretical estimates of electron cyclotron heating predict an increase in the perpendicular electron temperature.⁵ In reactors the anomalous loss of trapped alpha particles⁶ may produce $p_{\parallel} \neq p_{\perp}$ because the beta they introduce is comparable to the background plasma beta near the magnetic axis.

The efficient utilization of the magnetic field demands the operation of a tokamak at the highest possible beta. It is generally conceded that the limiting value of beta that can be achieved is imposed by stability to ballooning modes. These perturbations arise after a threshold pressure gradient is reached and are driven by the interaction of that gradient with bad magnetic field curvature. In those regions

where the curvature is unfavorable, the plasma bulges out, causing the magnetic field lines to bend; this acts as a restoring force.

The ballooning instability is analogous to the Rayleigh-Taylor instability, which is driven by a gravitational force.⁷ From a particle point of view, the gravitational force in a tokamak geometry can be broken up into a centrifugal force and a force on the magnetic moment due to the gradient of the magnetic field,⁸ and both are approximately in the same direction. These forces are balanced by the tension caused by the bending and stretching of the magnetic field lines they induce. The centrifugal force experienced by ions and electrons as they move along the field lines is proportional to $v_{\parallel}^2$, and the gradient $|\vec{B}|$ force is proportional to $1/2v_{\perp}^2$, with roughly the same constant of proportionality. Parallel injection would tend to enhance the centrifugal force as the beta of the plasma is raised, and perpendicular injection would enhance the gradient $|\vec{B}|$ force on the magnetic moment. From a fluid point of view, the centrifugal force is related to the interaction of the parallel pressure gradient with the magnetic field line curvature, and the gradient $|\vec{B}|$ force is related to the interaction of the perpendicular pressure gradient with radial gradients in $|\vec{B}|$.

A study of the stability of tokamak plasmas to ballooning modes with large wavelengths in the toroidal direction requires the use of large spectral⁹ or initial value¹⁰ computer codes. Ballooning instabilities that display a lot of structure in the toroidal direction seem to set the limiting beta once the plasma becomes slightly diamagnetic.¹¹

The investigation of ballooning modes with large toroidal mode number n initiated by Dobrott et al.¹² for ideal MHD plasmas consists of minimizing an energy principle that has been expanded in powers of $1/n$. The problem then reduces to solving one second order ordinary differential equation on each flux surface. To obtain a solution Dobrott et al. assumed that the perturbation vanishes at the point on each flux surface where the magnitude of the magnetic field is largest. Connor et al.¹³ improved on these results by introducing a transformation that extended to infinity the periodic domain of the poloidal angle. In this manner they were able to construct a perturbation in an eikonal form⁷ (i.e., a sum of periodic perturbations that itself need not be

periodic) that varies slowly along the field lines and rapidly across them in a magnetic field structure with shear without having to impose any constraints on the perturbation.¹⁴ The structure of the Euler equation and hence of the instability is modified very little by inclusion of nonideal effects,⁷ kinetic effects,¹⁵ or pressure anisotropy.¹⁶

Fielding and Haas¹⁶ have found that injecting particles almost perpendicularly to the magnetic field (but with enough parallel velocity so that the beam is not trapped) yields a distribution function that can sustain a pressure anisotropy where the perpendicular pressure varies on a flux surface even in a large aspect ratio device. They obtained an Euler equation from a version of the guiding center plasma energy principle that neglects a usually small term known as the kinetic energy integral.¹⁷ After expanding in large aspect ratio, they found that the gradient of the perpendicular pressure along the poloidal angle on a flux surface can be stabilizing, provided the pressure is largest at the inner edge of the torus. The resulting inward shift of the perpendicular pressure surfaces can be sustained because the injected fast particles move more slowly along the field lines on the inner edge of the torus than on the outer edge where the curvature is bad.

In this report we derive a force balance relation and expressions for the poloidal beta and the current densities in an axisymmetric torus with tensor pressure and with arbitrary aspect ratio and cross section. The results obtained do not vary too significantly from, and when $p_{\parallel} = p_{\perp}$, reduce to, the comparable ideal MHD results presented by Callen and Dory.¹⁸ From a distribution function that models the injection of passing fast particles in a tokamak,¹⁹ we evaluate perpendicular and parallel beam pressure components. In addition, we obtain a double adiabatic energy principle by adding correction terms to the Kruskal-Oberman energy principle.¹ If the Kruskal-Oberman (or guiding center plasma²⁰) energy principle predicts a plasma to be stable, this ensures that a double adiabatic (or guiding center fluid²⁰) energy principle will also predict stability provided the distribution function is monotonically decreasing in energy. By solving the Euler equation with and without the double adiabatic contribution, we can bracket the marginal stability point of the guiding center plasma without ever

having to evaluate the effect of the complicated kinetic energy integral.

In the first section of this report, we examine the equilibrium of an axisymmetric tensor pressure tokamak and obtain a force balance relation and expressions for the poloidal beta and the current densities. From a distribution function that models neutral particle injection, we obtain perpendicular and parallel beam pressure components. In the section on high toroidal mode number stability, we derive an Euler equation from a double adiabatic energy principle.

2. AXISYMMETRIC TOROIDAL TENSOR PRESSURE EQUILIBRIA

2.1 BASIC EQUILIBRIUM EQUATIONS

The macroscopic equilibrium equations are given by

$$\vec{\nabla} \cdot \vec{P} = \vec{j} \times \vec{B} , \quad (2.1)$$

$$\vec{\nabla} \times \vec{B} = \vec{j} , \quad (2.2)$$

$$\vec{\nabla} \cdot \vec{B} = 0 , \quad (2.3)$$

where the tensor pressure $\vec{P} = p_{\perp} \vec{I} + (p_{\parallel} - p_{\perp}) \hat{b} \hat{b}$ and $\hat{b}$ is the unit vector along a magnetic field line. With this expression we can write Eq. (2.1) as

$$\vec{\nabla} \cdot \vec{P} = \vec{\nabla} p_{\perp} + \hat{b} (\hat{b} \cdot \vec{\nabla}) (p_{\parallel} - p_{\perp}) + (p_{\parallel} - p_{\perp}) \left[\vec{\kappa} - \frac{\hat{b} (\hat{b} \cdot \vec{\nabla}) B^2}{2B^2} \right] = \vec{j} \times \vec{B} , \quad (2.4)$$

where $\vec{\kappa}$, the curvature of the field lines, is given by

$$\vec{\kappa} \equiv (\hat{b} \cdot \vec{\nabla}) \hat{b} = \frac{\vec{\nabla} B^2 - \hat{b} (\hat{b} \cdot \vec{\nabla}) B^2}{2B^2} + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{B^2} . \quad (2.5)$$

The component of Eq. (2.4) along the field line yields

$$(\hat{b} \cdot \vec{\nabla}) p_{\parallel} = \frac{p_{\parallel} - p_{\perp}}{2B^2} (\hat{b} \cdot \vec{\nabla}) B^2 , \quad (2.6)$$

so that, combining the last three equations, we get

$$\sigma (\vec{\nabla} \times \vec{B}) \times \vec{B} = \vec{\nabla}_{\perp} p_{\perp} + \frac{p_{\parallel} - p_{\perp}}{2B^2} \vec{\nabla}_{\perp} B^2 , \quad (2.7)$$

where the operator $\vec{\nabla}_\perp \equiv \vec{\nabla} - \hat{b}(\hat{b} \cdot \vec{\nabla})$ and $\sigma \equiv 1 - \frac{p_\parallel - p_\perp}{B^2}$. A rearrangement of terms results in

$$[\vec{\nabla} \times (\sigma \vec{B})] \times \vec{B} = \vec{\nabla}_\perp p_\parallel - \frac{p_\parallel - p_\perp}{2B^2} \vec{\nabla}_\perp B^2. \quad (2.8)$$

In a tokamak the toroidal field is

$$\vec{B} = B_\phi \hat{e}_\phi + B_\chi \hat{e}_\chi = I \vec{\nabla}_\phi + \vec{\nabla}_\phi \times \vec{\nabla}_\psi, \quad (2.9)$$

where $I \equiv RB_\phi$, ϕ is the toroidal angle, and $2\pi\psi$ is the poloidal flux function. In an orthogonal (ϕ, ψ, χ) coordinate system, the gradient operator is

$$\vec{\nabla} = \left(\frac{\hat{e}_\phi}{R} \right) \frac{\partial}{\partial \phi} + \hat{e}_\psi RB_\chi \frac{\partial}{\partial \psi} + \left(\frac{\hat{e}_\chi}{JB_\chi} \right) \frac{\partial}{\partial \chi}, \quad (2.10)$$

where J is the Jacobian.

If we expand Eq. (2.8) using this flux coordinate system and noting the requirement of axisymmetry, we get two equations:²¹

$$\frac{\partial}{\partial \chi} (\sigma I) = - \frac{I}{B^2} \left[\frac{\partial p_\parallel}{\partial \chi} - \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \chi} \right] \quad (2.11)$$

and

$$\frac{1}{J} \frac{\partial}{\partial \psi} (\sigma JB_\chi^2) = - \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) - \frac{\partial p_\parallel}{\partial \psi} + \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \psi}. \quad (2.12)$$

If we expand Eq. (2.6), we obtain

$$\frac{\partial p_\parallel}{\partial \chi} = \frac{(p_\parallel - p_\perp)}{2B^2} \frac{\partial B^2}{\partial \chi}, \quad (2.13)$$

which implies that σI is a function only of ψ . After rearranging some terms and writing the left-hand side of Eq. (2.12), we can obtain the basic equilibrium equation

$$\Delta^* \psi + \frac{1}{\sigma} \vec{\nabla} \psi \cdot \vec{\nabla} \sigma = - \frac{I}{\sigma} \frac{\partial}{\partial \psi} (\sigma I) - \frac{R^2}{\sigma} \left[\frac{\partial p_{\parallel}}{\partial \psi} - \frac{(p_{\parallel} - p_{\perp})}{2B^2} \frac{\partial B^2}{\partial \psi} \right], \quad (2.14)$$

where the operator $\Delta^* \equiv R^2 \vec{\nabla} \cdot \left(\frac{1}{R^2} \vec{\nabla} \right)$. If we take the pressure to be functional forms of ψ and $B \equiv |\vec{B}|$, then

$$\left. \frac{\partial p_{\parallel}}{\partial \psi} \right|_{\chi} = \left. \frac{\partial p_{\parallel}}{\partial \psi} \right|_B + \left. \frac{\partial p_{\parallel}}{\partial B} \right|_{\psi} \frac{\partial B}{\partial \psi} \Big|_{\chi}. \quad (2.15)$$

With $p_{\parallel} = p_{\parallel}(\psi, B)$, we get from Eq. (2.6) that $\frac{\partial p_{\parallel}}{\partial B} = \frac{p_{\parallel} - p_{\perp}}{B}$; therefore,

$$\left. \frac{\partial p_{\parallel}}{\partial \psi} \right|_B = \left. \frac{\partial p_{\parallel}}{\partial \psi} \right|_{\chi} - \frac{(p_{\parallel} - p_{\perp})}{B} \left. \frac{\partial B}{\partial \psi} \right|_{\chi}, \quad (2.16)$$

and the analog of the Grad-Shafranov equation with tensor pressure is

$$\Delta^* \psi + \frac{1}{\sigma} \vec{\nabla} \psi \cdot \vec{\nabla} \sigma = - \frac{I}{\sigma} \frac{\partial}{\partial \psi} (\sigma I) - \frac{R^2}{\sigma} \left. \frac{\partial p_{\parallel}}{\partial \psi} \right|_B. \quad (2.17)$$

This form of the equilibrium equation was shown by Grad²⁰ and requires that the parallel pressure be treated as a function of ψ and B .

