A DESIGN STUDY FOR A HIGH-CURRENT, STEADY-
STATE AUTO-RESONANT ACCELERATOR

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Final Report
for Period July 14, 1979-November 14, 1979
Report #I-ARA-79-U-136 (ARA-368)

W. E. Drummond, et al.

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December 1979

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I. INTRODUCTION AND SUMMARY OF RESEARCH PROGRESS

During the past year and a half, both theoretical and engineering investigations have been carried forward in a research program to assess the feasibility, and provide conceptual designs and engineering design studies for high current, steady state Auto-Resonant Accelerators. The behavior of each of the various components of an Auto-Resonant Accelerator has been carefully examined for the case of moderately relativistic electron energy and high electron current which will be appropriate for steady state operation. In general, a very considerable amount of progress has been achieved on the front-end components of the accelerator, with design options narrowed significantly and preliminary design parameters even determined for the power supplies, the electron beam source, the radiation shielding, the convergent Pierce gun diode, and the electron cyclotron wave exciter. For the downstream accelerator components consisting of the gas cell and electrode structures which are used to generate and load

the ions into the cyclotron wave and to eliminate secondary electrons, conceptual designs have been realized; however, some further work remains to be done before an optimized engineering design of these components can be constructed for a first experimental test of a steady state Auto-Resonant Accelerator.

Specifically, the progress during this design study may be summarized as follows:

1. The fundamental theory of electron beam equilibria and of the electron cyclotron waves which may be supported upon the electron beam was extended to the nonultrarelativistic electron energy range required for steady state accelerator operation, with no deleterious phenomena being encountered;
2. The SURVEY computer code used to generate design parameters for Auto-Resonant Accelerators was generalized to the nonultrarelativistic regime and was exercised extensively to provide design parameter inputs for the engineering design studies;

3. In accordance with the SURVEY indications that a first experimental test of the steady state Auto-Resonant Accelerator might best be performed in a system of moderate voltage and current (e.g., 400 kV, 400 amperes), engineering design studies were carried out for such an electron beam accelerator, resulting in a single gap, vacuum insulated high voltage system. Using an empirically conservative electric stress level, a very compact electron injector design was realized. Both dc requirements of high voltage and thermionic cathode design (and power dissipation) were achieved. At present, the electron injector design and cathode design are felt to be well in hand. The proposed rep-rated mode of prototype operation should be easily realized with conventional power supplies and capacitive energy storage;
4. Engineering design studies of conventional impregnated tungsten matrix cathodes have indicated that cathode operation at 5 A/cm^2 in a convergent

beam, single gap geometry is feasible, and that power requirements are minimal;

5. Cost trade-off studies have suggested that use of modern superconducting magnets may be preferable in the high field sections of the accelerator; however, maximum flexibility is achieved with conventional water cooled systems;
6. A computer code PGUN was developed and exercised to explore the electron beam optics in the Pierce gun and subsequent magnetic compression region. Satisfactory beam characteristics were obtained with realizable Pierce gun designs: Beam compression of the order desired ($10^2 - 10^3$ area compression) was found to be feasible, with no major problems with beam emittance. Transverse beam heating was also acceptable;
7. Theoretical examination of cyclotron wave generation revealed that a wave of the desired amplitude can be achieved via amplification in a slow wave helical structure in the region of uncompressed magnetic field, without severe technological or engineering problems;

8. The possibility of encountering a Landau resonance in the transverse electron cyclotron motion for certain accelerator configurations was observed on the SURVEY code and was explored theoretically. Although no divergences are present within linearized cold fluid theory, some wave damping is possible in such cases due to electron thermal effects. This does not appear to jeopardize the successful functioning of an Auto-Resonant Accelerator, although additional studies are in order;
9. An examination of a neutral gas cell in the electron beam path as a means of supplying the ions to be accelerated by the cyclotron wave has revealed the greatest unanticipated complexity in these design studies. The gas cell chemistry favors the predominance of molecular ion species, such as H_2^+ or D_2^+ . We have determined that the wave should be initiated at a high phase velocity before reaching the gas cell, and then decelerated to the desired ion injection velocity via magnetic compression downstream of the gas cell. Meanwhile,

the ions which are formed by electron impact ionization of the neutral gas, should be pre-accelerated to the injection velocity by a dc potential drop applied across and also downstream of the gas cell. With proper tuning, these pre-accelerated ions may then be trapped in the cyclotron wave prior to the final, principal wave-ion acceleration;

10. A method of extracting slow secondary electrons from the primary beam path via application of a transverse electric field to the beam has been examined and found to be feasible, although it requires some complexity of the electrode geometry near the gas cell.

We believe that the progress achieved during this design study has been sufficient that a first experimental test of a steady state Auto-Resonant Accelerator may now be attempted with a reasonably high level of confidence of success. Although further analysis will undoubtedly be necessary in realizing a finalized engineering design, which

is optimized to the specific goals adopted for such an experimental test, such further analysis might well be most fruitfully performed in tandem with experimental preparations.

II. DESIGN CONSIDERATIONS FOR A HIGH CURRENT, STEADY STATE AUTO-RESONANT ACCELERATOR

The initial conceptual design of a high current, steady state Auto-Resonant Accelerator, with which we began this contract study, is illustrated in the schematic of Figure 1. Because we first anticipated using a moderately large electron beam source of 700 to 800 amperes of 500 keV electrons, it was thought most reasonable to use a hot cathode Pierce type convergent gun as the electron current source. Once formed in the Pierce gun at an initial particle energy of perhaps 20 keV, the beam would be brought to full energy by post-acceleration down a resistively graded acceleration tube.

However, as this study progressed, it was recognized that the most logical range of beam parameters for a first experimental test of such a steady state Auto-Resonant Accelerator are somewhat more modest. We now anticipate electron beam source requirements in the range of 300 to 400 amperes of 300 to 400 keV electrons, transported along a

Figure 1. Schematic for Steady State Auto-Resonant Accelerator

magnetic field whose axial strength ranges from 0.1 kG to 40 kG throughout the system. Such a beam source is common to several of the more promising applications for such a steady state ion accelerator, mentioned earlier (reference 1).

The goal of this first experimental module might be to provide a meaningful, yet cost effective, proof-of-principle demonstration of the experimental feasibility of a steady state Auto-Resonant Accelerator. It should be possible to accelerate ions during this first experiment to an energy of some 3 MeV per ion (i.e., about ten times the energy per beam electron), at a current level of 0.1 to 1 ampere. The envisioned experiment would undoubtedly not be operated steady state, but would have some reasonable duty factor of perhaps 10^{-4} , such that the essential physics would still be effectively steady state in nature. A sample SURVEY output which illustrates the parameters envisioned for the wave acceleration section of this module is shown in Table 1.

1. W. E. Drummond, Proposal for "A Design Study for a High Current, Steady State Auto-Resonant Accelerator," December 1977. ARA Report No. I-ARA-77-U-102 (ARA-296).

TABLE 1. Possible Design Parameters for Steady State
Proof-of-Principle Experiment.

INITIAL E-BEAM ENERGY (MV) =	0.300	INITIAL ION ENERGY (MV) =	0.100
INITIAL E-BEAM CURRENT (KAMPS) =	0.400	FINAL ION ENERGY (MV) =	3.00
INITIAL BEAM RADIUS (CM) =	0.250	ION CURRENT (KAMPS) =	0.00100
WALL TO BEAM RADIUS RATIO =	2.000	ION CHARGE NUMBER =	1.0
INITIAL MAGNETIC FIELD (KG) =	40.00	MASS NUMBER =	2.0
FREQUENCY (MHZ) =	1012.	WAVE PARAMETER ALPHA =	2.510
SAFETY FACTOR FOR ACCELERATION =	0.670	INITIAL KZ/KP VALUE =	1.646089

	LENGTH (METERS)	TIME (NS)	ELECTRON ENERGY (MV)	MAGNETIC FIELD (KG)	POTENTIAL (MV)	ION ENERGY (MV)	ELECTRIC FIELD (MV/M)	WORST INEQUALITY
ET	0.000	0.000	0.269	40.000	0.0010	0.100	2.043	DELV =0.54
	0.115	28.400	0.269	24.499	0.0018	0.263	2.197	DELV =0.54
	0.221	47.060	0.268	19.182	0.0024	0.423	2.335	DELV =0.56
	0.323	61.900	0.268	16.193	0.0029	0.588	2.450	DELV =0.58
	0.418	73.814	0.267	14.304	0.0034	0.746	2.543	DELV =0.60
	0.513	84.515	0.267	12.892	0.0039	0.911	2.625	DELV =0.62
	0.601	93.613	0.266	11.857	0.0044	1.068	2.693	DELV =0.63
	0.696	102.600	0.266	10.959	0.0048	1.240	2.758	DELV =0.65
	0.780	110.126	0.265	10.285	0.0053	1.397	2.811	DELV =0.67
	0.865	117.298	0.265	9.702	0.0057	1.558	2.861	DELV =0.68
	0.949	124.056	0.264	9.198	0.0060	1.722	2.906	DELV =0.69
	1.032	130.341	0.263	8.765	0.0064	1.883	2.945	DELV =0.70
	1.111	136.101	0.263	8.395	0.0068	2.039	2.973	DELV =0.71
	1.195	142.046	0.262	8.040	0.0071	2.208	2.989	DELV =0.72
	1.274	147.380	0.262	7.740	0.0074	2.367	3.025	DELV =0.73
	1.357	152.847	0.261	7.450	0.0078	2.538	3.063	DELV =0.74
	1.433	157.547	0.261	7.208	0.0081	2.694	3.094	DELV =0.75
	1.513	162.545	0.260	6.974	0.0085	2.860	3.125	DELV =0.76
	1.582	166.701	0.260	6.784	0.0087	3.005	3.148	DELV =0.77

If successful, such an experimental module could then represent the first stage of a two stage accelerator, for which the second stage might boost the ion energy from 3 MeV to 30 MeV per ion. Alternatively, the first module might represent the initial portion of a longer single stage accelerator which could likewise achieve higher ion energy per particle.

Adopting this tentative accelerator goal for the sake of delimiting the design parameter domain, we carried forward engineering design studies of all of the major components of such an experimental module. In the course of this study, a number of improvements and alterations were evolved to the initial conceptual design of Figure 1.

The results of the design considerations for each of the system components, which constitute the conceptual design of a high current, steady state Auto-Resonant Accelerator, are presented in the remainder of this section.

A. Engineering Considerations of the Electron Beam Source and Transport Magnet System

1. General Discussion

To give some perspective to the engineering studies described in this section, some guidelines used for design calculation purposes are listed:

- (a) the electron beam energy is not to exceed 500 kilovolts.
- (b) the electron beam current is to lie in the range 300 to 400 amperes.
- (c) maximum magnetic field strengths are substantial, i.e., 50 kilogauss.
- (d) although a prototype system would operate at a low duty cycle, as many components as possible are scoped assuming true d.c. operation.
- (e) whenever possible, existing and reliable technology is employed.

2. System Description

Figure 2 shows schematically the components required for a prototype electron beam source. For voltages not

exceeding 500 kilovolts, the use of a single vacuum gap for accelerating the electron beam to full energy is preferable. The real advantage of this approach is that it considerably simplifies the problem of beam focusing. In addition, this system is much more compact, and less expensive, than the use of a graded acceleration tube system. Details of how one sizes such a system for a particular voltage is given in Appendix A.

A hot cathode Pierce type convergent gun is used as the current source. Once formed, the beam is brought to full energy by acceleration across a spherically convergent vacuum gap. The system is evacuated using a turbo-molecular pump. This type of pump is chosen because it produces a "clean" vacuum, in the sense that oil back-streaming is virtually nonexistent. This minimizes the problem of cathode poisoning. Also, this pump is equally efficient in pumping all types of gases.

To provide the fast energy needed for a pulsed prototype system, a low inductance capacitive store is used. Its value is

$$C = \frac{\Delta Q}{\Delta V} = 0.1 \frac{I\tau}{\eta V_0} \quad (\mu F)$$

where

I = beam current in kiloamps

τ = beam pulse width in μ seconds

η = percentage voltage droop allowable

V_0 = main accelerating voltage

To achieve this capacitance, and total voltage hold-off, a series--parallel arrangement of capacitors in a radial array is used.

For the prototype system, the Pierce gun is gated on and off by a control grid. To estimate the gun size, a crude approximation based on the parallel plate Child-Langmuir law is adequate:

$$J = 2.34 \times 10^{-6} \frac{V^{3/2}}{d^2}$$

For

$$J = 5 \text{ A/cm}^2$$

and

$$V = 2 \times 10^4 \text{ volts}$$

then

$$d \approx 1.15 \text{ cm}$$

Having a grid control separation of this size is desirable since mechanical tolerances due to thermal expansion are less stringent. It should be noted that achieving current densities of 5 A/cm^2 is easily realized using conventional barium impregnated dispenser cathodes. Care must be taken to insure that all parts which become hot are made of, or plated with, molybdenum. Stainless steel, for example, when heated will evaporate chromium atoms which quickly poisons a dispenser cathode.

A quantity of importance in the study of beam optics is the beam perveance. This is defined as

$$P = \frac{I}{V^{3/2}}$$

and is given a dimension of pervals (= amperes per volt^{1.5}). For parameters of interest, the final beam has a perveance of

$$P \approx 1.6 \times 10^{-6} \text{ pervals} = 1.6 \text{ } \mu\text{pervs}$$

Beams of this perveance are commonly used in traveling wave tube microwave sources; further, electron beam optical components such as magnet systems can readily manipulate beams of this perveance.

Prior to acceleration to full energy, the situation is quite different. The Pierce gun perveance is

$$P = \frac{I}{V^{3/2}} = 2.34 \frac{A}{d^2} \text{ } \mu\text{pervs}$$

where

A = cathode area in cm²

For a typical case,

I = 400 amps

V = 20 kV

then

P ≈ 140 μpervs

which is rather large. The cathode diameter to gap spacing ratio is

$$D/d = \sqrt{\frac{4}{\pi} \left(\frac{P}{2.34} \right)} \approx 8.77$$

giving a cathode diameter of

D ≈ 10 cm

Assuming power loss only from the front surface of the cathode, and using unit emissivity and a working temperature of 1200°C, the minimum heater power required is

$$\dot{W} = 5.67 (1.5)^4 \text{ A}$$

~ 2.3 kilowatt

Without cooling being provided, such power levels would very quickly heat up the entire high voltage terminal structure. Since a low voltage, high current tungsten wire array is the normal method of heating the cathode, i.e., mounted on the back surface of the cathode, substantial a.c. (or d.c.) magnetic fields are generated near the cathode surface by the heater current. To minimize this, a bi-filar heater design will be used. This will provide minimal perturbation of the normal Child law laminar flow.

Referring again to Figure 2, the high voltage supply is sized to provide average power only. High peak power is supplied by the capacitive store. To scope the power supply

cost versus average power, Table 2 gives some illustrative numbers.

TABLE 2

<u>Voltage (kV)</u>	<u>Average Power (kW)</u>	<u>Price</u>
500	50	\$ 25 K
500	500	\$ 150 K

For intermediate power levels, a linear scaling is used.

Also housed inside the high voltage supply are two additional structures. One is a Faraday enclosure housing a pulse forming network (PFN) for gating the cathode on and off. It also houses a step-down transformer for supplying cathode heater power. Note that a "video" signal is supplied to the PFN to control the current pulse length and duty cycle. Secondly, a high voltage isolation transformer is used to supply "line level" - 120 V, 60 Hz - power to the components in the Faraday enclosure.

The system described above has all but the power supply and beam current operating under d.c. conditions. Nothing exceeding conventional technology is involved. The resultant system is quite compact, and relatively inexpensive.

3. Magnet System

The accelerator parameters under consideration require magnetic guide fields ranging from < 0.5 to ≥ 40 kilogauss; without knowing exact dimensions it is difficult to make quantitative cost trade-off estimates between conventional water cooled systems requiring high power consumption, and superconducting systems (including a cryostat to re-cycle the helium) where operating costs are low, but with a larger initial fabrication cost. For a system involving moderately long high field regions, initial estimates based on 2000 hours/year operation indicate that the superconducting system may be cheaper. The main drawback of this approach is that reconfiguring the field profile is costly. If flexibility in charging the magnetic field profile is desirable, a likely situation for a prototype system, then a conventional design is better.

B. Computer Simulations of a Steady State Pierce Gun

A computer code (PGUN) has been developed which numerically simulates the steady state behavior of a cylindrically symmetric electron beam accelerator. The code runs on ARA's in-house VAX computer, and makes use of both the GENISCO color graphics display and VERSAPLOT hard copy graphics. A brief description of the programming algorithms, along with preliminary results, is given in Appendix B. These code results indicate that a beam with the desired characteristics, e.g., $I = 400$ A, $B_z = 1.8$ kG, $a = 1.2$ cm can be achieved with a realizable design of the emitting Pierce gun. Confidence in this code is high because the results are in good agreement with theory, and the monitored conservations are well satisfied.

C. Technological Requirements for Travelling Wave Generation

In any travelling wave-type accelerator, a major requirement is the generation and maintenance of a travelling wave of sufficient amplitude to efficiently accelerate particles to high energy. In the Auto-Resonant Accelerator

concept, this wave is the lower Doppler-shifted electron cyclotron wave, which must be first externally impressed on the electron beam before ion acceleration can be initiated. There are, conceptually, two distinct sets of beam parameters where wave generation can occur, corresponding to two different points along the electron beam transport system:

1. Generation of the wave in a relatively low magnetic field region (e.g., following extraction of the beam from the Pierce gun diode), with magnetic fields in the 1 to 10 kG range and beam radii on the order of 1 centimeter, and with the wave phase velocity on the order of 0.1 c, or

2. Generation of the wave in a relatively strong magnetic field region, as, for example, following compression on the beam to the 0.25 cm radius where the guide magnetic field is on the order of 40 to 50 kG, and the wave phase velocity is about 0.01 c.

For purposes of investigating technological requirements for the generation of the desired amplitude cyclotron wave, both these regimes have been considered:

(1) Compressed Beam Case

Magnetic Guide Field	40 kG
Beam Energy	300 kV
Beam Current	300 amps
Beam Radius	0.25 cm
Wave Frequency	1 GHz
Wave Phase Velocity	0.01 c
Wave Potential ($\tilde{\Phi}$)	1 kV

(2) Uncompressed Beam Case

Magnetic Guide Field	7 kG
Beam Energy	300 kV
Beam Current	300 amps
Beam Radius	0.60 cm
Wave Frequency	1 GHz
Wave Phase Velocity	0.06 c
Wave Potential ($\tilde{\Phi}$) *	3 kV

*The wave potential is chosen higher in the uncompressed case to compensate for the loss in potential during subsequent compression.

Two specific methods have been considered for the generation of this wave:

1. Direct Drive, through use of a half wavelength inductive strap (or other similar structure).
2. Amplification, through use of a slow wave helical structure.

The results of this preliminary investigation of cyclotron wave generation are given below. When the electron beam-wave parameter design is finalized, these calculations can then be carried out in greater detail to provide more precise quantitative design parameters; however, these calculations indicate clearly which direction must be taken.

Method 1--Direct Drive

Direct drive of the cyclotron wave to the proper amplitude appears totally impractical, both for the compressed and the uncompressed case.

For the compressed beam case, with $B_z = 40$ kG, the cyclotron wavelength is calculated to be 0.32 cm. For the half wavelength inductive strap drive (0.16 cm), the amount of circulating power is calculated to be

$$P \sim 10^9 \tilde{\Phi}^2 \text{ (kV) watts} \quad . \quad (1)$$

For the 1 kV wave potential, this yields

$$P \sim 10^9 \text{ watts} \quad (2)$$

which is prohibitive, both from the point of view of required drive power and electric field stresses.

For the uncompressed case, the length of the structure is longer (1.0 cm), but again the required power is high:

$$P \sim 5 \cdot 10^7 \tilde{\Phi}^2 \text{ (kV) watts} \quad (3)$$

which, for the required 3 kV potential, yields

$$P \sim 5 \cdot 10^8 \text{ watts} \quad . \quad (4)$$

which is again prohibitive.

Thus, in both the compressed and uncompressed cases, direct drive of the cyclotron wave requires power levels on the order of 10^9 watts to be sustained on a physically very small wave drive structure. Direct drive therefore

appears to be an impractical approach to cyclotron wave generation.

Method 2--Amplification by a Helical Slow Wave Structure

The use of a slow wave structure for amplification of the desired cyclotron wave offers several advantages. First, the power amplification down the structure is given by

$$P_f \sim P_o \exp 2L/L_g \quad (5)$$

where P_o is the input power; P_f the final power; L the physical length of the structure; and L_g the amplification length. Although the power P_f must be terminated into a load of some type, the initial driving power P_o can be relatively small, provided that $L/L_g > 1$.

