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NUMERICAL METHODS IN FUSION RESEARCH

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John Killeen

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NUMERICAL METHODS IN FUSION RESEARCH

John Killeen
Magnetic Fusion Energy Computer Center
Lawrence Livermore Laboratory
University of California
Livermore, California 94550

Abstract

The numerical models used in fusion research are briefly reviewed. The application of implicit difference techniques to problems in resistive magnetohydrodynamics, transport and the Fokker-Planck equation is discussed.

1. INTRODUCTION

Numerical calculations have had an important role in controlled fusion research since its beginning in the early 1950's, but the application of computers to plasma physics has advanced rapidly in the last few years. One reason for this is the increasing sophistication of the mathematical models of plasma behavior, and another is the increased speed and memory of the computers which made it reasonable to consider numerical simulation of plasmas.

The behavior of a plasma confined by a magnetic field is simulated by a variety of numerical models. Some models used for short times give detailed knowledge of the plasma on a microscopic scale, while other models used for much longer times compute macroscopic properties of the plasma dynamics. All of these models are under continual development, and in the next few years there will be a substantial increase in the use of numerical models in order to meet the needs of the fusion power program.

Prior to 1973, research in computational plasma physics primarily consisted of calculations with particle codes used to simulate collisionless plasmas. The primary emphasis of these studies is to contribute to the understanding of plasma theory, particularly velocity space instabilities and wave-particle interactions. In order to simulate fusion devices computational plasma physicists are now developing many new macroscopic codes. Equilibrium, magnetohydrodynamic, and Tokamak transport codes have already given much insight into the understanding of experimental results, and have aided in the design of future reactors.

For the purpose of discussing the physical models, and also the numerical methods, it is convenient to consider the following categories of computer codes used to model the physics of fusion devices.

1. Time-dependent magnetohydrodynamics.
2. Plasma transport in a magnetic field.
3. MHD and guiding-center equilibria.

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4. MHD stability of confinement systems.
5. Vlasov and particle models.
6. Multi-species Fokker-Planck codes.
7. Hybrid codes.

In a short paper it is impossible to review all of the above topics, so a brief description of the models will be given followed by a more detailed discussion of two of the categories.

1.1 Time-dependent magnetohydrodynamics

Detailed comparison of experimental data from pulsed high-beta devices with theory, taking due account of experimental complications depends on the application of 2-D and 3-D (two- and three-dimensional) versions of codes analogous to the 1-D Hain-Roberts code [1]. Most of the physical phenomena important here lie in the fast MHD time scale (nanoseconds to microseconds). For Tokamak configurations, the corresponding effects occur on longer time scales-milliseconds; however, the questions of stability of Tokamak discharges toward MHD modes are very important. One example of great interest is the study of the early stages of a Tokamak discharge and the formation and destruction of magnetic surfaces. Here also the relevant times are on the fast MHD time scale.

The article by Roberts and Potter [2] gives a good review of the role of MHD computations and discusses methods for the solution of time-dependent problems. I have also written a review article for Information Processing 71 which considers both time-dependent and time-independent [3] problems. Recent research in this area is described in three review papers [Refs. 4, 5, and 6], and in the paper by Roberts in these proceedings [7].

There are a great variety of MHD codes being developed. Within the fluid theory various degrees of complexity are considered. The so-called ideal MHD is an infinite conductivity approximation. The more realistic models include the transport coefficients, e.g., thermal conductivity and electrical resistivity, and these can be scalars or tensors [1, 8]. Two-dimensional codes are now fairly standard and there are several three-dimensional codes. In some cases perturbation theory is used and the equations are then linearized and Fourier analyzed in one or two coordinates.

The choice of coordinate system varies among these codes. A fixed Eulerian grid is the usual choice, but Lagrangian descriptions, particularly using magnetic flux surfaces as coordinate surfaces [5], are proving useful in certain problems as are particle-in-cell methods [9]. In the work of Brackbill [4] a moving grid is used

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A variety of difference schemes are being used, ranging from fully explicit using a Lax-Wendroff or a leap-frog scheme, to implicit methods employing the ADI scheme or "splitting" (the method of fractional time steps). In a later section of this paper we shall describe the application of implicit difference methods to problems in resistive magnetohydrodynamics.

