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Drift Hamiltonian in Magnetic Coordinates
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Drift Hamiltonian in Magnetic Coordinates

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A Hamiltonian formulation of the guiding-center drift in arbitrary, steady state, magnetic and electric fields is given. The canonical variables of this formulation are simply related to the magnetic coordinates. The modifications required to treat ergodic magnetic fields using magnetic coordinates are explicitly given in the Hamiltonian formulation.

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The calculation of the orbits of individual charged particles in given electric and magnetic fields is one of the oldest problems in plasma physics. The practical importance of the problem is most obvious in the magnetic fusion program. The evaluation of the confinement of high energy ions and the diffusion coefficients of thermal particles involves following individual particle orbits for long times. To integrate the equations of motion for the required period of time, the smallness of the gyroradii of the particles compared to the system size must be used to obtain the guiding-center equations for the trajectories of the particles. These equations have played an important role in plasma physics since their development by Alfvén.¹ They have also been the subject of continual investigation.²⁻⁷

The traditional form for the guiding-center velocity, in given magnetic and electric fields $\vec{B}$ and $-\vec{\nabla}\Phi$, is

$$\vec{v} = \frac{v_{\parallel}}{B} \vec{B} + \frac{c\vec{B}}{eB^2} \times (mv_{\parallel}^2 \frac{\vec{B}}{B} \cdot \vec{\nabla} \frac{\vec{B}}{B} + \mu \vec{\nabla} B + e \vec{\nabla} \Phi) \quad (1)$$

with $\mu = mv_{\perp}^2/(2B)$, the magnetic moment, c the speed of light, e the particle charge, and $v_{\parallel}$ and $v_{\perp}$ the components of the particle velocity parallel and perpendicular to the magnetic field. The particle energy can be written

$$E = \frac{1}{2} mv_{\parallel}^2 + \mu B + e\Phi \quad (2)$$

Morozov and Solov'ev⁸ have pointed out that if $\rho_{\parallel}(E, \mu, \vec{x}) \equiv v_{\parallel}/\omega_C$, then the traditional form for the guiding-center velocity can be written

$$\dot{\vec{r}} = \frac{v_{\parallel}}{B} [\vec{B} + (\vec{v} \times \rho_{\parallel} \vec{B})_{\perp}] \quad (3)$$

with $\vec{r}$ the position of the guiding center and $\omega_c = eB/mc$, the cyclotron frequency. Although this form is extensively used in the neoclassical transport theory, it has a number of serious deficiencies. For example, Liouville's theorem is not obeyed nor is a canonical momentum conserved in a symmetric system unless $\vec{B} \cdot (\vec{v} \times \vec{B}) = 0$. These problems can be eliminated by writing⁵

$$\dot{\vec{r}} = \frac{v_{\parallel}}{B} \frac{1}{1 + \rho_{\parallel} \hat{b} \cdot (\vec{v} \times \hat{b})} (\vec{B} + \vec{v} \times \rho_{\parallel} \vec{B}) \quad (4)$$

with $\hat{b} = \vec{B}/B$. This equation contains some of the so-called second order terms of Northrop and Rome.⁴ These terms must be kept to retain the conservation laws which are obeyed by the particle trajectories. Littlejohn⁹ has noted that Eq. (4) gives a Hamiltonian system of equations. In this paper, we will show $\dot{\vec{r}}$ is manifestly Hamiltonian by giving both the Hamiltonian and the canonical variables. The canonical variables are closely related to the magnetic coordinates of the field $\vec{B}$.

In most guiding-center orbit problems, the time required to traverse a certain portion of a trajectory is not required to high accuracy. In these problems, the dividing factor $1 + \rho_{\parallel} \hat{b} \cdot \vec{v} \times \hat{b}$ can be eliminated. Although this factor does not affect the constants of the motion f , which are defined by $\vec{v} \cdot \vec{\nabla} f = 0$, it is essential to the Hamiltonian formulation.

The most useful coordinate system in which to integrate the drift

equations is a magnetic coordinate system.⁵ The primary reason is, we are interested in the motion of particles relative to the plasma structure, which is determined by the magnetic field. There are other advantages such as the separation of slowly and rapidly varying coordinates and the fact that less information is required about the magnetic field during the integration.

The magnetic coordinates we will use are ψ , θ , χ with

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta, \quad \vec{H} = \vec{\nabla}\chi + \beta \vec{\nabla}\psi, \quad \vec{B} = \lambda \vec{H} \quad (5)$$

Unfortunately, the standard nomenclature from electrodynamics is confusing. $\vec{B}$ is called the magnetic induction, $\vec{H}$ the magnetic field, and λ the permeability. However, $\vec{B}$ is the real magnetic field and $\vec{H}$ is an artificial quantity which simplifies the treatment of currents. The permeability λ is dimensionless and is determined by the condition that $\vec{\nabla}\psi \cdot (\vec{\nabla} \times \vec{H}) = 0$. In scalar pressure equilibria the pressure is a function of ψ alone and $\vec{\nabla}\psi \cdot (\vec{\nabla} \times \vec{H}) = 0$; so λ is unity. In tensor pressure equilibria¹⁰

$$\frac{1}{\lambda} = 1 + 4\pi \frac{p_{\perp} - p_{\parallel}}{B^2} \quad (6)$$

with $p_{\parallel}$ and $p_{\perp}$ the parallel and perpendicular pressures. The form $\vec{B} = \lambda(\vec{\nabla}\chi + \beta \vec{\nabla}\psi)$ is adequate to represent an arbitrary vector field since there are three independent scalar functions λ , χ , and β .