2.2 FORCE BALANCE RELATION AND POLOIDAL BETA

In a tensor pressure plasma, it is convenient to define poloidal beta as one-third of an average of the trace of the pressure tensor divided by the mean poloidal magnetic field energy density at the plasma boundary. For that purpose we need to rearrange the terms in Eq. (2.12) to get

$$\frac{1}{J} \frac{\partial}{\partial \psi} (\sigma J B_{\chi}^2) = - \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) - \frac{1}{4} \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) + B^2 \frac{\partial \sigma}{\partial \psi} - \frac{1}{4} \frac{\partial p_{\parallel}}{\partial \psi}.$$

Then using Eq. (2.12) again to replace the last term on the right-hand side, we obtain

$$\frac{1}{J} \frac{\partial}{\partial \psi} (\sigma J B_{\chi}^2) = - \frac{I}{R^2} \frac{\partial}{\partial \psi} (\sigma I) - \frac{1}{3} \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) + \frac{4}{3} B^2 \frac{\partial \sigma}{\partial \psi} - \frac{(p_{\parallel} - p_{\perp})}{6B^2} \frac{\partial B^2}{\partial \psi}. \quad (2.18)$$

We now integrate this equation over ϕ and χ on a flux surface to get

$$\begin{aligned} \frac{1}{V'(\psi)} \frac{\partial}{\partial \psi} [V'(\psi) \langle \sigma B_{\chi}^2 \rangle_{\psi}] = & - \frac{1}{3} \left\langle \frac{\partial}{\partial \psi} (p_{\parallel} + 2p_{\perp}) \right\rangle_{\psi} - \left\langle \frac{1}{R^2} \frac{\partial}{\partial \psi} (\sigma I) \right\rangle_{\psi} \\ & + \frac{4}{3} \left\langle B^2 \frac{\partial \sigma}{\partial \psi} \right\rangle_{\psi} - \left\langle \frac{p_{\parallel} - p_{\perp}}{6R^2} \frac{\partial B^2}{\partial \psi} \right\rangle_{\psi}, \quad (2.19) \end{aligned}$$

where

$$\langle X \rangle_{\psi} \equiv \frac{2\pi}{V'(\psi)} \oint \frac{d\ell}{B_{\chi}} X, \quad (2.20)$$

$$V'(\psi) \equiv \frac{dV}{d\psi} \equiv 2\pi \oint \frac{d\ell}{B_{\chi}},$$

and $d\ell = J B_{\chi} d\chi$. To obtain a force balance relation, we follow very closely the derivation that Callen and Dory did for a scalar pressure plasma.¹⁸ The next step is to integrate Eq. (2.19) from ψ to ψ_b , where ψ_b is the value of the flux at any arbitrary flux surface most generally taken to be that at the boundary. We then multiply the equation by $V'(\psi) d\psi$ and integrate it from ψ_a , the flux at the magnetic axis, to ψ_b to get the force balance equation. The averaging of the left-hand side of Eq. (2.19) as outlined above is after an integration by parts,

$$\int_{\psi_a}^{\psi_b} d\psi \frac{dV}{d\psi} \int_{\psi}^{\psi_b} \frac{du}{V'(\psi)} \frac{\partial}{\partial u} [V'(u) \langle \sigma B_X^2 \rangle_u] = \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V'(\psi)} \left[V''(\psi) \langle \sigma B_X^2 \rangle_\psi + V'(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_\psi \right],$$

where we have made use of the fact that $V(\psi_a) = 0$. The right-hand side of this equation can be written as

$$\frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V'^2(\psi)} \left[2V'(\psi)V''(\psi) + V'^2(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_\psi \right] + \frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi V(\psi) \frac{\partial}{\partial \psi} \langle \sigma B_X^2 \rangle_\psi.$$

An integration by parts on the last term transforms the expression to

$$\frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi \frac{V^2(\psi)}{V'^2(\psi)} \left\{ \frac{1}{V(\psi)} \frac{\partial}{\partial \psi} [V'^2(\psi) \langle \sigma B_X^2 \rangle_\psi] - \frac{V'^2(\psi)}{V^2(\psi)} V'(\psi) \langle \sigma B_X^2 \rangle_\psi \right\} + \frac{V(\psi_b)}{2} \langle \sigma B_X^2 \rangle_{\psi_b},$$

so that after some algebraic manipulation, we obtain

$$\frac{1}{2} \int_{\psi_a}^{\psi_b} d\psi V(\psi) \langle \sigma B_X^2 \rangle_\psi \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_X^2 \rangle_\psi}{V(\psi)} \right] \right\} + \frac{1}{2} V(\psi_b) \langle \sigma B_X^2 \rangle_{\psi_b}. \quad (2.21)$$

The integration of the right-hand side of Eq. (2.19) is considerably easier, and the force balance relation can be written as

$$\begin{aligned}
\frac{1}{3} [\overline{P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)}] &= \frac{1}{2} \langle \sigma B^2 \rangle_{\chi \psi_b} \\
+ \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V(\psi_b)} &\left(\left\langle \frac{1}{\sigma R^2} \right\rangle_{\psi} \frac{\partial}{\partial \psi} (\sigma I)^2 + \langle \sigma B^2 \rangle_{\chi \psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B^2 \rangle_{\chi \psi}}{V(\psi)} \right] \right\} \right. \\
&\left. - \frac{8}{3} \left\langle B^2 \frac{\partial \sigma}{\partial \psi} \right\rangle_{\psi} + \frac{1}{3} \left\langle \frac{P_{\parallel} - P_{\perp}}{B^2} \frac{\partial B^2}{\partial \psi} \right\rangle_{\psi} \right), \quad (2.22)
\end{aligned}$$

where we have defined

$$\begin{aligned}
[\overline{P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)}] &= \\
&= \frac{\int_{\psi_a}^{\psi_b} d\psi V'(\psi) \int_{\psi}^{\psi_b} du \left\langle \frac{\partial}{\partial u} (p_{\parallel} + 2p_{\perp}) \right\rangle_u}{V(\psi_b)}. \quad (2.23)
\end{aligned}$$

An expression for poloidal beta β_p can be obtained from the force balance relation if we define

$$\beta_p \equiv \frac{2}{3 \langle \sigma B^2 \rangle_{\chi \psi_b}} [\overline{P_{\parallel}(\psi) - P_{\parallel}(\psi_b) + 2P_{\perp}(\psi) - 2P_{\perp}(\psi_b)}]. \quad (2.24)$$

Then

$$\begin{aligned}
\beta_p \equiv 1 - \frac{\left\langle \frac{p_{\parallel} - p_{\perp}}{B^2} B_{\chi}^2 \right\rangle_{\psi_b}}{\langle B_{\chi}^2 \rangle_{\psi_b}} \\
+ \int_{\psi_a}^{\psi_b} d\psi \frac{V(\psi)}{V(\psi_b) \langle B_{\chi}^2 \rangle_{\psi_b}} \left(\left\langle \frac{1}{\sigma R^2} \right\rangle_{\psi} \frac{\partial}{\partial \psi} (\sigma I)^2 + \langle \sigma B_{\chi}^2 \rangle_{\psi} \frac{\partial}{\partial \psi} \left\{ \ln \left[\frac{V'^2(\psi) \langle \sigma B_{\chi}^2 \rangle_{\psi}}{V(\psi)} \right] \right\} \right. \\
\left. - \frac{8}{3} \left\langle B^2 \frac{\partial \sigma}{\partial \psi} \right\rangle_{\psi} + \frac{1}{3} \left\langle \frac{p_{\parallel} - p_{\perp}}{B^2} \frac{\partial B^2}{\partial \psi} \right\rangle_{\psi} \right). \quad (2.25)
\end{aligned}$$

When this expression is evaluated at the plasma boundary, the second term on the right-hand side must vanish. The last two terms under the integral sign are absent in the scalar pressure case. In the limit $p_{\parallel} = p_{\perp}$, the force balance relation and, consequently, β_p are equivalent to those derived by Callen and Dory. In a tensor pressure plasma, the pressures are functions of the flux and the magnitude of the magnetic field; therefore, a strict separation of pressure and magnetic contributions to the force balance relation is no longer possible. The pressures can be considered to be balanced not only by the fields, but also by the degree of anisotropy. The diamagnetic or paramagnetic character of the plasma is still determined mainly by the sign of $\frac{\partial}{\partial \psi} (\sigma I)^2$, although for very anisotropic pressure plasmas the contribution of $\left\langle \sigma B \frac{\partial \sigma}{\partial \psi} \right\rangle_{\psi}$ could conceivably be significant. A very diamagnetic plasma must have $\frac{\partial}{\partial \psi} (\sigma I)^2$ large and positive.

2.3 THE CURRENT DENSITIES

When we express Ampere's law in flux coordinates, it shows the toroidal current density to be

$$j_\phi = \frac{R}{J} \frac{\partial}{\partial \psi} (JB_\chi^2) .$$

Using the equilibrium equation, Eq. (2.12), we can easily write the toroidal current density as

$$j_\phi = - \frac{R}{\sigma} \left(\frac{\partial p_\perp}{\partial \psi} + \frac{p_\parallel - p_\perp}{2B^2} \frac{\partial B^2}{\partial \psi} \right) - \frac{I}{R} \frac{\partial I}{\partial \psi} . \quad (2.26)$$

From Ampere's law we also get, for the poloidal current density,

$$j_\chi = -B_\chi \frac{\partial I}{\partial \psi} \quad (2.27)$$

and, for the radial current density,

$$j_\psi = - \frac{B_\phi}{\sigma B^2 J B_\chi} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) . \quad (2.28)$$

It is immediately noted that for scalar pressure plasmas $j_\psi = 0$. The radial or normal current density in tensor pressure axisymmetric toroidal devices is driven by the variation of the pressures along the poloidal angle on a flux surface; when the pressure is scalar, it is independent of the poloidal angle, which implies that $j_\psi = 0$.

The parallel current $\vec{j}_\parallel = (\vec{j} \cdot \vec{B}/B^2)\vec{B}$; using Eqs. (2.26) and (2.27), we get

$$\vec{j}_\parallel = - \left[\left(\frac{\partial p_\perp}{\partial \psi} + \frac{p_\parallel - p_\perp}{2B^2} \frac{\partial B^2}{\partial \psi} \right) \frac{1}{\sigma B^2} + \frac{\partial I}{\partial \psi} \right] \vec{B} , \quad (2.29)$$

and the perpendicular current density is obtained from $\vec{j} - \vec{j}_\parallel$:

$$\vec{j}_\perp = - \left(\frac{\partial p_\perp}{\partial \psi} + \frac{p_\parallel - p_\perp}{2B^2} \frac{\partial B^2}{\partial \psi} \right) \frac{B_\chi (RB_\chi \hat{e}_\phi - I \hat{e}_\chi)}{\sigma B^2} . \quad (2.30)$$

The poloidal current density is unaltered from the case of scalar pressure. On the other hand, the anisotropy does modify the toroidal current density as well as the parallel and perpendicular current densities. The extent of this modification consists of replacing $p'(\psi)$ of the scalar pressure case by $1/\sigma\{\partial p_{\perp}/\partial\psi + [(p_{\parallel} - p_{\perp})/2B^2]\partial B^2/\partial\psi\}$ wherever it appears, which, as can be seen reduces to the appropriate term when $p_{\parallel} = p_{\perp}$. As mentioned above, the anisotropy introduces a normal current density as well.

2.4 PRESSURE MOMENTS FROM A BEAM DISTRIBUTION FUNCTION

The steady-state distribution function obtained by Cordey and Houghton¹⁹ for very energetic injected hot ions in a uniform magnetic field is²²

$$f_b = \frac{1}{2\pi} \frac{I_b \tau_s S(\psi)}{B_o(\psi)} \frac{(0.5m_h)^{3/2}}{E^{3/2} + E_c^{3/2}} \delta\left[\frac{\mu}{E} - \mu_e(\psi)\right], \quad (2.31)$$

where m_h is the mass of the beam particle, τ_s is the Spitzer slowing-down time, I_b is the beam current per unit charge, $h(\psi)$ is the number of injected particles per unit volume within a flux surface ψ , and $E_c = 1/2 m_h/m_i (3\sqrt{\pi}/4)^{2/3} (m_i/m_e)^{1/3} T_e$ (where m_i = ion mass, m_e = electron mass, and T_e = electron temperature). The magnetic moment μ_c (normalized to the beam particle injected energy E_o) of the fast ion and B_o (the magnitude of the magnetic field) are evaluated at the equatorial plane on the outer edge of each flux surface. We point out that at any location in the device this distribution function conserves pitch angle. Because Cordey²³ has also derived a more accurate, yet more complicated, distribution function in a nonuniform magnetic field, the effect of trapping of beam particles could be considered. Nevertheless, for the injection of passing particles, the distribution function can be well approximated by Eq. (2.31).