1.2 Plasma transport in a magnetic field

In order to simulate the transport of a plasma in a magnetic confinement device over most of its lifetime--from tens to hundreds of milliseconds--a set of partial differential equations of the diffusion type must be solved. Typical dependent variables are the number densities and temperatures of each particle species, current densities, and magnetic fields. The transport coefficients such as thermal conductivity, electrical resistivity, and diffusion coefficients are obtained from the best available theories, but the codes also have the capability of easily changing the form of the coefficients in order to develop phenomenological models. In the past years, a considerable effort has been devoted to the numerical solution of these equations for toroidal plasmas which provides an excellent means of comparing theory with experiment. Reviews of these models and their application to Tokamaks are given in Refs. [10-13].

In all of these codes implicit difference methods are used for the solution of the coupled diffusion equations. In the final section of this paper we shall describe the coupling of such a system to a set of kinetic equations in order to model a two-energy component plasma.

1.3 MHD and guiding-center equilibria

It is necessary to develop time-independent codes to support the design and operation of each major fusion experiment. These include codes used to compute and study prospective equilibrium plasma configurations. Experimental devices incorporating the idea of axial symmetry in a torus appear to be capable of plasma confinement for times which are of great interest. One reason for this result is the assurance of equilibria in such devices, as predicted by MHD and guiding-center theories. The computation of these two-dimensional equilibria involves the solution of an elliptic partial differential equation. Iterative methods such as SOR, ADI, and ICCG, and direct methods including cyclic reduction and FFT have all been used in these codes.

Several containment schemes, including mirror devices do not possess an ignorable coordinate. Three dimensional codes [14, 15] have been developed for these cases.

Equilibrium computational methods are reviewed in Refs. [15, 16], and the coupling of an equilibrium code to the Fokker Planck-Transport model is described in the last section.

1.4 MHD stability of confinement systems

Another important class of time-independent code is used to determine the ideal MHD stability of equilibrium configurations by calculating eigenvalues of the linearized perturbed equations. Variational methods are used, and finite elements and Fourier series are used to represent eigenmodes. Reviews of these computations have been given by Grimm et al [17] and Tsyon and Gruber in these proceedings [18].

1.5 Vlasov and particle models

Particle codes are fundamental in that they compute in detail the motion of particles under the influence of their self-consistent electric and magnetic fields, as well as any externally imposed fields. These codes give phase-space distribution functions, fluctuation and wave spectra, and orbits of individual particles. They are ideal for providing detailed information on the growth and saturation of strong instabilities and the effects of turbulence.

Particle codes are usually classified as either "electrostatic" or "electromagnetic". In the first type only the self-consistent electric field is computed via Poisson's equation and the magnetic field is either absent or constant in time. Recent methods and results are reviewed in Ref. [19]. In the last five years there has been a considerable development in electromagnetic codes. They are either relativistic and fully electromagnetic, i.e., the particle equations of motion are relativistic and the electric and magnetic fields are obtained from the full Maxwell equations (wave equations) as in Ref. [20] or they are in the nonradiative limit where the equations are nonrelativistic and displacement currents are neglected as in Ref. [21].

1.6 Multi-species Fokker-Planck codes

In the simulation of magnetically confined plasmas where the ions are not Maxwellian and where a knowledge of the distribution functions is important, kinetic equations must be solved. At number densities and energies typical of mirror machines, the end losses are due primarily to the scattering of charged particles into the loss cones in velocity space by classical Coulomb collisions. The kinetic equation describing this process is the Boltzmann equation with Fokker-Planck collision terms [22]. The use of this equation is not restricted to mirror systems. The heating of plasmas by energetic neutral beams, the thermalization of α -particles in DT plasmas, the study of runaway electrons and ions in Tokamaks, and the performance of two-energy component fusion reactors are other examples where the solution of the Fokker-Planck equation is required [23].

