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INTEGRAL FORMULATION FOR THE
TWO-DIMENSIONAL SPATIAL
STRUCTURE OF DRIFT AND
TRAPPED-ELECTRON MODES

BY

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PLASMA PHYSICS
LABORATORY

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ABSTRACT

Previous calculations of the linear growth rate and two-dimensional spatial structure of trapped-electron modes are made more general and more accurate in several ways. First, an integral equation formulation of the eigenmode problem allows arbitrary values of $k_r \rho_i$ (where k_r is the radial wavenumber and ρ_i is the ion gyroradius) to be treated. Second, the ion response is generalized so that arbitrary ratios of the ion magnetic drift frequency to the mode frequency are allowed. Finally, the electron and ion collision operators have been improved to allow consideration of the plateau and Pfirsch-Schlüter regimes in addition to the usual banana regime. It is therefore possible to follow the transition in toroidal geometry from the trapped-electron mode to the collisionless and collisional drift modes. The method used involves expansion of the perturbed electrostatic potential in complete sets of radial and poloidal basis functions to convert the quasineutrality integral equation into a matrix equation.

I. INTRODUCTION

It has been emphasized recently in several papers¹⁻⁴ that a proper evaluation of the linear growth rate and the two-dimensional spatial structure of drift and trapped-electron modes in toroidal geometry requires a two-dimensional analysis. For example, terms_{ai} associated with trapped-electron contributions¹ and with ion magnetic drifts³ can lead to a "ballooning" structure along the magnetic field lines. As discussed in some detail in Ref. 5, when this ballooning is sufficiently strong, the stabilizing effect of magnetic shear can be substantially reduced and possibly eliminated.

A general approach to the two-dimensional problem introduced in earlier work^{1,2} involves the expansion of the perturbed electrostatic potential in complete sets of radial and poloidal basis functions. This effectively converts the quasineutrality integral equation into a matrix equation. In the limit $k_r \rho_i < 1$ (where k_r is the radial wavenumber and ρ_i is the ion gyroradius), the radial dependence is primarily governed by a radial differential operator. Hence, an alternate approach to the method just described is to directly solve a large system of poloidally coupled radial differential equations.⁴ Results for the eigenfunctions and eigenvalues obtained using these different approaches were found to be in reasonable agreement when equal numbers of poloidal harmonics were retained. However, it has also been emphasized that, to determine the actual two-dimensional mode structure, it is necessary to take into account the radial variation of the equilibrium quantities. In fact, in the absence of such variations,

it is possible to construct a very broad mode extending over the entire plasma if a very large number of poloidal harmonics are retained.⁴ Nevertheless, provided that outgoing energy boundary conditions (far from the radial region of interest) on the mode are appropriate, accurate estimates for the eigenvalues can be obtained with a limited number of poloidal harmonics.^{1,2,4}

At shorter wavelengths ($k_r \rho_i \geq 1$), the integrodifferential equation formulation of the two-dimensional problem must be replaced by the more general purely integral equation formulation. This procedure was introduced briefly in Ref. 2 and will be presented in detail in this paper. Also, in the short-wavelength regime, the ion magnetic drift frequency can no longer be assumed to be small and expandable with respect to the mode frequency for typical parameters. For both the two-dimensional treatment in the present paper and a recent radially local calculation,⁶ it has been found that using the unexpanded form of the ion response leads to enhanced stabilization at short wavelengths ($k_i \rho_i > 1$, where $k_i = (k_r^2 + k_\theta^2)^{1/2}$ and k_θ is the poloidal wavenumber).

In order to extend the two-dimensional calculation into higher collisionality regimes than the banana regime considered in Ref. 1, an energy and pitch-angle dependent Krook operator has been introduced to account for electron collisions. Although this model operator gives only a somewhat crude approximation to the actual behavior of the Fokker-Planck operator, the general features of the dominant growth rates as a function of collisionality are qualitatively reproduced. In any case, the primary intent

here is to examine the two-dimensional modifications to the conventional one-dimensional theory of drift and trapped-electron modes.

The notation defined here is the same as that used in Ref. 1; but the definitions are given here in full in order to make this paper independently readable. The remainder of the paper is organized as follows: In Sec. II, the notation and the assumptions made about the equilibrium are explained, the form used for the perturbed electrostatic potential is discussed, and the final forms of the perturbed electron and ion densities are given. The quasi-neutrality integral equation is then presented in Sec. III, and the method of solution is discussed. In Sec. IV, numerical results are given and discussed, and in Sec. V, conclusions are given. The derivation for the perturbed electron density is presented in Appendix A, and the derivation for the perturbed ion density is given in Appendix B. In Appendix C, computationally convenient forms for the elements of the matrix form of the mode equation are derived.

II. PERTURBED ELECTRON AND ION DENSITIES

A. Notation and Equilibrium

In the coordinate system employed, as in Ref. 1, ζ denotes the toroidal angle, θ the poloidal angle, r the minor radius of a magnetic surface, and R_0 the major radius of the magnetic axis, with $\theta = 0$ at the outside of the torus. The form for the

magnetic field corresponding to the assumption of concentric, circular cross-section magnetic surfaces is

$$\underline{B} = [B_0 \underline{e}_\zeta + B_\theta^0(r) \underline{e}_\theta] / h(\theta) , \quad (1)$$

where $h(\theta) \equiv 1 + \epsilon_0 \cos \theta$, and the inverse aspect ratio is $\epsilon_0 \equiv r/R_0 \ll 1$. Along a magnetic field line

$$d\ell_{||} / B = R d\zeta / B_\zeta = r d\theta / B_\theta , \quad (2)$$

where $\ell_{||}$ indicates length along a field line and $R = R_0 h(\theta)$, so that the "safety factor" can be expressed as

$$q(r) \equiv r B_0 / R_0 B_\theta^0 \approx d\zeta / d\theta , \quad (3)$$

where $d\zeta/d\theta$ is evaluated along a field line. Since $B_\theta^0 \ll B_0$, q is of order unity and $B \equiv |\underline{B}| \approx B_0 / h(\theta)$. In velocity space two different coordinate systems will be used; namely, $v_{||}$ and $v_\perp$, indicating velocity components parallel and perpendicular to the equilibrium magnetic field, as well as the quantities $\epsilon \equiv (m_j/2) (v_{||}^2 + v_\perp^2)$, the kinetic energy, and $\Lambda \equiv \mu B_0 / \epsilon$, the dimensionless pitch angle variable, where $\mu \equiv (m_j/2) v_\perp^2 / B$ is the magnetic moment, m_j is the particle mass, and $j = e, i$ denote electrons and ions, respectively. The ions have charge $e_i = +|e|$ and the electrons have charge $e_e = -|e|$. The parallel velocity has the form

$$v_{||} = \sigma_{||} [(2/m_j)(\epsilon - \mu B)]^{1/2} = \sigma_{||} v [1 - \Lambda/h(\theta)]^{1/2}, \quad (4)$$

where $\sigma_{||} = \text{sgn}(v_{||})$. Circulating particles correspond to $0 \leq \Lambda < 1 - \epsilon_0 \equiv \Lambda_c$ and trapped particles to $1 - \epsilon_0 < \Lambda \leq 1 + \epsilon_0$. The trapped particle turning points are $\pm \theta_0 = \pm \arccos [(\Lambda - 1)/\epsilon_0]$.

The equilibrium electric field is assumed equal to zero.

Then the single particle constants of the motion are ϵ , Λ , and the toroidal canonical angular momentum

$$P_\zeta \approx m_j R (v_\zeta - \Omega_{\theta j} r) \approx m_j R_0 (v_{||} - \Omega_{\theta j}^0 r) , \quad (5)$$

where $\Omega_{\theta j} \equiv e_j B_\theta / m_j c$, and $\Omega_{\theta j}^0 \equiv e_j B_\theta^0 / m_j c$.

The bounce period τ_b for a trapped particle is given by

$$\tau_b = \oint dt = q R_0 \oint \frac{d\theta}{v_{||}(\theta)} = 2q R_0 \int_{-\theta_0}^{\theta_0} \frac{d\theta}{|v_{||}(\theta)|} \quad (6)$$

and the transit period for a circulating particle by

$$\tau_t = q R_0 \int_{-\pi}^{\pi} d\theta / v_{||}(\theta) . \quad (7)$$

The corresponding bounce and transit frequencies are defined as

$\omega_b \equiv 2\pi/\tau_t$ and $\omega_t \equiv 2\pi/\tau_b$ respectively. Note that $\tau_b, \omega_b > 0$, while $\text{sgn}(\tau_t, \omega_t) = \text{sgn}(v_{||}) = \sigma_{||}$, by convention, and that τ_b is equivalent to $2|\tau_t|$ at the boundary between trapped and circulating particles.

The equilibrium distribution function for the ions and the trapped electrons is taken to be $f_j^{(0)} = f_{mj} (1 + \hat{f}_j)$, where $\hat{f}_j \ll 1$,

$$f_{mj} \equiv \frac{n(r)}{[2\pi T_j(r)/m_j]^{3/2}} \exp[-\epsilon/T_j(r)] , \quad (8)$$

and

$$\hat{f}_j \equiv -\frac{v_\zeta}{\Omega_{\theta j}} \left[\frac{1}{n} \frac{dn}{dr} + \frac{1}{T_j} \frac{dT_j}{dr} \left(\frac{\epsilon}{T_j} - \frac{3}{2} \right) \right] , \quad (9)$$

where T_j is the temperature in energy units. For the circulating electrons, in order to investigate current-driven modes, the Maxwellian will be taken to be shifted by the constant velocity u_0 . Thus, for circulating electrons $f_e^{(0)} = f_{me}^C (1 + \hat{f}_e)$, with $\hat{f}_e$ given by Eq. (9), and

$$f_{me}^C = \frac{n(r)}{[2\pi T_e(r)/m_e]^{3/2}} \exp\{-[(v_{||} - u_0)^2 + v_\perp^2] (m_e/2T_e)\} \\ \approx \left(1 + \frac{2u_0 v_{||}}{v_e^2} \right) f_{me} \quad (10)$$

where $v_j \equiv (2T_j/m_j)^{1/2}$, through first order in u_0/v_e for $u_0/v_e \ll 1$. However, in order to make the equilibrium distribution function a function of the constants of the motion, $v_{||}$ will be replaced by its time average along a circulating electron orbit $v_{||}^{(0)} = q R_0 \omega_t$. A superscript zero will always mean that a quantity is time averaged along a single particle orbit. This

replacement was shown in Ref. 7 to give a good approximation to the actual neoclassical current term in the neoclassical distribution function calculated in Ref. 8. The other "neoclassical" corrections for the circulating particles are ignored. These corrections were shown in Ref. 9 to have a negligible effect on the results.

In terms of the variables defined above, the integral over velocity space becomes

$$\int d^3v = 2\pi \int_0^\infty dv_\perp v_\perp \int_{-\infty}^\infty dv_\parallel = \frac{\pi}{2} \left(\frac{2}{m_j}\right)^{3/2} \sum_{\sigma_\parallel = \pm 1} \int_0^\infty d\epsilon \epsilon^{1/2}$$

$$\int_0^{h(\theta)} d\Lambda \{ h(\theta) [1 - \Lambda/h(\theta)]^{1/2} \}^{-1} \quad (11)$$

at a fixed value of θ . The notation

$$\langle \dots \rangle_j \equiv \frac{1}{n} \int d^3v f_{mj} \dots \quad (12)$$

will be used. The integration will sometimes be over only the trapped (or circulating) particles, as will be clear from the context. Note that

$$\langle |\omega_t| \rangle_j \approx \omega_{tj} \equiv v_j / q R_0, \quad (13)$$

and

$$\langle \omega_b \rangle_j \approx (\epsilon_0/2)^{1/2} \bar{\omega}_{tj}. \quad (14)$$

B. Mode Properties

Considering $\beta = 8\pi n(T_e + T_i)/B_0^2 \ll 1$, the electrostatic approximation can be made for the perturbed electric field

$$\tilde{E}^{(1)} = -\nabla \tilde{\phi} . \quad (15)$$

By virtue of axisymmetry, the perturbed electrostatic potential for the linear stage of the mode can be written as

$$\tilde{\phi}(r, \theta, \zeta, t) = \tilde{\phi}_\ell(\theta, r) \exp(-i\omega t + i\ell \zeta) , \quad (16)$$

where ℓ , the toroidal mode number, is an integer. It is convenient to choose a "reference" mode-rational surface at $r = r_0$ in the radial region of interest, around which the mode is to be localized, such that $q(r_0) = m^0/\ell$. The "slow" and "fast" dependence can then be separated without loss of generality, i.e.,

$$\tilde{\phi}(r, \theta, \zeta, t) = \tilde{\phi}(\theta, r) \exp(-i\omega t + i\ell \zeta - im^0 \theta) , \quad (17)$$

where m^0 , the poloidal mode number, is an integer, and $\tilde{\phi}(\theta, r)$ must be periodic in θ with period 2π , and $|(\partial/\partial\theta)\ln \tilde{\phi}| \ll m^0$, and $k_\theta \equiv -m^0/r_0$.

The radial distance from r_0 in units of the spacing of mode-rational surfaces is now defined by the new radial variable

$$S(r) \equiv \ell q(r) - m^0 = \ell[q(r) - q(r_0)] , \quad (18)$$

with $S(r_0) = 0$. Note that $S(r) \approx (r - r_0)/\Delta r_s$, where $\Delta r_s \equiv [\ell(dq/dr)]^{-1} \equiv (\ell q')^{-1}$ is the spacing of mode rational surfaces, and $d^2 q/dr^2$ is neglected. In terms of $S(r)$,

$$\tilde{\Phi} = [\tilde{\phi}(\theta, S) \exp(iS\theta)] \exp\{i\ell[\zeta - q(r)\theta] - i\omega t\}. \quad (19)$$

Since $[\zeta - q(r)\theta]$ is approximately constant along a field line, the variation of $\tilde{\Phi}$ along a field line is given by $[\tilde{\phi}(\theta, S) \exp(iS\theta)]$. In particular,

$$\begin{aligned} \frac{\partial}{\partial \ell_{||}} \tilde{\Phi} &= \frac{1}{qR_0} \exp\{i\ell[\zeta - q(r)\theta] - i\omega t\} \frac{\partial}{\partial \theta} [\tilde{\phi}(\theta, S) \exp(iS\theta)] \\ &= \frac{1}{qR_0} \exp\{i\ell[\zeta - q(r)\theta] - i\omega t\} \exp(iS\theta) \left(\frac{\partial}{\partial \theta} + iS \right) \tilde{\phi}(\theta, S) \end{aligned} \quad (20)$$

to lowest order in ε_0 .