The contribution of the beam to the pressure can be obtained by taking the appropriate moments of the distribution function. The perpendicular pressure due to the beam is

$$p_{\perp b} = 2\pi \sum_h m_h^{-3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E - \mu B)}} \mu B f_b$$

$$p_{\perp b} = \frac{1}{2} P(\psi) \frac{\mu_e(\psi) B^2}{\sqrt{1 - \mu_e(\psi) B}} H[1 - \mu_e(\psi) B] , \quad (2.32)$$

and the beam parallel pressure is

$$p_{\parallel b} = 2\pi \sum_h m_h^{-3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E - \mu B)}} 2(E - \mu B) f_b$$

$$p_{\parallel b} = P(\psi) B \sqrt{1 - \mu_e(\psi) B} H[1 - \mu_e(\psi) B] , \quad (2.33)$$

where

$$P(\psi) = \frac{1}{2} \frac{I_b \tau_s S(\psi)}{B_o(\psi)} \int_0^1 dE \frac{E^{3/2}}{E^{3/2} + E_c^{3/2}}$$

and $H[1 - \mu_e(\psi) B]$ is a step function. We have assumed that the beam contains only one species of particles.

In a high beta, large aspect ratio device, $1 - \mu_e B \sim 1$ for preferentially parallel injection. Then to get $p_{\parallel b}$ of order of the inverse aspect ratio, we need $P(\psi)$, which is a measure of the source strength, to be $O(\epsilon)$. On the other hand, with mostly perpendicular injection but enough parallel velocity to avoid trapping the beam, we could have $1 - (\mu_e B)$ of order of the inverse aspect ratio ϵ . In this case, for $p_{\perp b}$ to be $O(\epsilon)$, the same as the background plasma pressure, we only need $P(\psi)$ to be of order $\epsilon^{3/2}$. Therefore, a significant anisotropy

can most effectively be sustained with almost perpendicular injection. We also notice, particularly for perpendicular injection, that the beam pressures can be made larger on the inner edge of the torus, where the magnetic field is highest, and consequently away from the bad curvature region, where ballooning instabilities can be driven. From a particle point of view, this is possible because the hot ions move more slowly along the field lines on the inner side of the torus and more rapidly on the outer edge; they therefore spend a larger fraction of time in the high field region.²⁴ Assuming a Maxwellian background, the perpendicular plasma pressure is $p_o(\psi) + p_{\perp b}(\psi, B)$, and the parallel plasma pressure is $p_o(\psi) + p_{\parallel b}(\psi, B)$.

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3. HIGH n -MODE STABILITY OF A TENSOR PRESSURE PLASMA

3.1 THE ENERGY PRINCIPLE

Energy principles that allow pressure anisotropy have a multiplicity of names which are used interchangeably and can cause confusion. Effectively they can be divided into two categories:

- (1) the guiding center plasma,²⁴ Kruskal-Oberman,¹ collisionless,²⁵ or kinetic²⁶ energy principle;
- (2) the guiding center fluid,²⁷ hydromagnetic,¹ Chew-Goldberger-Low,²⁸ or double adiabatic energy principle.

The Kruskal-Oberman energy principle δW_{KO} can be written as

$$\delta W_{KO} = \delta W_U + \delta W_K, \quad (3.1)$$

where δW_U , commonly referred to as the universal term, contains all the purely fluid elements and δW_K is the kinetic energy integral. The double adiabatic energy principle δW_{DA} can be expressed as

$$\delta W_{DA} = \delta W_U + \delta W_{DAC}, \quad (3.2)$$

where δW_{DAC} is a double adiabatic correction term.

By means of a Schwartz inequality, it can be shown¹ that the kinetic energy integral δW_K can be reduced to a fluid form less than or equal to δW_{DAC} provided the distribution function is monotonically decreasing the energy. Consequently, $\delta W_{KO} \leq \delta W_{DA}$, implying that any criteria derived from the guiding center fluid can be viewed as necessary for the stability of a guiding center plasma. It is reasonable to expect this, even though δW_{DA} does not include the stabilizing effects of trapped particles²⁹ in a tokamak, because in deriving the adiabatic laws governing the evolution of the pressures in the double adiabatic theory, heat flow along the magnetic field lines, which is by no means small, was ignored.

The double adiabatic energy principle is

$$\begin{aligned}
\delta W_{DA} = & \frac{1}{2} \int d^3x \left(Q^2 + \vec{J} \cdot (\vec{\xi} \times \vec{Q}) + (\vec{\xi} \cdot \vec{\nabla}) p_{\perp} (\vec{\nabla} \cdot \vec{\xi}) \right. \\
& + p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})^2 - 4(\vec{\nabla} \cdot \vec{\xi})q + 2q^2] - \hat{C}(q - \vec{\nabla} \cdot \vec{\xi})^2 \\
& + (p_{\parallel} - p_{\perp}) \{ -(\vec{\xi} \cdot \vec{\nabla})q + q^2 - \hat{b} \cdot [(\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot \vec{\nabla} \vec{\xi} \\
& + \hat{b} \cdot [(\vec{\xi} \cdot \vec{\nabla})\hat{b} \cdot \vec{\nabla} \vec{\xi} - [(\hat{b} \cdot \vec{\nabla})\vec{\xi}]^2 + (\hat{b} \cdot \vec{\nabla})\vec{\xi} \cdot (\vec{\xi} \cdot \vec{\nabla})\hat{b}] \} \\
& \left. + \frac{1}{2} \int d^3x \{ p_{\perp} [2(\vec{\nabla} \cdot \vec{\xi})q + q^2] + 3(p_{\parallel} - p_{\perp})q^2 + \hat{C}(q - \vec{\nabla} \cdot \vec{\xi})^2 \} \right), \quad (3.3)
\end{aligned}$$

where the first integral is δW_U and the second is δW_{DAC} and where $\hat{b}$ is the unit vector along the field lines, ξ is the perturbation, $\vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B})$ is the perturbed magnetic field, and $q = \hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}$.

The parallel and perpendicular pressures, we recall, are given by the appropriate moments of the distribution function:

$$[p_{\parallel}, p_{\perp}] = 2\pi \sum_i m_i^{-3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B}{\sqrt{2(E - \mu B)}} [2(E - \mu B), \mu B] f_i, \quad (3.4)$$

and the pressure-like moment $\hat{C}$ is given by

$$\hat{C} = 2\pi \sum_i m_i^{-3/2} \int_0^1 dE \int_0^{E/B} d\mu \frac{B(\mu B)^2}{\sqrt{2(E - \mu B)}} \left(\frac{\partial f_i}{\partial E} \right), \quad (3.5)$$

where $f_i = f_i(E, \mu, \psi)$.

The motivation for writing the double adiabatic energy principle as the universal term of the guiding center plasma energy principle plus double adiabatic corrections is that δW_U is independent of the component of the perturbation along the field lines. In that case, we need only be concerned with $\xi_{\parallel}$ when treating δW_{DAC} .

The next procedure is to try to obtain δW_{DA} in a more concise and transparent form. We closely follow the derivation in previous publications.³⁰ To this effect we use such relations as

$$\vec{Q}_{\parallel} = \vec{B}\hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_1 - \hat{b}(\vec{\xi}_1 \cdot \vec{\nabla})B - \vec{B}(\vec{\nabla} \cdot \vec{\xi}_1)$$

and

$$\frac{\vec{Q}_1}{B} = (\hat{b} \cdot \vec{\nabla})\vec{\xi}_1 - (\vec{\xi}_1 \cdot \vec{\nabla})\hat{b} - \hat{b}[\hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_1],$$

and we expand $\vec{j} = j_{\parallel}\hat{b} + \vec{j}_1$, where $\vec{j}_1 = [\vec{B} \times (\vec{\nabla} \cdot \vec{\overline{P}})/B^2]$ ($\vec{\overline{P}}$ is the pressure tensor), and note that $\hat{b} \cdot (\hat{b} \cdot \vec{\nabla})\vec{\xi}_1 = -\xi_1 \cdot \vec{\kappa}$. With this taken into account, we can rewrite the double adiabatic energy principle in the intermediate form

$$\begin{aligned} \delta W_{DA} = \frac{1}{2} \int d^3x \left[\sigma Q_1^2 + Q_{\parallel}^2 + \sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_1 \times \vec{Q}_1) - \hat{c}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa})^2 \right. \\ + \left(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa} - \frac{Q_{\parallel}}{B} \right) (\vec{\xi}_1 \cdot \vec{\nabla})p_1 - (\vec{\xi}_1 \cdot \vec{\kappa})(\vec{\xi}_1 \cdot \vec{\nabla})p_{\parallel} \\ + 2p_1(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa})^2 - (p_{\parallel} - p_1) \left(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa} \right. \\ \left. \left. + \frac{Q_{\parallel}}{B} \right) (\vec{\xi}_1 \cdot \vec{\kappa}) \right] \\ + \delta W_{DAC}. \end{aligned} \quad (3.6)$$

Expanding $Q_{\parallel}$ in the form shown above and then making use of the equilibrium relation

$$\vec{\nabla} \left(p_1 + \frac{B^2}{2} \right) = \hat{b}(\hat{b} \cdot \vec{\nabla}) \left(p_1 + \frac{B^2}{2} \right) + \sigma B^2 \cdot \vec{\kappa}, \quad (3.7)$$

where $\sigma = 1 - (p_{\parallel} - p_{\perp})/B^2$ and $\vec{\kappa} = (\hat{b} \cdot \vec{\nabla})\hat{b}$ is the magnetic field curvature, then we can get an expression for $(\vec{\xi}_1 \cdot \vec{\nabla})B^2$. Then the energy principle reduces to

$$\begin{aligned} \delta W_{DA} = & \frac{1}{2} \int d^3x \left\{ \sigma Q_1^2 + \sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_1 \times \vec{Q}_1) + \sigma B^2 \left(1 - \frac{\sigma}{\tau_0} \right) (\vec{\xi}_1 \cdot \vec{\kappa})^2 \right. \\ & - \hat{C}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa})^2 + \tau_0 B^2 \vec{\nabla} \cdot \vec{\xi}_1 + \left(1 + \frac{\sigma}{\tau_0} \right) (\vec{\xi}_1 \cdot \vec{\kappa})^2 \\ & \left. - (\vec{\xi}_1 \cdot \vec{\kappa})(\vec{\xi}_1 \cdot \vec{\nabla})(p_{\parallel} + p_{\perp}) \right\} \\ & + \delta W_{DAC}, \end{aligned} \quad (3.8)$$

where $\tau_0 \equiv 1 + 2p_{\perp}/B^2$.