The problem is to solve a nonlinear partial differential equation for the distribution function of each charged species in the plasma, as functions of seven independent variables (three spatial coordinates, three velocity coordinates, and time). Such an

equation, even for a single species, exceeds the capability of any present computer so several simplifying assumptions are therefore required to treat the problem. In the final section of this paper we shall discuss the solution to the Fokker-Planck equations in a two-dimensional velocity space and the coupling to a spatially dependent Tokamak transport code.

1.7 Hybrid codes

There is a need for codes which can best be described as Hybrid Codes; these are codes which combine the good features of fluid codes with the good features of particle codes. The advantage of a particle code is that it contains the most complete treatment of the physics. Its disadvantage also stems from this feature because it is forced to follow the development of the plasma on the fastest time scale and shortest space scale at which significant plasma phenomena occur. These scales are typically much shorter than the time and size scale in fusion devices. The feature of fluid codes which is attractive is that they treat the plasma on a coarser scale and hence need many fewer time steps and spatial points; however, the motions of certain classes of particles are often crucial. It is clear that proper treatment of such phenomena requires an accurate description of the important class of particles. On the other hand, it is possible to treat the rest of the plasma by means of fluid equations.

Another class of hybrid code which is useful is the coupling of a Fokker-Planck code to a plasma transport (diffusion) code which we describe in the last section of this paper.

2. RESISTIVE MAGNETOHYDRODYNAMICS

The existence of sheared or reversed magnetic fields in many controlled fusion devices has led to interest in the tearing mode, one of a class of finite resistivity instabilities which were first studied in detail by Furth, Killeen, and Rosenbluth [24]. In addition, the tearing mode has been studied for several years in connection with the problem of solar flares where it has been advanced as a candidate for the onset of the flare event. Also, there is currently considerable interest in the double-tearing mode which can arise when two singular ($k \cdot B = 0$) surfaces are in close proximity. The resulting interaction can lead to enhanced cross-field diffusion and field annihilation.

Concurrently, the rise in speed and memory size of computers has made feasible the solution of more and more complex problems by numerical, rather than analytical, means. For example, ideal MHD instabilities are now being studied as initial value problems in more than one space dimension. In this way the equations need only be modified for physical, not analytical, reasons, and non-linearities and complex

geometries can be treated.

This section describes the application of this approach to the study of tearing modes. The problem here is more complex than for ideal MHD, as we now have phenomena occurring on two often widely separated time scales: the hydromagnetic time and the resistive diffusion time. Thus, to make the problem computationally realizable, implicit methods should be used. This was first done by Killeen [25] who obtained growth rates by solving the linearized, Fourier decomposed equations in one space dimension. This work was extended by Dibiase [26] who included the effects of compressibility, viscosity and thermal conductivity. In this section we consider the non-linear model which is used to study the growth and saturation of unstable modes found by the linear codes. [27,28]

The equations appropriate for the description to low frequency phenomena in a conducting fluid are the MHD equations

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \frac{nc^2}{4\pi} \nabla \times \underline{B}) \quad (1)$$

$$\rho \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla \cdot \underline{P} + \frac{1}{4\pi} (\nabla \times \underline{B}) \times \underline{B} \quad (2)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v}) \quad (3)$$

$$\begin{aligned} \frac{\partial (\rho \underline{v})}{\partial t} = & -\rho \nabla \cdot \underline{v} - \nabla \cdot (\rho \underline{v} \underline{v} - \kappa \nabla T) \\ & + \eta \left(\frac{c^2}{4\pi} \right) (\nabla \times \underline{B})^2 - \underline{v} \underline{v} : \nabla \underline{v} \end{aligned} \quad (4)$$

along with the equation of state $p = \rho T$. Here, $c = kT/m_i(\gamma-1)$ is the internal energy per unit mass, η is the resistivity, κ is the thermal conductivity, ν is the viscosity, and $\underline{P}$ is the pressure tensor. $\underline{v}$ is the viscous stress tensor.