C. Perturbed Electron Density

A form of the perturbed electron density response is needed which will be valid in the limits $|\Omega_e| \gg \omega_b \sim \omega$ and $k_{\perp} \rho_e \ll 1$, which are the usual parameter ranges of interest for dealing with low-frequency drift modes. Here, $\Omega_e \equiv -|e|B_0/m_e c$, $\rho_e \equiv v_e/|\Omega_e|$, and $k_{\perp}$ is the wavenumber perpendicular to the magnetic field. To obtain the perturbed electron distribution function, the linearized drift kinetic equation is solved by means of an expansion in bounce or transit frequency harmonics and an integration along

unperturbed particle (guiding center) orbits; this procedure was carried out in detail in Ref. 1.

To account for the effects of collisions, an energy and pitch angle dependent Krook model collision operator has been employed, with a number conserving term, specifically

$$C_e = -v_f(\epsilon, \Lambda) [f_e^{(1)} + e\tilde{\phi}(\partial f_{me}/\partial \epsilon)] + v_f^h \frac{f_{me}}{n} [n_e^{(1)} - (e\tilde{\phi}/T_e)n] , \quad (21)$$

where $f_e^{(1)}$ is the perturbed electron distribution function, $n_e^{(1)} \equiv \int d^3v f_e^{(1)}$,

$$v_f(\epsilon, \Lambda) \equiv \{v_f^l + v_f^h \frac{v_e^*}{X^3} [1 + H(X)]\} / \{1 + \frac{v_e^*}{X^3} [1 + H(X)]\} , \quad (22)$$

$$v_f^l \equiv \frac{v_{ei}}{X^3} \frac{1 + H(X)}{|(1 - \epsilon_0) - \Lambda|} \quad (23)$$

$$v_f^h \equiv 1.08 v_{ei} , \quad (24)$$

$$H(X) \equiv [\pi^{1/2} X]^{-1} \exp(-X^2) + (1 - \frac{1}{2X^2}) \text{erf}(X) \quad (25)$$

erf is the usual error function, $X \equiv v/v_e = [\epsilon/T_e]^{1/2}$, $v_{ei} \equiv 4\pi n e^2 \lambda / [(2T_e)^{3/2} m_e^{1/2}]$, and $v_e^* \equiv v_{ei} / [\epsilon_0^{3/2} \bar{\omega}_{te}]$. Note that this $v_{ei} = (3\pi^{1/2}/4)/\tau_e$, where τ_e is the standard collision time of Braginskii.¹⁰

In the low ν_e^* limit of ν_f , namely ν_f^l , the factor $|(1 - \epsilon_0) - \Lambda|^{-1}$ provides the well-known enhancement of the collision frequency which simulates the effect of the second-derivative operator in pitch angle in a more exact collision operator, such as a Lorentz operator, when acting on the localization of the perturbed distribution function in the range of pitch angle corresponding to trapped and barely circulating particles. This factor also displays the sort of divergence at the boundary between trapped and circulating particles which obtains for a Lorentz operator.¹¹ The term $H(X)$ accounts for electron-electron collisions¹¹ while the unity term accounts for electron-ion collisions.

In the high- ν_e^* limit of ν_f , namely ν_f^h , no such localization in the perturbed distribution function can develop, so there is no enhancement of the collision frequency. Also in this limit, the number conserving term is needed to recover the standard result for the collisional (or dissipative) drift mode,¹² while the term has negligible effect in the low- ν_e^* limit. The energy dependence of ν_f is suppressed in the high- ν_e^* limit so that the number conserving term will not involve any moments of $f_e^{(1)}$ other than $n_e^{(1)}$. However, the correction factor 1.08 is included in ν_f^h in order to recover exactly the collisional drift mode growth rate derived in Ref. 12 from fluid equations in the absence of equilibrium temperature gradients, and this correction factor thus partially compensates for the lack of energy dependence. The total collision frequency ν_f makes

a transition between these limits v_f^l and v_f^h which is local in (ϵ/T_e) . This operator is expected to provide at least qualitatively correct results over a wide range of collision frequencies.

The form of the perturbed electron density which will be employed is derived in Appendix A and is given by $n_e^{(1)} = \tilde{n}_e \exp(-i\omega t + i l \zeta - i m^0 \theta)$, where

$$\begin{aligned} \tilde{n}_e = & \frac{en}{T_e} \left[\tilde{\phi}(\theta, S) - \left[\frac{1}{n} \int d^3v f_{me} \left[1 + \frac{2u_0 v_{||}^{(0)}}{v_e^2} H(\Lambda_c - \Lambda) \right] \right. \right. \\ & \times \left. \left\{ \omega \left[1 - \frac{u_0}{v_{||}^{(0)}} H(\Lambda_c - \Lambda) \right] - \omega_{*e}^T \right\} \sum_p \right. \\ & \times \left. \frac{\tilde{\phi}^{(p)} \exp[ip\omega_{b,t} \hat{t}(\theta)]}{\omega + i v_f - \omega_{de}^{(0)} - [p + SH(\Lambda_c - \Lambda)] \omega_{b,t}} \left. \left[1 + \frac{i v_f^h}{k_{||} v_e} z \left(\frac{\omega - k_{||} u_0 + i v_f^h}{k_{||} v_e} \right) \right]^{-1} \right] \right], \end{aligned} \quad (26)$$

$H(\Lambda_c - \Lambda)$ is the Heaviside step function which is one for circulating particles and zero for trapped particles, $\omega_{*e}^T \equiv \omega_{*e} \times \left[1 + \eta_e \{ [\epsilon - m_e u_0 v_{||}^{(0)}] H(\Lambda_c - \Lambda) \} / T_e - 3/2 \right]$, $\omega_{*e} \equiv [m^0 c T_e / (|e| r_0 B_0)] d \ln n / dr$, $\eta_e \equiv (d \ln T_e / dr) / (d \ln n / dr)$, the summation is over all integer values of p , $\omega_{b,t}$ means ω_b for trapped particles and ω_t for circulating particles, $\omega_{de}^{(0)}$ is the electron time-average magnetic drift frequency whose detailed form is given in Appendix A, $\tilde{\phi}^{(p)}$ is the coefficient function of the p -th bounce or transit frequency harmonic whose detailed

form is given in Appendices A and B, the parametric time function $\hat{t}(\theta)$ is defined in Appendix A, $k_{||}$ is a constant which will be taken generally as $k_{||} = 1/qR_0$, and Z is the usual plasma dispersion function defined in Appendix A. The factor $\{1 + i[v_f^h/(k_{||}v_e)]Z\}^{-1}$ from the electron collision operator number-conserving term has been evaluated approximately in a slab limit, for reasons given in Appendix A. This form in Eq. (26) for the electron response is valid for $\omega \geq \langle \omega_b \rangle_e$ as well as in the usual limit $\omega < \langle \omega_b \rangle_e$.

D. Perturbed Ion Density

The perturbed ion density response $\tilde{n}_i$ is needed for arbitrary $k_r \rho_i$ and for $\omega \sim \bar{\omega}_{di} \gg \bar{\omega}_{ti}$, where $\bar{\omega}_{di}$ is the average ion magnetic drift frequency. In Ref. 1 $\tilde{n}_i$ is obtained for $\omega \gg \bar{\omega}_{di} \sim \bar{\omega}_{ti}$ and for $k_r \rho_i \ll 1$ by expanding in $k_r \rho_i$ and then regarding k_r as a differential operator $-i(\partial/\partial r)$. The result of Ref. 1 may be generalized by instead regarding k_r as a Fourier transform variable; that is, the perturbed potential as a function of r must be explicitly Fourier transformed, the perturbed density response operator for a single pure Fourier harmonic $\exp[ik_r(r - r_0)]$ must be applied, and finally the result must be inverse Fourier transformed. Note that this is not a WKB approximation. The results for this integral formulation were compared numerically with results from the differential formulation of Ref. 1, and the differences were found to be of the order of the first omitted term $b_{ir}^3 \equiv (\rho_i^2 k_r^2/2)^3$.

in the differential formulation, if k_r is taken to be of order k_θ , for $k_\theta \rho_i < 1$. For $k_\theta \rho_i \gtrsim 1$ the differential formulation gives meaningless results. The result of Ref. 1 for $\omega \gg \bar{\omega}_{di}$ may be generalized to $\omega \sim \bar{\omega}_{di}$ by expanding to all orders in $\bar{\omega}_{di}/\omega$ and then resumming the series. Also, effects of finite ion radial excursions (banana widths), which were omitted in Ref. 1, will be retained.

The form to be used for the perturbed ion density response is derived in Appendix B. It is $n_i^{(1)} = \tilde{n}_i \exp\{-i\omega t + i l \zeta - i m^0 \theta\}$, where $\tilde{n}_i = \tilde{n}_i^{nr} + \tilde{n}_i^c + \tilde{n}_i^r$. The "nonresonant" response $\tilde{n}_i^{nr}$, which actually includes the ion magnetic drift resonances but not the circulating ion transit resonances, is

$$\begin{aligned} \tilde{n}_i^{nr} = & -\frac{en}{T_i} \{ \tilde{\phi}(\theta, S) - \frac{1}{n} \int d^3v f_{mi}(\omega - \omega_{*i}^T) \\ & \times \left[\frac{1}{\omega - \omega_{di}(1)} - \frac{\hat{v}_i^2 \bar{\omega}_{ti}^{-2}}{[\omega - \omega_{di}(1)]} \left(\frac{\partial}{\partial \theta} + iS \right) \frac{1}{\omega - \omega_{di}(1)} \right. \\ & \times \left(\frac{\partial}{\partial \theta} + iS \right) \frac{1}{\omega - \omega_{di}(1)} + i \left(\frac{\bar{\omega}_{ti} \hat{v}_i^2 q \rho_i}{2 \Delta r_s} \right) \frac{\sin \theta}{[\omega - \omega_{di}(1)]^2} \frac{\partial}{\partial S} \\ & \left. - \left(\frac{\bar{\omega}_{ti} \hat{v}_i^2 q \rho_i}{2 \Delta r_s} \right)^2 \frac{\sin^2 \theta}{[\omega - \omega_{di}(1)]^3} \frac{\partial^2}{\partial S^2} \right] (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_r \\ & \times \exp[ik_r(r - r_0)] J_0^2[\rho_i \hat{v}_i(k_\theta^2 + k_r^2)^{1/2}] \\ & \times (2\pi)^{-1/2} \int_{-\infty}^{\infty} d(r - r_0) \exp[-ik_r(r - r_0)] \tilde{\phi}[\theta, S(r)] \} , \quad (27) \end{aligned}$$

where $\omega_{*i}^T = \omega_{*i} \{1 + \eta_i [\hat{v}_{||}^2 + \hat{v}_\perp^2 - 3/2]\}$, $\omega_{*i} = - (T_i/T_e) \omega_{*e}$,
 $\eta_i = (d \ln T_i / dr) / (d \ln n / dr)$, $\hat{v}_{||} \equiv v_{||} / v_i$, $\hat{v}_\perp \equiv v_\perp / v_i$,
 $\omega_{di} = \bar{\omega}_{di} (\hat{v}_\perp^2 + 2\hat{v}_{||}^2) \cos \theta$, $\bar{\omega}_{di} = (r_n / R_0) \omega_{*i}$, $r_n = -[d \ln n / dr]^{-1}$,
 $\rho_i \equiv v_i / \Omega_i$, $\Omega_i \equiv |e| B_0 / m_i c$, and J_0 is the Bessel function of the first kind.

The part of the ion response due to ion-ion collisions $\tilde{n}_i^C$ is

$$\begin{aligned} \tilde{n}_i^C = & - \frac{en}{T_i} (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_r \exp[ik_r(r - r_0)] \\ & \left[H(b_i - 1) i \frac{v_{ii}}{\omega^2} \frac{3(\pi+1)}{8\pi^{1/2}} [\omega - \omega_{*i} (1 - \frac{3\pi+2}{4\pi+4} \eta_i)] b_i^{1/2} \right. \\ & + H(1 - b_i) i \frac{v_{ii}}{\omega^2} \frac{7}{10} (1 + 0.2517 b_i) \{\omega - \omega_{*i} [1 \\ & - \frac{3}{28} (1 + 5.4366 b_i) \eta_i]\} b_i^2 \left. \right] (2\pi)^{-1/2} \int_{-\infty}^{\infty} d(r - r_0) \\ & \exp[-ik_r(r - r_0)] \phi[\theta, S(r)] , \end{aligned} \quad (28)$$

where $v_{ii} = v_{ei} (T_e/T_i)^{3/2} (m_e/2m_i)^{1/2}$, $b_i \equiv \rho_i^2 (k_\theta^2 + k_r^2)/2$
 $\equiv b_{i\theta} + b_{ir}$, and H is the Heaviside step function.

The "resonant" ion response $\tilde{n}_i^r$, which actually includes only the circulating ion transit resonances, is

$$\tilde{n}_i^r = - \frac{|e|n}{T_i} i \frac{\pi^{1/2}}{h(\theta)} \sum_{\sigma_{||}} \int_0^{1-\epsilon_0} d\Lambda [1 - \Lambda/h(\theta)]^{-1/2} \sum_{p=-\infty}^{\infty} \sum_{j''=1}^2 \tilde{\phi}^{(p)} \times \exp[ip\omega_t \hat{t}(\theta)] |(p + S)\hat{\omega}_t - 2\hat{S}^{(\sim)}\hat{\omega}_t x_{j''}|^{-1} x_{j''}^2 \exp(-x_{j''}^2) \quad (29)$$

$$\times \{\omega - \omega_{*i} [1 + \eta_i (x_{j''}^2 - 3/2)]\} J_0^2 \{ [2\Lambda b_{i\theta} (1 + \bar{k}_r^2/k_{\theta}^2)]^{1/2} x_{j''} \} ,$$

where the explicit form of the coefficient function $\tilde{\phi}^{(p)}$ of the p-th transit harmonic is given in Appendices B and C, the parametric time function $\hat{t}(\theta)$ is defined in Appendix A, $\hat{\omega}_t$ is ω_t with the energy dependence $x \equiv (\epsilon/T_i)^{1/2}$ factored out, $\hat{S}^{(\sim)}$ is defined in Appendix B, the two resonant values $x_{j''}$ of x are given in Appendix B, and $\bar{k}_r$ is a rough average value of k_r .