We now examine the double adiabatic correction δW_{DAC} and break up the perturbation into parallel and perpendicular components to the magnetic field:

$$\begin{aligned} \delta W_{DAC} = & \frac{1}{2} \int d^3x \left\{ -2p_{\perp} (\vec{\xi}_1 \cdot \vec{\kappa})(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa}) + 3p_{\parallel} (\vec{\xi}_1 \cdot \vec{\kappa})^2 \right. \\ & + \hat{C}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa})^2 + 2(\hat{b} \cdot \vec{\nabla})\xi_{\parallel} [p_{\perp}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa}) \\ & - 3p_{\parallel}(\vec{\xi}_1 \cdot \vec{\kappa}) + p_{\perp}\xi_{\parallel}(\vec{\nabla} \cdot \hat{b})] + \hat{C}(\vec{\nabla} \cdot \hat{b})^2\xi_{\parallel}^2 + 2\xi_{\parallel}(\vec{\nabla} \cdot \hat{b}) \\ & \left. \times [\hat{C}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa}) - p_{\perp}(\vec{\xi}_1 \cdot \vec{\kappa})] + 3p_{\parallel}[(\hat{b} \cdot \vec{\nabla})\xi_{\parallel}]^2 \right\}. \end{aligned} \quad (3.9)$$

We note that the term $\hat{C}(\vec{\nabla} \cdot \vec{\xi}_1 + \vec{\xi}_1 \cdot \vec{\kappa})^2$ in δW_{DAC} and Eq. (3.8) will cancel each other out. We now complete the square on the term $(\hat{b} \cdot \vec{\nabla})\xi_{\parallel}$ and then complete the square on $\xi_{\parallel}(\vec{\nabla} \cdot \hat{b})$ in δW_{DAC} so that

$$\begin{aligned}
\delta W_{\text{DAC}} = & \frac{1}{2} \int d^3x \left(-2p_{\perp} (\vec{\xi}_{\perp} \cdot \vec{\kappa}) (\vec{\nabla} \cdot \vec{\xi}_{\perp} + \vec{\xi}_{\perp} \cdot \vec{\kappa}) + 3p_{\parallel} (\vec{\xi}_{\perp} \cdot \vec{\kappa})^2 + \hat{c} (\vec{\nabla} \cdot \vec{\xi}_{\perp} \right. \\
& + \vec{\xi}_{\perp} \cdot \vec{\kappa})^2 - \frac{p_{\perp}^2}{3p_{\parallel}} \left[\vec{\nabla} \cdot \vec{\xi}_{\perp} + \left(1 - \frac{3p_{\parallel}}{p_{\perp}} \right) (\vec{\xi}_{\perp} \cdot \vec{\kappa}) \right]^2 \\
& + \frac{p_{\perp}^2}{3p_{\parallel}} \left[\vec{\nabla} \cdot \vec{\xi}_{\perp} + \left(1 - \frac{3p_{\parallel}}{p_{\perp}} \right) (\vec{\xi}_{\perp} \cdot \vec{\kappa}) + \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) \right. \\
& + \left. \left. \frac{3p_{\parallel}}{p_{\perp}} (\hat{b} \cdot \vec{\nabla}) \xi_{\parallel} \right]^2 + \left(\hat{c} - \frac{p_{\perp}^2}{3p_{\parallel}} \right) \{ [\vec{\nabla} \cdot \vec{\xi}_{\perp} + \vec{\xi}_{\perp} \cdot \vec{\kappa} \right. \\
& + \left. \left. \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) \right]^2 - (\vec{\nabla} \cdot \vec{\xi}_{\perp} + \vec{\xi}_{\perp} \cdot \vec{\kappa})^2 \} \right). \quad (3.10)
\end{aligned}$$

The first four terms in δW_{DAC} contain only components of the perturbation perpendicular to the magnetic field. After combining these with the second, third, and last terms of Eq. (3.8), we can express the double adiabatic energy principle as

$$\begin{aligned}
\delta W_{\text{DA}} = & \frac{1}{2} \int d^3x \left(\sigma Q_{\perp}^2 + \sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_{\perp} \times \vec{Q}_{\perp}) + \sigma B^2 \left(1 - \frac{\sigma}{\tau_o} \right) (\vec{\xi}_{\perp} \cdot \vec{\kappa})^2 \right. \\
& + \tau_d B^2 \left[\vec{\nabla} \cdot \vec{\xi}_{\perp} + \left(1 + \frac{\sigma}{\tau_d} \right) (\vec{\xi}_{\perp} \cdot \vec{\kappa}) \right]^2 (\vec{\xi}_{\perp} \cdot \vec{\kappa}) (\vec{\xi}_{\perp} \cdot \vec{\nabla}) \\
& + (p_{\parallel} + p_{\perp}) + \frac{p_{\perp}^2}{3p_{\parallel}} \left[\vec{\nabla} \cdot \vec{\xi}_{\perp} + \left(1 - \frac{3p_{\parallel}}{p_{\perp}} \right) (\vec{\xi}_{\perp} \cdot \vec{\kappa}) + \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) \right. \\
& + \left. \left. \frac{3p_{\parallel}}{p_{\perp}} (\hat{b} \cdot \vec{\nabla}) \xi_{\parallel} \right]^2 + \left(\hat{c} - \frac{\mu_{\perp}^2}{3p_{\parallel}} \right) \{ [\vec{\nabla} \cdot \vec{\xi}_{\perp} + \vec{\xi}_{\perp} \cdot \vec{\kappa} \right. \\
& + \left. \left. \xi_{\parallel} (\vec{\nabla} \cdot \hat{b}) \right]^2 - (\vec{\nabla} \cdot \vec{\xi}_{\perp} + \vec{\xi}_{\perp} \cdot \vec{\kappa})^2 \} \right), \quad (3.11)
\end{aligned}$$

$$\begin{aligned}
& + \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) (\vec{\xi}_1 \cdot \vec{\kappa})^2 \\
& \quad \text{anisotropic-double adiabatic} \\
& \quad + \sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_1 \times \vec{Q}_1) - (\vec{\xi}_1 \cdot \vec{\kappa}) (\vec{\xi}_1 \cdot \vec{v}) (p_{\parallel} + p_{\perp}) \Bigg\} , \\
& \quad \text{kink} \qquad \qquad \qquad \text{ballooning}
\end{aligned} \tag{3.12}$$

where τ_0 has been defined earlier and $\gamma \equiv 1 - p_{\parallel}/B^2$. The first term, σQ_1^2 , is commonly referred to as the field-bending term and acts to stabilize ballooning modes. We note that it is slightly modified by anisotropy through σ , the modification being of order β . It is of interest to recognize that because the pressures are functions not only of the poloidal flux but also of the magnitude of the magnetic field, the pressures and the fields are no longer independent of one another. Therefore, when a magnetic field line bends in response to a perturbation, it also stretches, causing the magnitude of the field to decrease, which in turn affects the pressures.

The next term, $\tau_0 B^2 [\vec{v} \cdot \vec{\xi}_1 + (1 + \gamma/\tau_0)(\vec{\xi}_1 \cdot \vec{\chi})]^2$ is a field-compression element, which, like the field-bending term, is also modified by the anisotropy. The compression of the magnetic field alters its magnitude and consequently the pressures. In the limit that $p_{\parallel} = p_{\perp}$, we do not recover the scalar pressure field compression term, which is $B^2 [\vec{v} \cdot \vec{\xi}_1 + 2(\vec{\xi}_1 \cdot \vec{\chi})]^2$. This is expected because we cannot obtain the ideal MHD energy principle from δW_{DA} . Consequently, it must be and is slightly different from the field-compression term of the guiding center plasma energy principle (from which the ideal MHD is recovered). We notice that the field compression is positive definite and hence stabilizing.

The term $\sigma B^2 (1 + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma \tau_0) (\vec{\xi}_1 \cdot \vec{\chi})^2$ combines the effects of anisotropy and double adiabaticity and has no equivalent in ideal MHD. The anisotropic part is $\sigma B^2 (1 - \sigma/\tau_d)$, which is zero when $p_{\parallel} = p_{\perp}$. A very similar term that is due to the anisotropy¹⁶ appears in the guiding center plasma energy principle. The double adiabatic part can be written as $\sigma B^2 (\sigma/\tau_d + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma \tau_0)$ and is finite even if we take

where $\tau_d \equiv 1 + (2p_\perp - \hat{C}_1)/B^2$ (where we identify $\hat{C} = \hat{C}_\eta = \eta p_\perp^2/3p_\parallel$ so that $\hat{C}_1 = p_\perp^2/3p_\parallel$).

For a Maxwellian or bi-Maxwellian plasma, the evaluation of $\hat{C}$ from Eq. (3.5) gives $\hat{C} = 2p_\perp^2/p_\parallel$, which implies $\eta = 6$ for such distributions. On the other hand, if the distribution function is a delta function in magnetic moment and a step function in energy, then $\hat{C} = p_\perp^2/3p_\parallel$; as a consequence, $\eta = 1$. For a guiding center plasma, $1 \leq \eta \leq 6$. The energy principle obtained from the universal term δW_U of the guiding center plasma energy principle¹⁶ can easily be obtained from δW_{DA} by replacing $\hat{C}_1$ with $\hat{C}_\eta$ in τ_d and by neglecting the last two terms, which are unique to the double adiabatic theory.

The minimization of this guiding center fluid energy principle [Eq. (3.11)] with respect to the component of the perturbation along the magnetic field lines is exceedingly difficult. Our interest in deriving a double adiabatic energy principle and examining its stability properties is that any criteria derived from it are necessary for the stability of a guiding center plasma. From this vantage we are fully justified in assuming that the perturbation has only perpendicular components to the magnetic field. This is equivalent to imposing the constraint on the perturbation that $\xi_\parallel = 0$. Therefore, any criteria obtained from the double adiabatic energy principle with this constraint imposed will predict more optimistic stability conditions than when the constraint is absent. Consequently, it must also be necessary for the stability of a guiding center plasma. Taking $\xi_\parallel = 0$ considerably simplifies δW_{DA} and the task of minimizing it. The last term in δW_{DA} disappears, and the other term peculiar to the double adiabatic theory can be combined with the second term of δW_{DA} so that the energy principle reduces to

$$\delta W_{DA} = \frac{1}{2} \int d^3x \left\{ \begin{array}{ll} \sigma Q_1^2 & + \tau_0 B^2 \left[\vec{\nabla} \cdot \vec{\xi}_\perp + \left(1 + \frac{\gamma}{\tau_0} \right) (\vec{\xi}_\perp \cdot \vec{\kappa}) \right]^2 \end{array} \right.$$

field bending field compression

$p_{\parallel} = p_{\perp}$. The double adiabatic part can be shown to be positive definite and hence stabilizing, but the anisotropic part is not necessarily so.

In δW_{DA} , $\sigma j_{\parallel} \hat{b} \cdot (\vec{\xi}_{\perp} \times \vec{Q}_{\perp})$ is an anisotropy-modified term that drives kink instabilities, and $(\vec{\xi}_{\perp} \cdot \vec{\chi})(\vec{\xi}_{\perp} \cdot \vec{\nabla})(p_{\parallel} + p_{\perp})$ drives ballooning modes and interchanges. If we take $p_{\parallel} = p_{\perp}$, these two terms reduce to the comparable expressions that appear in ideal MHD. Despite all the modifications, the structure of the guiding center fluid energy principle δW_{DA} is remarkably similar to the ideal MHD energy principle.

3.2 THE ENERGY PRINCIPLE IN AXISYMMETRIC TORI

We now expand the different terms appearing in δW_{DA} in an orthogonal (ϕ, ψ, χ) flux coordinate system with the metric $(d\ell)^2 = (Rd\phi)^2 + (d\psi/RB_{\chi})^2 + (JB_{\chi} d\chi)^2$, where J is the Jacobian. Then

$$Q_{\perp}^2 = \frac{1}{J^2 R^2 B_{\chi}^2} \left(\frac{\partial X}{\partial \psi} + v \frac{\partial X}{\partial \phi} \right)^2 + \frac{R^2 B_{\chi}^2}{J^2 B^2} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} - X \frac{\partial v}{\partial \psi} \right)^2.$$

Here $X \equiv \vec{\xi}_{\perp} \cdot \vec{\nabla} \psi$ is related to the component of the perturbation normal to the flux surface, $U \equiv [\vec{\xi}_{\perp} \cdot (\vec{\nabla} \psi \times \vec{B})]/|\vec{\nabla} \psi|^2$ is the normalized component of the perturbation that lies on a flux surface in the geodesic direction, and $v \equiv JB_{\phi}/R$, where the safety factor $q(\psi) = (1/2\pi)\oint v d\chi$.