For computational purposes, it is preferable to recast the equations in conservation form. Equations (1-4) can be combined to yield (in dimensionless variables)

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B} - \frac{\eta}{S} \nabla \times \underline{B}) \quad (5)$$

$$\frac{\partial(\rho \underline{v})}{\partial t} = -\nabla \cdot \left[\rho \underline{v} \underline{v} + \frac{1}{2} (\rho + B^2) \underline{I} - \underline{B} \underline{v} - \frac{\nu}{R} \underline{\nabla} \right] \quad (6)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v}) \quad (7)$$

$$\begin{aligned} \frac{\partial u}{\partial t} = -\nabla \cdot \left[(u + p) \underline{v} + \frac{2\gamma}{S} (\underline{B} \cdot \nabla \underline{B} - \nabla \underline{B} \cdot \underline{B}) \right. \\ \left. + (B^2 \underline{J} - 2 \underline{B} \underline{B}) \cdot \underline{v} - \frac{2\nu}{R} \underline{v} \cdot \underline{\nabla} - \frac{\kappa}{K} \nabla T \right] \end{aligned} \quad (8)$$

which express the conservation of magnetic flux, momentum, mass, and energy. The quantity $u = \rho v^2 + B^2 + p/(\gamma-1)$ is the total energy density of the fluid. The nondimensional numbers R , S , and K appearing in (5) thru (8) are the hydrodynamic, magnetic, and thermal Reynolds numbers, defined as the ratios of the corresponding diffusion times to the hydromagnetic transit time. Equations (5), (6), (7), and (8), along with the equation of state and the definition of u , define a set of 8 equations in 8 unknowns: three components of the magnetic flux density $\underline{B}$, three components of the momentum density $\rho \underline{v}$, the mass density, and the total energy density.

These eight equations together with the appropriate boundary and initial conditions, define an initial value problem which we solve by finite difference techniques on an Eulerian mesh. The spatial differencing is fully conservative on a variably zoned grid with densities defined at grid points and fluxes at half grid points [28]. The temporal differencing is Alternating Direction Implicit. The mixed derivatives are always placed at the "old" time step. This differs from the method used by Killeen and Marx [22] and Lindemuth [6], but it is conservative at each half step.

The resulting difference equations are non-linear, and are solved by iterating the solution several times over a time step. Since the momenta are taken as the dependent variables, all velocities appearing in the difference equations are considered known, and their values at the appropriate time level from the most recent iteration are used. The boundary points are advanced by an implicit algorithm which assures total conservation on the difference mesh.

The above model has been applied to several geometries. For the case of the sheet pinch in cartesian coordinates, the model gives good agreement with previous

analytic [24] and numerical results [29] for the linear regime. We also study the nonlinear saturation of the tearing mode and find that it saturates when the magnetic island width becomes comparable with the width of the singular layer, in agreement with theoretical predictions [30]. In addition we have studied the linear and non-linear properties of the double tearing mode [27]. We find that the interaction of the singular surfaces can lead to linear growth rates which are greatly enhanced over that of the single tearing mode, while its nonlinear evolution can result in magnetic field configurations (see Fig. 1) which are consistent with enhanced particle and thermal transport perpendicular to the zero order magnetic field.

The model has also been applied to the tearing mode in cylindrical geometry - in particular the $m = 0, 1$, and 2 modes in a Reversed Field Pinch [28]. For $m > 0$ we employ a transformation to helical coordinates which allows us to solve the full three dimensional equations under the assumption that helical symmetry is preserved. The cylindrical linear MHD model [25] is used to generate unstable modes which are then followed into the non-linear regime (see Fig. 2).

Fig. 1 Flux surfaces in region between singular surfaces for the double tearing mode

Fig. 2 Helical flux contours for the $m=1$ tearing mode in a cylindrical Reversed Field Pinch

A fully three dimensional version of the model also is running [31]. The spatial differencing techniques are the same as just described and the temporal differencing scheme is the Douglas-Gunn algorithm [32]. Problems have been run on the MFECC 7600 with a $30 \times 30 \times 25$ mesh with portions of the code written in assembly language. A code of this kind clearly requires a class VI computer for its implementation.