III. INTEGRODIFFERENTIAL EQUATION AND SOLUTION METHOD

Since drift and trapped-electron modes have wavelengths long compared with the Debye length, the appropriate normal mode equation is just the quasineutrality condition $\sum_j e_j \tilde{n}_j = 0$, or $\tilde{n}_e = \tilde{n}_i = \tilde{n}_i^{nr} + \tilde{n}_i^c + \tilde{n}_i^r$, with $\tilde{n}_e$, $\tilde{n}_i^{nr}$, $\tilde{n}_i^c$, and $\tilde{n}_i^r$ given by Eqs. (26), (27), (28), and (29), respectively. This general integral equation contains information about several different length scales, specifically about variation over the length scale of the spacing of mode rational surfaces Δr_s and over the

equilibrium length scale r_n . For realistic parameters for the mode with $(q'r/q) \sim 1$, $\Delta r_s \ll r_n$. This fact may be exploited by making a multiple radial length scale expansion of the quasi-neutrality integral equation in the parameter $\Delta r_s/r_n$ and treating only the lowest order equation in the hierarchy of equations that results. This is equivalent to taking all of the radial equilibrium gradients to be constant, so that the effects of the real variation of n , T_i , T_e , and q over the tokamak are simplified. With this approximation, only the explicit S dependence in $\tilde{n}_e$ and $\tilde{n}_i$ is to be considered in solving the integral equation.

The integral equation $\tilde{n}_e = \tilde{n}_i$ may be written as

$$L\tilde{\phi} = L(\omega)\tilde{\phi} \equiv [T_e/(|e|n)](\tilde{n}_e - \tilde{n}_i) = 0, \quad (30)$$

where L is a complex, linear, non-Hermitian integral operator. In order to solve Eq. (30), $\tilde{\phi}(\theta, S)$ is expanded in complete sets of basis functions in both the S and θ directions, thus converting the integral equation into a matrix equation. Specifically,

$$\tilde{\phi}(\theta, S) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \tilde{\phi}_{jn} g_j(\theta) h_n(S). \quad (31)$$

Then, using the decomposition Eq. (31) and applying the operator

$$\int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} dS g_j^*(\theta) h_n(S)$$

to Eq. (30) yields the desired matrix equation

$$\sum_{j,n} L_{jj',nn'} \tilde{\phi}_{jn} = 0, \quad (32)$$

with

$$L_{jj',nn'} = \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} dS [g_{j'}^*(\theta) h_n(S)] L[g_j(\theta) h_n(S)] , \quad (33)$$

as is shown in Ref. 1 .

Equation (32) is not in the form of a standard eigenvalue-eigenvector problem, but it can be converted into that form by the following procedure. The function $C(\omega) \tilde{\phi}_{j',n'}$ is subtracted from both sides of Eq. (32), where the function $C(\omega)$ is to some extent arbitrary. In this case, $C(\omega)$ has been chosen to be the same as in Ref. 1:

$$C(\omega) = 1 + \frac{T_e}{T_i} - \left\{ \left(\frac{T_e}{T_i} + \frac{\omega^* e}{\omega} \right) I_0(b_{i\theta}) + \eta_i \frac{\omega^* e}{\omega} b_{i\theta} [I_1(b_{i\theta}) - I_0(b_{i\theta})] \right\} \exp(-b_{i\theta}) , \quad (34)$$

where I_n denotes a modified Bessel function of order n .

Then Eq. (32) becomes

$$\sum_{j,n} [L_{jj',nn'} - C(\omega) \delta_{jj'} \delta_{nn'}] \tilde{\phi}_{jn} = - C(\omega) \tilde{\phi}_{j',n'} . \quad (35)$$

Since the frequency ω enters Eq. (35) in a complicated fashion, an approximate value ω_0 for the frequency can be "corrected" by a frequency shift $\delta\omega \equiv \omega - \omega_0 < \omega_0$ obtained perturbatively. When the matrix $[L_{jj}, nn' - C \delta_{jj}, \delta_{nn'}]_{\omega=\omega_0}$ is solved numerically, a set of eigenvalues and eigenvectors is obtained. Choosing some particular eigenvalue $(-\hat{C})$, then Eq. (35) becomes

$$[-\hat{C}(\omega_0)] + \delta\omega \left[\frac{\partial}{\partial\omega} (-\hat{C}) \right]_{\omega=\omega_0} = -C(\omega_0) + \delta\omega \left[\frac{\partial}{\partial\omega} (-C) \right]_{\omega=\omega_0},$$

or

$$\delta\omega = - \frac{-\hat{C}(\omega_0) + C(\omega_0)}{[(\partial/\partial\omega)C + (\partial/\partial\omega)(-\hat{C})]_{\omega=\omega_0}}. \quad (36)$$

Now $(\partial/\partial\omega)C$ can be evaluated explicitly from Eq. (34), but $(\partial/\partial\omega)(-\hat{C})$ must be determined approximately from the difference of two recursions in ω . Labeling the recursion by (n) , Eq. (36) becomes

$$\omega^{(n+1)} = \omega^{(n)} - \{C[\omega^{(n)}] - \hat{C}[\omega^{(n)}]\} \left\{ \{\omega_{*e}/[\omega^{(n)}]^2\} [I_0(b_{i\theta}) + \eta_i b_{i\theta} (\bar{I}_1 - I_0) \exp(-b_{i\theta}) + \frac{-\hat{C}[\omega^{(n)}] - \{-\hat{C}[\omega^{(n-1)}]\}}{\omega^{(n)} - \omega^{(n-1)}}] \right\}^{-1}. \quad (37)$$

This is just the variant of Newton's method for root finding known as the method of false position.¹³

To be more specific, the choices that will be made for $g_j(\theta)$ and $h_n(S)$ are

$$g_j(\theta) = (2\pi)^{-1/2} \exp(ij\theta) \quad (38)$$

a Fourier series, and

$$h_n(S) = H_n(\sigma^{1/2} S) \exp(-\sigma S^2/2) M_n^{-1/2}, \quad (39)$$

a Hermite function series, where H_n denotes a Hermite polynomial and $M_n \equiv (\pi/\sigma)^{1/2} 2^n n!$. Here σ is a parameter [with $\text{Re}(\sigma) > 0$] which can be adjusted to minimize the number of radial basis functions which need to be retained. Considering that the equation to be solved is the lowest in a hierarchy in $(\Delta r_s/r_n)$, the limits for the S integration may be extended from those corresponding to the magnetic axis and the limiter to $\pm\infty$. The boundary conditions imposed are $\tilde{\phi} \rightarrow 0$ as $S \rightarrow \pm\infty$, which are automatically satisfied for $\text{Re } \sigma > 0$. This corresponds to "globally" localized modes which may spread over a number of mode-rational surfaces, with outgoing energy at $S \rightarrow \pm\infty$.

The radial basis functions in Eq. (39) have the important property, besides completeness and orthonormality, that their Fourier transforms are known analytically¹⁴ and are just the

original basis functions with argument k_r , within constants, i.e.,

$$\begin{aligned}
 (2\pi)^{-1/2} \int_{-\infty}^{\infty} d(r - r_0) \exp[-ik_r(r - r_0)] h_n\left(\frac{r - r_0}{\Delta r_s}\right) \\
 = \frac{\Delta r_s}{\sigma^{1/2}} (-i)^n h_n\left(\frac{\Delta r_s k_r}{\sigma}\right), \quad (40)
 \end{aligned}$$

recalling that $S \approx (r - r_0)/\Delta r_s$. This property will be useful in computing matrix elements for the nonresonant ion term.

The matrix element in Eq. (33) may be broken up into pieces corresponding to the different terms in the quasineutrality equation

$$\frac{T_e}{|e|n} [\tilde{n}_e^T + \tilde{n}_e^C - \tilde{n}_i^{nr} - \tilde{n}_i^C - \tilde{n}_i^r] = 0, \quad (41)$$

where $\tilde{n}_e^T$ refers to trapped electrons and $\tilde{n}_e^C$ to circulating electrons. These matrix elements are put into computationally useful form in Appendix C.

IV. NUMERICAL RESULTS

The computer code of Ref. 1 has been revised to implement the procedure described in the preceding section. A number of cases have been investigated, and the results will be presented and discussed in this section. For all of these cases, all of the terms in the mode equation [Eq. (41)] are retained, except as noted below. In general, only results for the lowest mode, that is, the one with the least number of radial and poloidal nodes, are given for each case. The results obtained are consistent with the conjecture that this lowest mode is the one with the largest linear growth rate for a given set of parameters. The results presented are for nine radial basis functions ($n = 0$ through 8) and nine poloidal basis functions ($j = -4$ through 4), with eleven bounce or transit frequency harmonics in the trapped and circulating electron terms and in the circulating ion resonance term ($p = -5$ through 5). This means that (81×81) matrices were solved numerically for their eigenvalues and eigenvectors. The value of σ was varied to obtain the best convergence of the coefficients of the $j=0$ radial basis functions. The value of ω was computed recursively using Eq. (37) until satisfactory convergence was achieved. It is expected that the results for ω obtained with this truncated set of basis functions will be reasonably accurate, but that the radial extent of the eigenfunction will be underestimated, as was explained in the introduction.

The results of the two-dimensional code will be compared with those from the local code originally described in Ref. 16. As stated in Ref. 1, this local code has been modified by the addition of the ion magnetic drift correction terms given in Ref. 1 of first

and second order in $\bar{\omega}_{di}/\omega$ (recall that the two-dimensional code is now correct for arbitrary $\bar{\omega}_{di}/\omega$), and by the addition of the shear damping term obtained in Ref. 16 by solving the one-dimensional radial Weber's equation. More recently, this code (which will still be referred to as "local") has been improved by several additional changes. A circulating-electron term has been added of the same form as the trapped-electron time average term written out in Ref. 16, except with a resonant denominator of the form $(\omega + i\nu_f - k_{||}v)$. Here $k_{||}$ is a constant to be chosen to give the best agreement with the two-dimensional code results. An ion-ion collision term of the form given in Appendix B has also been added, except with $k_r \equiv 0$. The electron collision frequency has been replaced by one of the form of Eq. (22) except that $|(1 - \epsilon_0) - \Lambda|^{-1}$ has been replaced by the more usual enhancement factor ϵ_0^{-1} , since there is an energy integration but no pitch-angle integration in the local code. An enhancement factor of the same form as that in Eq. (26) from the number-conserving term in the electron collision operator has also been added, but with $u_0 \equiv 0$ only. These additions now allow treatment of the collisionless and collisional drift modes as well as the trapped electron mode with the local code. The fraction of trapped particles is generally taken as $(2\epsilon_0)^{1/2}$ in the local code.

A. Basic Case Results

Results for what will be referred to as the "basic case" will be presented in several different forms, and the effects of turning off different terms in the mode equation will be investigated.

Results of separately varying the parameters of the basic case will be given. The chosen parameters of this case are: $\epsilon_0 = 0.1$, $T_e/T_i = 2$, $q = 2.5$, $q'r/q = 1$, $\eta_i = \eta_e = 0$, $r_n/r = 1$, $m_i/m_e = 3672$, $v_e^* \equiv v_{ei}/(\epsilon_0^{3/2} \bar{\omega}_{te}) = 0.03$, $b_{i0} = 0.2$ (or $k_\theta \rho_i = 0.63$), and $u_0 = 0$. From these parameters, several other values of interest can be derived: $v_{ei}/(\epsilon_0 \omega_{*e}) = 0.0514$, $\bar{\omega}_{ti}/\omega_{*e} = 0.0633$, $\bar{\omega}_{te}/\omega_{*e} = 5.42$, and $\rho_i/\Delta r_s = 0.633$. The value of σ used is $\sigma = 0.70 + 0.15i$. The mode frequency is $\omega/\omega_{*e} = 0.319 + 0.175i$, or, in more physical units, $\omega/(\epsilon_0^{1/2} \bar{\omega}_{te}) = 0.186 + 0.102i$. This mode frequency is fairly well-centered in the frequency range between $\bar{\omega}_{ti}$ and $\langle \omega_b \rangle_e \approx (1/\sqrt{2}) (\epsilon_0^{1/2} \bar{\omega}_{te})$: $|\omega|/\bar{\omega}_{ti} = 5.75 = 1/0.174$ and $|\omega|/\langle \omega_b \rangle_e = 0.300 = 1/3.33$, so that neither species dominates the other in the terms in the mode equation.

In Fig. 1 are plotted the real and imaginary parts of $\tilde{\phi}(\theta, S)$ for the basic case. Only positive values of S are shown, since the lowest eigenfunction is symmetric in $(S\theta)$, in the absence of a current. In the figure, $\theta = 0$ (or 2π) corresponds to the outside of the torus, where the trapped particles are located. The eigenfunction is seen to be localized in this region; i.e., it exhibits the usual ballooning characteristic. The normalization for the eigenfunction is such that the coefficient of the $n=0, j=0$ basis function product is just $(1+0i)$. Radially this function drops off smoothly and monotonically, the same basic behavior seen in Ref. 1 for smaller $k_\theta \rho_i$. In Fig. 2 are plotted the real and imaginary parts of $\tilde{\phi}(\theta, S) \exp(iS\theta)$, which gives the variation of the eigenfunction along the magnetic field line. Note that this function is not periodic in θ except at integer values of S , that is at mode-rational surfaces. The shape of the eigenfunction for $0 \leq \theta \leq \pi$

is seen to be preserved as S increases from zero. Hence, as the sheared magnetic field rotates, the eigenfunction rotates along with it to preserve the same structure along the field, in a manner consistent with the periodicity in θ and ζ required of the total potential. In Fig. 3 are plotted the real and imaginary parts of the radial Fourier transform $\tilde{\phi}(\theta, k_r)$ of $\tilde{\phi}(\theta, S)$. Note that while $k_{\theta}\rho_i = 0.63$, the distribution of $k_r\rho_i$'s is quite broad, extending from zero to several times $k_{\theta}\rho_i$. However, $\bar{k}_r\rho_i = k_{\theta}\rho_i$ is acceptable as an approximate average value, and this is the value used in the ion "resonance" term generally.