The divergence of the perturbation in this geometry is

$$\vec{\nabla} \cdot \vec{\xi}_{\perp} = \frac{B^2}{B^2} \frac{\partial U}{\partial \phi} + \frac{1}{J} \frac{\partial}{\partial \psi} (JX) - \frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{RB_{\phi}}{B^2} U \right)$$

and

$$\vec{\xi}_{\perp} \cdot \vec{\kappa} = \frac{\kappa_{\psi}}{RB_{\chi}} X - \frac{RB_{\chi} \kappa_s}{B} U,$$

where $\kappa_\psi = (\vec{\nabla}\psi \cdot \vec{\kappa})/|\vec{\nabla}\psi|$ is the normal curvature and $\kappa_s = [(\vec{B} \times \vec{\nabla}\psi) \cdot \vec{\kappa}]/(B|\vec{\nabla}\psi|)$ is the geodesic curvature. For the kink term, we get

$$\hat{b} \cdot (\vec{\xi}_1 \times \vec{Q}_1) = \frac{U}{JB} \left(\frac{\partial X}{\partial \chi} + v \frac{\partial X}{\partial \phi} \right) - \frac{X}{JB} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} \right) + \frac{X^2}{JB} \frac{\partial v}{\partial \psi},$$

and for the driving term, we get

$$(\vec{\xi}_1 \cdot \vec{\nabla})(p_{\parallel} + p_{\perp}) = X \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{RB_\phi}{JB^2} U \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}).$$

As a consequence, the guiding center fluid energy principle is

$$\begin{aligned} \delta W_{DA} = \frac{1}{2} \int d^3x \left\{ \frac{\sigma}{J^2 R^2 B_\chi^2} \left(\frac{\partial X}{\partial \psi} + v \frac{\partial X}{\partial \phi} \right)^2 + \frac{\sigma R^2 B^2}{J^2 B^2} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} - X \frac{\partial v}{\partial \psi} \right)^2 \right. \\ + \tau_o B^2 \left[\frac{B^2}{B^2} \frac{\partial U}{\partial \phi} + \left(\frac{1}{J} \right) \frac{\partial}{\partial \psi} (JX) - \left(\frac{1}{J} \right) \frac{\partial}{\partial \chi} \left(\frac{RB_\phi}{B^2} U \right) \right. \\ \left. \left. + \left(1 + \frac{\gamma}{\tau_o} \right) \left(\frac{\kappa_\psi}{RB_\chi} X - \frac{RB_\chi \kappa_s}{B} U \right) \right]^2 \right. \\ + \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left(\frac{\kappa_\psi}{RB_\chi} X - \frac{RB_\chi \kappa_s}{B} U \right)^2 \\ + \sigma j_{\parallel} \left[\frac{U}{JB} \left(\frac{\partial X}{\partial \chi} + v \frac{\partial X}{\partial \phi} \right) - \frac{X}{JB} \left(\frac{\partial U}{\partial \chi} + v \frac{\partial U}{\partial \phi} \right) + \frac{X^2}{JB} \frac{\partial v}{\partial \psi} \right]^2 \\ \left. - \left[X \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{RB_\phi}{JB^2} U \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_\psi}{RB_\chi} X - \frac{RB_\chi \kappa_s}{B} U \right) \right\}. \end{aligned} \quad (3.13)$$

Because the system is axisymmetric, the different toroidal Fourier components of the perturbation are decoupled one from another. Therefore, we need consider only a single Fourier mode $\exp[-in\phi]$ of the displacement $\vec{\xi}_1$, where ϕ is the toroidal angle and n is the mode number. The energy principle then becomes

$$\begin{aligned}
\delta W_{DA} = & \frac{1}{2} \int d^3x \left\{ \frac{\sigma}{J^2 R^2 B^2 \chi} \left| \frac{\partial X}{\partial \psi} + invX \right|^2 + \frac{\sigma R^2 B^2 \chi}{J^2 B^2} \left| \frac{\partial U}{\partial \chi} + invU - X \frac{\partial v}{\partial \psi} \right|^2 \right. \\
& + \tau_o B^2 \left| in \frac{B^2}{B^2} U + \left(\frac{1}{J} \right) \frac{\partial}{\partial \psi} (JX) - \left(\frac{1}{J} \right) \frac{\partial}{\partial \chi} \left(\frac{RB}{B^2} \phi U \right) \right. \\
& \quad \left. \left. + \left(1 + \frac{\gamma}{\tau_o} \right) \left(\frac{\kappa_\psi}{RB_\chi} X - \frac{RB \chi \kappa_s}{B} U \right) \right|^2 \right. \\
& + \sigma B^2 \left(1 + \frac{3p_{||}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left| \frac{\kappa_\psi}{RB_\chi} X - \frac{RB \chi \kappa_s}{B} U \right|^2 \\
& + \frac{\sigma j_{||}}{2} \left[\frac{U^*}{JB} \left(\frac{\partial X}{\partial \chi} + invX \right) - \frac{X^*}{JB} \left(\frac{\partial U}{\partial \chi} + invU \right) + \frac{2|X|^2}{JB} \frac{\partial v}{\partial \psi} \right. \\
& \quad \left. + \frac{U}{JB} \left(\frac{\partial X^*}{\partial \chi} + invX^* \right) - \frac{X}{JB} \left(\frac{\partial U^*}{\partial \chi} + invU^* \right) \right] \\
& - \frac{1}{2} \left[X^* \frac{\partial}{\partial \psi} (p_{||} + p_\perp) \right. \\
& \quad \left. - \frac{RB}{JB^2} \phi U^* \frac{\partial}{\partial \chi} (p_{||} + p_\perp) \right] \left(\frac{\kappa_\psi}{RB_\chi} X - \frac{RB \chi \kappa_s}{B} U \right) \\
& - \frac{1}{2} \left[X \frac{\partial}{\partial \psi} (p_{||} + p_\perp) \right. \\
& \quad \left. - \frac{RB}{JB^2} \phi U \frac{\partial}{\partial \chi} (p_{||} + p_\perp) \right] \left(\frac{\kappa_\psi}{RB_\chi} X^* - \frac{RB \chi \kappa_s}{B} U^* \right) \Bigg\} ,
\end{aligned}
\tag{3.14}$$

where X^* and U^* are the complex conjugates of X and U , respectively. The first two terms in Eq. (3.14) correspond to the field bending, the third is the field compression, the fourth is the anisotropic-double adiabatic contribution, the fifth is the kink instability driving term, and the last two terms drive ballooning and flute instabilities.

We note at this point that the derivatives of the displacements X and U along the field lines are given by

$$(\vec{B} \cdot \vec{\nabla})X = \frac{1}{J} \left(\frac{\partial X}{\partial \chi} + i\nu X \right)$$

and (3.15)

$$(\vec{B} \cdot \vec{\nabla})U = \frac{1}{J} \left(\frac{\partial U}{\partial \chi} + i\nu U \right).$$

3.3 HIGH n PERTURBATIONS AND THE KINK TERM

We now consider only perturbations of large toroidal mode number n and expand the perturbations¹⁰ in powers of $(1/n)$. That is, we take $X = X_0 + \frac{1}{n} X_1 + \dots$ and $U = U_0 + \frac{1}{n} U_1 + \dots$. The double adiabatic energy principle to lowest order in the $1/n$ expansion is

$$\delta W_{DA} = \frac{1}{2} \int d^3x \, n^2 \left(\frac{\sigma}{J^2 R^2 B^2} |\nu X'_0|^2 + \frac{\sigma R^2 B^2}{J^2 B^2} |\nu U_0|^2 + \tau_0 B^2 \left| \frac{B^2}{B^2} U_0 \right|^2 \right),$$

which has contributions only from the field bending and the field compression. This becomes large and positive and therefore predicts stability in the limit $n \rightarrow \infty$ unless the following conditions are satisfied.

- (1) The gradient along the field lines of the perturbation is of order unity; i.e.,

$$(\vec{B} \cdot \vec{\nabla})X_o \sim 0(1) ,$$

$$(\vec{B} \cdot \vec{\nabla})U_o \sim 0(1) .$$

(2) The divergence of the lowest order component of the perturbation is also of order unity. An equivalent statement is that to lowest order the perturbation is incompressible. Then

$$\vec{\nabla} \cdot \vec{\xi}_{1o} = in \frac{B^2}{B^2} U_o + \frac{1}{J} \frac{\partial}{\partial \psi} (JX_o) - \frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{RB_\phi}{B^2} U_o \right) \sim 0(1) .$$

This can be put into the form

$$inU_o + \frac{\partial X_o}{\partial \psi} - \frac{RB_\phi}{JB^2} \left(\frac{\partial U_o}{\partial \chi} + invU_o \right) - \frac{U_o}{J} \frac{\partial}{\partial \chi} \left(\frac{RB_\phi}{B^2} \right) + \frac{X_o}{J} \frac{\partial J}{\partial \psi} \sim 0(1) .$$

Using the fact that $(\vec{B} \cdot \vec{\nabla})U_o$ is of order unity, it is clear then that if $\vec{\nabla} \cdot \vec{\xi}_{1o}$ is also to be of order unity, we must have to lowest order in $1/n$

$$U_o = \frac{i}{n} \frac{\partial X_o}{\partial \psi} . \quad (3.16)$$

Using this relation, the kink contribution to the energy principle, which we shall call δW_J , can be written as

$$\begin{aligned} \delta W_J = \frac{1}{2} \int d^3x \frac{\sigma_{j\parallel}}{2JB} & \left\{ - \frac{i}{n} \frac{\partial X_o^*}{\partial \psi} \left(\frac{\partial X_o}{\partial \chi} + invX_o \right) - \frac{iX_o^*}{n} \left[\frac{\partial}{\partial \chi} \left(\frac{\partial X_o}{\partial \psi} \right) + inv \frac{\partial X_o}{\partial \psi} \right] \right. \\ & \left. + 2|X_o|^2 \frac{\partial v}{\partial \psi} + \frac{i}{n} \frac{\partial X_o}{\partial \psi} \left(\frac{\partial X_o^*}{\partial \chi} - invX_o^* \right) \right\} \end{aligned}$$

$$+ \frac{iX_o}{n} \left[\frac{\partial}{\partial \chi} \left(\frac{\partial X_o^*}{\partial \psi} \right) - \text{inv} \frac{\partial X_o^*}{\partial \psi} \right] \Bigg\} .$$

By algebraic manipulations, we can reduce this expression to

$$\delta W_J = \frac{i}{2} \int d\phi \, d\chi \int d\psi \frac{\sigma j_{\parallel}}{2nB} \frac{\partial}{\partial \psi} \left[X_o \left(\frac{\partial X_o^*}{\partial \chi} - \text{inv} X_o^* \right) - X_o^* \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) \right] .$$

An integration by parts over ψ yields

$$\delta W_J = - \frac{i}{2} \int d^3x \frac{\partial}{\partial \psi} \left(\frac{\sigma j_{\parallel}}{2nB} \right) [X_o (\vec{B} \cdot \vec{\nabla}) X_o^* - X_o^* (\vec{B} \cdot \vec{\nabla}) X_o] . \quad (3.17)$$

We immediately note that δW_J is of order $(1/n)$ because the derivatives of equilibrium quantities with respect to ψ and χ are of order unity. We will keep only terms of $O(1)$ in the energy principle; therefore, the driving term for kink instabilities will be neglected because it is of higher order and vanishes in the limit $n \rightarrow \infty$. The implication of this result is that current-driven instabilities do not display a lot of structure in the ignorable direction of an axisymmetric toroidal device. That an order $(1/n)$ contribution from the kink term survives is a result of the particular form the energy principle was put into. In a different form the kink term can be made to vanish exactly in a large n expansion.