3. FOKKER-PLANCK/TRANSPORT MODEL FOR BEAM DRIVEN TOKAMAKS

Neutral beam heated Tokamaks [33] are characterized by a warm Maxwellian background plasma, whose evolution can be described by a set of macroscopic transport equations.

and one or more energetic species which are quite non-Maxwellian, whose evolution should be represented by Fokker-Planck equations. The coupling of these systems is by means of particle and energy sources in the multispecies transport equations and a Maxwellian target plasma in the multi-species Fokker-Planck equations.

The Fokker-Planck/Transport (FPT) Code [34] models the time-dependent behavior of such a system. The model assumes the existence of an arbitrary number of Maxwellian warm ion species which are described by their individual densities $n_a(\rho, t)$ and by a common temperature profile $T_i(\rho, t)$, where ρ is the average radius of a flux surface. The electrons are described by a separate temperature profile $T_e(\rho, t)$ and their density is determined by quasineutrality. The energetic species are represented by velocity space distribution functions $f_b(v, \theta, \rho, t)$, where v is speed and θ is pitch angle. Since the magnetic flux surface shapes will tend to change more slowly than the macroscopic plasma parameters, it is a valid approximation to allow the transport and collision processes to proceed through a sequence of magnetohydrodynamic equilibria. The poloidal flux ψ , which determines the radial coordinate ρ , is thus obtained from Ampere's law. The density and energy of the neutrals population are computed using the Monte Carlo code of Hughes and Post [35].

3.1 Fokker-Planck Equations

We assume that the distribution functions are azimuthally invariant in velocity space (about the direction of the magnetic field). This assumption reduces the velocity space from three to two dimensions. Each energetic species is described by an equation

$$\frac{\partial f_b}{\partial t} = \left(\frac{\partial f_b}{\partial t} \right)_c + H_b - S_{bc} + S_{ab} - C_b f_b + \hat{E}_b \quad (9)$$

Here, H_b is the source profile for species "b"; S_{aD} , $S_{a\alpha}$ and S_{aT} are source or loss terms describing the D-T- α reaction; S_{bc} represents the transfer of (low energy) particles from a hot species to its corresponding background.

The Fokker-Planck collision term for an inverse-square force was derived by Rosenbluth, et al. [36] in the form

$$\Gamma_a \left(\frac{\partial f}{\partial t} \right)_c = - \frac{\partial}{\partial v_i} \left(f_a \frac{\partial h_a}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f_a \frac{\partial^2 g_a}{\partial v_i \partial v_j} \right) \quad (10)$$

where $\Gamma_a = 4\pi Z_a^2 e^4 / m_a^2$ In the present work we write the "Rosenbluth potentials"

$$g_a = \sum_b \left(\frac{Z_b}{Z_a} \right)^2 \ln \Lambda_{ab} \int f_b(v') |v - v'| dv' \quad (11)$$

$$h_a = \sum_b \frac{m_a + m_b}{m_b} \left(\frac{z_b}{z_a} \right)^2 \ln h_{ab} \int f_b(\underline{v}') |\underline{v} - \underline{v}'|^{-1} d\underline{v}' \quad (12)$$

The functions g_a and h_a , defined by Eqs. (11) and (12), can be represented by expansions in Legendre polynomials [23]. Summation is over all species.

Equation (10) in (v, θ) coordinates, written in conservative form, is

$$\frac{1}{\Gamma_a} \left(\frac{\partial f_a}{\partial t} \right)_c = \frac{1}{v^2} \frac{\partial G_a}{\partial v} + \frac{1}{v^2 \sin \theta} \frac{\partial H_a}{\partial \theta} \quad (13)$$

where

$$G_a = A_a f_a + B_a \frac{\partial f_a}{\partial v} + C_a \frac{\partial f_a}{\partial \theta} \quad \text{and} \quad H_a = D_a f_a + E_a \frac{\partial f_a}{\partial v} + F_a \frac{\partial f_a}{\partial \theta}$$

The coefficients which involve derivatives of g_a and h_a are defined in Refs [23,34].

Equation (13) is integrated using the method of splitting, or fractional timesteps.