B. Influence of Different Terms

It is easy to determine the importance of the different terms in the mode equation for the basic case by turning them off, one at a time, and then finding the value of ω at convergence. Table I lists these values of ω in units of ω_{*e} . The circulating electron term $\tilde{n}_e^C$ is seen to be rather strongly destabilizing and to decrease the real frequency somewhat. The destabilization is presumably due mainly to the barely circulating electrons, whose transit frequency is either smaller than or of the same order as the average bounce frequency of trapped electrons. The trapped electron non-time-average [$p \neq 0$ in Eq. (26)] term is slightly stabilizing and decreases the real frequency also. The trapped electron time-average [$p = 0$ in Eq. (26)] term is of course the main destabilizing term. The electron collision number conserving term [the Z-function term in Eq. (26)] is seen to be very slightly stabilizing, as is expected deep in the banana regime [where the

small argument limit of the Z-function obtains]. For purposes of comparison, this case was also run using the familiar simpler electron collision operator with $\nu_f(\epsilon) = (\nu_{ei}/\epsilon_0)(T_e/\epsilon)^{3/2}$ with no number conserving term, as was used in Ref. 1. This simpler operator is seen to give a somewhat lower growth rate and real frequency than the normal pitch-angle dependent operator used here. However, the differences are not appreciably large. In higher collisionality regimes, the simple collision operator would be less satisfactory and the differences would be larger. The circulating ion "resonance" term is slightly stabilizing, as expected, and decreases the real frequency slightly. The circulating ion finite radial excursion terms [the $(\partial/\partial S)$ and $(\partial^2/\partial S^2)$ terms in Eq. (27)] are seen to be somewhat destabilizing and tend to increase the real frequency slightly. The ion sound term [the $(\partial/\partial \theta + iS)$ term in Eq. (27)] is stabilizing, as expected since it provides radial energy convection in certain circumstances (see below), and increases the real frequency very slightly. The ion-ion collision term is seen to be only slightly stabilizing, as is expected for low collisionality cases with $b_{i0} < 1$.

C. Effect of Parameter Variation

Figure 4 displays the growth rate in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ (approximately in units of the average trapped-electron bounce frequency) as the collisionality ν_e^* is varied for the basic case. By convention, the trapped-electron mode occurs in the banana regime ($\nu_e^* < 1$), the collisionless drift mode in the plateau regime ($1 < \nu_e^* < \epsilon_0^{-3/2}$), and the collisional drift mode in

the Pfirsch-Schlüter regime ($v_e^* > \epsilon_0^{-3/2}$). The transitions here are gradual with the dominant destabilizing mechanism changing from the trapped electron time-average term in the banana regime, to the resonant part of the circulating electron term in the plateau regime, to electron collisions in the Pfirsch-Schlüter regime. Positive $\eta_{e,i}$ is destabilizing for the trapped-electron mode while negative $\eta_{e,i}$ is destabilizing for the collisionless and collisional drift modes. Similarly larger ϵ_0 is destabilizing for the trapped-electron mode (keeping appropriate parameters constant) while smaller ϵ_0 is destabilizing for the collisionless and collisional drift modes. It should also be noted that the value of b_{i0} corresponding to the largest growth rate is smaller for the collisionless and collisional drift modes than for the trapped electron mode. The parameters of the basic case are somewhat artificial, representing a compromise in order to keep the modes unstable in all three regimes. From the curve in Fig. 4 for the two-dimensional (2-D) code, the trapped-electron mode has a higher growth rate than the collisionless or collisional drift modes, though this relationship would change for different parameters. This curve is to be compared with that for the local code for $b_{i0} = 0.2$ which shows a lower growth rate for the collisionless drift mode and a higher growth rate for the collisional drift mode, while the trapped-electron mode growth rates are very close. For the local code, the value of $k_{||}$ was used which made the maximum growth rate as a function of v_e^* occur for the same value of v_e^* as for the two-dimensional code, namely

$k_{||} = 0.8/qR_0$. This value of $k_{||}$ will be used for all local code results to be presented. In the two-dimensional analysis, there are many additional "degrees of freedom" available to the mode in the spatial variation of the eigenfunction over those in the local analysis. It is conjectured here that the mode is using these additional degrees of freedom to minimize the effect of the v_e^* parameter change on the growth rate. The cutoff at large v_e^* in the Pfirsch-Schlüter regime occurs when $v_f \approx k_{||}^2 v_e^2 / \omega$. Also shown in Fig. 4 is the curve for the local code for $b_{i0} = 0.8$. This curve illustrates the point that the maximum growth rate (as a function of b_{i0}) occurs for higher b_{i0} at low v_e^* than at high v_e^* (see below also). In particular, the ion-ion collision term, which is proportional to $v_e^* b_i^2$ (for fixed T_e/T_i and for $b_i < 1$) is strongly stabilizing at high v_e^* except for small b_{i0} .

Figures 5 and 6 show the real and imaginary parts of the eigenfunction $\tilde{\phi}(\theta, S)$ for $v_e^* = 10$ (plateau regime) and 120 (Pfirsch-Schlüter regime) respectively, and should be compared with Fig. 1 for $v_e^* = 0.03$ (banana regime). There is strong ballooning in all three cases. It might be thought at first that the ballooning should be reduced as v_e^* increases and the distinction between trapped and circulating particles becomes less meaningful. However, there are other relevant considerations. The number-conserving factor in Eq. (26), for example, serves to keep the trapped-electron term from falling off as v_{ei}^{-1} for large v_e^* ; asymptotically it is independent of v_{ei} as is required. Also, as noted in the introduction, ballooning comes from other terms in the mode equation besides the trapped-electron term. Examples are

the circulating-electron term (in particular, for the barely circulating electrons), and the ion magnetic drift term (proportional to $\cos \theta$). Any one of these terms can give strong ballooning in the eigenfunction. Also the θ -dependent finite radial excursion terms containing $S^{(\sim)}(\theta)$ in the ion resonance term can give weak ballooning. In short, toroidal effects can be important at any collisionality. At high collisionality the electron response is forced to be local, while at low collisionality free streaming occurs and the electron response can be strongly nonlocal. In either case, however, the toroidal magnetic field inhomogeneity is a local driving force which can cause ballooning. Another feature to notice in comparing Figs. 1, 5, and 6 is that there appears to be increased "smoothing" of the eigenfunction as v_e^* increases; this is presumably due to the ion-ion collision term with its explicit k_r -dependence in Eq. (28).

In Fig. 7 are shown growth rates γ in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ as functions of b_{i0} for the same three values of v_e^* . The decrease in the value of b_{i0} corresponding to the maximum value of γ as v_e^* increases is obvious here. Corresponding results from the local code are shown also, and there is rough quantitative agreement. The corresponding curves for γ/k_0^2 versus b_{i0} are shown in Fig. 8. Several different nonlinear analyses⁵ suggest that anomalous diffusion coefficients will be given roughly by γ/k_i^2 or $\gamma/2k_r^2$. For the present cases, a rough average value of k_r is given by $\bar{k}_r = k_0$, so that the quantity plotted in Fig. 8 is relevant. One particular strong turbulence theory of anomalous diffusion¹⁵ suggests that the relevant diffusion coefficient is not $(\gamma/2k_r^2)_{\max}$, but rather $(\gamma_{\max}/2k_r^2)$, where the subscript max

indicates the maximum value as $k_{\perp}$ is varied. The corresponding quantities $(\gamma_{\max}/k_{\theta}^2)$ for the two-dimensional code are indicated by the heavy dots in Fig. 8.

The basic case has $\epsilon_0 = 0.10$ and $\eta_{i,e} = 0$. The effects of changing these parameters, at fixed collisionality $\nu_e^* = 0.03$, is shown in Fig. 9. Changing $\eta_{i,e}$ from zero to one does not change the maximum value of the growth rate appreciably, but it does lower the value of $b_{i\theta}$ at which the maximum growth rate occurs. Further changing ϵ_0 from 0.10 to 0.25 has a dramatic stabilizing effect on the mode. This is explained by noting that while for the basic case $|\omega|/\bar{\omega}_{ti} = 5.75$, for the case with $\epsilon_0 = 0.25$, $\eta_{i,e} = 1$ it is found that $|\omega|/\bar{\omega}_{ti} = 2.33$ at $b_{i\theta} = 0.1$. The ion "resonant" term is stabilizing and is roughly proportional to $\exp(-\omega^2/\bar{\omega}_{ti}^2)$ and the ion sound term is also stabilizing and is proportional to $\bar{\omega}_{ti}^2/\omega^2$. Thus, as $|\omega|/\bar{\omega}_{ti}$ decreases, these stabilizing terms get much stronger.

The next case considered has the same basic parameters, with $\epsilon_0 = 0.25$ and $\eta_{i,e} = 1$. This corresponds to case (d) of Ref. 1. The difference in ν_e^* ($\nu_e^* = 0.03$ here versus $\nu_e^* = 0.06$ in Ref. 1) accounts approximately for the inclusion here of electron-electron collisions which were omitted in Ref. 1. For $b_{i\theta} = 0.025$, the present integral form of the two-dimensional code gives $\omega/\omega_{*e} = 0.875 + 0.075i$ with $\sigma = 1.98 + 0.426i$, which is to be compared with $\omega/\omega_{*e} = 0.831 + 0.178i$ with $\sigma = 1.98 + 0.501i$ in Ref. 1. The difference is due both to the different electron collision operators and to the difference between the expanded and unexpanded forms of the ion response in $\bar{\omega}_{di}/\omega$. For $b_{i\theta} = 0.1$,

the present two-dimensional code gives $\omega/\omega_{*e} = 0.501 + 0.137i$, with $\sigma = 1.40 + 0.213i$. The corresponding eigenfunction $\tilde{\phi}(\theta, S)$ is shown in Fig. 10. It is qualitatively the same as that for the basic case in Fig. 1 or that for case (d) in Ref. 1.

D. Shear Stabilization

The effect on the growth rate of varying the shear parameter $(q'r/q)$ for the basic case is shown in Fig. 11. The dashed curve in Fig. 11 gives the growth rate from the local code which contains the shear damping term obtained by solving the one-dimensional radial Weber's equation.¹⁶ This term is linear in $(q'r/q)$ and the local growth rate correspondingly decreases linearly with $(q'r/q)$. The solid curve in Fig. 11 gives the growth rate from the two-dimensional code. It decreases with increasing $(q'r/q)$ for $(q'r/q) \geq 1$ at roughly the same rate as that from the local code. However, for $(q'r/q) \leq 1$ the two-dimensional code growth rate is independent of $(q'r/q)$. This type of dependence of γ on $(q'r/q)$ was derived analytically from simplified two-dimensional differential equations valid for $k_r \rho_i < 1$ in Refs. 17 and 5, using the technique of Taylor,³ except with a transition at $(q'r/q) = 1/2$ rather than $(q'r/q) = 1$. However, the $k_r \rho_i < 1$ assumption is only marginally satisfied in the present case (see Fig. 3) and many additional terms are included, so that the different transition value of $(q'r/q)$ is not surprising. For all of the $(q'r/q)$ values run with the two-dimensional code (the heavy dots in Fig. 11), the mode balloons normally outward, in the manner shown in Fig. 1.

E. Electron Currents

The effect of an electron current, or equivalently, of a non-zero circulating-electron flow velocity u_0 , is in general destabilizing. However, the destabilization seems to be negligible even for u_0/v_e a substantial fraction of unity if $\omega/\bar{\omega}_{te}$ is very small. In particular, the increase in the growth rate for the basic case parameters is negligible even when u_0/v_e is increased from 0 to 0.25. A particularly simple way of increasing $(\omega/\bar{\omega}_{te})$ is to decrease r_n/r , so that ω_{*e} , and consequently ω , are increased. For the basic case parameters, except with $r_n/r = 1/2$, the two-dimensional code gives $\omega/\omega_{*e} = 0.257 + 0.110i$ for $u_0/v_e = 0$ and $\omega/\omega_{*e} = 0.257 + 0.118i$ for $u_0/v_e = 0.25$. From these sorts of results it appears unlikely that the destabilizing effect of an electron current will be important, since for typical tokamak parameters $u_0/v_e \ll 1$.

F. Odd Modes

Eigenmodes other than the lowest, (i.e., the one with the least number of poloidal and radial nodes) have also been investigated using the two-dimensional code. In particular, odd modes, whose perturbed electrostatic potential is odd in θ around $\theta = 0$ (and therefore, also around $\theta = \pi$) at $S = 0$, have been studied extensively by Coppi and co-workers (see Ref. 18 and references contained therein). Such an odd mode, obtained from the two-dimensional code for the basic case parameters, is shown in Fig. 12. It has a frequency $\omega/\omega_{*e} = 0.586 - 0.013i$ which

indicates a slightly damped mode. This is in fact expected since the basic case parameters do not satisfy the instability criterion for odd modes.¹⁸

G. "Ubiquitous" Modes

Recently, Coppi and co-workers have investigated "ubiquitous" modes, e.g., modes which for certain parameters can propagate in the ion, rather than the electron, diamagnetic direction, that is $\omega_r/\omega_{*e} < 0$ (see Ref. 19 and references contained therein). A numerical study of this kind of mode, and of the transition from $\omega_r/\omega_{*e} > 0$ to $\omega_r/\omega_{*e} < 0$ as b_{i0} increases, was made in Ref. 20 using a local code which, however, omitted completely the ion magnetic drift terms. The case considered in Figs. 6 and 7 of Ref. 20 had the parameters $\epsilon_0 = 0.28$, $T_e/T_i = 2$, $q = 2.5$, $q'r/q = 1$, $\eta_i = 1$, $\eta_e = 0.96$, $r_n/r = 0.288$, $m_i/m_e = 3672$, $v_e^* = 0.075$, $u_0 = 0$, and varying b_{i0} , in present notation. This case actually corresponds most closely to the $v_e^* = 0.15$ curves (with ion-ion collisions) in Ref. 20, since electron-electron collisions were omitted there, but are included in the two-dimensional code. The results for the real frequency and growth rate are shown in Fig. 13. The dashed curves are taken from Figs. 6 and 7 of Ref. 20; the solid curves are obtained with the two-dimensional code. The sign convention is such that positive $\omega_r/(\epsilon_0^{1/2}\omega_{te})$ corresponds to positive ω_r/ω_{*e} . The real frequency does go negative at about $b_{i0} = 0.5$ for the two-dimensional code, whereas that from Ref. 20 stays positive. The growth rates are qualitatively the same for $b_{i0} \lesssim 1$, but for $b_{i0} > 1$ the growth rate of

Ref. 20 increases as $b_{i\theta}$ increases, while that from the two-dimensional code approaches zero. These differences result both from the allowance for ballooning in the two-dimensional code, and from the inclusion there of the ion magnetic drift terms, both of which make the two-dimensional code results more realistic than those of Ref. 20. The eigenfunctions from the two-dimensional code balloon to the outside (around $\theta = 0$) for all of the cases run (the heavy dots in Fig. 13). This means that ion magnetic drift resonances can occur when $\omega_r/\omega_{*e} < 0$, since the resonant denominator is of the form $[\omega - \bar{\omega}_{di}(\hat{v}_\perp^2 + 2\hat{v}_\parallel^2)\cos\theta]$ in Eq. (27), and $\omega_{di}/\omega_{*e} < 0$ for this case, as is usual. For modes with $\omega_r/\omega_{*e} > 0$, this resonance cannot occur if the ballooning is to the outside. Results similar to those of the two-dimensional code have been recently obtained for this case with a one-dimensional, along-the-field-line computation.⁶

V. CONCLUSIONS

The present two-dimensional analysis of drift and trapped-electron modes represents a considerable improvement over previous calculations^{1,4} in generality and accuracy. The maximum linear growth rate typically occurs for $k_\theta \rho_i$ of order unity, and the results presented in Sec. IV show that the mean k_r is often comparable to k_θ . Therefore, consideration of the parameter range $k_r \rho_i \geq 1$ is of particular relevance, and this is not possible with the integrodifferential approach of Refs. 1 and 4. The present approach, using an explicit radial Fourier transformation and inverse transformation, allows treatment of arbitrary $k_r \rho_i$. For $k_r \rho_i < 1$, the two approaches give identical results to an accuracy of the order of the first omitted radial derivative term in the integrodifferential formulation.¹

Another major improvement is the use of an improved electron collision operator, specifically an energy and pitch-angle dependent Krook-type operator. With this operator, it is possible to treat a very wide range of collisionality in a unified fashion to follow the transition in toroidal geometry from the trapped-electron mode in the banana regime to the collisionless drift mode in the plateau regime, and then to the collisional drift mode in the Pfirsch-Schlüter regime. The mode balloons strongly in all three regimes. This model collision operator should give results which are accurate within a factor of two or so.