For the case of amplification in the compressed beam region, we again encounter engineering problems: To grow a 1 GHz wave at 40 kG requires that the pitch of the helix be of the order of 1° or less. With the helix radius chosen to be 1.5 times the beam radius (resulting in a 0.36 cm helix

radius), the turn to turn spacing of the helix is then calculated to be 0.04 cm! This is reflected in the high helical impedance ($Z_o \sim 3500$ ohms). Calculated electric field stresses are also rather high ($E \sim 30$ kV/cm). Thus, amplification in the compressed field region looks impractical.

Finally, if we consider wave amplification in the uncompressed field region, we obtain a structure and field levels which appear reasonable:

Helix radius	1.0 cm	
Pitch angle	4.5°	
Turn to turn spacing	0.5 cm	
Z_o	600 ohms	(6)
P_f	$5.8 \cdot 10^4 \tilde{\Phi}^2$ (kV) watts	
E_r	$5.4 \cdot 10^3 \tilde{\Phi}$ (kV) V/cm	
L_g	19 cm	

Taking values of

$$\tilde{\Phi} = 3 \text{ kV} \quad (7)$$

$$L = 3 L_g = 57 \text{ cm} \quad (8)$$

we obtain power levels of

$$P_f = 520 \text{ kW} \quad (9)$$

$$P_o = 1.3 \text{ kW} \quad (10)$$

and maximum electric field stress

$$E \sim 16 \text{ kV/cm} \quad (11)$$

In summary, generation of the desired amplitude cyclotron wave can be accomplished using a slow wave helical structure in the uncompressed magnetic field region without severe technological or engineering problems.

D. Ion Generation in a Gas Cell and Ion Injection Into the Wave

One straightforward way of supplying the ions to be accelerated in the cyclotron wave is via a cell of neutral gas placed in the beam path. If the neutral gas pressure and the dimensions of the gas cell are appropriately selected, the desired quantity of ions may be produced by electron impact

ionization of a small fraction of the neutrals by the primary beam electrons.

These ions may then be injected, or loaded, into the cyclotron wave if the relative velocity between the ions and the wave can be made low enough that the ion energy viewed from the wave frame is less than the wave potential amplitude (of ~ 1 keV).

There are several conceptual configurations for achieving ion generation and ion loading in a fashion compatible with wave growth (parameter definitions follow):

Method (1)--Let the ions be born in the wave and

picked up from rest; which requires

$e\phi_o > e\tilde{\phi} > \frac{1}{2} M v_{ph}^2$. This is achievable by either

(a) Growing a slow wave, or

(b) Growing a fast wave, then slowing it.

Method (2)--Accelerate ions to the injection energy,

then grow a slow wave around these moving

ions; requiring $e\phi_o > e\tilde{\phi} > \frac{1}{2} M (v_{ph} - v_i)^2$.

Method (3)--Preform the wave, then decelerate the

wave and also accelerate the ions to a common injection energy for loading;

requiring $e\phi_0 > e\tilde{\phi} > \frac{1}{2} M(v_{ph} - v_i)^2$.

This is achievable by either

(a) Growing a fast wave, then slowing the

wave to the injection energy, then

accelerating ions to the injection

energy, or

(b) Growing a fast wave, then accelerat-

ing ions to the injection energy,

then slowing the wave to the injec-

tion energy.

In order to understand these alternative concepts for ion

loading, it is helpful to recall from Table 1 that the

desired injection energy is $\frac{1}{2} M v_{ph}^2 \sim 100$ keV, correspond-

ing to a phase velocity of $v_{ph} \sim 0.01$ c for deuterons (D^+) or

hydrogen ions (H_2^+), with compressed beam parameters such as

$B_z \sim 40$ kG. This is what is meant by a "slow" wave. By com-

parison, "fast" wave refers to the uncompressed beam parameters,

such as $B_z \sim 7$ kG with $v_{ph} \sim 0.06$ c. Slowing the wave is

achieved by magnetic compression. The dc potential drop across the 0.4 kA beam of Table 1 is $e\phi_0 \sim 16$ keV, while the desired wave potential is $e\tilde{\phi} \sim 1$ keV.

It may be immediately observed that the simplest method (1) will not work in this case, because the wave potential of ~ 1 keV is far below the desired injection energy of ~ 100 keV. Consequently, the ions must be pre-accelerated to 100 keV as a part of the process of injecting them into the wave. This preacceleration may be accomplished by applying an external, axial dc electric field to the ions. A portion of this field must be applied across the gas cell to eject the ions therefrom, with the remainder of the preacceleration to ~ 100 keV (which we will term "injection") occurring downstream of the gas cell.

Method (2), of growing a slow cyclotron wave about these preaccelerated ions, might allow a very high trapping efficiency of ions into the wave. However, this method must be ruled out here also since such compressed beam growth has been seen to be impractical. Consequently, method (3) must be adopted, wherein a fast wave is grown in the uncompressed

beam region and then slowed to the injection velocity of $0.01\ c$ prior to ion loading. Since method (3) entails both preacceleration of the ions and slowing of the wave, there are two subsets (3a) and (3b) according to which of these is done first. Under method (3a), the beam would be compressed while passing through the gas cell and the wave would be slow in the gas cell; while under method (3b) the beam would be uncompressed in the gas cell and the wave would still be fast there.

It will be seen below that considerations relating to the elimination of the secondary electrons which are born in the gas cell dictate that the magnetic compression should occur downstream of the gas cell. Consequently the beam should be uncompressed in the gas cell, and method (3b) is found to be the optimum choice for loading the ions into the wave.

The parameters to be determined from the following considerations of ion generation and ion injection include the ion current, I_i ; the neutral gas density, n_0 ; the length of the gas cell, d ; the preacceleration potential applied across the gas cell, $\Delta\phi_{\text{gas}}$; the injection potential

applied downstream of the gas cell, $\Delta\phi_{\text{injection}}$; and the length of the injection region, L .

1. Ionization Chemistry in the Gas Cell.

Even for the simplest case of hydrogen gas, which is discussed here for illustration, there is a rich variety of possible processes of ionization, dissociation, stripping, and charge exchange which can occur in the gas cell. A non-exhaustive list of these processes includes the following:

- (a) Electron impact ionization of neutral H_2 via collisions with primary beam electrons, with a cross section $\sigma_b \sim 3 \times 10^{-19} \text{ cm}^2$ * for 300 keV primaries; the vast majority of these primary beam ionizations will yield the molecular ion H_2^+ -- no more than a few percent will be accompanied by the dissociation of H_2 to yield H^+ protons;
- (b) Electron impact ionization of neutral H_2 via collisions with lower energy secondary

*All of the cross sections given here are in units of cm^2 per molecule of neutral hydrogen gas.

electrons, with a cross section

$$\sigma_e \sim (0.3 - 1) \times 10^{-16} \text{ cm}^2 \text{ for } 0.1 - 1 \text{ keV}$$

secondaries; these ionizations also yield

predominately H_2^+ ;

The newly born H_2^+ ions and the few H^+ ions will rapidly be accelerated from their initial kinetic energy of $< 10 \text{ eV}$ to several tens of keV, as they oscillate in the radial confining potential well $\phi_0 \sim 16 \text{ kV}$ (for $I_b \sim 0.4 \text{ kA}$) and are accelerated axially out of the gas cell by $\Delta\phi_{\text{gas}}$ of a few tens of kV. These fast H_2^+ and H^+ ions may then undergo additional collisional interactions with the neutral H_2 background gas, as indicated in the continued list below:

(c) Ion impact ionization of neutral H_2 via collisions with accelerated H_2^+ and H^+ ions;

(1) The total ionization cross section, for the production of new secondary electrons, is $\sigma_i \sim (0.3 - 3) \times 10^{-16} \text{ cm}^2$ for 1 to 50 keV ions; the bulk of the increased secondary electron population

is associated with pure ionization of H_2 to produce new, slow H_2^+ ions;

- (2) The cross section for the production of new, slow secondary H^+ protons via ion impact ionization and dissociation of H_2 is

$$\sigma_{\text{slow } H^+} \sim (0.1 - 0.5) \times 10^{-16} \text{ cm}^2$$

for 1 to 50 keV ions;

- (d) Charge exchange of the fast H_2^+ and H^+ ions via collisions with neutral H_2 ;

- (1) The cross section for the charge exchange of fast H_2^+ is

$$\sigma_{\text{CE}/H_2^+} \sim (1 - 6) \times 10^{-16} \text{ cm}^2 \text{ for } 1 \text{ to } 50 \text{ keV ions; this reaction produces fast } H_2 \text{ neutrals and slow } H_2^+;$$

- (2) The cross section for the charge exchange of fast H^+ is

$$\sigma_{\text{CE}/H^+} \sim (1 - 9) \times 10^{-16} \text{ cm}^2 \text{ for } 1 \text{ to } 50 \text{ keV ions; this reaction produces fast } H \text{ neutral atoms and slow } H_2^+;$$

(3) The cross section for the production of slow H_2^+ from fast H_2^+ is the sum of the cross sections for charge exchange and pure ionization, and is $\sigma_{H_2^+} \sim (5 - 7.5) \times 10^{-16} \text{ cm}^2$ for 1 to 50 keV ions

(e) Dissociation of the fast H_2^+ ions via collisions with neutral H_2 ;

(1) The cross section for the production of fast H^+ fragment protons is $\sigma_{\text{fast } H^+} \sim (1 - 3) \times 10^{-16} \text{ cm}^2$ for 1 to 50 keV ions;

(2) The cross section for the production of fast H neutral atoms is

$\sigma_{\text{fast } H} \sim (5 - 8.5) \times 10^{-16} \text{ cm}^2$ for 1 to 50 keV ions.

There are other processes which we shall not enumerate whereby the fast H_2 and fast H neutrals may interact with the slow background H_2 neutrals to form additional charged particles. There are also processes for the creation

and destruction of fast H^+ , H , and H^- via the collisions between these particles and the background H_2 .

It is apparent that these collisional processes in the gas cell favor the predominance of molecular H_2^+ ions over the atomic H^+ ions. In fact, the strongest process for the generation of H^+ is the fifth process of dissociation of H_2^+ , which is a second order process. Therefore, the number of H^+ ions can never represent more than a small fraction of the number of H_2^+ ions.

Although one could in principle arrange to load and accelerate only the H^+ ions in the wave, leaving behind the larger numbers of H_2^+ ions, this could only be done by considerably reducing the current of accelerated H^+ such that the H_2^+ did not build up to a level which would disrupt the primary beam. The more likely possibility, which we shall assume, is that the accelerator applications of interest can be realized by accelerating the molecular ions formed in the gas cell. It is desired that the dominant mechanism for ionization should be the first process of primary beam ionization, so the ions can be formed at a predetermined rate fixed by the neutral density. In this case, the ratio of

the accelerated ion current to the primary electron beam current will be given by

$$\frac{I_i}{I_b} \approx F_T n_o \sigma_b d \quad (12)$$

where F_T is the fraction of ions which are successfully trapped in the wave. Thus, for $I_i \approx 0.1 - 1$ A, $I_b \approx 400$ A, $\sigma_b \approx 3 \times 10^{-19}$ cm², and $F_T \sim 0.5$, it is necessary for $n_o d$ to be in the range of $10^{15} - 10^{16}$ cm⁻². It is anticipated that d will be in the range of 1 - 10 cm, so that n_o will be in the range of $10^{14} - 10^{16}$ cm⁻³, corresponding to pressures of 3 - 300 mT. Even before these parameter ranges are further narrowed, it is clear that the neutral gas density will be much larger than the uncompressed beam density of $n_b \approx 10^{11}$ cm⁻³. Therefore, the desired level of ionization represents only a tiny fraction of the total neutral gas density, since the ion density n_i must be less than the electron beam density.

Therefore it is necessary to avoid cascading, or avalanching, which could cause a high level of ionization

of the neutrals via processes (b) and (c) listed above.

This requires that

$$n_o \sigma_e d < 1 \quad (13)$$

$$n_o \sigma_i d < 1 \quad (14)$$

and of these two processes, the ion impact ionization is probably the more important since σ_i tends to exceed σ_e somewhat.

It is further necessary to avoid charge exchange of the newly born H_2^+ ions with neutrals before the ions are ejected downstream from the gas cell. This requires that

$$n_o \sigma_{CE/H_2^+} d < 1 \quad (15)$$

and since σ_{CE/H_2^+} tends to exceed σ_i for the lower ion energies, this may be the most severe limitation on the obtainable ion current. The implied limitation is roughly that

$$\frac{I_i}{I_b} < \frac{F_T \sigma_b}{(\sigma_{CE/H_2^+}, \sigma_i, \sigma_e)_{\max}} \quad , \quad (16)$$

which should permit ion currents of 0.1 ampere, but may limit the ion current somewhat below 1 ampere. To determine this current limitation more precisely would require a careful computer analysis of the ion dynamics as the ions are accelerated out of the gas cell, to include the effect of the energy dependence of the cross sections which enter the various rate equations.

2. Applied Potential in the Gas Cell and Injector Region

The externally applied potential $\Delta\phi_{\text{gas}}$ is placed axially across the gas cell to eject the newly born ions downstream from the gas cell. Immediately downstream from the gas cell is the injector region, where a larger potential $\Delta\phi_{\text{injection}}$ is applied to preaccelerate the ions to the appropriate energy for loading into the wave. If the total preacceleration of the ions is to be ~ 100 keV prior to loading into the wave, then roughly

$$\frac{1}{2} e \Delta\phi_{\text{gas}} + e \Delta\phi_{\text{injection}} = \epsilon_{\text{loading}} \sim 100 \text{ keV} \quad (17)$$

where the $\frac{1}{2}$ factor reflects the fact that ions may be born at various axial positions in the gas cell, and may therefore receive various fractions of the total energy $e \Delta\phi_{\text{gas}}$ of the potential applied across the entire gas cell. For example, $e \Delta\phi_{\text{gas}}$ may be on the order of 20 keV and $e \Delta\phi_{\text{injection}}$ may be on the order of 90 keV. The radial potential drop $e \phi_0$ across the beam electrons may be about 16 keV, and there may be further significant radial potential variation between the beam edge and surrounding metal surfaces.

The parameters $\Delta\phi_{\text{gas}}$ and $\Delta\phi_{\text{injection}}$ are determined as follows. Knowledge of the desired ion loading energy $\epsilon_{\text{loading}}$ first establishes the relation of equation (17). An upper bound on $\Delta\phi_{\text{gas}}$ may be established, as described in Section 3 below, to insure that an adequate fraction of the ions may be trapped in the wave. Then a lower bound on $\Delta\phi_{\text{gas}}$ may be established to insure that the ion density in the gas cell does not build up to a

disruptive level. Thus $\Delta\phi_{\text{gas}}$ may be arrived at by trading off the upper and lower limiting conditions, as a function of other parameters such as I_i and $\tilde{\phi}$. Then $\Delta\phi_{\text{injection}}$ is given by equation (17).

The lower bound on $\Delta\phi_{\text{gas}}$ is derived as follows. As the ions are accelerated axially by the applied potential $\Delta\phi_{\text{gas}}$, the ion density will increase toward the downstream end of the gas cell, and the corresponding axial gradient in the ion space charge may be expected to produce some reduction or shielding of the axial electric field seen by the ions in the interior of the gas cell.

In order to maintain radial confinement of the ions, as well as to minimize disruption of the primary beam electrons, it is necessary that the ion density n_i be everywhere less than the beam density n_b . The most critical axial location for this constraint will be the downstream end of the gas cell, where n_i is highest. There, one may require that

$$1 > \frac{n_i}{n_b} \approx \frac{I_i \beta_{bz} A_b}{F_T I_B \beta_{iz} A_i} \quad (18)$$

where I_i/F_T is the total current of ions in the gas cell; A_i, A_b are the cross-sectional areas of the ion and beam electron currents; and $\beta_{iz} c, \beta_{bz} c$ are the axial flow velocities of the ions and beam electrons. For example, if $(A_i/A_b) \sim 1, I_b \sim 400 \text{ A}, \beta_{bz} \sim 0.76, F_T \sim 0.5, I_i \sim 0.1 - 1 \text{ A}$, then it is necessary that $\beta_{iz} > (0.38 - 3.8) \times 10^{-3}$, corresponding to H_2^+ drift energy of $E_i > 0.14 - 14 \text{ keV}$ at the downstream end of the gas cell. Ignoring shielding effects, and recalling that the average ion is only accelerated across half of the gas cell, this translates into a rough requirement that $e \Delta\phi_{\text{gas}} > 0.28 - 28 \text{ keV}$. Thus, at the higher ion current levels, it is required that a substantial potential gradient be applied to the gas cell to prevent an excessive build up of charge density there.

Alternatively, one may see from equation (19) below that for a wave potential of $\tilde{\phi} \sim 1 \text{ kV}$, one desires that $\Delta\phi_{\text{gas}} \leq 20 \text{ kV}$ to achieve a fair trapping efficiency. This translates into $\beta_{iz} < 3.3 \times 10^{-3}$, so that equation (18) may finally be used to imply that $I_i < 0.86 \text{ amperes}$ for the other parameter choices as given above.

3. Ion Trapping Into the Cyclotron Wave

Since the wave potential amplitude is only about $e\tilde{\Phi} \sim 1 \text{ keV}$ at the position of loading, while the ions are preaccelerated to $\frac{1}{2} M v_{ph}^2 \sim 100 \text{ keV}$ in the gas cell and in the injector, there cannot be too great a spread in the final ion velocities if a high fraction of the ions are to be trapped in the wave. However it is unavoidable that the ions will acquire some velocity spread in the gas cell, since they will be accelerated through a variable fraction of the potential drop $e \Delta\Phi_{gas}$, depending upon where they are born in the gas cell. Assuming that the final ion energies after injection range from $e \Delta\Phi_{injection}$ to $e \Delta\Phi_{injection} + e \Delta\Phi_{gas}$, and that the injection wave energy is $\frac{1}{2} M v_{ph}^2 = e \Delta\Phi_{injection} + \frac{1}{2} e \Delta\Phi_{gas}$, the rough trapping criterion $e\tilde{\Phi} > \frac{1}{2} M(v_{ph} - v_i)^2$ may be conservatively expressed as

$$e \Delta\Phi_{gas} < 2[(e\tilde{\Phi}) (M v_{ph}^2/2)]^{1/2} \sim 20 \text{ keV} \quad (19)$$

to achieve a fair trapping efficiency. This equation (19) is

the remaining design constraint which may be employed together with equations (17) and (18) to determine $\Delta\phi_{\text{gas}}$ and $\Delta\phi_{\text{injection}}$.

To achieve a careful estimate of the trapping efficiency, as well as the depth of trapping of the ions in the cyclotron wave, a one-dimensional computer code has been written to follow the ions all the way from their birth in the gas cell to the loading region after the injector. In this way, the perturbations of the ion dynamics by the wave fields, which occur as the ions approach the wave speed, can be included in the analysis. In addition, the scaling of the ion trapping efficiency and trapping depth with the parameters $\Delta\phi_{\text{gas}}$, $\Delta\phi_{\text{injection}}$, d , L , $\tilde{\phi}$ can be examined for optimization. This code and the results which it predicts are presented in Appendix C.

4. Dynamics and Removal of Secondary Electrons

Since this accelerator is to be operated in a steady state fashion, it is necessary to provide an avenue for the removal of the secondary electrons which are created in the gas cell; otherwise, they might accumulate to an intolerable level which would disrupt the primary beam.

After the secondaries are born in the gas cell, they will promptly be accelerated by the radial electric field of the primary beam to transverse energies of $E_{e\perp} \leq 1 \text{ keV}$, if the beam is still uncompressed in the gas cell. However, even with the beam uncompressed, the secondaries will be well tied to the axial magnetic field lines with $B_z \sim 7 \text{ kG}$ as has been previously envisioned.

The axial electric field $E_z \sim \Delta\phi_{\text{gas}}/d$ which is used to eject the ions downstream from the gas cell will act so as to carry the secondary electrons upstream from the gas cell, provided that E_z exceeds the axial wave electric field $\tilde{E}_z \approx k_z \tilde{\phi}$.

On the other hand, if $E_z < \tilde{E}_z$, then some of the secondary electrons will likely be trapped in the wave and carried downstream, since $\frac{1}{2} M v_{ph}^2 \sim 0.9 \text{ keV}$ is somewhat less than $e\tilde{\phi} \sim 3 \text{ keV}$ in the gas cell. However, to enhance the energy efficiency of this accelerator, which is very important for its steady state operation, the unused primary beam electron energy must be recovered by electrostatically stopping the beam electrons downstream of the wave acceleration section. This means that any secondary electrons which are

carried downstream will be axially reflected before reaching the beam dump, and their removal from the accelerator section might prove difficult.

Instead, a preferable design is to arrange for

$$E_z \approx \frac{\Delta\phi_{\text{gas}}}{d} > k_z \tilde{\phi} \quad . \quad (20)$$

Then the secondaries will be accelerated upstream from the gas cell, and probably trapped in an axial potential well between the gas cell and the cathode.