We first advance

$$\frac{1}{\Gamma_a} \frac{\partial f_a}{\partial t} \approx \frac{1}{v^2} \frac{\partial G_a}{\partial v} \quad (14)$$

using an implicit difference algorithm and then advance

$$\frac{1}{\Gamma_a} \frac{\partial f_a}{\partial t} \approx \frac{1}{v^2 \sin \theta} \frac{\partial H_a}{\partial \theta} \quad (15)$$

in an analogous manner. Eq. (14) is differenced as follows:

$$\begin{aligned} \frac{f_{i,j}^{n+1} - f_{i,j}^n}{\Gamma_a \Delta t} &= \frac{A_{i,j+1}^n (f_{i,j+1}^{n+1} - f_{i,j-1}^{n+1})}{2v_j^2 \Delta v_j} \\ &+ \frac{1}{v_j^2} \left[\frac{B_{i,j+\frac{1}{2}}^n (f_{i,j+\frac{1}{2}}^{n+1} - f_{i,j-\frac{1}{2}}^{n+1})}{\Delta v_{j+\frac{1}{2}}} - \frac{B_{i,j-\frac{1}{2}}^n (f_{i,j}^{n+1} - f_{i,j-1}^{n+1})}{\Delta v_{j-\frac{1}{2}}} \right] \\ &+ \frac{1}{2v_j^2 \Delta v_j} \left[\frac{C_{i,j+1}^n (f_{i+1,j+1}^{n+1} - f_{i-1,j+1}^{n+1})}{2\Delta \theta_j} - \frac{C_{i,j-1}^n (f_{i+1,j-1}^{n+1} - f_{i-1,j-1}^{n+1})}{2\Delta \theta_j} \right] \end{aligned} \quad (16)$$

We see that the terms of mixed second derivative type may not be written fully implicitly if we wish to maintain a tri-diagonal form. Eq. (15) is integrated in a similar manner, with the roles of v and θ reversed. We have also used the ADI algorithm for the solution of the Fokker-Planck equation [22], and we have a version in which Eq. (13) is differenced fully implicitly including the mixed second derivative and the resulting algebraic problem is solved by the ICCG method [37].

3.2 Transport Equations

The spatial independent variable for the transport model is $\rho = (V/2\pi^2 R_0^3)^{1/2}$, where V is the volume within the flux surface and R_0 is the major radius of the torus. We can consider an arbitrary number of warm ion species, each described by an equation of the form

$$\frac{\partial n_a}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \Gamma_a) + \int (S_{bc} + c_b f_b) d\mathbf{x} \cdot \delta(a,b) \quad (17)$$

The term Γ_a is the particle flux for species "a", and depends on the plasma transport model used [34,38]; S_{bc} represents the transfer of low energy particles from a hot species (see Eq. (9)) to its corresponding background; c_b is an inverse charge exchange time.

The electron density is determined by quasi-neutrality and $T_e(\rho, \tau)$ is governed by

$$\frac{\partial}{\partial t} \left(\frac{3}{2} n_e T_e \right) = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho Q_e) - Q_\Delta + \sum_b Q_{eb} - \frac{3}{2} n_e T_e / \hat{\tau}_e + \langle J_\phi E_\phi \rangle \quad (18)$$

Q_e is the electron energy flux [34,38]; Q_Δ represents energy transfer between ions and electrons; Q_{eb} represents heating of electrons by energetic species "b"; $\hat{\tau}_e$ is the electron energy confinement time, and $\langle J_\phi E_\phi \rangle$ is the Ohmic heating term, where $\langle \rangle$ denotes a flux surface average.

The warm ions are all assumed to be at the same temperature [38], and their temperature is determined by

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{3}{2} \sum_a n_a T_i \right) = & -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho Q_i) + Q_\Delta + \hat{Q} + \sum_{a,b} Q_{ab} \\ & + \frac{3}{2} T_i \sum_{a,b} \int (S_{bc} + c_b f_b) d\mathbf{x} \delta(a,b) \end{aligned} \quad (19)$$

Q_i is the ion energy flux; Q_Δ and $\hat{Q}$ represent energy transfer between particles; Q_{ab} is the heating of species "a" by energetic species "b", and the last term represents the energy transferred from the hot species to the warm plasma because

of particle transfer from the energetic species to the plasma.