The third major improvement is the generalization of the ion response to an arbitrary ratio of the ion magnetic drift frequency

to the mode frequency. This is in contrast to Ref. 1 where this ratio is assumed small, and Ref. 4 where ion magnetic drifts are omitted entirely. The absence of this generalization can give incorrectly large growth rates for $k_{\perp} \rho_i \gg 1$, as has been shown also in Ref. 6. A simple ion-ion collision term has now been added to the ion response, and finite ion radial excursion terms have also been retained. In addition, the electron response has been generalized, to allow treatment of cases where $\omega \approx \langle \omega_b \rangle_e$, as well as the more familiar cases where $\omega < \langle \omega_b \rangle_e$.

A significant result of the present two-dimensional code is that shear stabilization is inoperative for appropriate parameters provided the shear is less than a certain critical value. This had been predicted in Refs. 17 and 5 by a purely analytic solution of a highly simplified two-dimensional mode equation.

The present two-dimensional code can be used in interpreting nonlinear saturated fluctuation spectra measured by techniques such as microwave scattering²¹ in machines like PLT, which at present operate usually in the plateau regime. Features of the fluctuation spectra such as localization in poloidal angle and the dominant perpendicular wavelength can be compared with the linear results obtained from the two-dimensional code. Results from such a calculation can also be of utility in future transport codes, if an assumption like $D = \gamma / 2k_0^2$ (for $\gamma > 0$ only) is made. Here the actual linear growth rate for the instantaneous equilibrium parameters would be used, rather than a series of growth rate formulas based on very simplified limiting cases in the local approximation.

Future work will involve the addition of electromagnetic effects which are important for $(m_e/m_i) \approx \beta \ll 1$ by solving for both the perturbed electrostatic potential $\tilde{\Phi}$ and the component of the perturbed magnetic vector potential parallel to the equilibrium magnetic field $\tilde{A}_{||}$. This will allow consideration of both the drift branch and the shear-Alfvén branch, along with the coupling between them. Also the treatment of realistic profiles for the equilibrium quantities will be possible if a slight change is made in the basis functions employed. This will allow study of the realistic large-scale radial localization of these modes.

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APPENDIX A

The linearized drift kinetic equation, for a magnetic field independent of time, in terms of the quantities defined in Section II, is for circulating electrons

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (\underline{v}_{||} + \underline{v}_B) \cdot \nabla \right] f_e^{(1)} = - \underline{v}_E^{(1)} \cdot \nabla f_{me}^C - e (\underline{v}_{||} + \underline{v}_B) \cdot \nabla \tilde{\Phi} \frac{\partial f_{me}^C}{\partial \varepsilon} \\ - \underline{v}_f [f_e^{(1)} + e \tilde{\Phi} (\partial f_{me}^C / \partial \varepsilon)] + \underline{v}_f \frac{f_{me}^C}{n} [\mu_e^{(1)} - (e \tilde{\Phi} / T_e) n], \end{aligned} \quad (A1)$$

where $\underline{v}_E^{(1)}$ is the perturbed electric drift velocity and $\underline{v}_B$ is the magnetic drift velocity. For the circulating electrons

$$\frac{\partial f_{me}^C}{\partial \varepsilon} = - \frac{f_{me}^C}{T_e} \left(1 - \frac{u_o}{v_{||}^{(0)}} \right) \quad (A2)$$

and

$$\nabla f_{me}^C = \underline{e}_r \frac{\partial f_{me}^C}{\partial r} = \underline{e}_r f_{me}^C \left(\frac{1}{n} \frac{dn}{dr} + \frac{1}{T_e} \frac{dT_e}{dr} \left[\frac{\varepsilon - m_e u_o v_{||}^{(0)}}{T_e} - \frac{3}{2} \right] \right), \quad (A3)$$

to first order in u_o/v_e , if $\partial u_o / \partial r$ is taken to be zero.

Evaluating $\underline{v}_{Er}$ yields

$$\begin{aligned}
 \left. \frac{d}{dt} \right|_b f_e^{(1)} &\equiv \left[\frac{\partial}{\partial t} + (\underline{v}_{||} + \underline{v}_B) \cdot \nabla \right] f_e^{(1)} \\
 &= \frac{e}{T_e} f_{me} \left[1 - \frac{u_0}{v_{||}^{(0)}} \right] \left(\left. \frac{d}{dt} \right|_b + i\omega \right) \tilde{\Phi} - \frac{c}{Br} \text{im}^0 \tilde{\Phi} \frac{\partial f_{me}^C}{\partial r} \\
 &\quad - v_f \left[f_e^{(1)} - (e\tilde{\Phi}/T_e) f_{me}^C \left(1 - \frac{u_0}{v_{||}^{(0)}} \right) \right] \\
 &\quad + v_f \frac{f_{me}^C}{n} \left[n_e^{(1)} - (e\tilde{\Phi}/T_e) n \right] , \tag{A4}
 \end{aligned}$$

where $(d/dt)|_b$ indicates the total derivative along a guiding-center (banana) orbit. Thus

$$\begin{aligned}
 \left(\left. \frac{d}{dt} \right|_b + v_f \right) f_e^{(1)} &\approx \frac{e}{T_e} f_{me}^C \left(\left[1 - \frac{u_0}{v_{||}^{(0)}} \right] \left(\left. \frac{d}{dt} \right|_b + v_f + i\omega \right) \tilde{\Phi} - i\omega_{*e}^T \tilde{\Phi} \right) \\
 &\quad + v_f \frac{f_{me}^C}{n} \left[n_e^{(1)} - (e\tilde{\Phi}/T_e) n \right] , \tag{A5}
 \end{aligned}$$

where the definitions

$$\omega_{*e} \equiv [m^0 c T_e / (|e| r_0 B_0)] d \ln n / dr , \tag{A6}$$

$$\eta_e \equiv (d \ln T_e / dr) / (d \ln n / dr) , \tag{A7}$$

and

$$\omega_{*e}^T \equiv \omega_{*e} \left(1 + \eta_e \left[\frac{\epsilon - m_e u_0 v_{||}^{(0)}}{T_e} - \frac{3}{2} \right] \right) \tag{A8}$$

have been used. Equation (A5) may be integrated along unperturbed guiding-center orbits to obtain

$$\begin{aligned}
 f_e^{(1)} = & \frac{ef_{me}^C}{T_e} \left(\tilde{\phi} \left[1 - \frac{u_0}{v_{||}^{(0)}} \right] + i \left\{ \omega \left[1 - \frac{u_0}{v_{||}^{(0)}} \right] - \omega_{*e}^T \right\} \exp(-v_f t) \right. \\
 & \times \int_{-\infty}^t dt' \exp(v_f t') \tilde{\phi} \Bigg) + v_f^h \frac{f_{me}}{n} \exp(-v_f t) \\
 & \times \int_{-\infty}^t dt' \exp(v_f t') [\tilde{n}_e^{(1)} - (e\tilde{\phi}/T_e)n] . \quad (A9)
 \end{aligned}$$

A corresponding result holds for trapped electrons with u_0 set to zero. For typical parameters, the ion gyroradius ρ_i is large compared with the average trapped-electron banana width ρ_{be} , and terms of order $\rho_{be}/\rho_i \sim k_{\perp} \rho_{be}$ or smaller, will be dropped.

The first integration along unperturbed orbits in Eq. (A9) is worked out in detail in Ref. 1. It yields $f_e^{(1)} \equiv \tilde{f}_e \exp(-i\omega t + i\ell\zeta - im^0\theta)$, with $n_e^{(1)} \equiv \tilde{n}_e \exp(-i\omega t + i\ell\zeta - im^0\theta)$,

$$\begin{aligned}
 \tilde{f}_e = & \frac{ef_{me}}{T_e} \left[1 + \frac{2u_0 v_{||}^{(0)}}{v_e^2} H(\Lambda_c - \Lambda) \right] \left(\tilde{\phi} \left[1 - \frac{u_0 H(\Lambda_c - \Lambda)}{v_{||}^{(0)}} \right] - \left\{ \omega \left[1 - \frac{u_0 H(\Lambda_c - \Lambda)}{v_{||}^{(0)}} \right] - \omega_{*e}^T \right\} \right. \\
 & \times \sum_{p=-\infty}^{\infty} \frac{\tilde{\phi}^{(p)} \exp[ip\omega_{b,t} \hat{t}(\theta)]}{\omega + iv_f - \omega_{de}^{(0)} - [p + SH(\Lambda_c - \Lambda)]\omega_{b,t}} \Bigg) + v_f \frac{f_{me}}{n} \left[1 + \frac{2u_0 v_{||}^{(0)}}{v_e^2} H(\Lambda_c - \Lambda) \right] \\
 & \times \exp(-v_f t) \int_{-\infty}^t dt' \exp(v_f t') [\tilde{n}_e - (e\tilde{\phi}/T_e)n] . \quad (A10)
 \end{aligned}$$

Here H is the Heaviside step function and $\Lambda_c \equiv 1 - \epsilon_0$, so that $H(\Lambda_c - \Lambda)$ is one for circulating particles and zero for trapped particles. The summation in Eq. (A10) is over all integral values of p ; $\omega_{b,t}$ means ω_b for trapped particles and ω_t for circulating particles. The zero superscript indicates a time average along a particle orbit. Also

$$\hat{t}(\theta) \equiv qR_0 \int_0^\theta d\theta' / v_{||}(\theta') + \text{const} , \quad (\text{A11})$$

$$\begin{aligned} \tilde{\phi}(p) = & \frac{1}{\tau_{b,t}} \oint dt' \tilde{\phi}(\theta', S') \exp \{ iS(\theta' - \theta) - i\omega_t S[\hat{t}(\theta') - \hat{t}(\theta)] H(\Lambda_c - \Lambda) \} \\ & \times \exp \{ -ip\omega_{b,t} \hat{t}[\theta(t')] \} , \end{aligned} \quad (\text{A12})$$

$$\omega_{de} = \bar{\omega}_{de} x^2 \left\{ \left(1 + \frac{v_{||}^2}{v^2} \right) \cos \theta + \frac{q'r}{q} \frac{2}{\epsilon_0} \frac{v_{||} [v_{||} - v_{||}^{(0)}]}{v^2} \right\} , \quad (\text{A13})$$

where $\bar{\omega}_{de} \equiv \omega_{*e}(r_n/R_0)$, $r_n \equiv -(d \ln n / dr)^{-1}$, and $x^2 \equiv \epsilon / T_e$. The time average $\omega_{de}^{(0)}$ of ω_{de} is computed in detail in Ref. 18. For trapped electrons it is $\omega_{de}^{(0)} = \bar{\omega}_{de} x^2 G(\kappa)$, where $\kappa \equiv [1/2 + (1 - \Lambda)/2\epsilon_0]^{1/2}$,

$$G(\kappa) \equiv \left[2 \frac{E(\kappa)}{K(\kappa)} - 1 \right] + \frac{q'r}{q} \left[4 \frac{E(\kappa)}{K(\kappa)} + 4(\kappa^2 - 1) \right] , \quad (\text{A14})$$

$$K(\kappa) \equiv \int_0^{\pi/2} d\xi (1 - \kappa^2 \sin^2 \xi)^{-1/2} \quad (\text{A15})$$

is the complete elliptic integral of the first kind, and

$$E(\kappa) \equiv \int_0^{\pi/2} d\xi (1 - \kappa^2 \sin^2 \xi)^{1/2}, \quad (A16)$$

is the complete elliptic of the second kind. For circulating electron, $\omega_{de}^{(0)}$ is negligible.

The number-conserving term will be evaluated approximately in a slab limit. This is possible because it will eventually give a multiplicative, rather than an additive, correction factor which differs significantly from unity only in the large v_e^* limit, where toroidal effects are not dominant. Thus

$$\begin{aligned} f_e &\approx \frac{ef_{me}}{T_e} \left[1 + \frac{2u_o v_{||}^{(0)}}{v_e^2} H(\Lambda_c - \Lambda) \right] \left(\tilde{\phi} \left[1 - \frac{u_o H(\Lambda_c - \Lambda)}{v_{||}^{(0)}} \right] \right. \\ &\quad \left. - \left\{ \omega \left[1 - \frac{u_o H(\Lambda_c - \Lambda)}{v_{||}^{(0)}} \right] - \omega_{*e}^T \right\} \right. \\ &\quad \left. \times \sum_p \frac{\tilde{\phi}(p) \exp[ip\omega_{b,t} \hat{t}(\theta)]}{p\omega + iv_f - \omega_{de}^{(0)} - [p + SH(\Lambda_c - \Lambda)]\omega_{b,t}} \right) \\ &\quad + iv_f^h \frac{f_{me}}{n} \left[1 + \frac{2u_o v_{||}^{(0)}}{v_e^2} H(\Lambda_c - \Lambda) \right] [\tilde{n}_e - (e\tilde{\phi}/T_e)n] (\omega + iv_f^h - k_{||} v_{||})^{-1}. \quad (A17) \end{aligned}$$

Integrating over velocity space gives

$$\begin{aligned}
 \tilde{n}_e = \int d^3v f_e = \frac{en}{T_e} & \left\{ \tilde{\phi} - \left(\frac{1}{n} \right) \int d^3v f_{me} \left[1 + \frac{2u_o v_{||}^{(o)}}{v_e^2} H(\Lambda_c - \Lambda) \right] \right. \\
 & \times \{ \omega [1 - u_o H(\Lambda_c - \Lambda) / v_{||}^{(o)}] - \omega_{*e}^T \} \\
 & \times \sum_p \frac{\tilde{\phi}(p) \exp[ip\omega_{b,t} \hat{t}(\theta)]}{\omega + iv_f - \omega_{de}^{(o)} - [p + SH(\Lambda_c - \Lambda)]\omega_{b,t}} \Bigg) \\
 & \times \left[1 + \frac{iv_f^h}{k_{||} v_e} Z \left(\frac{\omega - k_{||} u_o + iv_f^h}{k_{||} v_e} \right) \right]^{-1} \Bigg\}, \quad (A18)
 \end{aligned}$$

where

$$Z(W) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} d\xi \frac{\exp(-\xi^2)}{\xi - W} \quad (\text{Im } W > 0), \quad (A19)$$

is the usual plasma dispersion function. In the number-conserving term $k_{||}$ is taken as $1/qR_o$, so that $k_{||} v_e = \bar{\omega}_{te}$, which gives the right qualitative scaling for circulating electrons.