The energy principle to order unity in the $(1/n)$ expansion is

$$\begin{aligned} \delta W_{DA} = \frac{1}{2} \int d^3x \Bigg\{ & \frac{\sigma}{J^2 R^2 B^2 \chi} \left| \frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right|^2 + \frac{\sigma R^2 B^2 \chi}{J^2 B^2} \left| \frac{\partial U_o}{\partial \chi} + \text{inv} U_o - X_o \frac{\partial v}{\partial \psi} \right|^2 \\ & + \tau_o B^2 \left| iU_1 + \frac{1}{n} \frac{\partial X_1}{\partial \psi} - \frac{1}{J} \left[\frac{\partial}{\partial \chi} \left(\frac{RB_{\phi}}{B^2} U_o \right) + \text{inv} \frac{RB_{\phi}}{B^2} U_o \right] \right|^2 \\ & + \frac{X_o}{J} \frac{\partial J}{\partial \psi} + \left(1 + \frac{\gamma}{\tau_o} \right) \left(\frac{\kappa_{\psi}}{RB_{\chi}} - \frac{RB_{\chi} \kappa_s}{B} U_o \right) \Bigg\}^2 \end{aligned}$$

$$\begin{aligned}
& + \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left| \frac{\kappa_{\psi}}{RB_{\chi}} X_o - \frac{RB_{\chi} \kappa_s}{B} U_o \right|^2 \\
& - \frac{1}{2} \left[X_o^* \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right. \\
& - \frac{RB_{\phi}}{JB^2} U_o^* \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \left. \right] \left(\frac{\kappa_{\psi}}{RB_{\chi}} X_o - \frac{RB_{\chi} \kappa_s}{B} U_o \right) \\
& - \frac{1}{2} \left[X_o \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right. \\
& \quad \left. - \frac{RB_{\phi}}{JB^2} U_o \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{RB_{\chi}} X_o^* - \frac{RB_{\chi} \kappa_s}{B} U_o^* \right) \Bigg\} .
\end{aligned}
\tag{3.18}$$

3.4 MINIMIZATION OF THE FIELD-COMPRESSION TERM

The field-compression term in δW_{DA} , which we refer to as δW_C , is to order unity

$$\begin{aligned}
\delta W_C = \frac{1}{2} \int d^3x \tau_o B^2 & \left[iU_1 + \frac{1}{n} \frac{\partial X_1}{\partial \psi} - \frac{1}{J} \left[\frac{\partial}{\partial \chi} \left(\frac{RB_{\phi}}{B^2} U_o \right) + \text{inv} \frac{RB_{\phi}}{B^2} U_o \right] \right. \\
& \left. + \frac{X_o}{J} \frac{\partial J}{\partial \psi} + \left(1 + \frac{\gamma}{\tau_o} \right) \left(\frac{\kappa_{\psi}}{RB_{\chi}} - \frac{RB_{\chi} \kappa_s}{B} U_o \right) \right]^2 .
\end{aligned}$$

We notice that the first order component of the perturbation in the geodesic direction, U_1 , appears in the energy principle in the field-compression term only. It does appear in the other terms, but only to higher order. Therefore, the field compression can be minimized algebraically to zero by taking U_1 to be

$$U_1 = -\frac{i}{n} \frac{\partial X_1}{\partial \psi} - i(\vec{B} \cdot \vec{\nabla}) \left(\frac{RB\phi}{B^2} U_0 \right) + \frac{iX_0}{J} \frac{\partial J}{\partial \psi} + i \left(1 + \frac{\gamma}{\tau_0} \right) \left(\frac{\kappa_\psi}{RB_\chi} X_0 - \frac{RB\chi\kappa_s}{B} U_0 \right).$$

Then using Eq. (3.16) for U_0 , the double adiabatic energy principle correct to order unity in $(1/n)$ is

$$\begin{aligned} \delta W_{DA} = & \frac{1}{2} \int d^3x \left\{ \frac{\sigma}{J^2 R^2 B^2} \left| \frac{\partial X_0}{\partial \chi} + i n v X_0 \right|^2 + \frac{\sigma R^2 B^2}{J^2 B^2} \left| \frac{1}{n} \frac{\partial}{\partial \psi} \left(\frac{\partial X_0}{\partial \chi} + i n v X_0 \right) \right|^2 \right. \\ & + \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_0} \right) \left| \frac{\kappa_\psi}{RB_\chi} X_0 - \frac{iRB\chi\kappa_s}{nB} \frac{\partial X_0}{\partial \psi} \right|^2 \\ & - \frac{\kappa_\psi}{RB_\chi} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) |X_0|^2 - \frac{R^2 B \phi \chi \kappa_s}{n^2 J B^3} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \frac{\partial X_0^*}{\partial \psi} \frac{\partial X_0}{\partial \psi} \\ & - \frac{i}{2n} \left[\frac{B \phi \kappa_\psi}{J B^2 B_\chi} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB\chi\kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_0 \frac{\partial X_0^*}{\partial \psi} \\ & \left. + \frac{i}{2n} \left[\frac{B \phi \kappa_\psi}{J B^2 B_\chi} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB\chi\kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_0^* \frac{\partial X_0}{\partial \psi} \right\}, \end{aligned} \quad (3.19)$$

where the first two terms correspond to the field bending, the third is the anisotropic double adiabatic contribution, and the remaining terms drive flute and ballooning modes, the field compression term having been eliminated by the minimization of the energy principle with respect to the first order component of the perturbation in the geodesic direction.

3.5 THE EULER EQUATION

To obtain an Euler equation from the guiding center fluid energy principle δW_{DA} , we need to minimize it with respect to X_0^* . To perform this minimization we would like only X_0^* and $\partial X_0^*/\partial \chi$ to appear; therefore,

we need to eliminate any terms like $\partial X_o^*/\partial\psi$ and $\partial(\partial X_o/\partial\chi)/\partial\psi$ by means of an integration by parts in the ψ variable. As these integrations are done, we neglect contributions that are of order $(1/n)$, and we assume that the perturbation vanishes at the plasma boundary, which it does if there is a wall present.

The second of the two field-bending terms, δW_{B2} , can be written as

$$\delta W_{B2} = \frac{1}{2} \int d^3x \frac{\sigma R^2 B^2}{J^2 B^2 n^2} \left[\frac{\partial}{\partial\psi} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right) \right] \left[\frac{\partial}{\partial\psi} \left(\frac{\partial X_o^*}{\partial\chi} - \text{inv} X_o^* \right) \right].$$

By integration by parts we get

$$\begin{aligned} \delta W_{B2} = & \frac{1}{2} \int d\phi \, d\chi \int d\psi \frac{\partial}{\partial\psi} \left[\frac{\sigma R^2 B^2}{J B^2 n^2} \left(\frac{\partial X_o^*}{\partial\chi} - \text{inv} X_o^* \right) \frac{\partial}{\partial\psi} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right) \right] \\ & - \frac{1}{2} \int d\phi \, d\chi \int d\psi \frac{\partial}{\partial\psi} \left(\frac{\sigma R^2 B^2}{J B^2 n^2} \right) \left(\frac{\partial X_o^*}{\partial\chi} - \text{inv} X_o^* \right) \frac{\partial}{\partial\psi} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right) \\ & - \frac{1}{2} \int d\phi \, d\chi \int d\psi \frac{\sigma R^2 B^2}{J B^2 n^2} \left(\frac{\partial X_o^*}{\partial\chi} - \text{inv} X_o^* \right) \frac{\partial^2}{\partial\psi^2} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right). \end{aligned}$$

The first integral vanishes as the perturbation vanishes at the plasma boundary, and the second term is of order $(1/n)$ so we neglect it; only the third term survives to make the field-bending δW_B :

$$\begin{aligned} \delta W_B = & \frac{1}{2} \int d^3x \left\{ \left[\frac{\sigma}{J^2 R^2 B^2 n^2} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right) - \left(\frac{\sigma R^2 B^2}{J^2 B^2 n^2} \right) \frac{\partial^2}{\partial\psi^2} \left(\frac{\partial X_o}{\partial\chi} + \text{inv} X_o \right) \right] \right. \\ & \left. \times \left(\frac{\partial X_o^*}{\partial\chi} - \text{inv} X_o^* \right) \right\}. \end{aligned} \quad (3.20)$$

The anisotropic-double adiabatic term δW_E is given by

$$\begin{aligned} \delta W_E = & \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left(\frac{\kappa_{\psi}}{RB_{\chi}} X_o - \frac{iRB_{\chi} \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} \right) \frac{\kappa_{\psi}}{RB_{\chi}} X_o^* \\ & + \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left(\frac{\kappa_{\psi}}{RB_{\chi}} X_o - \frac{iRB_{\chi} \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} \right) \frac{iRB_{\chi} \kappa_s}{nB} \frac{\partial X_o^*}{\partial \psi} . \end{aligned}$$

Integrating the second term of δW_E by parts and neglecting terms that are of order $(1/n)$ reduces δW_E to

$$\delta W_E = \frac{1}{2} \int d^3x \sigma B^2 \left(1 + \frac{3p_{\parallel}}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left[\left(\frac{\kappa_{\psi}}{RB_{\chi}} \right)^2 X_o - \frac{2i\kappa_{\psi} \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_{\chi} \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] X_o^* . \quad (3.21)$$

Finally, the ballooning mode driving term δW_D is

$$\begin{aligned} \delta W_D = & \frac{1}{2} \int d^3x \left\{ - \frac{\kappa_{\psi}}{RB_{\chi}} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) X_o X_o^* - \frac{R^2 B_{\chi} \kappa_s}{n^2 J B^3} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \frac{\partial X_o^*}{\partial \psi} \frac{\partial X_o}{\partial \psi} \right. \\ & - \frac{i}{2n} \left[\frac{B_{\chi} \kappa_{\psi}}{J B^2 B_{\chi}} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_{\chi} \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o \frac{\partial X_o^*}{\partial \psi} \\ & \left. + \frac{i}{2n} \left[\frac{B_{\chi} \kappa_{\psi}}{J B^2 B_{\chi}} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) + \frac{RB_{\chi} \kappa_s}{B} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] X_o^* \frac{\partial X_o}{\partial \psi} \right\} . \end{aligned}$$

We now integrate the second and third terms by parts in δW_D , which contain $\partial X_o^* / \partial \psi$, keeping terms only to order unity. The resulting form for δW_D is

$$\delta W_D = \frac{1}{2} \int d^3x \left\{ - \frac{\kappa_\psi}{RB_\chi} \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) X_o X_o^* + \frac{R^2 B_\phi \chi \kappa_s}{n^2 JB^3} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \frac{\partial^2 X_o}{\partial \psi^2} \right. \\ \left. + \frac{i}{n} \left[\frac{B_\phi \kappa_\psi}{JB^2 B_\chi} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) + \frac{RB_\chi \kappa_s}{B} \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) \right] \frac{\partial X_o}{\partial \psi} X_o^* \right\}. \quad (3.22)$$

Using Eqs. (3.20), (3.21), and (3.22), we can write the double adiabatic energy principle as

$$\delta W_{DA} = \frac{1}{2} \int d\phi \, d\psi \int d\chi \left[A_1 \frac{\partial X_o^*}{\partial \chi} + (-invA_1 + A_2) X_o^* \right], \quad (3.23)$$

where

$$A_1 \equiv \frac{\sigma}{JB^2 R_\chi^2} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right) - \left(\frac{\sigma R^2 B_\phi^2}{JB^2 n^2} \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + inv X_o \right)$$

and

$$A_2 \equiv J\sigma B^2 \left(1 + \frac{3p_\parallel}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left[\left(\frac{\kappa_\psi}{RB_\chi} \right)^2 X_o - \frac{2i\kappa_\psi \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_\chi \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] \\ - \frac{J\kappa_\psi}{RB_\chi} \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) X_o + \frac{R^2 B_\phi \chi \kappa_s}{n^2 B^3} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) \frac{\partial^2 X_o}{\partial \psi^2} \\ + \frac{iJ}{n} \left[\frac{B_\phi \kappa_\psi}{JB^2 B_\chi} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) + \frac{RB_\chi \kappa_s}{B} \frac{\partial}{\partial \psi} (p_\parallel + p_\perp) \right] \frac{\partial X_o}{\partial \psi}.$$

Note that A_1 can alternatively be written as

$$A_1 = \left(\frac{\sigma B}{R^2 B_\chi^2} \right) \frac{1}{JB} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) - \left(\frac{\sigma R^2 B^2}{B n^2} \right) \frac{\partial^2}{\partial \psi^2} \left[\frac{1}{JB} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) \right] \\ - \left(\frac{\sigma R^2 B^2}{JB^2 n^2} \right) \frac{\partial (JB)}{\partial \psi} \frac{\partial}{\partial \psi} \left[\frac{2}{JB} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) \right] - \left(\frac{\sigma R^2 B^2}{J^2 B^3 n^2} \right) \frac{\partial^2 (JB)}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right).$$