To provide for their elimination from the beam path, a transverse electric field E_y may be applied to the beam in the region upstream of the gas cell, so that the secondaries might be $E_y \times B_z$ drifted out of the beam to a nearby collecting baffle. This possibility is explored in Appendix D.

There it is shown that to successfully eliminate all of the secondaries from within the primary beam cross section, it is necessary that E_y exceed the self electric field E_r ($r = a$) of the primary beam electrons. For a 400 A, 300 keV primary beam, it is estimated that

$$E_r (r=a) \approx 53 \text{ kV/cm for } a \approx 0.6 \text{ cm (uncompressed beam)}$$

(21)

which is near the upper limit which might be achieved for E_y using standard technology. At the very least, this appears to provide a feasible "loss cone" avenue for removal of the secondary electrons.

It may be observed from equation (20), that for $\Delta\Phi_{\text{gas}} \sim 20 \text{ kV}$, $\tilde{\Phi} \sim 3 \text{ kV}$ in the gas cell, and $B_z \sim 7 \text{ kG}$, it is necessary that $d \leq 2 \text{ cm}$. However, the conditions expressed in both equations (20) and (21) could be eased a bit if the beam was slightly larger in and upstream of the gas cell (i.e., $B_z < 7 \text{ kG}$), which would facilitate removal of the secondaries and would also ease the wave growth slightly.

The various design considerations mentioned above for each of the accelerator system components constitute the conceptual design of a high current, steady state Auto-Resonant Accelerator. Naturally, once specific goals were adopted for a proposed accelerator, one should re-examine and optimize these constraints for that particular case.

III. SUPPORTING THEORETICAL ANALYSIS

The theoretical analysis of these steady state accelerators was able to get a fast start by exploiting the very large amount of previous work accomplished on pulsed Auto-Resonant Accelerators and relativistic electron beam interactions under several other contracts, including the most recent: DASG60-76-C-0045 and AFOSR F49620-76-C-0002. Several detailed theoretical calculations, which provided background support for the engineering design considerations desicussed earlier, are outlined below. These included analyses of the equilibria and linear eigenmodes supported by an unneutralized, relativistic electron beam; questions regarding the Landau resonance which can occur in the transverse electron cyclotron dynamics; and finally use of the SURVEY computer code to generate self-consistent accelerator design parameters.

A. Self-Consistent Equilibria and Linearized Cyclotron Eigenmodes of an Unneutralized Relativistic Electron Beam

The initial task undertaken in support of these design studies was to adapt previous analysis of both the self-consistent equilibria of such relativistic electron beams and of the linear waves which they support to the case of low - γ beams which are relevant for a steady state accelerator.

In order to understand the behavior of a steady state accelerator, it is first essential to determine the conditions under which a stationary beam equilibrium may exist in the downstream accelerator section (i.e., the region of Figure 1 from (4) to (5)), and further to determine the radial dependence of the various self-consistent equilibrium parameters. This question is examined in some detail in Appendix E.

Assumptions about the nature of the beam equilibrium are required, since many different equilibria are possible in general. It is most accurate to make these assumptions about the conditions of creation of the beam at the cathode (i.e., point (1) of Figure 1). Then conservation equations

are required to complete determination of the equilibrium quantities in the downstream accelerator section, which is considerably removed from the cathode. However, even with the aid of such conservation equations, the complete set of equations which determine the beam equilibrium are highly nonlinear in general and mathematically tractable only to numerical computer solution. Fortunately, there exist a number of small parameters which may be exploited to obtain an iterative analytical solution of the equilibrium, and these parameters remain small throughout the parameter space anticipated for steady state accelerator applications. These parameters, and the underlying physical phenomena which they reflect are itemized below.

$$(1) \quad \epsilon_1 \approx \left[\frac{I_e \text{ (kA)}}{5 B_z \text{ (kG)} a \text{ (cm)} \gamma \beta_z} \right]^2 \quad \text{---Axial Diamagnetism}$$

Because of the electromagnetic self-fields produced by the space charge and current of the unneutralized electron beam, an axial magnetic guide field must be present for equilibrium and the beam must precess in this field with velocity

$\beta_\phi(r)c$ to create a $\beta_\phi B_z$ pinch force. However, this ϕ -precession causes a diamagnetic reduction in the strength of the B_z field within the electron beam, and the relative magnitude of this reduction is characterized by ϵ_1 .

$$(2) \quad \beta_\phi(a) \approx \frac{\epsilon_1^{1/2}}{\gamma} \text{ -- } \beta_\phi \times \tilde{B} \text{ Forces, } \beta_\phi \tilde{n} \text{ Current Density}$$

Because the beam equilibrium contains a zero order ϕ -velocity, the linearized equations which describe the wave supported by the equilibrium will have a number of complicating terms traceable to this velocity. (In addition, since β_ϕ as seen in the beam frame of reference is simply

$\gamma\beta_\phi(a) \approx \epsilon_1^{1/2}$, the parameter $\epsilon_1^{1/2}$ may be seen to characterize effects of relativistic beam frame precession.)

$$(3) \quad \epsilon_2 \approx \left[\frac{I_e \text{ (kA)}}{0.7 B_z^2 \text{ (kG)} a^2 \text{ (cm)} \gamma \beta_z} \right] \text{ -- Centrifugal and Coriolis Forces}$$

Because of the precession of the electron beam, there are outward radial centrifugal forces $\gamma m \beta_\phi^2 / r$ which must be contained to achieve an equilibrium, and in the linearized

equations of motion, there are both centrifugal and Coriolis forces which influence the nature of the waves. The parameter ϵ_2 characterizes the relative magnitude of these effects.

$$(4) \quad \gamma^{-2} \approx \tilde{\beta}_z, \quad 1 - \beta_z \quad \text{Longitudinal Beam Compression}$$

This parameter is obviously small only for ultrarelativistic beams, and hence may not be exploited for most steady state accelerator applications. However, when it is small, the difference between the beam velocity $\beta_z c$ and c may be neglected; and if β_ϕ is also small, there can be no longitudinal compression of the beam within linear theory.

$$(5) \quad \frac{\Delta\gamma}{\gamma} \approx \frac{2 \epsilon_1}{\epsilon_2} \approx \frac{I_e \text{ (kA)}}{17 \gamma \beta_z} ; \quad \frac{\Delta\gamma}{\gamma^3 \beta_z^2} \approx \frac{\Delta\beta_z}{\beta_z} \quad \text{-- Space Charge Limitations}$$

These parameters characterize the relative magnitude of the radial variation in γ and β_z of the beam, which arises when the central electrons are preferentially slowed by the space charge forces which oppose the propagation of the

beam into a conducting waveguide. If these parameters are too large, it is even possible for a virtual cathode to be established in the waveguide, from which some electrons may be axially reflected.

When the parameters of (1), (2), (3), and (5) are negligible, the behavior of the electron beam is simplified to the maximum degree possible for most steady state accelerator applications. In this extreme limit, the electron density n , axial velocity $\beta_z c$ and kinetic energy $(\gamma - 1)mc^2$ will be uniform in radius. The angular precession will be rigid (i.e., $\beta_\phi \propto r$) and negligibly slow, and there will be no diamagnetism, centrifugal or Coriolis forces. However, to achieve all of these conditions, the electron current must clearly be rather small, which implies a corresponding limitation on the ion currents which may be accelerated.

The formal ultrarelativistic limit, which has often been invoked in prior analysis of pulsed electron beams, is mathematically realized by tending γ , B_z , and I_e to infinity in such a way that the ratio of any two of these quantities remains finite. In this limit the parameters

of (1), (2), (3), and (4) vanish, but those of (5) do not. Consequently the space charge effects of (5) are not unique to low- γ , steady state accelerator applications, and they have been previously examined for high- γ pulsed beam accelerators.

In order to give some feel for the relative importance of these five effects, the small parameters are evaluated in Table 3 for the case of the pulsed proof-of-principle experiment, for a large, first generation steady state accelerator experiment, and for a higher performance steady state accelerator. From this table, it seems clear that the most significant effects for a steady state accelerator are the (obvious) γ^{-2} longitudinal compression effects and the ϵ_2 centrifugal and Coriolis forces. The space charge effects are somewhat less significant than for high current pulsed beams, but may still be appreciable for a high current steady state accelerator. The ϵ_1 diamagnetism and the β_ϕ influence on linear waves appear generally small enough to be neglected.

In Appendix F, the theory of linearized waves supported by the unneutralized, relativistic electron beam

TABLE 3. Assessment of Small Parameters

	Pulsed ARA (P.O.P.E.) $I_e = 30$ kA $\gamma = 7$ $B_z a > 7.5$ kG-cm $b/a = 1.3$	Steady State ARA (Experiment) $I_e = 0.8$ kA $\gamma = 2$ $B_z a > 2.5$ kG-cm $b/a = 1.3$	Steady State ARA (High Performance) $I_e = 4$ kA $\gamma = 2.4$ $B_z a > 3$ kG-cm $b/a = 1.3$
(1) $\epsilon_1 = \left(\frac{I_e}{5 B_z a \gamma \beta_z} \right)^2$	0.013	0.0014	0.015
(2) $\beta_\phi(a) = \frac{\epsilon_1^{1/2}}{\gamma}$	0.016	0.018	0.051
(3) $\epsilon_2 = \frac{I_e}{0.7 B_z^2 a^2 \gamma \beta_z}$	0.105	0.101	0.28
(4) γ^{-2}	0.020	0.25	0.17
(5a) $\frac{\Delta \gamma}{\gamma} = \frac{I_e}{17 \gamma \beta_z}$	0.25	0.027	0.108
(5b) $\frac{\Delta \beta_z}{\beta_z} = \frac{I_e}{17 \gamma^3 \beta_z^3}$	0.0053	0.0091	0.023

is discussed. The basic dispersion relation for the negative energy cyclotron wave which is used in Auto-Resonant acceleration is rederived there for a case in which the ϵ_1 , β_ϕ , and $\Delta\gamma$ effects of (1), (2), and (5) are neglected, but the important γ^{-2} and ϵ_2 effects of (3) and (4) are retained. Consequently, this analysis incorporates those low- γ effects which are most important for steady state acceleration. These γ^{-2} and ϵ_2 effects do not introduce any new radial dependence to the coefficients of the radial differential equations which must be solved, and consequently solutions can still be obtained in terms of Bessel eigenfunctions, despite additional coupling which is now present between the relevant pair of second order differential equations. The result, subject to certain orderings discussed in Appendix F, may be approximately given by

$$\omega - k_z \beta_z c \approx - \left[\frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma^2} \frac{k_\perp^2}{k^2} \right]^{1/2} \quad (22)$$

where $\Omega = \frac{eB}{\gamma mc}$ is the relativistic cyclotron frequency, $\omega_p = (4\pi ne^2/\gamma m)^{1/2}$ is the relativistic (transverse) plasma frequency, and $k^2 = k_{\perp}^2 + k_z^2$. Here the $2\omega_p^2/\gamma^2$ correction term reflects the centrifugal and Coriolis forces, while the $\omega_p^2 k_{\perp}^2/\gamma^2 k^2$ correction reflects the longitudinal beam compression. These correction terms also indicate the coupling between the cyclotron wave and the longitudinal plasma (or space charge) wave on the electron beam. Upon expanding the square root, the net relative size of these longitudinal plasma wave corrections is about $\frac{1}{4} \epsilon_2$, which is small. Consequently, the cyclotron wave in most respects may be expected to be little modified from the previously examined ultrarelativistic case.

B. The Electron Cyclotron Landau Resonance

Because of the effects of longitudinal beam compression, which are significant for low- γ , the occurrence of an exact Landau resonance in the transverse electron motion is found to be possible, and could be of concern for certain steady state accelerator configurations. This resonance occurs when

$$\omega - k_z \beta_z c + \bar{\Omega} = 0 \quad (23)$$

where $\bar{\Omega} \equiv \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)^{1/2}$ is the modified cyclotron frequency variable, into which small centrifugal and Coriolis terms have been absorbed. Equations (22) and (23) imply that the Landau resonance will occur within fluid theory when

$$\frac{k^2 c^2}{2\gamma_o^2 \Omega^2} \approx 1 \quad (24)$$

Of course, this condition is not obeyed for the high- γ beams for which ultrarelativistic theory is appropriate, but it is occasionally satisfied for certain steady state accelerator configurations. In some of the early runs of the SURVEY code during this study, apparent divergences was observed near the resonance point.

Therefore, a careful theoretical analysis of the resonance was performed within fluid theory, and is presented in Appendix G. It was found that there are no divergences within the linearized Maxwell-fluid equations, even at the point of exact Landau resonance. Subsequently, improved

methods were devised by which SURVEY could be operated satisfactorily near a Landau resonance.

Nevertheless, it was realized that although no divergences are present within linearized cold fluid theory, some wave damping may be possible due to electron thermal effects. Consequently, the effects of a small thermal spread in axial beam velocity on the dispersion relation for the negative energy electron cyclotron beam mode have been investigated in Appendix H. A slab equilibrium model of the electron beam was considered in which the distribution function was taken to be a delta function in total energy but with a small spread in perpendicular energy. This distribution approximates the thermal spread produced by scattering of a cold monoenergetic beam after passage through a thin foil, and may also mock up the thermal spread which can arise in foilless diode operation.

The frequency ω of the electron cyclotron beam mode was determined to be

$$\omega_r \begin{cases} k_z u_0 - \Omega + \epsilon \Omega - \frac{k_z \hat{v}^2}{u_0} & , \quad 1 \gg \epsilon > 0 \\ k_z u_0 - \Omega - |\epsilon| \Omega - \frac{k_z \hat{v}^2}{u_0} - i\pi |\epsilon| \Omega Q_0^2 \exp(-Q_0) & , \\ & - 1 \gg \epsilon < 0 \end{cases} \quad (25)$$

where

$$\epsilon \equiv \frac{\omega_p^2}{k^2 c^2} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma_0^2 \Omega^2} \right)$$

$$|\epsilon| \ll 1$$

$$k^2 \equiv k_{\perp}^2 + k_z^2$$

$$Q_0 \equiv \frac{2|\epsilon| \Omega u_0}{k_z \hat{v}^2} \gg 1$$

$$\frac{k_{\perp}^2 \hat{v}^2}{\Omega^2} \ll 1$$

$$u_0 \equiv \beta_z c = \text{mean axial beam velocity}$$

and $\hat{v}$ characterizes the thermal spread in the transverse beam velocity distribution, which also corresponds to a small spread in the axial beam velocity, since the particle energy was assumed constant. Thus a small thermal spread in axial beam velocity results in a slight decrease in the wave phase velocity, and in certain regions of parameter space introduces wave damping. However, if

$$\frac{k_{\perp}^2 c^2}{2\gamma_0 \Omega^2} < 1$$

and

$$\frac{\hat{v}^2}{u_0^2} < \frac{2\omega_p^2}{k^2 c^2} \frac{\Omega}{k_z u_0} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma_0^2 \Omega^2} \right)$$

the negative energy cyclotron beam modes propagate undamped by thermal effects.

For the parameters of an accelerator configuration, ϵ will normally be small but positive in the compressed magnetic field region where the ions are loaded. As the magnetic field is flared to accelerate the ions, ϵ will

approach zero from the positive side. For some parameter choices, ϵ may pass through zero and become negative but small before the final wave velocity is achieved. However, in many of these cases, the accelerator should still function successfully if the thermal effects are not too large, because the wave damping rate given by equation (25) would be sufficiently small and the waves would only have a short time to remain in the accelerator. Notice that the wave damping rate vanishes exponentially in the thermal parameter $\hat{v}^2$, when this is small.

C. SURVEY Code Development

A new SURVEY code has been developed by a process of modification of the old ultrarelativistic SURVEY code to the point where it incorporates much of the finite β theoretical analysis discussed earlier. As a result the range of validity of the code results has been considerably expanded.

The code modifications that have been undertaken are of two basic types:

The first type of modification is the removal of the assumption of high- γ . Provided that the criteria for

the validity of the equilibrium and perturbation analysis are satisfied, the removal of this assumption extends the validity of the code to the complete range of electron energy. More directly, the removal of the related assumption that $\beta_z = 1$ improves the accuracy of the magnetic field profiles for the accelerator section.

The second type of modification is the inclusion of the new eigenmode analysis, which replaced the slab model eigenfunctions used in the old code. This has had a significant quantitative impact on design parameters for a DC Auto-Resonant Accelerator.

Generally we have found that the use of the more accurate eigenfunctions results in a reduction of total electron energy required to grow or maintain a given wave amplitude. Associated with this effect is a corresponding reduction in the magnitude of perturbations of the electron beam required for a given wave amplitude. Thus criteria such as restrictions on the magnitude of the radial excursions on the electron beam are effectively relaxed in some cases.

A description of the specific modifications to the SURVEY code is given in Appendix I.

A P P E N D I X A

DESIGN OF A SINGLE VACUUM GAP ELECTRON SOURCE

I. Introduction

In the voltage range at or below 500 kilovolts a preferred approach to the design of electron beam sources is the use of a single vacuum gap accelerator. Such an approach is commonly used for pulsed beams in high power laser research; commercially, 300 kV dc systems operating in the 10 to 20 kilowatt range are used for radiation processing applications on a 24 hour a day basis. Thus, the reliability of such systems is well established; the single vacuum gap then offers the great advantage of simplifying the beam focusing/optics problem. This will be more thoroughly reviewed below.

II. Electron Source Design Considerations

1. Coaxial Feed Design

Vacuum breakdown at high voltages is an area which has received a great deal of experimental and practical study, but is not thoroughly understood theoretically. It

does seem to be well established that the breakdown process initiates on the negative electrode surface, where microscopic whiskers of metal cause field enhancement and begin to field emit electrons (the quantum tunneling effect).

Based on the performance of a 250 kV commercial system, the macroscopic negative electrode stress should lie in the range of 34 to 40 kV/cm (E_-).

At present, it is anticipated that the electron beam will be formed and pre-accelerated to 20 kV before entering the vacuum gap. Since the structure feeding power to the accelerating gap will be a cylindrical transmission line, it can be shown that the outer cylinder radius is given by

$$r_o = \frac{(V_o - 20)}{E_-} (60/z) e^{(z/60)} \quad (A1)$$

where

V_o = final electron energy in kV

z = wave impedance of coaxial feed

Figure A1 shows a plot of the outer (r_o) and inner (r_i) cylinder radii versus working stress, for various values of wave impedance. Curve (a) is for 300 kV final energy, while (b) is for 400 kV final energy. It should be noted that a wave impedance of 60 ohms will always give the smallest radial size for any working stress, while a 30 ohm wave impedance gives maximum power flow. Vacuum breakdown is known to have an area dependence of the form

$$E_- \propto A^{-1/10} \quad (A2)$$

The stress levels quoted above correspond to areas of the order of 10^3 cm^2 , which is comparable to the sizes being considered. Thus, these values are safe to use (note that the surfaces are assumed to be conditioned, electropolished 304 stainless steel).

Because the 30 ohm geometry has a large inner conductor, this value is chosen for reasons which are related to beam optics considerations. Note that for a beam energy of 300 kV and a working stress of 36.5 kV/cm, the outer

(a)

(b)

Figure A1. Radius versus Stress

cylindrical radius is 10 inches, and the inner radius is 6 inches. Figure A2 shows a sectional schematic of what a 300 kV electron beam source would look like. The details of how the various dimensions are determined are given below.

2. Electron Source Design

Inside the inner cylinder of the 30 ohm coaxial line (which in effect is a Faraday cage) is a spherically curved thermionic cathode. The cathode itself is of the dispenser variety, that is, a low work function emitter formed by impregnating a sintered tungsten matrix with barium. At a temperature of 1100 °C, the saturated emission density exceeds 7 A/cm²; therefore, the cathode can be operated in a space charge limited mode. With ambient pressures of 10⁻⁵ Torr or less, gas poisoning of the cathode is negligible. With stainless steel present, care has to be taken to keep the stainless cool; if it heats up, it evaporates chromium which, even in small amounts, quickly poisons the cathode. Therefore, all elements which are going to be heated up should be made of, or plated with, molybdenum.

A tungsten mesh is required in front of the cathode to prevent the main accelerating voltage from "leaking" back

Figure A2. Schematic of Electron Beam Source

into the cathode. For numerical purposes, it will be assumed that the mesh is 80% transparent. It should be noted that the presence of the mesh introduces "lensing" fields, which increases the emittance (phase space volume) of the beam. The emittance in turn determines the minimum beam size which can be realized. While an exact estimate of emittance has not yet been completed, preliminary guesses indicate that compressing the beam to the desired size is not limited by the beam emittance.