APPENDIX B

For a single pure Fourier harmonic $\tilde{\phi} \propto \exp [ik_r(r - r_0)]$, the result from Ref. 1 for the perturbed ion density is

$$n_i^{(1)} = \tilde{n}_i \exp (-i\omega t + i\ell\zeta - im^0\theta) , \text{ where}$$

$$\tilde{n}_i = -\frac{en}{T_i} \left(\tilde{\phi}(\theta, S) - \frac{1}{n} \int d^3v f_{mi}(\omega - \omega_{*i}^T) J_0^2\left(\frac{k_\perp v_\perp}{\Omega_i}\right) \times \sum_p \tilde{\phi}(p) \frac{\exp[ip\omega_{b,t}\hat{t}(\theta)]}{\omega + iv_i - \omega_{di}^{(0)} - [p + S^{(0)}]H(\Lambda_c = \Lambda)\omega_{b,t}} \right) . \quad (B1)$$

Here $\omega_{*i} = -(T_i/T_e)\omega_{*e}$, $\omega_{*i}^T \equiv \omega_{*i}[1 + \eta_i(\epsilon/T_i - 3/2)]$, $\eta_i \equiv (d \ln T_i / dr) / (d \ln n / dr)$, $k_\perp \equiv (k_\theta^2 + k_r^2)^{1/2}$, $k_\theta \equiv (-m^0/r)$, J_0 is the Bessel function of the first kind, $\Omega_i \equiv |e|B_0/m_i c$, ω_{di} is the ion analog of Eq. (A13), $S^{(0)} \equiv \ell\{q[r^{(0)}] - q(r_0)\}$, and $r^{(0)} \equiv \oint dt r / \tau_{b,t}$. The perturbed potential has been decomposed into bounce and transit harmonics:

$$\tilde{\phi}(\theta', S') = \exp[-iP_{b,t}^{(\sim)}(t')] \sum_{p=-\infty}^{\infty} \tilde{\phi}(p) \exp[ip\omega_{b,t}\hat{t}(\theta')] , \quad (B2)$$

where t' denotes time along a single particle orbit, $\theta' \equiv \theta(t')$, $S' \equiv S(t')$,

$$P_b^{(\sim)}(t') = S^{(0)}(\theta' - \theta) + w_{di}^{(\sim)} \quad (B3)$$

for trapped particles,

$$P_t^{(\sim)}(t') = S^{(0)}(\theta' - \theta) - \omega_t S^{(0)}(t' - t) + w_{di}^{(\sim)} \quad (B4)$$

for circulating particles, and

$$\tilde{w}_{di}^{(\sim)} \equiv \int_t^{t'} dt'' [\omega_{di}(t'') - \omega_{di}^{(0)}] \quad . \quad (B5)$$

The conditions that $\theta' = \theta$ and $S' = S$ at $t' = t$ will be imposed later. The other quantities have definitions corresponding to those for electrons. Ion collisions will be neglected temporarily, but will be reinserted later to give a small correction term.

A useful expression for $S^{(\sim)} \equiv S - S^{(0)}$ can be obtained as follows. Using Eq. (5) gives

$$S^{(\sim)} \approx \frac{r - r^{(0)}}{\Delta r_s} \approx \frac{v_{||} - v_{||}^{(0)}}{\Omega_{\theta}^0 \Delta r_s} \quad . \quad (B6)$$

For circulating ions this is

$$S^{(\sim)} \approx \frac{\sigma_{||} v [1 - \Lambda/h(\theta)]^{1/2} - q R_0 \omega_t}{\Omega_{\theta}^0 \Delta r_s} = \sigma_{||} X \frac{\rho_i}{\Delta r_s} \frac{q}{\epsilon_0} \left([1 - \Lambda/h(\theta)]^{1/2} - \frac{\epsilon_0^{1/2}}{L_t(\Lambda)} \right) \quad , \quad (B7)$$

where $X \equiv (\epsilon/T_i)^{1/2}$ and

$$L_t(\Lambda) \equiv \frac{\epsilon_0^{-1/2}}{2\pi} \int_{-\pi}^{\pi} d\theta [1 - \frac{\Lambda}{h(\theta)}]^{-1/2} \quad . \quad (B8)$$

Expanding the denominator in Eq. (B1) gives

$$\begin{aligned} \tilde{n}_i = & -\frac{en}{T_i} \left\{ \tilde{\phi}(\theta, S) - \frac{1}{n} \int d^3v f_{mi} (\omega - \omega_{*i}^T) J_0^2 \sum_p \tilde{\phi}^{(p)} \exp[ip\omega_{b,t} \hat{t}] \right. \\ & \times \frac{1}{\omega} \left[1 + \frac{\omega_{di}^{(0)} + [p + S^{(0)}]_H \omega_{b,t}}{\omega} + \left(\frac{\omega_{di}^{(0)} + [p + S^{(0)}]_H \omega_{b,t}}{\omega} \right)^2 \right. \\ & \left. \left. + \dots - i\pi\omega\delta\{\omega - \omega_{di}^{(0)} - [p + S^{(0)}]_H \omega_{b,t}\} \right] \right\} \quad . \quad (B9) \end{aligned}$$

Sum rules can be derived which make it possible to perform the p summations by the following procedure for circulating ions. Trapped ions may be treated similarly. Differentiating Eq. (B2) with respect to θ' at constant θ and $S^{(0)}$, using the result from Ref. 1 that

$$\omega_{di} + S^{(0)} \frac{v_{||}}{qR_0} = \omega_{di(1)} + S \frac{v_{||}}{qR_0} , \quad (B10)$$

where $\omega_{di(1)} \equiv \bar{\omega}_{di} (\hat{v}_\perp^2 + 2\hat{v}_{||}^2) \cos \theta$, $\hat{v}_\perp \equiv v_\perp/v_i$, $\hat{v}_{||} \equiv v_{||}/v_i$, and $\bar{\omega}_{di} \equiv \omega_{*i}(r_n/R_0)$, setting $\theta' = \theta$ and $S' = S$, and rearranging gives

$$\begin{aligned} \{ \omega_{di(1)} - i \frac{v_{||}}{qR_0} \left[\frac{\partial}{\partial \theta} \right]_{S^{(0)}} + iS \} \tilde{\phi}(\theta, S) = \sum_p \tilde{\phi}^{(p)} \exp[ip\omega_t \hat{t}(\theta)] \\ \times \{ \omega_{di}^{(0)} + [p + S^{(0)}] \omega_t \} . \end{aligned} \quad (B11)$$

A standard result for partial derivatives, along with Eq. (B1), yields

$$\left. \frac{\partial}{\partial \theta} \right|_{S^{(0)}} = \left. \frac{\partial}{\partial \theta} \right|_S + \left. \frac{\partial S}{\partial \theta} \right|_{S^{(0)}} \left. \frac{\partial}{\partial S} \right|_\theta = \left. \frac{\partial}{\partial \theta} \right|_S - \Gamma \sin \theta \left. \frac{\partial}{\partial S} \right|_\theta , \quad (B12)$$

where $\Gamma \equiv (q\rho_i/2\Delta r_s) [X\Lambda v/v_{||}(\theta)]$, to lowest order in ϵ_0 , so that

$$\begin{aligned} [\omega_{di(1)} - i \frac{v_{||}}{qR_0} \left(\left. \frac{\partial}{\partial \theta} \right|_S + iS - \Gamma \sin \theta \left. \frac{\partial}{\partial S} \right|_\theta \right)] \tilde{\phi}(\theta, S) = \sum_p \tilde{\phi}^{(p)} \exp[ip\omega_t \hat{t}(\theta)] \\ \times \{ \omega_{di}^{(0)} + [p + S^{(0)}] \omega_t \} . \end{aligned} \quad (B13)$$

Then the nonresonant part $\tilde{n}_i^{nr}$ of $\tilde{n}_i$ from Eq. (B9) becomes, using also the result analogous to Eq. (B13) for trapped particles,

$$\begin{aligned} \tilde{n}_i^{nr} = & -\frac{en}{T_i} \left\{ \tilde{\phi}(\theta, S) - \frac{1}{n} \int d^3v f_{mi}(\omega - \omega_{*i}^T) J_0^2 \frac{1}{\omega} \right. \\ & \times \left[1 + \left(\frac{\omega_{di}(1)}{\omega} - i \frac{v_{||}}{qR_0 \omega} \left(\frac{\partial}{\partial \theta} + iS - \Gamma \sin \theta \frac{\partial}{\partial S} \right) \right) \right. \\ & \left. \left. + \left(\frac{\omega_{di}(1)}{\omega} - i \frac{v_{||}}{qR_0 \omega} \left(\frac{\partial}{\partial \theta} + iS - \Gamma \sin \theta \frac{\partial}{\partial S} \right) \right)^2 + \dots \right] \tilde{\phi}(\theta, S) \right\} \quad (B14) \end{aligned}$$

Notice that Γ is odd in $\sigma_{||} = \text{sign}(v_{||})$ and that terms which are odd in $\sigma_{||}$ will vanish identically on performing the velocity space integration. Then terms of all orders in $[\omega_{di}(1)/\omega]$ may be resummed to yield

$$\begin{aligned} \tilde{n}_i^{nr} = & -\frac{en}{T_i} \left\{ \tilde{\phi}(\theta, S) - \frac{1}{n} \int d^3v f_{mi}(\omega - \omega_{*i}^T) J_0^2 \right. \\ & \times \left[\frac{1}{\omega - \omega_{di}(1)} - \frac{\hat{v}_{||}^2 \bar{\omega}_{ti}}{[\omega - \omega_{di}(1)]^2} \left(\frac{\partial}{\partial \theta} + iS \right) \frac{1}{\omega - \omega_{di}(1)} \right. \\ & \times \left(\frac{\partial}{\partial \theta} + iS \right) \frac{1}{\omega - \omega_{di}(1)} + i \left(\frac{\hat{v}_{||}^2 q \rho_i \bar{\omega}_{ti}}{2 \Delta r_s} \right) \frac{\sin \theta}{[\omega - \omega_{di}(1)]^2} \frac{\partial}{\partial S} \\ & \left. \left. - \left(\frac{\hat{v}_{||}^2 q \rho_i \bar{\omega}_{ti}}{2 \Delta r_s} \right)^2 \frac{\sin^2 \theta}{[\omega - \omega_{di}(1)]^3} \frac{\partial^2}{\partial S^2} \right] \tilde{\phi}(\theta, S) \right\} \quad , \quad (B15) \end{aligned}$$

through order $\{\bar{\omega}_{ti}/[\omega - \omega_{di}(1)]\}^2$. In resumming the series in Eq. (B14), the $\omega = \omega_{di}(1)$ resonances have been included; therefore the corresponding resonances must be suppressed in the "resonant" part $\tilde{n}_i^r$ of $\tilde{n}_i$ to avoid "double counting". The result, Eq. (B15),

could also have been obtained from the original kinetic equation for ions corresponding to Eq. (A1) for electrons by solving perturbatively in the terms which order as $\bar{\omega}_{ti}/\omega$. Note that in Eq. (B15) the theta derivatives act on $\omega_{di(1)}(\theta)$ as well as on $\tilde{\phi}(\theta, S)$. Writing out the radial Fourier transforms explicitly, Eq. (B15) implies the final result given by Eq. (27).

Ion-ion collisions will be included approximately as a small correction term using the results of Ref. 22 for the two limits $k_{\perp}\rho_i < 1$ and $k_{\perp}\rho_i > 1$. For a pure radial Fourier harmonic, the result for $k_{\perp}\rho_i > 1$ in the present notation would correspond to an extra term in $\tilde{n}_i$ given by

$$\tilde{n}_i^c = -\frac{en}{T_i} \left\{ i \frac{v_{ii}}{\omega^2} b_i^{1/2} \frac{3(\pi+1)}{8\pi^{1/2}} [\omega - \omega_{*i} (1 - \frac{3\pi+2}{4\pi+4} \eta_i)] \right\} \tilde{\phi} \quad , \quad (B16)$$

where $b_i \equiv k_{\perp}^2 \rho_i^2 / 2$ and $v_{ii} \equiv v_{ei} (T_e/T_i)^{3/2} (m_e/2m_i)^{1/2}$. For $1 > k_{\perp}\rho_i$ ($> k_{\parallel} v_i / \omega$) the result is given by

$$\tilde{n}_i^c = -\frac{en}{T_i} \left\{ i \frac{v_{ii}}{\omega^2} b_i^2 \frac{7}{10} [\omega - \omega_{*i} (1 - \frac{3}{28} \eta_i)] \right\} \tilde{\phi} \quad . \quad (B17)$$

These results correspond to neglecting $\omega_{di(1)}$ relative to ω , which will give at least qualitatively correct estimates of the effect of ion-ion collisions for the cases of interest in which $\omega \gtrsim \omega_{di(1)}$. These terms will be made continuous at $b_i = 1$ by altering the $k_{\perp}\rho_i < 1$ form to

$$\tilde{n}_i^c = -\frac{en}{T_i} \left(i \frac{v_{ii}}{\omega^2} b_i^2 \frac{7}{10} (1 + 0.2517b_i) \times \left\{ \omega - \omega_{*i} \left[1 - \frac{3}{28} (1 + 5.4366b_i) \eta_i \right] \right\} \right) \tilde{\phi} \quad . \quad (B18)$$

Writing out the radial Fourier transforms explicitly, Eqs. (B17) and (B18) yield the final result given by Eq. (28).