The form of the flux terms is [34]

$$\Gamma_a = \sum_b D_{ab}^d \frac{\partial n_b}{\partial \rho} + D_a^i \frac{\partial T_i}{\partial \rho} + D_a^e \frac{\partial T_e}{\partial \rho} \quad (20)$$

$$Q_i = \sum_b L_b^d \frac{\partial n_b}{\partial \rho} + L_i^i \frac{\partial T_i}{\partial \rho} + L_i^e \frac{\partial T_e}{\partial \rho} \quad (21)$$

$$Q_e = \sum_b M_b^d \frac{\partial n_b}{\partial \rho} + M^i \frac{\partial T_i}{\partial \rho} + M^e \frac{\partial T_e}{\partial \rho} \quad (22)$$

where "b" is summed over warm ion species. If we substitute Eqs. (20)-(22) into Eqs. (17)-(19) we see that we have a coupled system of one-dimensional diffusion equations which are solved by standard implicit difference techniques which are discussed in Refs. [10-13].

In addition to the transport equations (17)-(19) we have the equation for the poloidal magnetic field which is changing in time

$$\frac{\partial B_p}{\partial t} = \frac{c}{R_0} \frac{\partial}{\partial \rho} \langle R E_\phi \rangle \quad (23)$$

E_ϕ is the flux averaged toroidal electric field. Eq. (23) is solved with Eqs. (17)-(19) as part of the transport section of the code.

3.3 Equilibrium Calculation

The Fokker-Planck/Transport code described in Ref. [34] models beam driven Tokamaks with circular cross-section, i.e., where the magnetic flux surfaces are concentric circles. The equations presented in the two preceding sections apply to that case with ρ replaced by the cylindrical coordinate r . A new version [39] of the code exists, called FPT II, which describes the important class of non-circular Tokamak plasmas where the flux surfaces are elongated in the vertical direction. In this case we solve a time-independent equilibria problem periodically during the transport calculation.

Since the plasma contains an energetic component which is highly directed (beams) we use the equation for toroidal equilibria with anisotropic pressure [3] in cylindrical coordinates (R, Z)

Fig. 3 Distribution function, $f(n, v, e)$, versus v and e at $\beta = 0$ for hot deuterium (counterstreaming beams) and the sum of hot and cold deuterium. Contour plots of the same distribution function in (v, e) space.

$$\frac{\partial^2 \psi}{\partial Z^2} + R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{1}{\sigma} \left(\frac{\partial \sigma}{\partial R} \frac{\partial \psi}{\partial R} + \frac{\partial \sigma}{\partial Z} \frac{\partial \psi}{\partial Z} \right) = - \frac{1}{\sigma} \frac{\partial g}{\partial \psi} - \frac{R^2}{\sigma} \frac{\partial p_{ii}}{\partial \psi} \quad (24)$$

where $\psi(R, Z)$ is the magnetic flux function, and

$$B_R = - \frac{1}{R} \frac{\partial \psi}{\partial Z} \text{ and } B_Z = \frac{1}{R} \frac{\partial \psi}{\partial R} \quad (25)$$

are the components of the poloidal field. We have

$$g = \frac{1}{2} \sigma^2 R^2 B_\phi^2; \quad \sigma = 1 + (p_i - p_{ii})/B^2$$

where p_i and p_{ii} are determined from the Fokker-Planck and transport calculations. The results of the equilibrium calculation are a new B_p , $\langle J_\phi E_\phi \rangle$, and $\langle RE_\phi \rangle$ to be used in the transport calculation.

Eq. (24) is an elliptic partial differential equation and is solved as a difference equation on an R-Z grid by the ICCG iterative method [39].

3.4 Applications

As mentioned earlier the FPT code is used to model the behavior of beam driven Tokamaks [34]. The code is particularly suitable for studying reactor operation with counterstreaming ion beams (CIT). In this work a simplified transport model is used [40]. This model has also been applied to counterstreaming deuterium beams in the PDX Tokamak [41]. Fig. 3 shows the distribution function of the hot deuterium plasma formed and also the sum of the hot component and the warm background plasma distribution functions.

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