To extract a 4 A/cm² beam through an 80% transparent mesh, the cathode must be run at 5 A/cm². To achieve this under space charge limited conditions, the Child-Langmuir equation

$$j = 2.334 \times 10^{-6} \frac{V_e^{3/2}}{d^2} \quad (A3)$$

must be satisfied, where

V_e = extraction voltage in V

d = cathode/anode spacing in cm

Using a realistic extraction voltage of 20 kV, the gap spacing is

$$d = \left(\frac{2.335 \times 10^{-6} V_e^{3/2}}{j} \right)^{1/2} = 1.15 \text{ cm} \quad (\text{A4})$$

The corresponding average gap stress is

$$\bar{E} = \frac{V_e}{d} \approx 17.4 \text{ kV/cm} \quad (\text{A5})$$

which is well below any vacuum breakdown thresholds. Note that for a fixed current density, the mean stress scales as

$$\bar{E} \propto d^{1/3} \quad (\text{A6})$$

which is a weak dependence.

Note from Figure A2 that at the mesh the beam has a uniform current density of 4 A/cm², and has a spherically disked profile with total area equal to

$$A = 2\pi r^2 (1 - \cos \theta) \quad (\text{A7})$$

where

r = radius of the spherical surface

θ = maximum angular position of the beam relative to the z-axis as measured from the center of the sphere.

To estimate the radius of the sphere, equation (A3) can be used to calculate the normal electric field at the mesh. Equation (A3) can be rewritten as

$$V = \left(\frac{j}{2.335 \times 10^{-6}} \right)^{2/3} x^{4/3} \quad (A8)$$

so that

$$E_o = \left. \frac{dV}{dx} \right|_{x=d} = \frac{4}{3} \bar{E} \approx 23.2 \text{ kV/cm} \quad (A9)$$

For concentric spheres with potential ΔV between them, the stress on the outer sphere is

$$E_o = \frac{\Delta V}{r} \frac{1}{(f-1)} \quad (A10)$$

where

f = ratio of outer-to-inner sphere radii

Then the radius of the sphere is given by

$$r = \frac{3}{4} \frac{\Delta V}{\frac{E}{f-1}} \quad (A11)$$

Taking

$$f \approx 2. \quad (A12)$$

then

$$r \approx \frac{3}{4} \frac{\Delta V}{\frac{E}{2-1}} \quad (A13)$$

which, for a 300 kV beam energy, gives

$$r \approx \frac{280}{23.2} = 12.07 \text{ cm} = 4.75 \text{ inches} \quad (A14)$$

Using 4 A/cm², equation (A7) gives for the total beam current

$$I = 8\pi r^2 (1 - \cos \theta) \quad (A15)$$

Using the 300 kV beam parameters, and assuming a 400 ampere beam current, equation (A15) gives

$$\begin{aligned} \theta &\approx 27^\circ \\ \text{and} \\ r_e &= r \sin \theta \approx 5.5 \text{ cm} = 2.16 \text{ inches} \end{aligned} \quad \left. \vphantom{\begin{aligned} \theta &\approx 27^\circ \\ \text{and} \\ r_e &= r \sin \theta \approx 5.5 \text{ cm} = 2.16 \text{ inches} \end{aligned}} \right\} \quad (A16)$$

3. Toroidal Section

The spherical beam focusing region is blended onto the inner cylinder by using a toroidal structure. To estimate its radius, r_T , and spacing from the ground plane, a , use can be made of the formula for the maximum stress of a cylinder above a ground plane. This is given by the approximate relation

$$E_{\max} = 0.9 \frac{\Delta V}{r_T \ln \left(\frac{r_T + a}{r_T} \right)} \quad (\text{A17})$$

As noted above, minimum dimensions result when a 60 ohm geometry is used, that is

$$\ln \left(\frac{r_T + a}{r_T} \right) = 1 \quad (\text{A18})$$

Since the stressed toroidal area is at least a factor of two smaller than the inner cylinder area, use of equation (A2) allows for a maximum stress of

$$E_{\max} \approx 2^{1/10} E_- \approx 38.5 \text{ kV/cm} \quad (\text{A19})$$

Using this in equation (A17) yields, for the 300 kV beam case,

$$\left. \begin{aligned} r_T &\approx 6.55 \text{ cm} = 2.58 \text{ inches} \\ a &\approx 11.25 \text{ cm} = 4.43 \text{ inches} \end{aligned} \right\} \quad (\text{A20})$$

4. Comments

This completes the review of considerations which went into the electrical design of the single vacuum gap electron source. Certain other items which require attention are discussed below.

III. Thermal Design Considerations

1. Cathode Heater/Power Requirements

Heating of the cathode is generally achieved by passing large currents through a tungsten wire which is thermally (but not electrically) attached to the back side of the cathode. Ohmic heating of the wire is thermally conducted to the cathode. The front surface of the cathode dissipates power by black body radiation according to the Stefan-Boltzmann relation

$$\frac{P}{A} = 5.67 \epsilon_r T^4 \text{ watts/cm}^2 \quad (\text{A21})$$

where

$$\left. \begin{array}{l} \epsilon_r = \text{emissivity} \leq 1 \\ T = \text{temperature in } 10^3 \text{ } ^\circ\text{K} \end{array} \right\} \quad (\text{A22})$$

Since

$$T \approx 1400 \text{ } ^\circ\text{K} \quad (\text{A23})$$

then

$$\frac{P}{A} \leq 21.8 \text{ watts/cm}^2 \quad (\text{A24})$$

From equations (A17) and (A14) one can estimate the total power requirement as

$$P \leq 2.2 \text{ kilowatts} \quad (\text{A25})$$

High temperature insulation can be used to minimize losses from the back surface of the cathode; however, conduction losses down heater leads and cathode supports are more difficult to estimate. None the less, it is probably reasonable to assume that total power dissipation is no more than a factor of two higher than given by equation (A25).

One final comment should be made regarding heating of the cathode. The current necessary for ohmic heating is usually tens of amperes (whether a.c. or d.c.). These currents cause magnetic fields at the cathode which can dramatically affect the electron beam flow. The simplest solution to this problem is to use a bifilar winding, that is closely spaced wires which carry current in opposite directions. In this way a dipole field is produced which has a low magnitude and falls off rapidly with distance. This solution has been found to be very effective in earlier work with thermionic guns.

2. Cooling

The power radiated by the cathode will very quickly heat up a vacuum system, causing outgassing and cathode

poisoning problems. Attaching Freon cooling lines on the inside of the inner cylindrical line, and on the outside of the containment cylinder, can very easily dissipate the heat and keep the overall system at a low temperature.

IV. Radiation Shielding

For a pulsed test system operating in the 10 to 20 kilowatt average power range, a substantial quantity of X-rays will be produced; shielding is obviously required. At low energies, the X-ray radiation is almost isotropic, so a totally enclosed concrete vault will be required. A worst case analysis will assume the beam stopping in a tungsten converter. The X-ray output at one meter is then given by

$$\dot{R} \left\{ \frac{\text{mR/Hr}}{\text{kilowatt}} \right\} \approx 1.0 \times 10^5 \quad (\text{at 1 meter}) \quad (\text{A26})$$

For a 20 kilowatt, 300 kV system the radiation must be attenuated by 6 orders of magnitude! For concrete, the 10th power attenuation length is 4 inches; thus concrete

walls two feet thick are required. Note that having over-estimated the dose rate by two orders of magnitude would only reduce the thickness by about one-half foot.

A P P E N D I X B

DIODE DESIGN CODE

This appendix contains a brief description of the code PGUN that was developed to facilitate the electron source design. Its capabilities and limitations are discussed and preliminary results are presented.

I. Code Description

The code is a 2-1/2 D, azimuthally symmetric, steady state relativistic code that includes provisions for arbitrary electrode shapes within the resolution of the grid. The electron emission from one electrode is specified in terms of current density, energy and trajectory angle. The electrode potentials are also given as well as the externally applied magnetic field profiles. The code solves for a stationary flow pattern and the consistent fields by an iterative process. The major features of PGUN are:

- 2-1/2 D azimuthally symmetric cylindrical geometry is used.

- Fully relativistic particle dynamics are computed.
- Poisson's equation is solved with flow through ($d\phi/dz = 0$) conditions on the ends and specified potentials on arbitrary electrode shapes.
- Applied B_z and B_r magnetic fields are optionally^z specified either by solenoidal coil sources or else by arbitrary axial profiles [i.e., $B_z(r=0, z)$].
- Diamagnetic B_z fields due to particle θ currents are neglected. (The consistency of this assumption has been checked for several cases of interest and has been found good to a few parts per thousand.)
- The magnetic fields due to the other particle currents are included.

The electrode geometry is input as a sequence of straightline segments in an $(r - z)$ coordinate system.

The code then calculates the intersections of these straightlines with a specified rectangular mesh. A boundary Green's function matrix (also known as a capacitance matrix, reference 1) is determined by repeated solution of Poisson's

1. R. W. Hockney, "The Potential Calculation and Some Applications," Methods in Computational Physics, Vol. 9, 1970, Academic Press, p. 162.

equation over a rectangular region which encloses all the electrodes. A fast transform solver is used in solving the Poisson equation in the rectangular $(r - z)$ region. The Green's function matrix needs to be computed only once for a given electrode configuration. Once this matrix is computed, a solution of Poisson's equation with arbitrary external sources requires two passes through the fast transform solver.

The Poisson equation is first solved with no sources, and the fields that would be "seen" by a zero density beam are calculated. Then a beam of electrons of artificially low density is injected and trajectories are calculated. On the basis of these trajectories, charge and current densities are allocated to mesh sites and the process is repeated while the current density is gradually raised to the full injected value. The iteration continues until convergence criteria are met or the maximum allowed number of iterations is exceeded.

The code allows proposed design configurations to be evaluated through determination of (1) the degree to

which the desired beam properties can be achieved, and
(2) the size of the zero frequency sausageing cyclotron waves
which are excited.

II. Results

The particle trajectories for a proposed diode configuration are shown in Figure B1. This corresponds to a 400 amp beam in a 280 kV accelerating potential. The externally applied axial magnetic field varies from 0.1 to 1.8 kG. The beam radius at the emitting surface is 5 cm. The theoretically conserved energy and canonical angular momentum are plotted in Figures B2 and B3, for electrons starting at different radial positions.

Figure B1. Pierce Gun Beam Trajectories

Figure B2. Energy Conservation

Figure B3. Canonical Angular Momentum Conservation

A P P E N D I X C ION TRAPPING CRITERIA

A travelling potential wave of amplitude ϕ and wave phase velocity v cannot directly pick up and trap an ion born at rest in the wave if $v^2 \geq 4 \frac{e}{M} \phi$; where $\frac{e}{M}$ is the ion charge to mass ratio. However, by pre-acceleration of the ion to a velocity comparable to the wave phase velocity, efficient trapping can occur.

To examine this process quantitatively, we consider a one dimensional configuration where there exists a d.c. accelerating electric field in the left plane of amplitude E and a travelling wave $\phi(x,t) = -\phi \cos(kx - \omega t + \psi)$. The equation of motion for an ion born at rest at $x = -x_0$ is given by:

$$\ddot{x} = \frac{e}{m} [k\phi \sin(kx - \omega t + \psi) + E H(x)] \quad (C1)$$

where

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

and

$$\dot{x}(0) \equiv 0 \quad x(0) \equiv -x_0$$

For $x < 0$ there exists a constant of the motion

$$\epsilon_0 = \frac{1}{2} (\dot{x} - v)^2 + \frac{e\phi}{M} \cos(kx - \omega t + \psi) - \frac{eE}{M} (x - vt) \quad (C2)$$

and similarly for $x > 0$

$$\epsilon = \frac{1}{2} (\dot{x} - v)^2 + \frac{e\phi}{M} \cos(kx - \omega t + \psi) \quad (C3)$$

where $v \equiv \omega/k$. Trapping of the ion will have occurred if

$$|\epsilon| < \frac{e\phi}{M}.$$

Now ϵ_0 may be evaluated at $t = 0$

$$\varepsilon_0 = \frac{1}{2} v^2 + \frac{e\phi}{M} \cos(\psi - kx_0) + \frac{eE}{M} x_0 \quad (C4)$$

and at $t = t_0$ where $x(t_0) \equiv 0$, we have

$$\varepsilon = \varepsilon_0 - \frac{eE}{M} vt_0 \quad (C5)$$

An approximate solution may be effected for

$E \gg k\phi$ and $1/2 Mv^2 \gg \phi$ by noting

$$x \approx \frac{eE}{M} t^2/2 - x_0 \quad (C6)$$

so that

$$t_0 = \left(\frac{2Mx_0}{eE} \right)^{1/2} \quad (C7)$$

and

$$\varepsilon = \frac{1}{2} v^2 + \frac{e\phi}{M} \cos(\psi - kx_0) + \frac{eE}{M} \left[x_0 - v \left(\frac{2Mx_0}{eE} \right)^{1/2} \right] \quad (C8)$$

An ion will then be trapped if

$$-\frac{e\phi}{M} < \frac{1}{2} v^2 + \frac{e\phi}{M} \cos(\psi - kx_0) + \frac{eE}{M} \left[x_0 - v \left(\frac{2 M x_0}{eE} \right)^{1/2} \right] < \frac{e\phi}{M} \quad (C9)$$

Thus ions in the range

$$\frac{Mv^2}{2} \left(1 - 2.8 \sqrt{\frac{2e\phi}{Mv^2}} \right) \leq eEx_0 \leq \frac{Mv^2}{2} \left(1 + 2.8 \sqrt{\frac{2e\phi}{Mv^2}} \right) \quad (C10)$$

may be trapped; ions born outside this range will not be trapped.

In order to examine this process in detail a simple computer program was written to solve equation (C1) numerically and determine those values of ϵ where trapping would occur, and furthermore determine the percentage of particles randomly born at various x_0 that would be trapped. (A listing of the program is given at the end of this appendix.)

Computed trapping efficiencies are given in Figure C1. For this example, the above estimated bound, equation (C10), yields $0.6 < (2eEx_0)/(Mv^2) < 1.6$, which is somewhat of an overestimate but still in relatively good agreement.

This calculation thus shows that a large fraction of ions can be trapped in the potential well, and that the fractional range in $(\Delta x_0)/(x_0)$ varies only as $\sqrt{\frac{e\phi}{Mv^2}}$ and is therefore fairly broad.

However, if we inquire where the ions are trapped in the wave, we find that most of them will be only marginally trapped. For example, the bulk of the ions trapped for the parameters of Figure C1 are trapped with a value of $1 > \left| \frac{\epsilon/e\phi}{M} \right| \gtrsim 0.85$, so that even modest acceleration could easily spill the ions from the well. Care will therefore have to be taken in the acceleration process, or further growth of the wave achieved, to avoid spilling the ions from the potential well during wave acceleration.

Figure C1. Ion Trapping Efficiency


```

PROGRAM IONTRAP
10  CONTINUE
    OPEN(UNIT=5,NAME='IONTRAP',DISPOSE='PRINT')
    TYPE 101
101  FORMAT(1X,'ENTER E/(K*M*V*V) AND RETURN, THEN PHI/(M*V*V)'
1, ' AND RETURN, BOTH IN F12.5 FORMAT')
    ACCEPT 102,E,PHI
102  FORMAT(F12.5,/,F12.5)
    WRITE(6,103)E,PHI
    TYPE 103, E,PHI
103  FORMAT(1H1,////,10X'ION TRAPPING FOR E/(K*M*V*V) = '
1,F12.5,' AND PHI/(M*V*V) = 'F12.5,/)
    YMAX=AMIN1(0.,-1+SQRT(4.*PHI))
    YMAX=-YMAX**2/E*.5
    YMIN=-(1.+SQRT(4.*PHI))**2/E*.5
    WRITE(6,333) YMAX,YMIN
    TYPE 333,YMAX,YMIN
333  FORMAT(1X,/,',***** YMAX = 'F12.5,' AND YMIN = 'F12.5,/)
    DELY=(YMAX-YMIN)/30.
    DO 1 J=1,31
        YINIT=YMIN+(J-1)*DELY
        DO 1 I=1,31
            Y=YINIT
            SINIT=(I-1)/31.*6.2832
            S=SINIT
            VY=0.
            DS=.1/(1.+VY+SQRT(PHI+E))
3        VY=PHI*SIN(Y-S)*DS+E*DS+VY
            S=S+DS
            Y=Y+VY*DS
            IF(Y.LE.0.) GO TO 3
            ITEST=.5*(1.+.5*(VY-1.))**2/PHI+COS(Y-S))*20.
            IF(ITEST.GE.20) GO TO 2
            TEST=ITEST*5.
            SPRT=SINIT/6.2832
            YPRT=-YINIT*E*2
            WRITE(6,104)SPRT,YPRT,TEST
            TYPE 104,SPRT,YPRT,TEST
104  FORMAT(1X,'W*T/2/PI = 'F12.5,5X'E*Y/(.5*M*V*V) = 'F12.5,5X
1, 'TRAPPING IN WELL (0-100) = 'F12.5)
2    CONTINUE
1    CONTINUE
    CLOSE(UNIT=5)
    GO TO 10
END

```

A P P E N D I X D

REMOVAL OF SECONDARY ELECTRONS PRODUCED IN "GAS CELL"

It has been proposed that the ions to be accelerated be generated by ionization due to passage of the energetic electron beam through a "gas cell." If a potential difference is applied between the ends of the "gas cell," the ions can be accelerated modestly downstream and their energy increased to a desirable value at which trapping in the potential wells of a low phase velocity cyclotron wave can occur.

However, the secondary electrons produced may be prevented from escaping radially because of the presence of a strong magnetic guide field. Some of these electrons are expected to move upstream and eventually reflect; they are essentially trapped in an axial electrostatic potential well.

The accumulation of these secondary electrons is undesirable since they lead to excitation of two-stream instabilities with consequent deterioration of the quality of the primary electron beam. One possible solution is to

impose an externally applied electric field upstream of the "gas cell" and to $\underline{E} \times \underline{B}$ drift the secondary electrons out of the beam.

In this appendix, the feasibility of this solution is investigated. The magnitude of the external electric field necessary to drift the secondary electrons out of the beam and the time scale involved are estimated.

I. Single Particle Trajectory

The single particle dynamics of an electron is determined in the presence of a uniform guide field B_0 in the z-direction, a uniform electric field E_0 in the y-direction, and a radial electric field $E_r(r)$ due to the beam electrons:

$$E_r(r) = \begin{cases} -E_1 \frac{r}{a_0} & r < a_0 \\ -E_1 \frac{a_0}{r} & r > a_0 \end{cases}$$

where $E_1 = 2\pi N_0 e a_0$ is the beam electric field at the beam radius a_0 .

In the limit of the drift approximation where electron inertia may be neglected, the electron velocity is:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{e}{B_0} (\mathbf{E} \times \hat{\mathbf{z}}) \quad (D1)$$

where the total electric field $\mathbf{E}$ is

$$\mathbf{E} = E_r \hat{\mathbf{r}} + E_0 \hat{\mathbf{y}}$$

Equation (D1) determines the electron trajectory and solutions are obtained for $r < a_0$ and $r > a_0$.

$$1. \quad r < a_0$$

In Cartesian coordinates (x, y) , equation (D1) takes the form

$$\frac{dx}{dt} = - \frac{c E_1}{a_0 B_0} y + \frac{c E_0}{B_0} \quad (D2)$$

$$\frac{dy}{dt} = \frac{c E_1}{a_o B_o} x \quad (D3)$$

Equations (D2) and (D3) have the following constant of motion

$$x^2 + \left(y - \frac{E_o}{E_1} a_o \right)^2 = \text{constant}$$

The electron trajectories are circles or arcs of circles about a stationary point $x = 0$, $y = (E_o/E_1) a_o$.

The condition that all the electrons drift to the surface of the beam is

$$\frac{E_o}{E_1} > 1 \quad .$$

The solutions of equations (D2) and (D3) are:

$$x = \alpha \cos \left(\frac{c E_1}{a_o B_o} t + \beta \right)$$

$$y = \frac{E_o}{E_1} a_o + \alpha \sin \left(\frac{c E_1}{a_o B_o} t + \beta \right)$$

where α and β are constants determined by initial conditions. The electrons move in a counterclockwise sense about the center $x = 0$, $y = \frac{E_0 a_0}{E_1}$. Those which reach the beam surface have coordinate $x > 0$, that is $\cos \theta > 0$.

If $\Delta\phi$ is the angle subtended at the center by the trajectory described by the electron in reaching the beam surface, the time interval $\Delta\tau$ taken is

$$\Delta\tau = \Delta\phi \frac{a_0 B_0}{c E_1}$$

For $\frac{E_0}{E_1} \geq 1$, $\Delta\phi_{\max} = 2 \sin^{-1} \frac{E_1}{E_0}$, and $\Delta\tau$ is bounded by:

$$2 \frac{a_0 B_0}{c E_1} \sin^{-1} \frac{E_1}{E_0} > \Delta\tau > 0$$

$$2. \quad r > a_0$$

In polar coordinates (r, θ) , equation (D1) takes the form:

$$\frac{dr}{dt} = \frac{c E_o}{B_o} \cos \theta \quad (D4)$$

$$r \frac{d\theta}{dt} = \frac{c E_l a_o}{B_o r} - \frac{c E_o}{B_o} \sin \theta \quad (D5)$$

The constant of the motion is

$$\frac{r}{a_o} \sin \theta - \frac{E_l}{E_o} \log \frac{r}{a_o} = \text{constant} = \sin \theta_o$$

where the constant is chosen so that the polar angle is

$\theta = \theta_o$ at the surface $r = a_o$.