The third term is of order $(1/n)$, and the fourth term is of order $(1/n^2)$ compared to the first two; therefore, we can neglect them. If we define the operator

$$ik_{\parallel} \equiv \frac{1}{JB} \left(\frac{\partial}{\partial \chi} + \text{inv} \right), \quad (3.24)$$

then we have the identity

$$\frac{\sigma}{JR^2 B_\chi^2} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) - \left(\frac{\sigma R^2 B^2}{JB^2 n^2} \right) \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial X_o}{\partial \chi} + \text{inv} X_o \right) = \left(\frac{\sigma B}{R^2 B_\chi^2} - \frac{\sigma R^2 B^2}{B n^2} \frac{\partial^2}{\partial \psi^2} \right) ik_{\parallel} X_o \\ + O\left(\frac{1}{n}\right). \quad (3.25)$$

By splitting the double adiabatic-anisotropic contribution in A_2 [that is, by taking $\sigma B^2(1 + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_o) = \sigma B^2(1 - \sigma/\tau_d) + \sigma B^2(\sigma/\tau_d + 3p_{\parallel}/\sigma B^2 - \gamma^2/\sigma\tau_o)$, where the first part is the anisotropic part and the second part is the double adiabatic term] and by redistributing the ballooning instability terms so that one can identify a purely poloidal pressure gradient contribution and a purely normal pressure gradient contribution, we get A_2 in the following form:

$$\begin{aligned}
A_2 = & -J \left(\frac{\kappa_\psi}{RB_\chi} X_o - \frac{iRB_\chi \kappa_s}{Bn} \frac{\partial X_o}{\partial \psi} \right) \left[\frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{\kappa_\psi}{RB_\chi} \sigma B^2 \left(1 - \frac{\sigma}{\tau_d} \right) \right] \\
& + J \left(\frac{i\kappa_\psi}{nRB_\chi} \frac{\partial X_o}{\partial \psi} + \frac{RB_\chi \kappa_s}{Bn^2} \frac{\partial^2 X_o}{\partial \psi^2} \right) \left[\frac{RB_\chi}{JB^2} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) - \frac{RB_\chi \kappa_s}{B} \sigma B^2 \left(1 - \frac{\sigma}{\tau_d} \right) \right] \\
& + J \sigma B^2 \left(\frac{\sigma}{\tau_d} + \frac{3p_\parallel}{\sigma B^2} - \frac{\gamma^2}{\sigma \tau_o} \right) \left[\left(\frac{\kappa_\psi}{RB_\chi} \right)^2 X_o - \frac{2i\kappa_\psi \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_\chi \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] .
\end{aligned} \tag{3.26}$$

Here the first two terms combine the pressure-driving contribution adiabatic effect. The term $\sigma B^2(\sigma/\tau_d + 3p_\parallel/\sigma B^2 - \gamma^2/\sigma \tau_o)$ can be transformed by algebraic manipulation into $(p_\perp^2/3p_\parallel \tau_o \tau_d)(\gamma + 3p_\parallel \tau_o/p_\perp)^2$. We notice that this term is positive because $\tau_d > 0$ for the equilibrium to be well posed.³¹ Otherwise, the equilibrium equation for the poloidal flux function is not elliptic. After some further algebra we may write $\sigma B^2(\sigma/\tau_d + 3p_\parallel/\sigma B^2 - \gamma^2/\sigma \tau_o) = (B^2/\tau_d)(p_\perp \sigma + 3p_\parallel \tau_d)^2/(p_\perp^2 + 3p_\parallel B^2 \tau_d)$.

The minimization of Eq. (3.23) with respect to X_o^* can be accomplished using the standard techniques of the calculus of variations to yield the Euler equation

$$\frac{\partial A_1}{\partial \chi} + \text{inv} A_1 - A_2 = 0 , \tag{3.27}$$

which then, using the previous relations, becomes

$$\begin{aligned}
& Bk_\parallel \left\{ \frac{\sigma B}{R^2 B_\chi^2} \left[1 - \left(\frac{R^2 B^2}{Bn} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] k_\parallel X_o \right\} \\
& - \left[\frac{\partial}{\partial \psi} (p_\parallel + p_\perp) - \frac{\kappa_\psi}{RB_\chi} \sigma B^2 \left(1 - \frac{\sigma}{\tau_d} \right) \right] \left(\frac{\kappa_\psi}{RB_\chi} X_o - \frac{iRB_\chi \kappa_s}{Bn} \frac{\partial X_o}{\partial \psi} \right) \\
& + \left[\frac{RB_\chi}{JB^2} \frac{\partial}{\partial \chi} (p_\parallel + p_\perp) - \frac{RB_\chi \kappa_s}{B} \sigma B^2 \left(1 - \frac{\sigma}{\tau_d} \right) \right] \left(\frac{i\kappa_\psi}{nRB_\chi} \frac{\partial X_o}{\partial \psi} + \frac{RB_\chi \kappa_s}{Bn^2} \frac{\partial^2 X_o}{\partial \psi^2} \right) \\
& + \frac{B^2 (p_\perp \sigma + 3p_\parallel \tau_d)^2}{\tau_d (p_\perp^2 + 3p_\parallel B^2 \tau_d)} \left[\left(\frac{\kappa_\psi}{RB_\chi} \right)^2 X_o - \frac{2i\kappa_\psi \kappa_s}{nB} \frac{\partial X_o}{\partial \psi} + \left(\frac{RB_\chi \kappa_s}{nB} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] = 0 .
\end{aligned} \tag{3.28}$$

From the equilibrium relation

$$\vec{\nabla} \left(p_{\perp} + \frac{B^2}{2} \right) = \hat{b} (\hat{b} \cdot \vec{\nabla}) \left(p_{\perp} + \frac{B^2}{2} \right) + \sigma B^2 \vec{\kappa}, \quad (3.29)$$

we can obtain a relationship between the curvatures and the other equilibrium quantities. In particular,

$$\kappa_{\psi} = \frac{RB_{\chi}}{\sigma B^2} \frac{\partial}{\partial \psi} \left(p_{\perp} + \frac{B^2}{2} \right), \quad (3.30)$$

$$\kappa_s = \frac{B_{\phi}}{\sigma B^3 J B_{\chi}} \frac{\partial}{\partial \chi} \left(p_{\perp} + \frac{B^2}{2} \right).$$

If we now consider the guiding center fluid as a limiting case of the guiding center plasma, we can use the relation

$$(\hat{b} \cdot \vec{\nabla}) p_{\perp} = \frac{2p_{\perp} - \hat{C}_1}{2B^2} (\hat{b} \cdot \vec{\nabla}) B^2, \quad (3.31)$$

which can be derived from Eqs. (3.4) and (3.5) which, when it is combined with the equilibrium relation

$$(\hat{b} \cdot \vec{\nabla}) p_{\parallel} = \frac{p_{\parallel} - p_{\perp}}{2B^2} (\hat{b} \cdot \vec{\nabla}) B^2,$$

will yield (after expanding in the orthogonal flux coordinate system)

$$\frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) = (\tau_d - \sigma) \frac{\partial}{\partial \chi} \left(\frac{B^2}{2} \right) = \left(1 - \frac{\sigma}{\tau_d} \right) \frac{\partial}{\partial \chi} \left(p_{\perp} + \frac{B^2}{2} \right).$$

Now using the expression for the geodesic curvature in Eq. (3.30), we can obtain the relation

$$\frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) = \sigma B^2 \left(1 - \frac{\sigma}{\tau_d} \right) \frac{B J B_{\chi}}{B_{\phi}} \kappa_s. \quad (3.32)$$

This implies that the third term of the Euler Eq. (3.28) is identically zero. Using Eq. (3.32) in the second term of Eq. (3.28) modifies the Euler equation, which becomes

$$\begin{aligned}
& Bk_{\parallel} \left\{ \frac{\sigma B}{R^2 B_{\chi}^2} \left[1 - \left(\frac{R^2 B_{\chi}^2}{Bn} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] k_{\parallel} X_o \right\} \\
& - \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{B_{\phi} (\kappa_{\psi} / \kappa_s)}{R B J B_{\chi}^2} \frac{\partial}{\partial \chi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{R B_{\chi}} X_o - \frac{i R B_{\chi} \kappa_s}{Bn} \frac{\partial X_o}{\partial \psi} \right) \\
& + \frac{B^2 (p_{\perp} \sigma + 3 p_{\parallel} \tau_d)^2}{\tau_d (p_{\perp}^2 + 3 p_{\parallel} B^2 \tau_d)} \left[\left(\frac{\kappa_{\psi}}{R B_{\chi}} \right)^2 X_o - \frac{2 i \kappa_{\psi} \kappa_s}{n B} \frac{\partial X_o}{\partial \psi} + \left(\frac{R B_{\chi} \kappa_s}{n B} \right)^2 \frac{\partial^2 X_o}{\partial \psi^2} \right] = 0,
\end{aligned}
\tag{3.33}$$

drives ballooning modes, and the last term is the double adiabatic addition. This equation is the analog of the ideal MHD equation first derived by Dobrott et al.,¹² which they solved numerically assuming that the perturbation vanished at the inner edge of the torus on every flux surface.

3.6 THE CONNOR, HASTIE, TAYLOR TRANSFORMATION¹³

It is extremely difficult to construct accurately a perturbation that is periodic in the poloidal angle variable and that varies slowly along the field lines but rapidly across them in a magnetic field with shear. Even the choice of the perturbation in the eikonal form is inappropriate if the requirement of periodicity in χ is to be satisfied concurrently with $dq/d\psi \neq 0$. This obstacle can be overcome if the perturbation is chosen to vanish at the inner edge of the torus.^{12,14} But this arbitrary constraint on the displacement implies that the resulting eigenfunction does not necessarily represent the most unstable perturbation that could be constructed. An alternative approach is to introduce the transformation described by Connor, Hastie, and Taylor,¹³

which allows us to extend the periodic domain of the poloidal angle into an infinite domain so that periodicity conditions on the perturbation are no longer required. The perturbation takes the form

$$Y(\psi, \chi) = \frac{1}{2\pi} \sum_m e^{-2\pi i m \chi / \oint d\chi} \int_{-\infty}^{\infty} e^{2\pi i m y / \oint d\chi} \hat{Y}(\psi, y) . \quad (3.34)$$

This transformation can be obtained by expanding the displacement $Y(\psi, \chi)$ in a Fourier series. The Fourier coefficients a_m represent a sequence of points in the complex plane that by analytic continuation can generate a function $a(m)$. Then the inverse Fourier transform of this function results in Eq. (3.34).