The resonant part $\tilde{n}_i^r$ of the perturbed ion density, from Eq. (B9), is

$$\tilde{n}_i^r = -\frac{en}{T_i} \frac{1}{n} \int_C d^3v f_{mi} (\omega - \omega_{*i}^T) J_0^2 \times \sum_p \tilde{\phi}^{(p)} \exp(ip\omega_t \hat{t}) i\pi \delta\{\omega - \omega_{di}^{(0)} - [p + s - s^{(\sim)}] \omega_t\} . \quad (B19)$$

The trapped-ion contribution to this resonant term has been dropped, since it does not contain the explicit S-dependence of the circulating-ion resonance term, and therefore will not contribute to the radial structure of the mode in a significant way. The time-average magnetic drift $\omega_{di}^{(0)}$ is negligible for circulating particles and will also be dropped. The delta function will be used to do the energy integration in Eq. (B19). Specifically,

$$\delta(\omega - [p + s - s^{(\sim)}] \omega_t) = \delta\{\omega - [p + s - \hat{s}^{(\sim)} X] \hat{\omega}_t X\} = \sum_{j''=1}^2 \delta(X - X_{j''}) | (p + s) \hat{\omega}_t - 2\hat{s}^{(\sim)} \hat{\omega}_t X_{j''} |^{-1} , \quad (B20)$$

using a standard formula for delta functions, where $X \equiv (\epsilon/T_i)^{1/2}$; a caret (^) specifies that the X-dependence has been factored out of a quantity, and

$$X_{\frac{1}{2}} = \frac{1}{2\hat{s}^{(\sim)} \omega_t} \{ (p + s) \hat{\omega}_t \pm [(p + s)^2 \hat{\omega}_t^2 - 4\omega \hat{\omega}_t \hat{s}^{(\sim)}]^{1/2} \} . \quad (B21)$$

In using this delta function to do the energy integration, negative values of $x_{j,1}^2$, must be excluded. Also, the resonant approximation used here breaks down if $\text{Im}(X_{j,1})$ becomes too large, and $X_{j,1}$'s with $\text{Im}(X_{j,1}) > |\text{Re } X_{j,1}|^{-1}$ are also excluded by analogy with the plasma dispersion function.²³ For circulating ions in general, from Ref. 1,

$$\begin{aligned} \tilde{\phi}(p) = \frac{1}{\tau_t} \int dt' \tilde{\phi}(\theta', S') \exp \{ i[S^{(n)}(\theta' - \theta) - \omega_t S^{(o)}(t' - t) \\ + w_{di}^{(-)} - ip\omega_t \hat{t}(\theta')] \} . \end{aligned} \quad (\text{B22})$$

To exclude the magnetic drift resonance $\omega = \omega_{di(1)}$, $w_{di}^{(-)}$ will be omitted in computing $\tilde{\phi}(p)$. What remain are the transit resonances $\omega = (p + S)\omega_t$ as modified by the energy dependent finite radial excursions of the circulating ions. To simplify the computation of the circulating ion resonance term, which is typically quite small, instead of doing explicit radial Fourier transforms as in the nonresonant ion term, a mean value $\bar{k}_r$ of k_r will be used, estimated from the results for a previous recursion for a given case (see Sec. IV). Thus the final form of the ion resonant term that will be used, noting that $(k_\theta v_{1i}/\Omega_i) = (2b_{1\theta}\Lambda)^{1/2}X$ to lowest order in ϵ_0 , is just that given by Eq. (29).

APPENDIX C

Using Eqs. (27) and (33), the nonresonant ion term matrix element may be written as $L_{jj',nn'}^{inr}$, where

$$\begin{aligned}
 L_{jj',nn'}^{inr} = & \frac{T_e}{T_i} \delta_{jj'} \delta_{nn'} - \int_{-\pi}^{\pi} d\theta \frac{\exp(-ij'\theta)}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dS h_{n'}(S) \\
 & \times \frac{2}{\pi^{1/2}} \int_0^{\infty} d\hat{v}_{\perp} \hat{v}_{\perp} \exp(-\hat{v}_{\perp}^2) \int_{-\infty}^{\infty} d\hat{v}_{\parallel} \exp(-\hat{v}_{\parallel}^2) \left\{ \frac{T_e}{T_i} \omega \right. \\
 & + \omega_{*e} [1 + \eta_i (\hat{v}_{\parallel}^2 + \hat{v}_{\perp}^2 - 3/2)] \} [(\omega - \omega_{di(1)})^{-1} \\
 & - \frac{\hat{v}_{\parallel}^2 \bar{\omega}_{ti}^{-2}}{[\omega - \omega_{di(1)}]} \left(\frac{\partial}{\partial \theta} + iS \right) (\omega - \omega_{di(1)})^{-1} \left(\frac{\partial}{\partial \theta} + iS \right) (\omega - \omega_{di(1)})^{-1} \\
 & + i \left(\frac{\hat{v}_{\perp}^2 \bar{\omega}_{ti} q_{\rho i}}{2\Delta r_s} \right) \frac{\sin \theta}{[\omega - \omega_{di(1)}]^2} \frac{\partial}{\partial S} - \left(\frac{\hat{v}_{\perp}^2 \bar{\omega}_{ti} q_{\rho i}}{2\Delta r_s} \right)^2 \frac{\sin^2 \theta}{[\omega - \omega_{di(1)}]^3} \\
 & \times \frac{\partial^2}{\partial S^2} \left. (2\pi)^{-1/2} \int_{-\infty}^{\infty} dk_s \exp[ik_s S] J_0^2[\rho_i \hat{v}_{\perp} (k_{\theta}^2 + k_r^2)^{1/2}] \right. \\
 & \times \frac{\exp(ij\theta)}{(2\pi)^{1/2}} (2\pi)^{-1/2} \int_{-\infty}^{\infty} dS \exp[-ik_s S] h_n(S) \quad (C1)
 \end{aligned}$$

and $k_s \equiv k_r \Delta r_s$ is dimensionless. Using Eq. (40), integrating by parts so that the S derivatives act on the $h_{n'}(S)$ function,

carrying out the θ differentiations, using the recursion and differentiation relations²⁴ for Hermite functions on $h_n(S)$, and noting that the order of integration may be changed so that the first S integral in Eq. (C1) just gives the radial Fourier transform of $h_n(S)$ yields finally

$$\begin{aligned}
 L_{jj',nn'}^{\text{inr}} &= \frac{T_e}{T_i} \delta_{jj'} \delta_{nn'} - \int_{-\pi}^{\pi} d\theta \frac{\exp[i(j-j')\theta]}{2\pi} \int_{-\infty}^{\infty} dk_s \\
 &\times \frac{(-i)^{n-n'}}{\sigma} h_n(k_s/\sigma) \frac{2}{\pi^{1/2}} \int_0^{\infty} d\hat{v}_{\perp} \hat{v}_{\perp} \exp(-\hat{v}_{\perp}^2) \int_{-\infty}^{\infty} d\hat{v}_{\parallel} \exp(-\hat{v}_{\parallel}^2) \\
 &\times \left\{ \frac{T_e}{T_i} \omega + \omega_{*e} [1 + \eta_i (\hat{v}_{\parallel}^2 + \hat{v}_{\perp}^2 - 3/2)] \right\} J_0^2[\rho_i \hat{v}_{\perp} (k_{\theta}^2 + k_r^2)^{1/2}] \\
 &\times \left\{ \frac{h_{n'}(k_s/\sigma)}{\omega - \hat{\omega}_{di} \cos \theta} + \frac{\hat{v}_{\parallel}^2 \bar{\omega}_{ti}^2}{(\omega - \hat{\omega}_{di} \cos \theta)^3} [j^2 h_{n'} + ij \sigma^{-1/2} (h_{n'+1} R_{n'+1,n'} \right. \\
 &- 2n' h_{n'-1} R_{n'-1,n'}) - \sigma^{-1} [\frac{1}{4} h_{n'+2} R_{n'+2,n'} - (n' + \frac{1}{2}) h_{n'} \\
 &+ n'(n' - 1) h_{n'-2} R_{n'-2,n'}) + \frac{\hat{\omega}_{Di} (\cos \theta + 3ij \sin \theta)}{[\omega - \hat{\omega}_{di} \cos \theta]} h_{n'} \\
 &+ \frac{\hat{\omega}_{Di} (3i \sin \theta)}{[\omega - \hat{\omega}_{di} \cos \theta]} \frac{i}{2\sigma^{1/2}} (h_{n'+1} R_{n'+1,n'} - 2n' h_{n'-1} R_{n'-1,n'})
 \end{aligned}$$

continued

$$\begin{aligned}
 & + \frac{\hat{\omega}_{Di}^2 \sin^2 \theta}{(\omega - \hat{\omega}_{Di} \cos \theta)^2}] - \left(\frac{\hat{v}_i^2 q \rho_i \bar{\omega}_{ti}}{2 \Delta r_s} \right) \frac{\sin \theta}{(\omega - \hat{\omega}_{Di} \cos \theta)^2} \\
 & \times \sigma^{1/2} \left(\frac{1}{2} h_{n'+1} R_{n'+1, n'} + n' h_{n'-1} R_{n'-1, n'} \right) + \left(\frac{\hat{v}_i^2 q \rho_i \bar{\omega}_{ti}}{2 \Delta r_s} \right)^2 \\
 & \times \frac{\sin^2 \theta}{(\omega - \hat{\omega}_{Di} \cos \theta)^3} \sigma \left[\frac{1}{4} h_{n'+2} R_{n'+2, n'} + \left(n' + \frac{1}{2} \right) h_{n'} \right. \\
 & \left. + n' (n' - 1) h_{n'-2} R_{n'-2, n'} \right] \} , \quad (C2)
 \end{aligned}$$

where $\hat{\omega}_{Di} \equiv \bar{\omega}_{Di} (\hat{v}_i^2 + 2 \hat{v}_{||}^2)$ and $R_{n_1, n_2} \equiv (M_{n_1} / M_{n_2})^{1/2}$. The θ integrals for all of the terms may, by simple trigonometric identities, be put into the form

$$\int_0^\pi d\theta \frac{\cos n \theta}{(b + \cos \theta)^m}$$

where n is an integer and $m = 1, 2, 3, 4$, or 5 . From Ref. 25,

$$\int_0^\pi d\theta \frac{\cos n \theta}{b + \cos \theta} = \pi \frac{[(b^2 - 1)^{1/2} - b]^n}{(b^2 - 1)^{1/2}} \equiv \pi D_0(b, n) . \quad (C3)$$

In Eq. (C3), the square roots take the sign of b . Differentiating Eq. (C3) successively with respect to b gives the other needed formulas:

$$\int_0^\pi d\theta \frac{\cos n\theta}{(b + \cos \theta)^2} = -\pi \frac{[(b^2 - 1)^{1/2} - b]^{n-1}}{(b^2 - 1)^{3/2}} \{b(n-1)[(b^2 - 1)^{1/2} - b] + n\} \equiv -\pi D_1(b, n), \quad (C4)$$

$$\int_0^\pi d\theta \frac{\cos n\theta}{(b + \cos \theta)^3} = \frac{\pi}{2} \frac{[(b^2 - 1)^{1/2} - b]^n}{(b^2 - 1)^{5/2}} [n^2(b^2 - 1) + 3nb(b^2 - 1)^{1/2} + 2b^2 + 1] \equiv \frac{\pi}{2} D_2(b, n), \quad (C5)$$

$$\int_0^\pi d\theta \frac{\cos n\theta}{(b + \cos \theta)^4} = \frac{\pi}{6} \frac{[(b^2 - 1)^{1/2} - b]^n}{(b^2 - 1)^{7/2}} [n^3(b^2 - 1)^{3/2} + 6n^2b(b^2 - 1) + n(11b^2 + 4)(b^2 - 1)^{1/2} + (6b^3 + 9b)] \equiv -\frac{\pi}{6} D_3(b, n), \quad (C6)$$

and

$$\int_0^\pi d\theta \frac{\cos n\theta}{(b + \cos \theta)^5} = \frac{\pi}{24} \frac{[(b^2 - 1)^{1/2} - b]^n}{(b^2 - 1)^{9/2}} [n^4(b^2 - 1)^2 + 10n^3b(b^2 - 1)^{3/2} + n^2(35b^2 + 10)(b^2 - 1) + n(50b^3 + 55b)(b^2 - 1)^{1/2} + (24b^4 + 72b^2 + 9)] \equiv \frac{\pi}{24} D_4(b, n). \quad (C7)$$

In all of these formulas, half-integer powers of $(b^2 - 1)$ take

the sign of b . Using these results, the nonresonant ion term matrix element may be written as

$$\begin{aligned}
 L_{jj',nn'}^{\text{inr}} &= \frac{T_e}{T_i} \delta_{jj'} \delta_{nn'} - \frac{(i)^{n-n'}}{\sigma} \int_{-\infty}^{\infty} dk_s h_n(k_s/\sigma) \frac{2}{\pi^{1/2}} \int_0^{\infty} d\hat{v}_\perp \hat{v}_\perp \\
 &\times \exp(-\hat{v}_\perp^2) J_0^2[\rho_i \hat{v}_\perp (k_\theta^2 + k_s^2/\Delta r_s^2)^{1/2}] \int_{-\infty}^{\infty} d\hat{v}_\parallel \exp(-\hat{v}_\parallel^2) \left\{ \frac{T_e}{T_i} \omega + \omega_{*e} \right. \\
 &\times [1 + \eta_i (\hat{v}_\parallel^2 + \hat{v}_\perp^2 - 3/2)] \} (-\hat{\omega}_{di})^{-1} \left[h_{n'}(k_s/\sigma) D_0(b, |j-j'|) \right. \\
 &- \frac{\hat{v}_\parallel^2 \omega_{ti}}{\hat{\omega}_{di}^2} \left(-\frac{h_{n'}}{12} [D_3(b, |j-j'+1|) + D_3(b, |j-j'-1|)] + \frac{h_{n'}}{32} \right. \\
 &\times [2D_4(b, |j-j'|) - D_4(b, |j-j'+2|) - D_4(b, |j-j'-2|)] + \frac{1}{4} [j h_{n'} + \\
 &\frac{i}{2\sigma^{1/2}} (h_{n'+1} R_{n'+1,n'} - 2n' h_{n'-1} R_{n'-1,n'})] [D_3(b, |j-j'-1|) - D_3(b, |j-j'+1|)] \\
 &+ \frac{1}{2} \{-j^2 h_{n'} - \frac{ij}{\sigma^{1/2}} (h_{n'+1} R_{n'+1,n'} - 2n' h_{n'-1} R_{n'-1,n'}) \\
 &+ \frac{1}{\sigma} [\frac{1}{4} h_{n'+2} R_{n'+2,n'} - (n' + \frac{1}{2}) h_{n'} + n'(n'-1) h_{n'-2} R_{n'-2,n'}]
 \end{aligned}$$

continued

$$\begin{aligned}
& \times D_2(b, |j-j'|) \} \Big\} - i \frac{\sigma^{1/2}}{4} \left(\frac{\hat{v}_i^2 \bar{\omega}_{ti} q \rho_i}{2\Delta r_s} \right) (-\hat{\omega}_{di})^{-1} (h_{n'+1} R_{n'+1, n'} \\
& + 2n' h_{n'-1} R_{n'-1, n'}) [D_1(b, |j-j'+1|) - D_1(b, |j-j'-1|)] \\
& + \frac{\sigma}{8} \left(\frac{\hat{v}_i^2 \bar{\omega}_{ti} q \rho_i}{2\Delta r_s} \right)^2 (-\hat{\omega}_{di})^{-2} \left[\frac{1}{4} h_{n'+2} R_{n'+2, n'} + (n' + \frac{1}{2}) h_{n'} \right. \\
& + n'(n'-1) h_{n'-2} R_{n'-2, n'} \Big] [2D_2(b, |j-j'|) - D_2(b, |j-j'+2|) \\
& \left. - D_2(b, |j-j'-2|) \right] , \tag{C8}
\end{aligned}$$

where $b \equiv (-\omega/\hat{\omega}_{di})$.