Substituting for θ in equation (D4)

$$\frac{d}{dt} \left(\frac{r^2}{a_o^2} \right) = \frac{2c E_o}{B_o a_o} \left[\frac{r^2}{a_o^2} - \left(\sin \theta_o + \frac{E_l}{E_o} \log \frac{r}{a_o} \right)^2 \right]^{1/2} \text{sign}(\cos \theta) \quad (D6)$$

Electrons with initial velocity $\frac{dr}{dt} > 0$ move away from the beam surface, and they continue to do so provided that

$$\frac{r^2}{a_o^2} > \left(\sin \theta_o + \frac{E_1}{E_o} \log \frac{r}{a_o} \right)^2$$

This condition is always satisfied if $\frac{E_o}{E_1} > 1$.

II. Discussion

The necessary requirement on the magnitude of the applied field $E_o \hat{y}$ so that all the secondary electrons be removed from the beam is $E_o > E_1$ (the radial self-field of the beam at the beam surface). This ensures that the electron velocity v_x in the x-direction is always positive.

In addition, the transit time T of the secondary electrons through the region of finite E_o must be long enough to allow the electrons to reach the beam surface. For $\frac{E_o}{E_1} > 1$, the time interval for electrons to reach the beam surface satisfies the inequality

$$\frac{2a_o B_o}{c E_1} \sin^{-1} \frac{E_1}{E_o} > \Delta \tau$$

This implies that

$$T > \frac{2a_o B_o}{c E_1} \sin^{-1} \frac{E_1}{E_o} \quad (D7)$$

is sufficient for all secondary electrons to escape from the beam. The corresponding axial extent of the region over which the E_o field must be applied is

$$L > \frac{2 v_z a_o B_o}{c E_1} \sin^{-1} \frac{E_1}{E_o} \quad (D8)$$

where v_z is the mean axial velocity of the secondary electrons.

A P P E N D I X E

SELF-CONSISTENT EQUILIBRIA OF UNNEUTRALIZED RELATIVISTIC ELECTRON BEAMS

In order to be able to exploit the travelling waves which may be supported by a beam of unneutralized, relativistic electrons, for the purpose of trapping and accelerating within such waves a species of positively charged ions, it is first necessary to understand the equilibrium properties of the electron beam itself. The most fundamental question is to determine the conditions under which an equilibrium of forces can be arranged such that the electron beam may propagate in vacuum away from the diode into a drift tube in a grossly stable configuration. If such a force equilibrium is not achieved, the unneutralized electron beam will rapidly blow itself apart by the powerful self electromagnetic fields generated by its own space charge. A second objective of such an equilibrium analysis is to determine the self consistent radial profiles of all equilibrium quantities, including the electromagnetic fields and the number density

and flow velocity of the electron beam. These equilibrium properties will in turn be needed to determine the linearized eigenmodes (or waves) which may be supported upon the electron beam.

The contribution of the subject ions (to be accelerated) to this equilibrium is neglected in this analysis. The first reason for this is that for many accelerator configurations, the electron beam must propagate stably for some distance in vacuum prior to the introduction of ions, so that an analysis of pure electron beam equilibria is required. Second, the accelerator can operate in principal in the "test ion" regime, where the ion contributions to the equilibria and waves are truly negligible. Finally, it is anticipated that the highest density of ions which may be accelerated will still be much less dense than the electrons, so that the ion effects may be treated as a perturbation on the fields supported by the electron beam.

The electron equilibrium may be examined by considering the single particle equations of motion together

with the Maxwell equations for the self-consistent electromagnetic fields.

$$\frac{d}{dt} \gamma \underline{\beta} = - \frac{e}{mc} (\underline{E} + \underline{\beta} \times \underline{B}) ; \quad \frac{d}{dt} \underline{x} = \underline{\beta} c \quad (E1)$$

$$\nabla \cdot \underline{B} = 0 = \nabla \times \underline{E} + \frac{1}{c} \frac{\partial}{\partial t} \underline{B} \quad (E2)$$

$$\nabla \cdot \underline{E} = - 4\pi e n \quad (E3)$$

$$\nabla \times \underline{B} = \frac{1}{c} \frac{\partial}{\partial t} \underline{E} - 4\pi e n \underline{\beta} \quad (E4)$$

Here $e = - |e|$, m are the electron charge and mass, and $\underline{\beta}c$, $(\gamma - 1)mc^2$ are its velocity and kinetic energy. Since the single particle equations of motion (E1) are the characteristics of the collisionless Vlasov equation, the equilibrium distribution function $f_o(\underline{x}, \underline{\beta})$ may be constructed from any constants of the single particle motion. We shall adopt a cold fluid model of the electron beam, in which the beam pressure is neglected, and n , $\underline{\beta}$ are the electron number density and flow velocity, which may in general be functions of $\underline{x}$, t .

The existence of steady state beam equilibrium solutions is by no means guaranteed. The crucial condition is the balance of radial forces, which may be expressed as

$$\frac{d}{dt} \gamma \beta_r = \frac{e}{mc} \left\{ -E_r + \beta_z B_\phi - \beta_\phi B_z + \frac{mc^2}{e} \frac{\gamma \beta_\phi^2}{r} \right\} = 0 \quad (E5)$$

in the cylindrical coordinates r, ϕ, z . The basic assumptions are that the equilibrium is steady state ($\partial/\partial t = 0$) and cylindrically symmetric ($\partial/\partial \phi = 0$) throughout the accelerator. It is further assumed that there is a downstream region away from the beam diode, for which the equilibrium is approximately independent of axial position ($\partial/\partial z = 0$). Practically speaking, this will be true throughout the "accelerator section" of the device, wherein the waves and ions are accelerated, if the z -variation in equilibrium quantities is small compared to the r -variation, and if the z -variation in equilibrium quantities is small (or adiabatic) compared to the z -variation in wave quantities, characterized by the axial wavelength.

The fields E_r and B_ϕ are the self-fields of the electron beam, determined in the downstream equilibrium region from the equations

$$\frac{1}{r} \frac{\partial}{\partial r} r E_r = - 4\pi e n \quad (E6)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r B_\phi = - 4\pi e n \beta_z \quad (E7)$$

It may be seen that both E_r and B_ϕ are negative, and that E_r is larger than B_ϕ in magnitude. Consequently there can be no equilibrium (in vacuum) for which $\beta_\phi = 0$, since the sum of the first two terms of equation (E5) are always positive, evidencing a net outward force on the electrons. This net outward force must be overcome by the $-\beta_\phi B_z$ pinch force, without the final centrifugal force destroying the balance. Here B_z is presumed to be an externally generated magnetic guide field, which is slightly reduced by the beam's diamagnetic precession in ϕ according to

$$\frac{\partial B_z}{\partial r} = 4\pi e n \beta_\phi \quad . \quad (E8)$$

If the fields E_r , B_ϕ , B_z are presumed to be known, equation (E5) may be examined for the conditions which these fields must obey in order that equilibria exist for which $\beta_r = 0 = \frac{d}{dt} \beta_r$. This analysis is most easily accomplished in an inertial frame of reference for which β_z locally vanishes, which we will loosely refer to as the beam frame, recognizing however that in general β_z will vary with r . In such a frame, the equilibrium condition (E5) may be written as

$$\beta_\phi^2 (1 - \beta_\phi^2)^{-1/2} = \frac{e}{mc^2} r E_r + \frac{e}{mc^2} r B_z \beta_\phi \quad . \quad (E9)$$

As a function of β_ϕ , the left-hand side of equation (E9) increases monotonically from zero and diverges as β_ϕ approaches one, while the right-hand side is a linear function of β_ϕ (if the diamagnetic β_ϕ -dependence of B_z is ignored) with a positive slope and negative intercept at $\beta_\phi = 0$. There will obviously be two solutions if the

functions intersect (i.e., two distinct equilibria) and no solutions if they do not.

A detailed algebraic analysis of equation (E9) reveals the following necessary and sufficient condition for the existence of equilibria:

$$\frac{2 \overline{\omega_{PB}^2(r)}}{\Omega_B^2(r)} Q \left[\frac{-E_{rB}(r)}{B_z(r)} \right] \leq 1 \quad (E10)$$

where

$$\overline{\omega_{PB}^2(r)} \equiv - \frac{2 e E_{rB}(r)}{m r} = \frac{4 \pi e^2}{m} \frac{1}{r^2} \int_0^{r^2} dr'^2 n_B(r'^2)$$

$$\Omega_B(r) \equiv \frac{e}{mc} B_z(r)$$

$$Q(x) \equiv \frac{\frac{1}{2} \left[1 - (1 + 8x^2)^{1/2} \right] + 2x^2}{4x \left[\frac{1}{2} (1 + 8x^2)^{1/2} - \frac{1}{2} - x^2 \right]^{3/2}} \geq 1$$

and the function $Q(x)$ is sketched in Figure (E1) below.

Figure E1. Function $Q(x)$

Here E_{rB} and n_B are the radial electric field and the electron density as seen from the local beam frame, and $B_z(r)$ is the total axial magnetic field--including diamagnetic corrections.

Equation (E10) immediately implies two necessary conditions for equilibrium. First is that the argument of Q , $-E_{rB}/B_z$ must be less than one, which is obvious from equations (E9) or (E5) since otherwise the net forces are manifestly positive. Second, since $Q \geq 1$, the coefficient of Q in equation (E10) must be less than one. This second condition ensures that the centrifugal forces do not destroy the equilibrium.

In order to express these two necessary criteria in terms of quantities observed in the laboratory frame of reference, it is necessary to make a Lorentz transformation, with the transformation velocity $c \beta_z(r)$. The results are

$$1 > \frac{-E_r(r)}{\gamma_T(r) B_z(r)} = \frac{\frac{1}{2} \frac{4\pi e^2}{mc} \frac{1}{r} \int_0^r dr'^2 n(r'^2)}{\gamma_T(r) \Omega_B(r)} \quad (\text{E11})$$

$$1 > \frac{2 \frac{4\pi e^2}{m} \frac{1}{r^2} \int_0^r dr'^2 n(r'^2)}{\gamma_T(r) \Omega_B^2(r)} \quad (\text{E12})$$

where $\gamma_T(r) = [1 - \beta_z^2(r)]^{-1/2}$. Both (E11) and (E12) must be obeyed for all r . Since the numerator of (E11) increases roughly as r , while the denominator of (E11) increases only weakly with r , condition (E11) will be most stringent near the beam edge $r = a$. The numerator of (E12) is roughly constant, while the denominator increases weakly with r , so that condition (E12) will be most stringent near the beam axis $r = 0$.

For virtually all cases of interest, both (E11) and (12) will be well satisfied and the equilibria may be readily analyzed in terms of the corresponding small parameters. In this case, the two equilibrium solutions for β_ϕ will be well separated; the "slow" equilibrium (for which β_ϕ is lowest) is the one desired for the accelerator applications.

In order to self-consistently compute the various equilibrium quantities, assumptions must be made about the conditions of creation of the beam. We shall assume that the electron beam is born at the cathode with $\beta_{\phi C} = 0$, $\gamma_C = 1$, and uniform current density $j_C = -ec n_0 \beta_{oz}$ throughout the initial beam radius a_0 . The cathode is assumed to be at a uniform, negative electric potential $\Phi_C < 0$. The beam is assumed to flow from the cathode through a convergent Pierce gun and then through a post-acceleration structure where it is raised to ground potential $\Phi = 0$ and then injected into a cylindrical waveguide whose conducting walls are also at ground potential.

The initial beam properties which are assumed above may be related to the corresponding beam properties in the

downstream, z-independent equilibrium through the use of three conservation equations. In this way, the chore of computing the actual beam trajectories all the way from the cathode to the downstream equilibrium region may be avoided.

The first equation is the conservation of canonical angular momentum, which follows from the assumption of azimuthal symmetry. It may be written as

$$\int_0^{r_0^2} dr'^2 B_{z0} = \int_0^{r^2} dr'^2 B_z - \frac{2 mc^2}{e} r \gamma \beta_\phi \quad (E13)$$

where r_0 is the initial radius of an electron at radius r in the downstream equilibrium, and B_{z0} is the external guide magnetic field in the cathode region, which is assumed to be known.

From the assumption that the equilibrium is steady-state in time, the continuity equation implies conservation of particles within annular rings:

$$n_0 \beta_{oz} = n \beta_z \frac{dr^2}{dr_0^2} \quad (E14)$$

while the force equations imply an energy constant:

$$\gamma - \frac{e}{mc^2} \phi = 1 - \frac{e}{mc^2} \phi_C \equiv \gamma_0 \quad (\text{E15})$$

where $(\gamma_0 - 1)mc^2$ is the final energy given to the electrons during post-acceleration.

Now the solution for the equilibrium may be obtained by iteration. Let us assume that

$$n = n_c + \delta n \quad (\text{E16})$$

$$\beta_z = \beta_c + \delta \beta_z \quad (\text{E17})$$

where n_c, β_c are the values of n, β_z at the center of the beam, and the ansatz is made that $|\delta n| \ll n_c,$

$|\delta \beta_z| \ll \beta_c$. Next one may define

$$\Omega_c \equiv \frac{e B_{zc}}{\gamma_c mc} \quad (\text{E18})$$

$$\omega_{pc}^2 \equiv \frac{4\pi e^2 n_c}{m \gamma_c} \quad (\text{E19})$$

$$\epsilon_1 = \left(\frac{\omega_{pc}^2 a}{2 \gamma_c \Omega_c c} \right)^2 \quad (\text{E20})$$

$$\epsilon_2 = \frac{2 \omega_{pc}^2}{\gamma_c^2 \Omega_c^2} \quad (\text{E21})$$

where $\gamma_c = (1 - \beta_c^2)^{-1/2}$ and it is assumed that $\epsilon_1 \ll 1$, $\epsilon_2 \ll 1$.

From equations (E6) - (E8) the electromagnetic field profiles are determined

$$B_\phi \approx \beta_c E_r \approx - 2\pi e n_c \beta_c r \quad (\text{E22})$$

$$B_z \approx B_{zc} \left(1 + \epsilon_1 \frac{r^2}{a^2} \right) \quad (\text{E23})$$

and from equation (E5) the equilibrium precessional velocity is determined

$$\beta_\phi \approx \frac{\epsilon_1^{1/2}}{\gamma_c} \frac{r}{a} \left[\frac{1 + \frac{1}{4} \epsilon_2}{1 + \epsilon_1 \frac{r^2}{a^2}} \right] \quad (\text{E24})$$

It is heuristically worthwhile to evaluate the necessary equilibrium conditions (E11) and (E12) at this point. To lowest order, (E11) becomes

$$1 > \frac{-E_{rB}(a)}{B_z} \approx \frac{\omega_{pe}^2 a}{2 \gamma_c \Omega_c c} = \epsilon_1^{1/2} \approx \frac{I_e \text{ (kA)}}{5 B_z \text{ (kG)} a \text{ (cm)} \gamma \beta_z} \quad (\text{E25})$$

and (E12) becomes

$$1 > \frac{2 \omega_{pe}^2}{\Omega_B^2} \approx \frac{2 \omega_{pe}^2}{\gamma_c^2 \Omega_c^2} = \epsilon_2 \approx \frac{I_e \text{ (kA)}}{0.7 B_z^2 \text{ (kG)} a^2 \text{ (cm)} \gamma \beta_z} \quad (\text{E26})$$

where γ and β_z have not been subscripted in the final expressions since we have not yet taken account of their radial variations. It may later be seen that $\gamma \beta_z @ r = a$ is appropriate for (E25) while $\gamma \beta_z @ r = 0$ is appropriate for (E26).

It is now apparent that $\epsilon_1^{1/2}$, which reflects the first necessary condition for equilibrium, also

characterizes the strength of the axial diamagnetism (which varies as ϵ_1) and of the beam-frame precessional velocity $[\beta_{\phi B}(r=a) \approx \epsilon_1^{1/2}]$. Likewise ϵ_2 , which reflects the second necessary condition for equilibrium, also characterizes the strength of centrifugal effects. Therefore, for the majority of cases where ϵ_1 and ϵ_2 are quite small, these complicating effects of diamagnetism, relativistic precession, and centrifugal effects may be treated as small corrections.

If the diamagnetic corrections are neglected in equation (E24), then $\beta_{\phi} \propto r$ and the electron beam precesses as a rigid body. Given the precessional velocity from equation (E24), then r^2/r_o^2 may be determined from equation (E13). One finds

$$\frac{dr^2}{dr_o^2} \approx \frac{B_{zo}}{B_{zc}} \left(1 + \frac{1}{2} \epsilon_2 - \epsilon_1 \frac{r^2}{a^2} \right) \quad (E27)$$

$$\frac{a^2}{a_o^2} = \frac{B_{zo}}{B_{zc}} \left(1 + \frac{1}{2} \epsilon_2 - \frac{1}{2} \epsilon_1 \right) \quad (E28)$$

where B_{z0} is assumed to be uniform across the cathode surface. To lowest order, equation (E13) implies that the electron beam propagates on magnetic flux surfaces; the small corrections due to the mechanical angular momentum acquired by the electrons are the $\frac{1}{2} \epsilon_2$ terms in equations (E27) and (E28).

From equation (E14), the downstream equilibrium particle density may now be expressed as

$$n = \frac{n_o \beta_{oz} a_o^2}{a^2} \frac{\left[1 - \epsilon_1 \left(\frac{1}{2} - \frac{r^2}{a^2} \right) \right]}{\beta_z} \quad (E29)$$

Finally, the radial dependence of γ may be determined from solving Poisson's equation in the downstream equilibrium region where the z -variation is negligible compared to the r -variation. The beam extends from $r = 0$ to $r = a$, with a vacuum gap from $r = a$ to the conducting waveguide walls at $r = b$. The third conservation equation (E15) may be used to define γ in terms of the electric potential ϕ for all r in the range $0 \leq r \leq b$. Hence the boundary condition $\phi(r=b) = 0$

is equivalent to $\gamma(r = b) = \gamma_0$. One thereby arrives at the following differential equation and boundary condition for $\gamma(r)$:

$$\frac{1}{r} \frac{d}{dr} r \frac{d\gamma}{dr} = \frac{4\pi e^2}{mc^2} n \quad (\text{E30})$$

$$\left(\gamma + \ln \frac{b}{a} r \frac{d\gamma}{dr} \right)_{r=a} = \gamma_0 \quad (\text{E31})$$

where the Poisson equation has already been integrated through the vacuum gap to yield the modified boundary condition (E31) at the beam edge. Equations (E30) and (E31) now determine the amount of γ -reduction (i.e., slowing) which the electron beam undergoes because of the space charge in the drift tube.

Inserting equation (E29) into equation (E30), with $\beta_z \approx \beta_c$, one then obtains the results

$$\begin{aligned} \gamma(r) &\approx \gamma_c + \Delta\gamma \frac{r^2}{a^2} \left[1 - \frac{1}{4} \epsilon_1 \left(1 - \frac{r^2}{a^2} \right) \right] \\ &\approx \gamma_c + \Delta\gamma \frac{r^2}{a^2} \end{aligned} \quad (\text{E32})$$

where

$$\gamma_c = \gamma_o - \frac{v}{\beta_c} \left(2 \ln \frac{b}{a} + 1 - \frac{1}{4} \epsilon_1 \right) \quad (\text{E33})$$

$$\Delta\gamma = \frac{v}{\beta_c} \left(1 - \frac{1}{4} \epsilon_1 \right)$$

$$v \equiv \frac{\pi n_o e^2 a^2 \beta_{oz}}{mc^2} = \frac{I_e}{mc^3/e} = \frac{I_e}{17 \text{ kA}} \quad (\text{E34})$$

and we shall assume that

$$\begin{aligned} 1 \gg \frac{\Delta\gamma}{\gamma_c} &= \frac{v \left(1 - \frac{1}{4} \epsilon_1 \right)}{\gamma_c \beta_c} = \frac{\omega_{pc}^2 a^2}{4c^2} \left(1 + \frac{1}{4} \epsilon_1 \right) \\ &= \frac{2 \epsilon_1 \left(1 + \frac{1}{4} \epsilon_1 \right)}{\epsilon_2} \end{aligned} \quad (\text{E35})$$

Although condition (E35) is motivated mainly for theoretical simplicity, it does not significantly restrict the parameter space of most accelerator applications, since such large electron currents are not generally needed.

If equation (E33) is solved for $v(\gamma_c)$, it is found to have a maximum value when $\gamma_c = \gamma_o^{1/3}$. The corresponding maximum value of I_e is called I_L , the limiting current. It is the largest electron current which may be propagated within the drift tube, and is given by

$$I_L \approx \frac{(\gamma_o^{2/3} - 1)^{3/2} mc^3/e}{2 \ln \frac{b}{a} + 1 - \frac{1}{4} \epsilon_1} \quad (E36)$$

We shall also assume for simplicity that $I_e \ll I_L$, which is somewhat similar to condition (E35).