In toroidal magnetic configurations, a single valued function ψ may not exist with $\vec{B} \cdot \vec{\nabla}\psi = 0$. A straight forward application of magnetic coordinates can not be made in such cases. However, the exact magnetic field $\vec{B}_e$ can be represented by¹¹

$$\vec{B}_e = \vec{B} + \vec{V} \times \alpha \vec{H} \quad (7)$$

with the field $\vec{B}$ having exact nested ψ surfaces. At least in magnetic fusion applications, α , which has the units of length, is very small compared to the system size. Assuming α is small, the guiding-center drift in the exact field $\vec{B}_e$ is given by

$$\vec{v} = \frac{v_{\parallel}}{B} \frac{1}{1 + (\rho_{\parallel} + \alpha/\lambda) \hat{b} \cdot \vec{\nabla} \times \hat{b}} [\vec{B} + \vec{V} \times (\lambda \rho_{\parallel} + \alpha) \vec{H}] \quad (8)$$

This expression for the guiding-center motion is Hamiltonian as will be demonstrated by giving the Hamiltonian and its canonical variables.

The Hamiltonian description of the guiding-center motion requires four canonical variables. These form two canonical pairs χ , ρ_c , and θ_c , ψ . The variables χ and ψ are magnetic coordinates. The canonical variable

$$\rho_c = \lambda \rho_{\parallel} + \alpha \quad (9)$$

The fourth canonical variable is

$$\theta_c = \theta_0 - \beta \rho_c \quad (10)$$

The Hamiltonian is c/e times the energy and can be written

$$H = \frac{1}{2} \frac{e}{mc} \left(\frac{B}{\lambda} \right)^2 (\rho_c - \alpha)^2 + \frac{\mu c}{e} B + c\Phi \quad (11)$$

Hamilton's equations are

$$\begin{aligned}\dot{\chi} &= \frac{\partial H}{\partial p_c} & \dot{p}_c &= -\frac{\partial H}{\partial \chi} \\ \dot{\phi}_c &= \frac{\partial H}{\partial \psi} & \dot{\psi} &= -\frac{\partial H}{\partial \phi_c}\end{aligned}\quad (12)$$

The demonstration that the Hamiltonian description and Eq. (8) are identical involves significant algebra. The equality of the two forms will be demonstrated for $\dot{\chi} = \vec{v} \cdot \vec{\nabla} \chi$ by starting from Eq. (8). First note that

$$\vec{B} \cdot (\vec{v} \times \hat{b}) = \frac{1}{B^2} \vec{B} \cdot (\vec{v} \times \vec{B}) = \frac{\lambda}{B^2} \vec{B} \cdot (\vec{v} \times \lambda \vec{B}) = -\lambda \frac{\partial \beta}{\partial \theta_0} \quad (13)$$

We used the identity

$$(\vec{v} \times \vec{\nabla}_{\theta_0}) \cdot \vec{\nabla} \chi = \frac{B^2}{\lambda} \quad (14)$$

It is then easy to show Eq. (8) implies

$$\vec{v} \cdot \vec{\nabla} \chi = \frac{v_{\parallel}^2}{\lambda} \left(1 - \frac{\beta}{1 - p_c (\partial \beta / \partial \theta_0)} \frac{\partial p_c}{\partial \theta_0} \right) \quad (15)$$

with $\partial p_c / \partial \theta_0$ meaning that ψ , χ , and the energy are held constant. The Hamiltonian, Eq. (11), is c/e times the energy. Using Eq. (11) to find $\partial p_c / \partial \theta_0$ while holding the Hamiltonian constant, one finds

$$\begin{aligned}
-\frac{e}{mc} \left(\frac{B}{\lambda}\right)^2 (\rho_c - \alpha) \frac{\partial \rho_c}{\partial \theta_0} &= \frac{1}{2} \frac{e}{mc} (\rho_c - \alpha)^2 \frac{\partial B^2/\lambda^2}{\partial \theta_0} + \frac{c\mu}{e} \frac{\partial B}{\partial \theta_0} + c \frac{\partial \Phi}{\partial \theta_0} \\
&\quad - \frac{e}{mc} \left(\frac{B}{\lambda}\right)^2 (\rho_c - \alpha) \frac{\partial \alpha}{\partial \theta_0} .
\end{aligned} \tag{16}$$

Equation (9), the definition of ρ_c , can be used to show

$$\frac{e}{mc} \left(\frac{B}{\lambda}\right)^2 (\rho_c - \alpha) = v_{\parallel} \frac{B}{\lambda} . \tag{17}$$

The identity

$$\left(\frac{\partial f}{\partial \rho_c}\right)_{\theta_c} = \frac{\beta}{1 - \rho_c (\partial \beta / \partial \theta_0)} \frac{\partial f}{\partial \theta_0}$$

with f any function of position follows from the rules of partial derivatives and the definition of ρ_c . Equations (15) through (18) imply, with $\partial/\partial \rho_c$ meaning θ_c is held constant,

$$\begin{aligned}
\vec{v} \cdot \vec{\nabla} \chi &= \frac{e}{mc} \left(\frac{B}{\lambda}\right)^2 (\rho_c - \alpha) \left(1 - \frac{\partial \alpha}{\partial \rho_c}\right) + \frac{1}{2} \frac{e}{mc} (\rho_c - \alpha)^2 \frac{\partial B^2/\lambda^2}{\partial \rho_c} \\
&\quad + \frac{c\mu}{e} \frac{\partial B}{\partial \rho_c} + c \frac{\partial \Phi}{\partial \rho_c} .
\end{aligned} \tag{19}$$

It is then obvious that $\vec{v} \cdot \vec{\nabla} \chi = \partial H / \partial \rho_c$, which was the desired result.

The equivalence of the Hamiltonian description and the expression for the drift velocity is much easier to demonstrate for curl free magnetic fields. This was done by Boozer⁵ for $\alpha = 0$ and Boozer and White¹¹ for finite α .

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