In like manner, the ion collision term matrix element is

$$\begin{aligned}
T_{jj', nn'}^{ic} &= i \frac{v_{ii}}{\omega^2} \delta_{jj'} \frac{(-i)^{n-n'}}{\sigma} \int_{-\infty}^{\infty} dk_s h_n(k_s/\sigma) h_{n'}(k_s/\sigma) \\
&\times \left[H(b_i - 1) \frac{3(\pi+1)}{8\pi^{1/2}} \left[\frac{T_e}{T_i} \omega + \omega_{*e} \left(1 - \frac{3\pi+2}{4\pi+4} \eta_i \right) \right] b_i^{1/2} + H(1 - b_i) \right. \\
&\left. + \frac{7}{10} (1 + 0.2517 b_i) \left\{ \frac{T_e}{T_i} \omega + \omega_{*e} \left[1 - \frac{3}{28} (1 + 5.4366 b_i) \eta_i \right] b_i^2 \right\} \right] , \tag{C9}
\end{aligned}$$

where $b_i \equiv (\rho_i^2/2) (k_\theta^2 + k_s^2/\Delta r_s^2)$ and H is the Heaviside step function.

For the circulating ion resonance term, note that Eq. (B22) for $\tilde{\phi}(p)$ can be put into the form, as in Ref. 1,

$$\begin{aligned} \tilde{\phi}(p) = & \sum_{j,n} \tilde{\phi}_{jn} h_n(S) \frac{\epsilon_0^{1/2}}{2\pi L_t(\Lambda)} \int_{-\pi}^{\pi} d\theta' [1 - \Lambda/h(\theta')]^{-1/2} \\ & \times \frac{\exp(ij\theta')}{(2\pi)^{1/2}} \exp\{-ip\omega_t \hat{t}(\theta') + i[S - S^{(\sim)}(\theta)](\theta' - \theta) - i\omega_t [S - S^{(\sim)}(\theta)] \\ & \times [\hat{t}(\theta') - \hat{t}(\theta)] + w_{di}^{(\sim)}\} , \end{aligned} \quad (C10)$$

where

$$L_t \equiv \frac{\epsilon_0^{1/2}}{2\pi} \int_{-\pi}^{\pi} d\theta [1 - \frac{\Lambda}{h(\theta)}]^{-1/2} , \quad (C11)$$

and a term involving $[S^{(\sim)}(\theta') - S^{(\sim)}(\theta)]$ has been neglected. Then using the explicit form for $S^{(\sim)} = \hat{S}^{(\sim)}X$ given by Eq. (B7) and for $w_{di}^{(\sim)}$ given in Ref. 1, noting a cancellation in lowest order in ϵ_0 between $[\theta S^{(\sim)}(\theta)]$ and one term in $w_{di}^{(\sim)}$, and using the symmetries in θ and j'' of the quantities involved, the circulating ion resonance term matrix element can be written as

$$\begin{aligned} L_{jj',nn'}^{ir} = & \frac{2i}{\pi^{1/2}} \int_{-\infty}^{\infty} dS h_n(S) h_{n'}(S) \int_0^{1-\epsilon_0} d\Lambda \int_0^{\pi} d\theta \{h(\theta) [1 - \Lambda/h(\theta)]^{1/2}\}^{-1} \\ & \times \sum_{p=-\infty}^{\infty} \sum_{\sigma_{||}=\pm 1} H(X_1^2) X_1^2 \exp(-X_1^2) \frac{(T_e/T_i)\omega + \omega_* e [1 + \eta_i (X_1^2 - 3/2)]}{|(p+S)\hat{\omega}_t - 2S^{(\sim)}\hat{\omega}_t X_1|} \cos\left[(j' + S)\theta\right] \end{aligned}$$

$$\begin{aligned}
 & - (p + s) \omega_t \hat{t}(\theta) + \sigma_{||} X_1 (\bar{\omega}_{di} / \bar{\omega}_{ti}) \int_0^\theta d\theta'' \{ [1 - \Lambda/h(\theta'')] \}^{-1/2} \\
 & \times [2 - \Lambda/h(\theta'')] \cos \theta'' \} + \omega_t \hat{t}(\theta) \hat{S}^{(\sim)}(\theta) X_1 \Big\} J_0^2 \{ X_1 [2 \Lambda b_{i\theta} (1 \\
 & + \bar{k}_r^2 / k_\theta^2)]^{1/2} \} \frac{\epsilon_0^{1/2}}{\pi L_t(\Lambda)} \int_0^\pi d\theta' [1 - \Lambda/h(\theta')]^{-1/2} \cos \left\{ (j + s) \theta' \right. \\
 & - (p + s) \omega_t \hat{t}(\theta) + \sigma_{||} X_1 (\bar{\omega}_{di} / \bar{\omega}_{ti}) \int_0^{\theta'} d\theta'' [1 - \Lambda/h(\theta'')]^{-1/2} [2 \\
 & \left. - \Lambda/h(\theta'')] \cos \theta'' \} + \omega_t \hat{t}(\theta') \hat{S}^{(\sim)}(\theta') X_1 \Big\} , \tag{C12}
 \end{aligned}$$

where X_1 is given by Eq. (B21).

In like manner, recalling that $S^{(\sim)}$ and $w_{de}^{(\sim)}$ can be neglected for electrons, the circulating electron term matrix element may be written as

$$\begin{aligned}
 L_{jj',nn'}^{ec} &= \delta_{jj'} \delta_{nn'} - \pi^{-3/2} \left\{ \int_{-\infty}^{\infty} dS h_n(S) h_{n'}(S) \int_0^{1-\epsilon_0} d\Lambda \int_0^\pi d\theta \right. \\
 & \times \{ h(\theta) [1 - \Lambda/h(\theta)]^{1/2} \}^{-1} \sum_{p=-\infty}^{\infty} \cos[(j' + s)\theta - (p + s) \omega_t \hat{t}(\theta)] \sum_{\sigma_{||}=\pm 1} \int_0^\infty dx x^2 \\
 & \times \exp(-x^2) \frac{\omega - \omega_{*e} [1 + \eta_e (x^2 - 3/2)]}{\omega + i\nu_f(X, \Lambda) - (p + s) \hat{\omega}_t X} \frac{\epsilon_0^{1/2}}{\pi L_t(\Lambda)} \int_0^\pi d\theta' [1 - \Lambda/h(\theta')]^{-1/2}
 \end{aligned}$$

continued

$$\times \cos[(j + s)\theta' - (p + s)\omega_t \hat{t}(\theta')]\left[1 + \frac{iv_f^h}{k_{||}v_e} z\left(\frac{\omega - k_{||}u_o + iv_f^h}{k_{||}v_e}\right)\right]^{-1} . \quad (C13)$$

Also the trapped electron term matrix element is

$$\begin{aligned} L_{jj',nn'}^{eT} = & -\pi^{-3/2} \left(\int_{-\infty}^{\infty} ds h_n(s) h_{n'}(s) \int_{1-\epsilon_o}^{1+\epsilon_o} d\Lambda \int_0^{\theta_o(\Lambda)} d\theta \right. \\ & \times \{h(\theta) [1 - \Lambda/h(\theta)]^{1/2}\}^{-1} \sum_{p=-\infty}^{\infty} \sum_{\sigma_{||}=\pm 1} \cos[(j' + s)\theta - p\omega_b \hat{t}(\theta)] \\ & \times \int_0^{\infty} dx x^2 \exp(-x^2) \frac{\omega - \omega_{*e} [1 + \eta_e (x^2 - 3/2)]}{\omega - \hat{\omega}_{de}^{(o)} x^2 + iv_f(X, \Lambda) - p\hat{\omega}_b x} \frac{\epsilon_o^{1/2}}{2\pi L_b(\Lambda)} \int_0^{\theta_o(\Lambda)} d\theta' \\ & \times [1 - \Lambda/h(\theta')]^{-1/2} \{ \cos[(j + s)\theta' - p\omega_b \hat{t}(\theta')] + (-1)^p \cos[(j \\ & + s)\theta' + p\omega_b \hat{t}(\theta')] \} \left[1 + i \frac{v_f^h}{k_{||}v_e} z\left(\frac{\omega - k_{||}u_o + iv_f^h}{k_{||}v_e}\right) \right] , \end{aligned} \quad (C14)$$

where

$$L_b(\Lambda) \equiv \frac{\epsilon_o^{1/2}}{2\pi} \int_{-\theta_o(\Lambda)}^{\theta_o(\Lambda)} d\theta [1 - \frac{\Lambda}{h(\theta)}]^{-1/2} . \quad (C15)$$

Table I. Effects on $\omega = \omega_r + i\gamma$ of turning off terms in mode equation for basic case parameters. Here ω_r and γ are in units of ω_{*e} .

Term turned off	ω_r	γ
none	0.319	0.175
circulating electron	0.326	0.160
trapped electron ($p \neq 0$)	0.321	0.173
electron collision number conservation	0.319	0.177
simple electron collision operator (see text)	0.313	0.170
ion "resonance"	0.321	0.178
ion finite radial excursion	0.317	0.169
ion sound	0.318	0.179
ion collision	0.319	0.177

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FIGURE CAPTIONS

Figure 1. Real and imaginary parts of the perturbed electrostatic potential eigenfunction $\tilde{\phi}(\theta, S)$ for the basic case. Note that the poloidal angle θ is zero at the outside of the torus, and that the radial variable S is $S = (r - r_0)/\Delta r_s$, where r_0 is the radius of the mode-rational surface around which the mode is centered, and Δr_s is the spacing of mode-rational surfaces. Only positive values of S are shown since the eigenfunction is symmetric in $(S\theta)$. The parameters are $\epsilon_0 = 0.1$, $T_e/T_i = 2$, $q = 2.5$, $q'r/q = 1$, $n_i = n_e = 0$, $r_n/r = 1$, $m_i/m_e = 3672$, $v_e^* = 0.03$, $b_{i\theta} = 0.2$, and $u_0 = 0$.

Figure 2. Real and imaginary parts of $\tilde{\phi}(\theta, S) \exp(iS\theta)$, which gives the non-periodic variation of the potential along the magnetic field lines, for the basic case.

Figure 3. Real and imaginary parts of the radial Fourier transform $\tilde{\phi}(\theta, k_r)$ of $\tilde{\phi}[\theta, S(r)]$ for the basic case.

Figure 4. Growth rate γ in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ (approximately the average trapped electron bounce frequency) versus $v_e^* \equiv v_{ei}/(\epsilon_0^{3/2} \bar{\omega}_{te})$ for $b_{i\theta} = 0.2$ and for the local code for $b_{i\theta} = 0.2$ and 0.8 . The local code includes the shear damping term obtained by solving the one-dimensional radial Weber's equation.

Figure 5. Real and imaginary parts of $\tilde{\phi}(\theta, S)$ as in Fig. 1, except that $v_e^* = 10$.

Figure 6. Real and imaginary parts of $\tilde{\phi}(\theta, S)$ as in Fig. 1, except that $v_e^* = 120$.

Figure 7. Growth rate γ in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ versus $b_{i\theta}$ for several different collisionalities v_e^* for basic case parameters. The solid curves give results from the two-dimensional code, while the dashed curves give results from the local code.

Figure 8. Growth rate γ divided by k_θ^2 in arbitrary units corresponding to the curves in Fig. 7. The dots give the value of γ/k_θ^2 for the maximum value of γ as k_θ varies in Fig. 7, for the two-dimensional code.

Figure 9. Growth rate γ in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ versus $b_{i\theta}$ for basic case parameters, varying ϵ_0 and $\eta_{i,e}$. The solid curves are from the two-dimensional code and the dashed curves from the local code.

Figure 10. Real and imaginary parts of $\tilde{\phi}(\theta, S)$ as in Fig. 1, except that $b_{i\theta} = 0.1$, $\epsilon_0 = 0.25$ and $\eta_{i,e} = 1$.

Figure 11. Growth rate γ in units of $\epsilon_0^{1/2} \bar{\omega}_{te}$ versus the shear parameter $q'r/q$. The solid curve was obtained from the two-dimensional code and the dashed curve from the local code.

Figure 12. Real and imaginary parts of $\tilde{\phi}(\theta, S)$ as in Fig. 1, except for a mode other than the lowest mode, in particular for a mode odd in θ around $\theta = 0$ (and π) at $S = 0$.

Figure 13. Real frequency ω_r and growth rate γ in units of $\epsilon_0^{1/2} \omega_{te}$ versus $b_{i\theta}$, for the parameters of Ross, Tang, and Adam²⁰ (see text). The dashed curves are taken from Ref. 20, while the solid curves give the two-dimensional code results.

Fig. 1. 772322

Fig. 2. 772320

Fig. 3. 772323

Fig. 4. 772310

Fig. 5. 772328

Fig. 6. 772327

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Fig. 8. 772326

Fig. 9. 772324

Fig. 10. 772289

Fig. 11. 772331

Fig. 12. 772332

Fig. 13. 772382