Using the expressions for β_ϕ and γ in equations (E24) and (E32), one may solve for β_z from

$\beta_z = (1 - \gamma^{-2} - \beta_\phi^2)^{1/2}$. The result is

$$\beta_z \approx \beta_c \left[1 + \left(\frac{\Delta\gamma}{\gamma_c^3 \beta_c^2} - \frac{\epsilon_1}{2 \gamma_c^2 \beta_c^2} \right) \frac{r^2}{a^2} \right] \quad (E37)$$

which exhibits the corrections to β_c . In equation (E35), observe that the $\epsilon_1/2 \gamma_c^2 \beta_c^2$ correction is smaller than

the $\Delta\gamma/\gamma_c^3 \beta_c^2$ correction by a factor of $\frac{1}{4} \epsilon_2$. Hence the dominant radial dependence of β_z derives from the space charge slowing, which affects the center electrons relatively more than those on the edge of the beam. The ansatz $1 \gg |\delta\beta_z|/\beta_c \approx \Delta\gamma/\gamma_c^3 \beta_c^2$ follows from the assumption $I_e \ll I_L$.

If equation (E37) is inserted in equation (E29), an improved expression for the radial dependence of the beam density is obtained:

$$n \approx n_c \left[1 - \left(\frac{\Delta\gamma}{\gamma_c^3 \beta_c^2} - \epsilon_1 - \frac{\epsilon_1}{2 \gamma_c^2 \beta_c^2} \right) \frac{r^2}{a^2} \right] \quad (\text{E38})$$

Again, our previous assumptions imply that the $\epsilon_1/2 \gamma_c^2 \beta_c^2$ correction is smaller than the $\Delta\gamma/\gamma_c^3 \beta_c^2$ correction, and that the ansatz $1 \gg |\delta n|/n_c$ is obeyed. However the relative size of the ϵ_1 diamagnetic correction and the $\Delta\gamma/\gamma_c^3 \beta_c^2$ space charge correction is not determined from the preceding assumptions.

At this point, one could continue with a second major iterative cycle through the equations to obtain still higher order corrections. However, since we expect our various assumptions of smallness to be fairly well obeyed for the applications of interest, we will not proceed further.

If the radial dependences herein derived are inserted into the necessary condition (E11), there results

$$1 > \epsilon_1^{1/2} \frac{r}{a} \left[1 - \frac{r^2}{a^2} \left(\frac{\Delta\gamma}{\gamma_c} + \frac{\Delta\gamma}{2 \gamma_c^3 \beta_c^2} \right) \right] \quad (E39)$$

Since our previous assumptions insure that

$[(\Delta\gamma/\gamma_c) + (\Delta\gamma/2 \gamma_c^3 \beta_c^2)] < \frac{1}{3}$, the right-hand side of equation (E39) is largest at $r = a$ as we previously hypothesized. The two small correction terms to the zero order result $1 > \epsilon_1^{1/2}$ are only present because we chose to define ϵ_1 in terms of on-axis quantities in equation (E20).

Likewise, condition (E12) is found to be most restrictive at the center of the beam, and it reduces simply to $1 > \epsilon_2$ since ϵ_2 was originally defined in terms of on-axis quantities.

NOTE: Some of the material in this appendix has been previously discussed elsewhere, and is repeated here for completeness. See W. E. Drummond, J. R. Thompson, et al., "Interaction of Relativistic Charged Particle Beams," Final Scientific Report for AFOSR Contract F49620-76-C-0002, November 1978. I-ARA-78-U-78.

A P P E N D I X F

THEORY OF THE LINEAR CYCLOTRON WAVE SUPPORTED BY AN UNNEUTRALIZED RELATIVISTIC ELECTRON BEAM

Among the travelling waves which may be supported by an unneutralized, relativistic electron beam is one wave whose phase velocity is readily variable over a wide range: from zero to almost $\beta_z c$, the velocity of the electron beam. This is the negative energy electron cyclotron wave, which has been selected for use in the scheme of Auto-Resonant ion acceleration. We will examine herein the derivation of the dispersion relation for these waves, paying attention to certain effects of centrifugal forces and longitudinal compression which are negligible when the electrons are ultrarelativistic (i.e., $\gamma_0^{-2} \ll 1$), but which must be retained for the case of lower energy electrons.

The mathematical complications which arise within linear theory may be traced to the complications inherent in the equilibrium of such unneutralized relativistic

electron beams. The self-fields E_r and B_ϕ of the electron beam combine to produce a net outward force $-e(E_r - \beta_z B_\phi) \approx -e\gamma_0^{-2} E_r$ acting upon the electrons, which can only be counterbalanced if the electrons have a precessional velocity $\beta_\phi c$ in an axial magnetic guide field, so that the magnetic pinch force $-e\beta_\phi B_z$, reduced somewhat by the outward centrifugal force $\gamma_0 mc^2 \beta_\phi^2/r$, restores the radial force equilibrium. This precession of the electrons in turn produces a diamagnetic reduction in the axial magnetic field within the beam, and slightly alters the radial distribution of the electrons relative to that at the cathode before the precession is established. Finally, the injection of the electron beam into an evacuated waveguide is accompanied by some r -dependent slowing of the beam due to space charge forces. This produces a slight outward radial gradient to the axial beam velocity, and corresponding r -dependence for the beam density and relativistic γ -factor. All of the above mentioned effects produce complicated radial dependence in the beam density and velocity parameters, which in turn

cause the self-fields E_r and B_ϕ and the diamagnetic correction to the axial magnetic field B_z to have complicated r -dependence.

These complicating features in the self-consistent equilibrium quantities produce a corresponding complication in the radial eigenvalue problem which must be solved to yield the linear wave dispersion relation. The extent of the complication is apparent from examining the linearized version of the relativistic force equation:

$$\frac{d}{dt} \gamma \underline{\beta} = - \frac{e}{mc} (\underline{E} + \underline{\beta} \times \underline{B}) \quad (F1)$$

where $e = -|e|$, m are the electron charge and mass, $\gamma_0 = (1 - \beta_0^2)^{-1/2}$, $\underline{\beta}_0 = \beta_z \hat{z} + \beta_\phi \hat{\phi}$ is the normalized, zero-order velocity vector of the electrons, and $\underline{E}_0 = E_r \hat{r}$, $\underline{B}_0 = B_z \hat{z} + B_\phi \hat{\phi}$ are the zero-order electromagnetic fields. In zero-order (i.e., equilibrium), the radial component of the force equation (1) is

$$-\gamma_0 \frac{c \beta_\phi^2}{r} = - \frac{e}{mc} (E_r - \beta_z B_\phi + \beta_\phi B_z) \quad (F2)$$

while in first-order, the force equation may be written as

$$\underbrace{\tilde{\beta} \left(\frac{d}{dt} \gamma \right)}_0 + \beta_0 \left(\frac{d}{dt} \gamma \right) + \tilde{\gamma} \left(\frac{d}{dt} \beta \right)_0 + \gamma_0 \left(\frac{d}{dt} \beta \right) =$$

$$- \frac{e}{mc} (\tilde{\mathbf{E}} + \tilde{\beta} \times \mathbf{B}_0 + \beta_0 \times \tilde{\mathbf{B}}) \quad (\text{F3})$$

where a tilde denotes first-order (i.e., wave) quantities.

The $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ components of equation (F3) are as follows:

$$\tilde{\gamma} \left(\frac{-e}{mc\gamma_0} \right) [\mathbf{E}_r - \beta_z \mathbf{B}_\phi + \beta_\phi \mathbf{B}_z] + \gamma_0 \left[\frac{\partial}{\partial t} + \frac{c\beta_\phi}{r} \frac{\partial}{\partial \phi} + c\beta_z \frac{\partial}{\partial z} \right] \tilde{\beta}_r$$

$$- 2\gamma_0 c \frac{\beta_\phi \tilde{\beta}_\phi}{r} = \frac{-e}{mc} [\tilde{\mathbf{E}}_r - \beta_z \tilde{\mathbf{B}}_\phi + \tilde{\beta}_\phi \mathbf{B}_z + \beta_\phi \tilde{\mathbf{B}}_z - \tilde{\beta}_z \mathbf{B}_\phi]$$

(F4)

$$\begin{aligned}
& \beta_{\phi} \left(\frac{-e}{mc} \right) [\tilde{\beta}_r E_r + \beta_{\phi} \tilde{E}_{\phi} + \beta_z \tilde{E}_z] + \gamma_0 \left[\frac{\partial}{\partial t} + \frac{c\beta_{\phi}}{r} \frac{\partial}{\partial \phi} + c\beta_z \frac{\partial}{\partial z} \right] \tilde{\beta}_{\phi} \\
& + \gamma_0 \frac{c\beta_{\phi}}{r} \tilde{\beta}_r + \gamma_0 c \tilde{\beta}_r \frac{\partial}{\partial r} \beta_{\phi} = \frac{-e}{mc} [\tilde{E}_{\phi} - \tilde{\beta}_r B_z + \beta_z \tilde{B}_r]
\end{aligned}$$

(F5)

$$\begin{aligned}
& \beta_z \left(\frac{-e}{mc} \right) [\tilde{\beta}_r E_r + \beta_{\phi} \tilde{E}_{\phi} + \beta_z \tilde{E}_z] + \gamma_0 \left[\frac{\partial}{\partial t} + \frac{c\beta_{\phi}}{r} \frac{\partial}{\partial \phi} + c\beta_z \frac{\partial}{\partial z} \right] \tilde{\beta}_z \\
& + \gamma_0 c \tilde{\beta}_r \frac{\partial}{\partial r} \beta_z = \frac{-e}{mc} [\tilde{E}_z + \tilde{\beta}_r B_{\phi} - \beta_{\phi} \tilde{B}_r]
\end{aligned}$$

(F6)

Here, the downward pointing arrows indicate terms which are present because of the equilibrium precession of the beam, while the upward pointing arrows indicate terms which are present because of the self-fields of the beam equilibrium. There are seventeen such terms in these equations, all of

which have complicated r -dependence and all of which are missing for the case of a charge- and current-neutralized beam or for an ultrarelativistic beam.

It is clear that the arrowed terms vanish individually for a neutralized beam. The ultrarelativistic limit may be taken formally by tending γ_0, B_z, n (hence E_r, B_ϕ) to infinity in such a way that the ratio of any two such quantities remains finite. In this way, the relativistic plasma and cyclotron frequencies remain finite, but longitudinal mass terms and precessional terms, which vary as γ_0^{-3} , tend to zero. Thus the β_ϕ terms of equations (F4) - (F6) tend to zero and the remaining self-field terms cancel to within $O(\gamma_0^{-2})$ and may be omitted. One also finds that $\tilde{\beta}_z \propto \gamma_0^{-3} \rightarrow 0$, so that there is no longitudinal beam compression in the ultrarelativistic limit.

Although neither of these limits is appropriate for a steady-state Auto-Resonant Accelerator, one can still define parameters ϵ_1, ϵ_2 , and $\beta_\phi(a) \approx \epsilon_1^{1/2}/\gamma_0$ as in Appendix E, which are small for all cases of interest and thus simplify the theory. (Notice that each of these parameters would vanish within the ultrarelativistic limit.)

Of the above three small parameters, the largest for steady-state accelerator applications is ϵ_2 , reflecting the centrifugal force effects. The parameters ϵ_1 and $\beta_\phi(a)$ represent effects of axial diamagnetism, $\beta_\phi \times \tilde{\underline{B}}$ forces, and relativistic beam-frame precession which have a magnitude of only a few percent or less for most steady-state accelerator applications. We shall neglect these latter effects, but retain the larger ϵ_2 centrifugal effects within the linear theory. In addition, it is necessary to retain effects such as longitudinal compression which have a relative magnitude of γ_0^{-2} , and are not at all small for many steady-state accelerator applications.

Finally, there are the small parameters $\Delta\gamma/\gamma_c$ and $\Delta\gamma/\gamma_c^3\beta_c^2$, also discussed in Appendix E. These parameters reflect the radial variation in beam energy due to effects of space-charge slowing. They are small only if the electron beam current is well below the Alfvén-Lawson limits, which--fortunately--is the usual case. Nevertheless, these corrections can sometimes be as large as ten percent, in which case they should be included in the analysis. In addition, these parameters do not formally vanish in the ultrarelativistic

limit, and they consequently have relevance for pulsed accelerator applications as well as for steady-state accelerator applications. However, because these effects produce radial dependence which greatly complicates the mathematical eigenvalue problem within linear theory, we shall neglect them for the present calculation. This will allow the linear theory to be solved in terms of relatively simple eigenfunctions (i.e., Bessel functions), and will allow us to examine the modifications which arise due to the retention of the ϵ_2 and γ_0^{-2} terms which are omitted for ultrarelativistic beams. The results however, will only be of high accuracy for the lower current regime of steady-state accelerators, and further analysis may later be required for high accuracy within the higher current regime of parameter space.

The approximations discussed above for this linear theory calculation imply that the electron density n , the axial velocity $\beta_z c$, and the kinetic energy $(\gamma - 1)mc^2$ may be treated as constants with respect to r . Hence we shall denote them as $n = n_0$, $\beta_z = \beta_0$, $\gamma = \gamma_0$. In addition we may let $\gamma_0 = (1 - \beta_0^2)^{-1/2}$ since the β_ϕ corrections to

.. γ_0 will be negligible under these approximations.

Furthermore, the axial magnetic field B_z may be treated as uniform in r , and we may define the usual characteristic frequencies

$$\Omega \equiv \frac{e B_z}{\gamma_0 m c} \quad (F7)$$

$$\omega_p^2 \equiv \frac{4\pi n_0 e^2}{\gamma_0 m} \quad , \quad (F8)$$

and the small parameters

$$\epsilon_1 \equiv \left(\frac{\omega_p^2 a}{2 \gamma_0 \Omega c} \right)^2 \ll 1 \quad (F9)$$

$$\epsilon_2 \equiv \left(\frac{2 \omega_p^2}{\gamma^2 \Omega^2} \right) \ll 1 \quad . \quad (F10)$$

The equilibrium self-fields of the beam are then given by

$$B_\phi \approx \beta_0 E_r \approx -2\pi e n_0 \beta_0 r \quad (F11)$$

and the equilibrium velocity of precession/c is

$$\beta_{\phi} \approx \frac{\omega^2 r}{2 \gamma_0^2 \Omega c} = \frac{\epsilon_1^{1/2}}{\gamma_0} \frac{r}{a} \quad . \quad (F12)$$

Among our approximations are the orderings

$$\epsilon_1 \ll \epsilon_2, \quad \beta_{\phi}(a) \ll 1 \quad (F13)$$

which also implies that

$$1 \gg \gamma_0 \epsilon_1^{1/2} \approx \gamma_0^2 \beta_{\phi}(a) \approx \frac{E_r(a)}{B_z} \quad . \quad (F14)$$

Returning again to the linear theory of waves supported by such an equilibrium, we shall restrict attention to the ϕ -independent sausage mode, for which $\frac{\partial}{\partial \phi} = 0$. The force equations, continuity equation, and Maxwell's field equations are linearized and transformed with respect to the basis functions $\exp i(k_z z - \omega t)$. It is convenient to define the following wave field variables

$$A^+ \equiv \tilde{E}_r + i \tilde{E}_\phi + i \beta_o \tilde{B}_r - \beta_o \tilde{B}_\phi \quad (F15)$$

$$A^- \equiv \tilde{E}_r - i \tilde{E}_\phi - i \beta_o \tilde{B}_r - \beta_o \tilde{B}_\phi \quad (F16)$$

For this case in which β_o is constant and $\frac{\partial}{\partial \phi} = 0$, Maxwell's equations imply the following pair of equations for $A^\pm$

$$\left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - k_z^2 + \frac{\omega^2}{c^2} \right) A^\pm = - \frac{4\pi i \Delta \omega}{c^2} (\tilde{J}_r \pm i \tilde{J}_\phi) + 4\pi \frac{\partial}{\partial r} \left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z \right) \quad (F17)$$

where $\Delta \omega \equiv \omega - k_z \beta_o c$ and $\tilde{\rho}, \tilde{J}$ are the beam charge and current density. In general, there will be boundary conditions at the conducting wall at $r = b$ to ensure that $\tilde{E}_z, \tilde{E}_\phi, \tilde{B}_r$ vanish there, and in addition there must be jump conditions at the beam edge $r = a$ to properly patch the beam interior field solutions to the vacuum gap field solutions. However, a great deal of this complexity may be avoided when the vacuum gap is narrow (i.e., $b - a \ll a$),

since then only the beam interior solutions are required and the boundary conditions may be applied at the beam edge. We shall make this assumption, in which case the boundary conditions reduce to

$$(A^+ - A^-)_{r=a} = 0 \quad (F18)$$

$$\left[\frac{1}{r} \frac{\partial}{\partial r} r (A^+ + A^-) \right]_{r=a} = 8\pi \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right)_{r=a} \quad (F19)$$

From the variables $A^\pm$, the perturbed electric and magnetic fields are found as follows

$$\tilde{E}_z = \frac{i}{2 \left(k_z - \beta_0 \frac{\omega}{c} \right)} \left[\frac{1}{r} \frac{\partial}{\partial r} r (A^+ + A^-) - 8\pi \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right) \right] \quad (F20)$$

$$\tilde{E}_r = \frac{-1}{2 \left(k_z - \beta_0 \frac{\omega}{c} \right)} \left[2 \beta_0 i \frac{\partial \tilde{E}_z}{\partial r} + \frac{\omega}{c} (A^+ + A^-) \right] \quad (F21)$$

$$\tilde{E}_\phi = \frac{i}{2 \left(k_z \beta_o - \frac{\omega}{c} \right)} \left[\frac{\omega}{c} (A^+ - A^-) \right] \quad (F22)$$

$$\tilde{B}_z = \frac{1}{2 \left(k_z \beta_o - \frac{\omega}{c} \right)} \left[\frac{1}{r} \frac{\partial}{\partial r} r (A^+ - A^-) \right] \quad (F23)$$

$$\tilde{B}_r = \frac{-1}{2 \left(k_z \beta_o - \frac{\omega}{c} \right)} \left[i k_z (A^+ - A^-) \right] \quad (F24)$$

$$\tilde{B}_\phi = \frac{-1}{2 \left(k_z \beta_o - \frac{\omega}{c} \right)} \left[2i \frac{\partial \tilde{E}_z}{\partial r} + k_z (A^+ - A^-) \right] \quad (F25)$$

The perturbed beam charge and current densities are related to the perturbed velocities and number density as follows

$$\tilde{\rho} = -e \tilde{n} \quad (F26)$$

$$\tilde{J}_z = -ec (\beta_o \tilde{n} + n_o \tilde{\beta}_z) \quad (F27)$$

$$\tilde{J}_r = - ec n_o \tilde{\beta}_r \quad (F28)$$

$$\tilde{J}_\phi = - ec (n_o \tilde{\beta}_\phi + \tilde{n} \beta_\phi) \approx - ec n_o \tilde{\beta}_\phi \quad (F29)$$

where the β_ϕ term in equation (F29) is neglected according to the previously discussed approximation (F13).

The force equations (F4) - (F6) simplify under our approximations to the form

$$i \frac{\Delta\omega}{\Omega} \tilde{\beta}_r - \left(1 - \frac{2 c\beta_\phi}{r\Omega}\right) \tilde{\beta}_\phi = \frac{1}{B_z} (\tilde{E}_r - \beta_o \tilde{B}_\phi) = \frac{A^+ + A^-}{2 B_z} \quad (F30)$$

$$\left(1 - \frac{2 c\beta_\phi}{r\Omega}\right) \tilde{\beta}_r + \frac{i\Delta\omega}{\Omega} \tilde{\beta}_\phi = \frac{1}{B_z} (\tilde{E}_\phi + \beta_o \tilde{B}_r) = \frac{A^+ - A^-}{2i B_z} \quad (F31)$$

$$i\Delta\omega \tilde{\beta}_z = \frac{e}{\gamma_o mc} [\gamma_o^{-2} \tilde{E}_z - \beta_\phi (\tilde{B}_r + \beta_o \tilde{E}_\phi)] \approx \frac{e}{\gamma_o^3 mc} \tilde{E}_z \quad (F32)$$

where approximation (F14) is invoked to simplify equation (F32).