We next outline some of the properties of this transformation. The product of two functions is transformed as

$$a(\psi, \chi) Y(\psi, \chi) = \frac{1}{2\pi} \sum_m e^{-2\pi i m \chi / \oint d\chi} \int_{-\infty}^{\infty} dy e^{2\pi i m y / \oint d\chi} a(\psi, y) \hat{Y}(\psi, y) , \quad (3.35)$$

where, for instance, $a(\psi, \chi)$ contains just equilibrium quantities and $Y(\psi, \chi)$ is the perturbation. The derivative of Y with respect to ψ transforms as

$$\frac{\partial}{\partial \chi} [Y(\psi, \chi)] = \frac{1}{2\pi} \sum_m e^{-2\pi i m \chi / \oint d\chi} \int_{-\infty}^{\infty} dy e^{2\pi i m y / \oint d\chi} \frac{\partial}{\partial y} [\hat{Y}(\psi, y)] . \quad (3.36)$$

Using the properties described by Eqs. (3.35) and (3.36), we now proceed to transform the Euler Eq. (3.33). For the field-bending term we transform the left-hand side of Eq. (3.25) rather than using the form it appears as in Eq. (3.33); thus, we have

$$\begin{aligned}
& - \frac{1}{J} \left(\frac{\partial}{\partial y} + \text{inv} \right) \left\{ \frac{\sigma B}{JR^2 B_X^2} \left[1 - \left(\frac{R^2 B_X^2}{Bn} \right)^2 \frac{\partial^2}{\partial \psi^2} \right] \left(\frac{\partial \hat{X}_0}{\partial y} + \text{inv} \hat{X}_0 \right) \right\} \\
& - \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{B_{\phi} (\kappa_{\psi} / \kappa_s)}{RBJB_X^2} \frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{RB_X} \hat{X}_0 - \frac{iRB_X \kappa_s}{Bn} \frac{\partial \hat{X}_0}{\partial \psi} \right) \\
& + \frac{B^2 (p_{\perp} \sigma + 3p_{\parallel} \tau_d)^2}{\tau_d (p_{\perp}^2 + 3p_{\parallel} B^2 \tau_d)} \left[\left(\frac{\kappa_{\psi}}{RB_X} \right)^2 \hat{X}_0 - \frac{2i\kappa_{\psi} \kappa_s}{nB} \frac{\partial \hat{X}_0}{\partial \psi} + \left(\frac{RB_X \kappa_s}{nB} \right)^2 \frac{\partial^2 \hat{X}_0}{\partial \psi^2} \right] = 0
\end{aligned} \tag{3.37}$$

in the infinite domain $-\infty \leq y \leq \infty$.

Because the transformed perturbation need not be periodic in the infinite domain, it can be expressed in a quasi-mode or eikonal form,⁷

$$\hat{X}_0(\psi, y) = F(\psi, y) \exp \left[-in \int^y v \, dy \right], \tag{3.38}$$

where F is a slowly varying function of ψ and y and all the rapid cross-field variations are contained in the exponent.

Using Eq. (3.38) for the perturbation, we can write the field-bending term of Eq. (3.37) as

$$\begin{aligned}
& - \frac{1}{J} \frac{\partial}{\partial y} \left\{ \frac{\sigma}{JR^2 B_X^2} \left[1 + \left(\frac{R^2 B_X^2}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] \frac{\partial F}{\partial y} + \frac{i\sigma}{Jn} \left(\frac{R^2 B_X^2}{B^2} \int^y dy \frac{\partial v}{\partial \psi} \right) \frac{\partial^2 F}{\partial \psi \partial y} \right. \\
& \quad \left. + \frac{i\sigma}{Jn} \left(\frac{R^2 B_X^2}{B^2} \int^y dy \frac{\partial^2 v}{\partial \psi^2} \right) \frac{\partial F}{\partial y} - \frac{\sigma R^2 B_X^2}{JB^2 n^2} \frac{\partial^3 F}{\partial \psi^2 \partial y} \right\} e^{-in \int^y v \, dy}.
\end{aligned}$$

Here the last three terms are of order $(1/n)$, $(1/n)$, and $(1/n^2)$, respectively; therefore, they are neglected compared to the first term. The evaluation of the pressure gradient driving term and the double adiabatic contribution can be done in a manner similar to the treatment of the field-bending term where any expression of order $(1/n)$ or higher is not kept.

The Euler equation then is given by

$$\begin{aligned}
& \frac{1}{J} \frac{\partial}{\partial y} \left\{ \frac{\sigma}{JR^2 B_X^2} \left[1 + \left(\frac{R^2 B_X^2}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 \right] \frac{\partial F}{\partial y} \right\} \\
& \quad \text{field bending} \\
& + \left[\frac{\partial}{\partial \psi} (p_{\parallel} + p_{\perp}) - \frac{B_{\phi} (\kappa_{\psi} / \kappa_s)}{RB_X B_X} \frac{\partial}{\partial y} (p_{\parallel} + p_{\perp}) \right] \left(\frac{\kappa_{\psi}}{RB_X} - \frac{RB_X \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right) F \\
& \quad \begin{array}{cc} \text{normal} & \text{poloidal} \\ \text{pressure} & \text{gradients} \end{array} \quad \begin{array}{cc} \text{normal} & \text{geodesic} \\ \text{curvatures} & \end{array} \\
& + \frac{B^2 (p_{\perp} \sigma + 3p_{\parallel} \tau_d)^2}{\tau_d (p_{\perp}^2 + 3p_{\parallel} B^2 \tau_d)} \left(\frac{\kappa_{\psi}}{RB_X} - \frac{RB_X \kappa_s}{B} \int^y dy \frac{\partial v}{\partial \psi} \right)^2 F(\psi, y) = 0. \quad (3.39) \\
& \quad \begin{array}{cc} \text{double} & \text{adiabatic} \end{array}
\end{aligned}$$

Here, as indicated, the first term is the field bending and is only slightly modified by the factor σ from the scalar pressure case. The second term shows the coupling between the pressure gradients and the curvatures, which drives ballooning instabilities. The main difference from ideal MHD is that because the pressures are no longer constant on a flux surface, an additional term due to the gradient of the pressures in the poloidal direction appears. The last term is a double adiabatic contribution that stabilizes whenever $\tau_d > 0$, a requirement that must be met for the equilibrium to be mirror stable, that is, to be well posed. If we neglect the double adiabatic contribution, the Euler equation is virtually identical to that derived by Fielding and Haas¹⁶ from the universal term of the Kruskal-Oberman energy principle. Without the stabilizing double adiabatic contribution, the solutions of the Euler equation will give criteria for marginal stability that are sufficient for the stability of a guiding center plasma; with it, the criteria are necessary for the stability of the guiding center plasma. In that sense, we can bracket the actual marginal stability point.

The boundary conditions are obtained by requiring that the perturbation be square integrable, i.e., that $\int_{-\infty}^{\infty} dy |F|^2$ be finite. If $|F|^2$

is proportional to $1/y$, the integral diverges logarithmically. If it is proportional to some power of the inverse of y that is slightly larger than unity, then the integral converges. That gives the natural boundary condition

$$|y|^{1/2} F \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty .$$

Although $\hat{X}_0(\psi, y)$ is not the actual perturbation, a real and periodic $X_0(\psi, \chi)$ can be constructed from it from a superposition of quasi-modes.

4. CONCLUSIONS

In the first section on the equilibrium of tensor pressure plasmas in axisymmetric tori, we present a force balance relation, a representation for β_p , and expressions for the current densities. In general, the modifications due to the pressure anisotropy are of order β and reduce to the ideal MHD result in the limit $p_{\parallel} = p_{\perp}$. From the pressure moments of a distribution function that models neutral beam injection, we find that the most effective way to generate a pressure anisotropy is to inject the beams mainly in the perpendicular direction, but with enough parallel velocity to prevent trapping. In the section on the stability of an anisotropic pressure plasma, we obtain a double adiabatic energy principle and apply it to an arbitrary aspect ratio, arbitrary cross-section axisymmetric torus to examine ballooning modes of large toroidal mode number. We derive an Euler equation that is necessary for the stability of a guiding center plasma. Although the field-bending term is only slightly modified from ideal MHD, the pressure gradient driving term is altered because the pressures no longer are constant on a flux surface. If the gradients of the pressures on a flux surface in the poloidal direction are such as to maximize the pressures on the inner edge of the torus and minimize them at the bad curvature edge, the effect is stabilizing. An additional double adiabatic term that is positive definite causes any marginal stability criterion derived to be necessary for a guiding center plasma; neglecting it altogether yields a criterion that is sufficient because the resulting Euler equation is identical to that obtained from the universal term of the Kruskal-Oberman energy principle. In this respect, we are able to bracket the actual marginal stability point for a guiding center plasma without ever having to evaluate the complicated kinetic energy integral.

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1. M. D. Kruskal and C. R. Oberman, *Phys. Fluids* 1, 275 (1958).
2. L. A. Berry et al., *Proc. 6th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-35/A4-1, Vol. 1, p. 49 (1977); Equipe TFR, *Proc. 6th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-35/A4-2, Vol. 1, p. 69 (1977); J. W. M. Paul et al., *Proc. 6th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-35/A17, Vol. 2, p. 269 (1977); V. S. Vlasenkov et al., *Proc. 6th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-35/A4-3, Vol. 1, p. 85 (1977); H. P. Eubank et al., *Bull. Am. Phys. Soc.* 23, 745 (1978).
3. H. P. Eubank (Princeton Plasma Physics Laboratory), private communication, 1978.
4. L. Dupas, *Bull. Am. Phys. Soc.* 23, 914 (1978) (post deadline paper); G. Tonon et al., EUR-CEA-FC 914 (1977).
5. O. Eldridge (Oak Ridge National Laboratory), private communication, 1978.
6. D. J. Sigmar and H. C. Chan, *Nucl. Fusion* 18, 1569 (1979).
7. K. V. Roberts and J. B. Taylor, *Phys. Fluids* 8, 315 (1965).
8. M. N. Rosenbluth and C. L. Longmire, *Ann. Phys. (N.Y.)* 1, 120 (1957).
9. D. Berger et al., *Proc. 6th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-35/B11-4, Vol. 2, p. 411 (1977); A. M. M. Todd et al., *Phys. Rev. Lett.* 38, 826 (1977).
10. G. Bateman and Y-K. M. Peng, *Phys. Rev. Lett.* 38, 829 (1977).
11. D. Dobrott et al., *Proc. 7th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-37/P4, Vol. 1, p. 717 (1979).
12. D. Dobrott et al., *Phys. Rev. Lett.* 39, 938 (1977).
13. J. W. Connor, R. J. Hastie, and J. B. Taylor, *Phys. Rev. Lett.* 40, 396 (1978).

14. B. Coppi, Phys. Rev. Lett. 39, 938 (1977); P. Rutherford et al., *Proc. 3rd Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-24/C-5, Vol. 1, p. 367 (1969); G. Bateman and D. B. Nelson, Oak Ridge National Laboratory Report ORNL/TM-6629, Oak Ridge, Tennessee (1978).
15. M. S. Chu et al., Phys. Rev. Lett. 41, 247 (1978).
16. P. J. Fielding and F. A. Haas, Phys. Rev. Lett. 41, 801 (1978).
17. P. H. Rutherford, L. Chen, and M. N. Rosenbluth, Princeton Plasma Physics Laboratory Report PPPL-1418, Princeton, New Jersey (1978).
18. J. D. Callen and R. A. Dory, Phys. Fluids 15, 1523 (1972).
19. J. G. Cordey and M. J. Houghton, Nucl. Fusion 13, 215 (1973).
20. H. Grad, p. 162 in *Magneto-Fluid and Plasma Dynamics*, American Mathematical Society, Providence, Rhode Island, 1967.
21. J. A. Rome and J. F. Lyon (Oak Ridge National Laboratory), private communication, 1978; C. Mercier and M. Cotsaftis, Nucl. Fusion 1, 12 (1961).
22. J. T. Hogan, Oak Ridge National Laboratory Report ORNL/TM-6049, Oak Ridge, Tennessee (1978).
23. J. G. Cordey, Nucl. Fusion 16, 499 (1976).
24. A. Sykes, M. F. Turner, P. J. Fielding, and F. A. Haas, *Proc. 7th Int. Conf. Plasma Physics and Controlled Nuclear Fusion Research*, IAEA-CN-37/K-5, Vol. 1, p. 625 (1979).
25. M. N. Rosenbluth and N. Rostoker, Phys. Fluids 2, 23 (1959).
26. J. W. Connor and R. J. Hastie, Phys. Fluids 19, 1727 (1976).
27. H. Grad, Phys. Fluids 9, 225 (1966).
28. G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. R. Soc. London, Ser. A, 236, 112 (1956).
29. J. W. Connor and R. J. Hastie, Phys. Rev. Lett. 33, 202 (1974).
30. G. E. Guest and C. L. Hedrick, Oak Ridge National Laboratory Report ORNL-TM-3943, Oak Ridge, Tennessee (1972); J. B. Taylor and R. J. Hastie, Phys. Fluids 8, 323 (1965).
31. H. Grad, Phys. Fluids 9, 498 (1966).

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