Finally, the continuity equation is employed to relate $\tilde{n}$ to $\tilde{\beta}$ as follows

$$\tilde{n} = n_o \left(\frac{k_z c}{\Delta \omega} \tilde{\beta}_z + \frac{c}{i \Delta \omega} \frac{1}{r} \frac{\partial}{\partial r} r \tilde{\beta}_r \right) . \quad (F33)$$

It may be observed that the ϵ_2 effects are represented through the centrifugal and Coriolis force terms in equations (F30) and (F31), which are the terms with coefficients $\frac{2 c \beta_\phi}{r \Omega} = \frac{1}{2} \epsilon_2$. These effects may be conveniently retained by introducing the modified variables

$$\bar{B}_z \equiv B_z \left(1 - \frac{1}{2} \epsilon_2 \right) ; \quad \bar{\Omega} \equiv \Omega \left(1 - \frac{1}{2} \epsilon_2 \right) \quad (F34)$$

in terms of which equations (F30) and (F31) become

$$i \frac{\Delta \omega}{\bar{\Omega}} \tilde{\beta}_r - \tilde{\beta}_\phi = \frac{A^+ + A^-}{2 \bar{B}_z} \quad (F35)$$

$$\tilde{\beta}_r + \frac{i\Delta\omega}{\Omega} \tilde{\beta}_\phi = \frac{A^+ - A^-}{2i \bar{B}_z} \quad (\text{F36})$$

The source terms $\tilde{J}_r + i \tilde{J}_\phi$ in equation (F17) require only $\tilde{\beta}_r, \tilde{\beta}_\phi$ which may be readily related to $A^\pm$ via the $\hat{r}, \hat{\phi}$ force equations (F35) and (F36). However, the source terms $\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z$ require $\gamma_0^{-2} \tilde{n} - n_0 \tilde{\beta}_z$, which couples the continuity equation and the $\hat{z}$ force equation to Maxwell's equations. This is where the γ_0^{-2} terms arise which are independent of the beam current magnitude. From equations (F20), (F26), (F27), (F32), (F33), (F35), and (F36), one finds that

$$8\pi \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right) = - \left[\frac{\frac{\omega^2 p}{\gamma_0^2 \Delta\omega^2}}{1 - \frac{\omega^2 p}{\gamma_0^2 \Delta\omega^2}} \right] \frac{1}{r} \frac{\partial}{\partial r} r \left[\frac{A^+}{1 + \frac{\Delta\omega}{\Omega}} \right. \\ \left. + \frac{A^-}{1 - \frac{\Delta\omega}{\Omega}} \right] \quad (\text{F37})$$

while from equations (F28), (F29), (F35), and (F36), one finds

$$-\frac{4\pi i}{c^2} \Delta\omega (\tilde{J}_r \pm i \tilde{J}_\phi) = \frac{\omega^2}{c^2} \frac{p}{\Delta\omega \pm \bar{\Omega}} A^\pm \quad (F38)$$

One may now insert equations (F37) and (F38) into the pair of equations (F17) to obtain a pair of coupled, second-order differential equations for the wave field variables $A^\pm$. To shorten the notation, we may define

$$\delta = \frac{\Delta\omega}{\bar{\Omega}} \quad (F39)$$

$$p_\pm^2 = k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega^2}{c^2} \frac{p}{\Delta\omega \pm \bar{\Omega}} \quad (F40)$$

$$T = \frac{\frac{\omega^2}{2 \gamma_o^2 \Delta\omega^2} p}{1 - \frac{\omega^2}{\gamma_o^2 \Delta\omega^2} p} \quad (F41)$$

and from equations (F17) - (F19) there results

$$\left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - p_{\pm}^2 \right) A^{\pm} = - T \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r \left(\frac{A^+}{1+\delta} + \frac{A^-}{1-\delta} \right) \quad (F42)$$

$$(A^+ - A^-)_a = 0 \quad (F43)$$

$$\left[\frac{1}{r} \frac{\partial}{\partial r} r (A^+ + A^-) \right]_a = - 2T \left[\frac{1}{r} \frac{\partial}{\partial r} r \left(\frac{A^+}{1+\delta} + \frac{A^-}{1-\delta} \right) \right]_a \quad (F44)$$

The pair of equations (F42) would be uncoupled in the ultrarelativistic limit where T vanishes, but otherwise they are coupled due to the γ_0^{-2} compressional effects. This coupled set of second order equations is equivalent to the following uncoupled pair of fourth order equations

$$\left[\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - \frac{p_+^2 (1+\delta) (1-\delta + T)}{1 - \delta^2 + 2T} \right] \left[\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - \frac{p_-^2 (1-\delta) (1+\delta+T)}{1 - \delta^2 + 2T} \right] A^{\pm} = \frac{T^2 p_+^2 p_-^2 (1-\delta^2)}{(1-\delta^2+2T)^2} A^{\pm} \quad (F45)$$

which may be solved in terms of Bessel functions since the coefficients are independent of r . If one makes the ansatz

$$A^+ = \alpha_1^+ J_1(k_{11}r) + \alpha_2^+ J_1(k_{12}r) \quad (F46)$$

$$A^- = \alpha_1^- J_1(k_{11}r) + \alpha_2^- J_1(k_{12}r) \quad (F47)$$

then equation (F45) reduces to a quadratic equation for k_1^2 which must be obeyed by both k_{11}^2 and k_{12}^2 , which are therefore the two roots of the quadratic:

$$\left[k_1^2 + \frac{p_+^2 (1 + \delta) (1 - \delta + T)}{1 - \delta^2 + 2T} \right] \left[k_1^2 + \frac{p_-^2 (1 - \delta) (1 + \delta + T)}{1 - \delta^2 + 2T} \right] - \frac{T^2 p_+^2 p_-^2 (1 - \delta^2)}{(1 - \delta^2 + 2T)^2} = 0 \quad (F48)$$

If equations (F46) and (F47) are inserted into (F42) and the boundary conditions (F43) and (F44), one finally arrives

at the dispersion relation, which may be cast in the form

$$G_1 \frac{k_{11} a J_0(k_{11} a)}{J_1(k_{11} a)} + G_2 \frac{k_{12} a J_0(k_{12} a)}{J_1(k_{12} a)} = 0 \quad (\text{F49})$$

where

$$G_1 = \frac{\left(1 + \frac{2T}{1+\delta}\right) + \left(1 + \frac{2T}{1-\delta}\right) \left\{ \frac{\frac{T p_+^2 (1-\delta)}{1-\delta^2 + 2T}}{\left[k_{11}^2 + \frac{p_-^2 (1-\delta) (1+\delta+T)}{1-\delta^2 + 2T} \right]} \right\}}{1 - \left\{ \frac{\frac{T p_+^2 (1-\delta)}{1-\delta^2 + 2T}}{\left[k_{11}^2 + \frac{p_-^2 (1-\delta) (1+\delta+T)}{1-\delta^2 + 2T} \right]} \right\}} \quad (\text{F50})$$

and

$$G_2 = - G_1 (k_{11}^2 + k_{12}^2) \quad (\text{F51})$$

For the case of the negative energy cyclotron mode, we may anticipate that $\Delta\omega \approx -\Omega$, hence $\delta \approx -1$ and $T \approx \frac{1}{4} \epsilon_2 \ll 1$. Now let k_{11}^2 be that solution of

equation (F48) which is near the zero of the first square bracket

$$k_{11}^2 \approx - \frac{p_+^2 (1 + \delta) (1 - \delta + T)}{1 - \delta^2 + 2T} + \dots \quad (F52)$$

and likewise k_{12}^2 will be the remaining solution which is near the zero of the second square bracket

$$k_{12}^2 \approx - \frac{p_-^2 (1 - \delta) (1 + \delta + T)}{1 - \delta^2 + 2T} + \dots \quad (F53)$$

Referring to equation (F40), one notes that $p_+^2 < 0$ is associated with the negative energy cyclotron mode, and therefore $k_{11}^2 > 0$ and k_{11} will be the corresponding perpendicular wave number for that mode. Likewise, when $\Delta\omega \approx -\Omega$, $p_-^2 > 0$ and thus $k_{12}^2 < 0$. Consequently, the Bessel functions in equation (F49) of argument $k_{12}a$ actually behave like the corresponding modified Bessel functions I_0 and I_1 of real arguments. Furthermore, for $|k_{12}a|^2 \approx p_-^2 a^2 \lesssim 1$, these Bessel functions may be expanded as

$$\frac{k_{12} a J_0(k_{12} a)}{J_1(k_{12} a)} \approx 2 - \frac{1}{4} k_{12}^2 a^2 \approx 2 + \frac{1}{4} p_-^2 a^2 . \quad (\text{F54})$$

The functions G_1 and G_2 may also be expanded for $\delta \approx -1$, $|p_-^2/p_+^2| \ll 1$, $T \ll 1$ to yield

$$G_1 \approx 1 \quad (\text{F55})$$

$$G_2 \approx 1 + T \left[1 + \frac{p_-^2}{p_+^2} \left(1 + \frac{T}{1+\delta} \right) \right]$$

so that the dispersion relation (F49) becomes

$$\frac{k_{11} a J_0(k_{11} a)}{J_1(k_{11} a)} \approx -2 \left\{ 1 + \frac{p_-^2 a^2}{8} + T \left[1 + \frac{p_-^2}{p_+^2} \left(1 + \frac{T}{1+\delta} \right) \right] \right\}$$

$$\approx -2 \quad (\text{F56})$$

Since the right hand side is approximately constant, equation (F56) approximately determines the numerical value of $k_{11} a$, and then equation (F48) becomes the dispersion relation relating ω and k_2 . The smallest solution of equation (F56) is given by

$$k_{\perp 1} a \approx 2.95 + \frac{1}{5.66} \left\{ \frac{p_-^2 a^2}{4} + 2T \left[1 + \frac{p_-^2}{p_+^2} \left(1 + \frac{T}{1+\delta} \right) \right] \right\}$$

$$\approx 2.95 \quad (\text{F57})$$

Finally, if equations (F34) and (F39) - (F41) are inserted into equation (F48), the dispersion relation for linear waves becomes

$$0 = \left(\Omega^2 - \frac{2\omega^2}{\gamma_0^2} \frac{p}{\gamma_0^2} \right) (k^2 c^2 - \omega^2) \left[\Delta\omega^2 (k^2 c^2 - \omega^2) \right. \\ \left. - \frac{\omega^2}{\gamma_0^2} \frac{p}{\gamma_0^2} (k_z^2 c^2 - \omega^2) \right] \\ - \Delta\omega^2 \left(\Delta\omega^2 - \frac{\omega^2}{\gamma_0^2} \frac{p}{\gamma_0^2} \right) (k^2 c^2 + \omega_p^2 - \omega^2)^2 \quad (\text{F58})$$

where $k^2 = k_{\perp 1}^2 + k_z^2$ and $\Delta\omega = \omega - k_z \beta_0 c$. This dispersion relation contains eight waves: four light waves, two longitudinal plasma waves, and two cyclotron waves. If equation (F58) is examined for the case of $\omega^2 \ll k_z^2 c^2 < k^2 c^2$ for which the four light waves are

decoupled, it is quadratic in $\Delta\omega^2$, and the solution which represents the two cyclotron waves may be written as

$$\begin{aligned} \Delta\omega^2 = & \frac{1}{2} \left[\frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma_o^2} \right] \\ & + \frac{1}{2} \left\{ \left[\frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma_o^2} \right]^2 \right. \\ & \left. - \frac{4\omega_p^2 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right) k^2 k_z c^4}{\gamma_o^2 (k^2 c^2 + \omega_p^2)^2} \right\}^{1/2} \end{aligned} \quad (F59)$$

If equation (F59) is further expanded for

$$2 \frac{\omega_p}{\gamma_o} \frac{\left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)^{1/2} k k_z c^2}{(k^2 c^2 + \omega_p^2)} \ll \frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma_o^2} \quad (F60)$$

one obtains

$$\Delta\omega^2 \approx \frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma_o^2} \frac{k_{\perp}^2}{k^2} \quad (F61)$$

Finally, for $\omega_p^2 \ll k^2 c^2$, $\gamma_o^2 \Omega^2$ one obtains

$$\Delta\omega = \omega - k_z \beta_o c \approx -\Omega \left[1 - \frac{\omega_p^2}{k^2 c^2} - \frac{\omega_p^2}{\gamma_o^2 \Omega^2} \left(1 - \frac{k_{\perp}^2}{2k^2} \right) \right] \quad (F62)$$

as the dispersion relation for the negative energy cyclotron mode. Here the correction term $-\omega_p^2/\gamma_o^2 \Omega^2$ is due to the centrifugal and Coriolis forces on the electrons, while the correction $+\omega_p^2 k_{\perp}^2 / 2k^2 \gamma_o^2 \Omega^2$ is due to the longitudinal compression of the electrons which couples longitudinal plasma modes to the cyclotron modes through the T terms.

The dispersion relation (F62) is the starting point for all subsequent linear analysis which must be performed to examine the behavior of the negative energy cyclotron wave in an Auto-Resonant Accelerator. For example, one may use (F62) to compute the linear wave fields A^+ and A^- and

then the wave electric and magnetic fields from equations
(F20) - (F25).

NOTE: Some of the material in this appendix has been previously discussed elsewhere, and is repeated here for completeness. See W. E. Drummond, J. R. Thompson, et al., "Interactions of Relativistic Charged Particle Beams," Final Scientific Report for AFOSR Contract F49620-76-C-0002, November 1978. I-ARA-78-U-78.

A P P E N D I X G

EXAMINATION OF THE LANDAU RESONANCE FOR A NEGATIVE ENERGY ELECTRON CYCLOTRON WAVE WITHIN FLUID THEORY

In Appendix F, the linear fluid theory was developed for the axisymmetric negative energy electron cyclotron wave, which has been selected for use in the scheme of Auto-Resonant ion acceleration. In this analysis, effects of centrifugal and Coriolis forces and of longitudinal beam compression were included. Although these effects are negligible when the electrons are ultrarelativistic (i.e., $1 \gg \gamma_0^{-2}$, ϵ_1 , ϵ_2 in the prior notation), they must be retained for lower energy electrons which are of relevance to many steady-state accelerator applications. In turn, these finite- γ_0 effects have been found to allow an exact resonance to occur between the perturbed radial motion of the electrons and the fields of the negative energy electron cyclotron wave, which is not possible in the previously explored ultrarelativistic approximation.

The possibility of such a resonance is apparent from equation (F61) for the approximate cyclotron wave dispersion relation:

$$\Delta\omega^2 \approx \frac{k^4 c^4 \left(\Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \right)}{(k^2 c^2 + \omega_p^2)^2} + \frac{\omega_p^2}{\gamma_o^2} \frac{k_{\perp}^2}{k^2} \quad (G1)$$

which may be simplified further for $\omega_p^2 \ll k^2 c^2$, $\gamma_o^2 \Omega^2$ to the form

$$\Delta\omega = \omega - k_z \beta_o c \approx -\bar{\Omega} \left[1 - \frac{\omega_p^2}{k^2 c^2} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma_o^2 \bar{\Omega}^2} \right) \right], \quad (G2)$$

where the centrifugal and Coriolis effects may be absorbed in the modified cyclotron frequency variable

$$\bar{\Omega} \equiv \Omega(1 - 1/2 \epsilon_2); \quad \bar{\Omega}^2 \approx \Omega^2 - \frac{2\omega_p^2}{\gamma_o^2} \quad (G3)$$

The presence of the correction term $+ \omega_p^2 k_{\perp}^2 / 2k^2 \gamma_0^2 \bar{\Omega}^2$ due to longitudinal velocity perturbations now makes it possible for the resonance condition

$$0 = 1 + \delta \equiv \frac{\Delta\omega + \bar{\Omega}}{\bar{\Omega}} \approx \frac{\omega_p^2}{k^2 c^2} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma_0^2 \bar{\Omega}^2} \right) \quad (\text{G4})$$

to be satisfied when the magnetic field is reduced to the point that $\bar{\Omega} = e\bar{B}_z / \gamma_0 mc = k_{\perp} c / \sqrt{2} \gamma_0$. The significance of this resonance condition is apparent from the linearized equations for transverse velocity perturbations

$$i \frac{\Delta\omega}{\bar{\Omega}} \tilde{\beta}_r - \tilde{\beta}_\phi = \frac{A^+ + A^-}{2 \bar{B}_z} \quad (\text{G5})$$

$$\tilde{\beta}_r + \frac{i\Delta\omega}{\bar{\Omega}} \tilde{\beta}_\phi = \frac{A^+ - A^-}{2 \bar{B}_z} \quad (\text{G6})$$

from which it follows that

$$i(\tilde{\beta}_r + i\tilde{\beta}_\phi) = \frac{A^+}{\bar{B}_z(1+\delta)} \quad (G7)$$

$$i(\tilde{\beta}_r - i\tilde{\beta}_\phi) = -\frac{A^-}{\bar{B}_z(1-\delta)} \quad (G8)$$

with corresponding solutions for $\tilde{\beta}_r, \tilde{\beta}_\phi$ individually obtained from sums and differences of equations (G7) and (G8). Here the wave field variables $A^\pm$ are related to the perturbed electromagnetic fields according to

$$A^\pm \equiv \tilde{E}_r \pm i\tilde{E}_\phi \pm i\beta_o \tilde{B}_r - \beta_o \tilde{B}_\phi \quad . \quad (G9)$$

It may be observed that when the resonance condition (G4) is satisfied, the determinant of the coefficients of $\tilde{\beta}_r, \tilde{\beta}_\phi$ in equations (G5) - (G6) vanishes, and consequently the combination $(\tilde{\beta}_r + i\tilde{\beta}_\phi)$ having the helicity of the negative energy cyclotron wave must either diverge or else the resonant eigenfunction A^+ must vanish, according to equation (G7).

In fact, the empirical evidence obtained from exercising the SURVEY code in the neighborhood of resonance has been that $|\tilde{\beta}_r|$ and $|\tilde{\beta}_\phi|$ become large--suggesting the disturbing possibility of a divergence within the fluid theory. However, this is not to be expected on theoretical grounds, since bounded, nontrivial solutions to the linearized fluid equations should exist even when the resonant eigenfunction A^+ vanishes. In this appendix we shall confirm this theoretical expectation, and in so doing shall lay the groundwork for a reformulation of the linearized equations in SURVEY which may be satisfactorily employed even near the conditions of resonance.

We pose the problem by assuming that the resonance condition (G4) is obeyed, and seeking whether bounded, nontrivial solutions to the Maxwell-fluid equations can be obtained. Continuing with the nomenclature and approximations of Appendix F, we recall that the linearized Maxwell equations imply the following pair of equations for the field variables $A^\pm$:

$$\begin{aligned}
\left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - k_z^2 + \frac{\omega^2}{c^2} \right) A^\pm = - \frac{4\pi i \Delta \omega}{c^2} (\tilde{J}_r \pm i \tilde{J}_\phi) \\
+ 4\pi \frac{\partial}{\partial r} \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right) .
\end{aligned}
\tag{G10}$$

We have already seen from equation (G7) that when
 $1 + \delta = 0$,

$$A^+ = 0 \tag{G11}$$

is required to avoid divergence in the perturbed velocity. Therefore only the A^- equation (G10) remains to be solved. This equation requires the source term $\tilde{J}_r - i\tilde{J}_\phi = -en_0 c(\tilde{\beta}_r - i\tilde{\beta}_\phi)$, and a solution for this particular combination is available in equation (G8), despite the fact that $\tilde{\beta}_r$ and $\tilde{\beta}_\phi$ are now individually indeterminable. Using equations (G8) and (G4), one obtains

$$- \frac{4\pi i \Delta \omega}{c^2} (\tilde{J}_r - i \tilde{J}_\phi) = \frac{\omega^2}{2c^2} A^- \quad . \quad (G12)$$

Equation (G10) also requires the source term

$$\begin{aligned} \left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z \right) &= - e (\gamma_o^{-2} \tilde{n} - n_o \beta_o' \tilde{\beta}_z) \\ &= - e \gamma_o^{-2} \left(\tilde{n} - \frac{i n_o \beta_o}{\bar{B}_z} \tilde{E}_z \right) \end{aligned} \quad (G13)$$

where $\tilde{\beta}_z$ is eliminated via the axial equation of motion

$$\tilde{\beta}_z = \frac{e}{i \Delta \omega \gamma_o^3 m c} \tilde{E}_z = \frac{i}{\gamma_o^2} \frac{\tilde{E}_z}{\bar{B}_z} \quad . \quad (G14)$$

If we attempt to follow the prior derivation at this point by employing the continuity equation to eliminate $\tilde{n}$

$$\tilde{n} = n_o \left(\frac{k_z c}{\Delta \omega} \tilde{\beta}_z + \frac{c}{i \Delta \omega} \frac{1}{r} \frac{\partial}{\partial r} r \tilde{\beta}_r \right) \quad (G15)$$

and by using the expression

$$\tilde{E}_z = \frac{i}{\left(k_z - \beta_o \frac{\omega}{c} \right)} \left[\frac{1}{2r} \frac{\partial}{\partial r} r A^- - 4\pi \left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z \right) \right] \quad (G16)$$

which follows from Maxwell's equations when $A^+ = 0$, and by using equations (G4) and (G8), then one arrives at the result

$$4\pi \left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z \right) = T \frac{1}{r} \frac{\partial}{\partial r} r (2\bar{B}_z \tilde{\beta}_\phi) \quad (G17)$$

where

$$T \equiv \frac{\frac{\omega_p^2}{2 \gamma_o^2 \Delta \omega^2}}{1 - \frac{\omega_p^2}{\gamma_o^2 \Delta \omega^2}} \quad (G18)$$

However, the equation (G17) is not useful to insert into equation (G10) since $\tilde{\beta}_\phi$ is so far indeterminable. Therefore, we adopt the alternate approach of using the Poisson equation to relate $\tilde{n}$ to the perturbed fields and thence to A^- . Using also equation (G13), Poisson's equation takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} r \tilde{E}_r + k_z (i \tilde{E}_z) = \gamma_o^2 \left[4\pi \left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z \right) \right] - \frac{\omega^2 \beta_o}{c \Omega} (i \tilde{E}_z) \quad . \quad (G19)$$

The $\hat{\phi}$ -component of the $\nabla \times \underline{\tilde{E}}$ Maxwell equation also implies that

$$\tilde{E}_r = - \frac{\beta_o c}{\Omega} \frac{\partial}{\partial r} (i \tilde{E}_z) - \frac{\omega}{2\Omega} A^- \quad . \quad (G20)$$

Finally, equations (G10) and (G12) may be combined in the form

$$\left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - p_-^2\right) A^- = \frac{\partial}{\partial r} \left[4\pi \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right) \right]. \quad (G21)$$

where

$$p_-^2 \equiv k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega^2 p}{2c^2}. \quad (G22)$$

It may now be observed that equations (G16), (G19), (G20), and (G21) comprise a set of four equations for the four unknowns $\tilde{E}_z$, $\tilde{E}_r$, A^- , and $4\pi[\tilde{\rho} - (\beta_0/c) \tilde{J}_z]$. One may next eliminate $\tilde{E}_r$ from equations (G19) and (G20) to obtain

$$\begin{aligned} \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - p_-^2 - \frac{\omega^2 p}{2c^2} \right) (i\tilde{E}_z) = \\ - \gamma_0^2 \left(k_z - \beta_0 \frac{\omega}{c} \right) \left[4\pi \left(\tilde{\rho} - \frac{\beta_0}{c} \tilde{J}_z \right) \right] \end{aligned} \quad (G23)$$

thus trimming the set to three equations and three unknowns. These equations are in turn subject to the boundary conditions that

$$A^-(r=a) = 0 \quad (G24)$$

and

$$\tilde{E}_z(r=a) = 0 \quad , \quad (G25)$$

where it is again presumed that a conducting wall exists around the beam edge at $r = a$.

One may now proceed to combine the equations (G16), (G21), and (G23) until one finally arrives at the single fourth order equation

$$\left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r + \frac{2\gamma_o^2 \bar{\Omega}^2}{c^2} - \frac{3\omega^2 p}{2c^2} \right) \left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r - p_-^2 \right) A^- = - \frac{\omega^2 p}{2c^2} p_-^2 A^- \quad (G26)$$

This equation may now be seen to agree precisely with equation (F45), in the limit that condition (G4) is invoked. As before, it may be solved in terms of Bessel functions. The various perturbed quantities may be expanded in the fashion

$$A^- = \alpha_1^- J_1(k_{11}r) + \alpha_2^- J_1(k_{12}r) \quad (G27)$$

$$i\tilde{E}_z = E_{z1} J_0(k_{11}r) + E_{z2} J_0(k_{12}r) \quad (G28)$$

$$4\pi\left(\tilde{\rho} - \frac{\beta_o}{c} \tilde{J}_z\right) = Q_1 J_0(k_{11}r) + Q_2 J_0(k_{12}r) \quad (G29)$$

$$\tilde{\beta}_\phi = \beta_{\phi 1} J_1(k_{11}r) + \beta_{\phi 2} J_1(k_{12}r) \quad (G30)$$

$$-i\tilde{\beta}_r = \beta_{r1} J_1(k_{11}r) + \beta_{r2} J_1(k_{12}r) \quad (G31)$$

and as before, equation (G26) reduces to a quadratic equation for k_1^2 which must be obeyed by both k_{11}^2 and k_{12}^2 , which are therefore the two roots of the quadratic

$$\left(k_{11}^2 - \frac{2\gamma_o^2 \bar{\Omega}^2}{c^2} + \frac{3\omega^2}{2c^2} \right) (k_{11}^2 + p_-^2) + \frac{\omega^2}{2c^2} p_-^2 = 0 \quad . \quad (G32)$$

If the two roots of equation (G32) are well separated, as for the case $\gamma_o \gg 1$, we may develop expansions such as

$$k_{11}^2 \approx \frac{2\gamma_o^2 \bar{\Omega}^2}{c^2} - \frac{3\omega^2}{2c^2} - \frac{\omega^2}{4\gamma_o^2 \bar{\Omega}^2} p_-^2 + \dots \quad (G33)$$

$$k_{12}^2 \approx -p_-^2 + \frac{\omega^2}{4\gamma_o^2 \bar{\Omega}^2} p_-^2 + \dots \quad (G34)$$

where $|k_{11}^2| \gg |k_{12}^2|$. It may be observed that condition (G33) agrees with the anticipated resonance condition (G4), as is to be expected.

The boundary conditions (G24) and (G25) may be expressed as

$$\alpha_1^- J_1(k_{11}a) + \alpha_2^- J_1(k_{12}a) = 0 \quad (G35)$$

$$E_{z1} J_0(k_{11}a) + E_{z2} J_0(k_{12}a) = 0 \quad (G36)$$

and from equations (G16), (G21), and (G23), one may relate α_1^- and E_{z1} and also α_2^- and E_{z2} as follows:

$$\begin{aligned} \frac{E_{z1}}{\alpha_1^-} &= \frac{k_{11}^2 + 2p_-^2}{2k_{11} \left(k_z - \beta_o \frac{\omega}{c} \right)} \\ &= \frac{\frac{1}{2} k_{11}^2 \gamma_o^2 \left(k_z - \beta_o \frac{\omega}{c} \right)}{k_{11}^2 + p_-^2 + \frac{\omega^2 p_-^2}{2c^2} - \gamma_o^2 \left(k_z - \beta_o \frac{\omega}{c} \right)^2} \end{aligned} \quad (G37)$$

$$\begin{aligned} \frac{E_{z2}}{\alpha_2^-} &= \frac{k_{12}^2 + 2p_-^2}{2k_{12} \left(k_z - \beta_o \frac{\omega}{c} \right)} \\ &= \frac{\frac{1}{2} k_{12}^2 \gamma_o^2 \left(k_z - \beta_o \frac{\omega}{c} \right)}{k_{12}^2 + p_-^2 + \frac{\omega^2 p_-^2}{2c^2} - \gamma_o^2 \left(k_z - \beta_o \frac{\omega}{c} \right)^2} \end{aligned} \quad (G38)$$

From the determinant of the coefficients of E_{z1} , E_{z2} , α_1^- , α_2^- in equations (G35) - (G38), the dispersion relation is derived:

$$\frac{(k_{11}^2 + 2p_-^2)J_0(k_{11}a)}{k_{11}J_1(k_{11}a)} = \frac{(k_{12}^2 + 2p_-^2)J_0(k_{12}a)}{k_{12}J_1(k_{12}a)} \quad (G39)$$

If the usual assumption that $|k_{12}^2 a^2| \approx |p_-^2 a^2| < 1$ is made, the dispersion relation reduces to the familiar zero order result

$$\frac{k_{11}a J_0(k_{11}a)}{J_1(k_{11}a)} \approx -2 + \dots \quad (G40)$$

With solutions now in hand for A^- and $\tilde{E}_z$, one may return to equations (G16), (G21), or (G23) to obtain $\tilde{\rho} = \frac{\beta_0}{c} \tilde{J}_z$. Then equation (G17) may be used to solve for the previously indeterminable quantity $\tilde{\beta}_\phi$, and equation (G8) will then give the corresponding value for $\tilde{\beta}_r$. The results for the amplitude coefficients of equations (G30) and (G31) are

$$\beta_{\phi 1} = \left(\frac{k_{11}^2 + p_-^2}{T k_{11}^2} \right) \frac{\alpha_1^-}{2\bar{B}_z} \quad (\text{G41})$$

$$\beta_{\phi 2} = \left(\frac{k_{12}^2 + p_-^2}{T k_{12}^2} \right) \frac{\alpha_2^-}{2\bar{B}_z} \quad (\text{G42})$$

$$\beta_{r1} = \left(\frac{k_{11}^2 + p_-^2}{T k_{11}^2} + 1 \right) \frac{\alpha_1^-}{2\bar{B}_z} \quad (\text{G43})$$

$$\beta_{r2} = \left(\frac{k_{12}^2 + p_-^2}{T k_{12}^2} + 1 \right) \frac{\alpha_2^-}{2\bar{B}_z} \quad (\text{G44})$$

If $\tilde{\beta}_\phi$ and $\tilde{\beta}_r$ are examined near the edge of the beam where the J_1 eigenfunctions are large, one finds that

$$\left| \frac{\beta_{\phi 1} J_1(k_{11} a)}{\beta_{\phi 2} J_1(k_{12} a)} \right| \approx \left| \frac{\beta_{r 1} J_1(k_{11} a)}{\beta_{r 2} J_1(k_{12} a)} \right|$$

$$\approx \left| \frac{\beta_{\phi 1} \alpha_2^-}{\beta_{\phi 2} \alpha_1^-} \right| \approx \frac{2}{T} \approx \frac{4\gamma_o^2 \bar{\Omega}^2}{\omega_p^2} \gg 1 \quad (G45)$$

so that the k_{11} pieces of $\tilde{\beta}_{\phi}$ and $\tilde{\beta}_r$ tend to be dominant. If we now return to the equation (G7) we notice that it is now possible to solve for the eigenfunction $A^+/(1+\delta)$, which is finite even though A^+ and $(1+\delta)$ individually vanish at the resonance point. Therefore, the primary effect of approaching the condition of resonance (G4) is simply that the eigenfunction A^+ shrinks in amplitude by the scalar factor $(1+\delta)$, but it retains its functional dependence on r . In this light, it is apparent that all of our previous equations in Appendix F will remain valid as the resonance is approached, and it is only necessary to group the variables together in the combinations $A^+/(1+\delta)$ and $p_+^2(1+\delta)$ where possible--since these combinations remain finite. In some of the equations, there may

be simplifications due to certain terms vanishing, but there will never be any divergences. Consequently, all of the results obtained in this appendix may be obtained by taking the appropriate resonance limit of the equations of Appendix F.

One finds for example, that in general,

$$\left| \frac{\alpha_1^- / (1 - \delta)}{\alpha_1^+ / (1 + \delta)} \right| \approx \left| \frac{\frac{T p_+^2 (1 + \delta)}{1 + 2T - \delta^2}}{k_{11}^2 + \frac{p_-^2 (1 - \delta) (1 + T + \delta)}{1 + 2T - \delta^2}} \right|$$

$$\approx \left| \frac{T p_+^2 (1 + \delta)}{k_{11}^2 (1 + 2T - \delta^2)} \right| \approx \left| \frac{T}{1 - \delta + T} \right| \ll 1$$

(G46)

which agrees with the resonance result $\frac{1}{2} T$ obtained from equations (G7), (G41), and (G43). Therefore, the right hand side of equation (G7) tends to exceed that of equation (G8) in magnitude, and therefore the resonant eigenfunction A^+ continues to dominate the transverse electron velocity

perturbations--even at the point of resonance--since the combination $A^+/(1+\delta)$ appears as the driving term.

In conclusion, it now seems clear that there are no divergences in the linear Maxwell-fluid equations, even at the point of Landau resonance where $\omega - k_z \beta_o c + \bar{\Omega} = 0$. It should be possible to remedy the anomalous behavior of the SURVEY code near resonance by simply computing the quantity $\alpha_1^+/(1+\delta)$ as a single entity rather than by computing α_1^+ and then dividing it where necessary by $(1+\delta)$ as is presently the case. In fact, all physical quantities should vary smoothly and continuously as the parameters are varied past the resonance point.

A P P E N D I X H

EXAMINATION OF THE LANDAU RESONANCE FOR

A NEGATIVE ENERGY ELECTRON CYCLOTRON

WAVE WITHIN KINETIC THEORY

For a cold relativistic electron beam propagating with velocity u along a magnetic guide field B_{oz} in the z -direction, the frequency of the electron cyclotron beam mode with wave number k_z is

$$\omega \approx k_z u - \Omega + \frac{\omega_p^2 \Omega}{k^2 c^2} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma^2 \Omega^2} \right)$$

where

$$k^2 = k_z^2 + k_{\perp}^2$$

$$\gamma = \left(1 - \frac{u^2}{c^2} \right)^{-1/2}$$

$$\frac{\omega_p^2}{k^2 c^2} \ll 1$$

$$\omega < k_z c$$

Ω and ω_p are the relativistic electron cyclotron and plasma frequencies respectively. The perturbed electron perpendicular velocity due to the cyclotron beam mode is

$$v_{\perp} \propto \frac{1}{(\omega - k_z v_z + \Omega)}$$

Thus a thermal spread in v_z of magnitude Δv_z

$$v_z \rightarrow u + \Delta v_z$$

such that

$$|k_z \Delta v_z| \sim \left| \frac{\omega_p^2 \Omega}{k^2 c^2} \left(1 - \frac{k_{\perp}^2 c^2}{2\gamma^2 \Omega^2} \right) \right|$$

will lead to significant corrections to the cold beam dispersion relation.

A relativistic electron beam will acquire a thermal spread in v_z after passage through an anode foil. If the foil is thin enough, the scattering of the electron beam by

the foil results in a negligibly small spread in energy but rotates the electron velocity vector in velocity space. A finite spread in v_z occurs.

The effects of a small thermal spread in v_z on the dispersion relation for the electron cyclotron beam mode has previously been discussed in the limit of $\gamma \rightarrow \infty$ (see reference 1). In this Appendix, the investigation of these effects are extended to include the regime of finite γ .

I. Equilibrium

We consider a slab equilibrium of the electron beam, infinite in extent in the z - and y -directions and bounded by conducting walls at $x = \pm a$. The beam propagates along a magnetic guide field B_{oz} in the z -direction. The beam equilibrium is described by the distribution function

1. W. E. Drummond, J. R. Thompson, et al., "Interaction of Relativistic Charged Particle Beams", Final Scientific Report for AFOSR Contract F49620-76-C-0002, November 1978. I-ARA-78-U-78.

$$f_o(\mathcal{E}, p_z) = \begin{cases} D_o \delta(\mathcal{E} - \mathcal{E}_o) g_o[p_\perp^2(\mathcal{E}, p_z)], & p_z > 0 \\ 0 & , p_z < 0 \end{cases}$$

(H1)

where $\mathcal{E}$ and p_z are the energy and z-component of canonical momentum respectively:

$$\mathcal{E} = m\gamma c^2 - e\phi_o(x)$$

$$p_z = p_z - \frac{e}{c} A_{oz}(x)$$

$$\gamma = \left[1 + \frac{p^2}{m^2 c^2} \right]^{1/2}$$

$$\underline{p} = m\gamma \underline{v}$$

and

$$g_o(p_\perp^2) = \exp \left(- \frac{p_\perp^2}{\hat{p}^2} \right)$$

$$p_z^2(\mathcal{E}, p_z) = \frac{\mathcal{E}^2}{c^2} - m^2 c^2 - p_z^2$$

$\hat{p}^2 \ll m^2 \gamma^2 c^2$ is a measure of the thermal spread.

The equilibrium beam density is

$$\begin{aligned} N_0 &= \int d^3p f_0 \\ &= 2\pi \int_{-\infty}^{+\infty} dp_z \int_{\mathcal{E}_b(p_z)}^{\infty} d\mathcal{E} \frac{(\mathcal{E} + e\phi_0)}{c^2} f_0 \end{aligned}$$

where

$$\mathcal{E}_b(p_z) = \left[\left(p_z + \frac{e}{c} A_{oz} \right)^2 c^2 + m^2 c^4 \right]^{1/2} - e\phi_0$$

We restrict the discussion to the case of a low density electron beam where $\mathcal{E} \gg e\phi_0, eA_{oz}$.

In this limit

$$N_o = 2\pi D_o m\gamma_o \int_0^{p_o} dp_z \exp \left\{ - \frac{(p_o^2 - p_z^2)}{\hat{p}^2} \right\}$$

$$\approx \frac{\pi D_o m\gamma_o}{p_o} \hat{p}^2 \quad (H2)$$

where

$$p_o^2 = \frac{\mathcal{E}^2}{c^2} - m^2 c^2 \gg \hat{p}^2$$

Equation (H2) relates the normalization constant D_o to the beam density N_o .

Similarly, the equilibrium current is

$$J_{oz} = -e \int d^3p \frac{p_z}{m\gamma} f_o \approx - \frac{eN_o p_o}{m\gamma_o}$$

From Maxwell's equations, the equilibrium electric potential Φ_o and magnetic potential A_{oz} can be determined.

If the effects of the beam self-fields are neglected, the electrons move along the magnetic guide field and the equilibrium electron trajectories may be approximated by

$$p'_z = p_z$$

$$p'_x = p_{\perp} \cos \{ \Omega(t' - t) + \psi \}$$

$$p'_y = p_{\perp} \sin \{ \Omega(t' - t) + \psi \}$$

$$z' = z + \frac{p_z}{m\gamma} (t' - t)$$

$$x' - x = \frac{p_{\perp}}{m\gamma\Omega} [\sin \{ \Omega(t' - t) + \psi \} - \sin \psi]$$

where

$$\Omega = \frac{eB_{oz}}{m\gamma c}$$

Dispersion Relation

Let the perturbed electromagnetic fields have the form

$$\underline{E}(\underline{r}, t) \rightarrow \underline{E}(\underline{x}) \exp (-i\omega t + i k_z z), \text{ etc.}$$

The linearized perturbed distribution function f_1 is given by

$$f_1(\underline{p}, \underline{x}) = e \int_{-\infty}^t dt' \left[\underline{E}(\underline{x}') + \frac{\underline{p}' \wedge \underline{B}(\underline{x}')}{m\gamma c} \right] \cdot \frac{\partial f_0}{\partial \underline{p}'} \exp \left\{ -i \left(\omega - \frac{k_z p_z}{m\gamma} \right) (t' - t) \right\} \quad (H3)$$

where the time integral is taken over the equilibrium trajectories.

Assuming small electron Larmor radii $\left(\frac{k_{\perp}^2 p_{\perp}^2}{m^2 \gamma^2 \Omega^2} \ll 1 \right)$

$$\begin{aligned}
f_1(\underline{p}, \underline{x}) &= \frac{e}{m\gamma} \frac{\partial f_0}{\partial \mathcal{E}} \int_{-\infty}^t dt' \exp \left\{ -i \left(\omega - \frac{k_z p_z}{m\gamma} \right) (t' - t) \right\} \\
&\quad \left[\underline{p}' \cdot \left\{ \underline{E}(\underline{x}) + (\underline{x}' - \underline{x}) \frac{\partial}{\partial \underline{x}} \underline{E}(\underline{x}) + \dots \right\} \right] \\
&+ e \frac{\partial f_0}{\partial p_z} \int_{-\infty}^t dt' \exp \left\{ -i \left(\omega - \frac{k_z p_z}{m\gamma} \right) (t' - t) \right\} \\
&\quad \left[E_z(\underline{x}) + (\underline{x}' - \underline{x}) \frac{\partial E_z}{\partial \underline{x}}(\underline{x}) + \dots \right] \\
&+ \frac{1}{m\gamma c} \hat{z} \cdot \underline{p}' \wedge \left\{ \underline{B}(\underline{x}) + (\underline{x}' - \underline{x}) \frac{\partial}{\partial \underline{x}} \underline{B}(\underline{x}) + \dots \right\} \quad (H4)
\end{aligned}$$

The perturbed perpendicular currents are

$$\begin{aligned}
J^{\pm}(\mathbf{x}) &= -e \int d^3p f_1 \frac{(p_x \pm i p_y)}{m\gamma} \\
&= -e \int_0^{2\pi} d\psi \int_{-\infty}^{+\infty} dP_z \int_{\mathcal{C}_b(P_z)}^{\infty} d\mathcal{C} P_{\perp} e^{\pm i\psi} f_1
\end{aligned}
\tag{H5}$$

The electron cyclotron beam modes with low phase velocity, traveling in the direction of beam propagation, have frequency $\omega \sim (k_z P_z)/(m\gamma) - \Omega$ [see equations (H16) and (H17)] such that $|\omega - (k_z P_z)/(m\gamma) + \Omega| \ll |\omega - (k_z P_z)/(m\gamma)| \sim \Omega$. Substituting equation (H4) for f_1 in equation (H5),

$$J^+ \approx 2\pi i e^2 \int_{-\infty}^{+\infty} dP_z \int_{\mathcal{E}_b}^{\infty} d\mathcal{E} \frac{f_o(\mathcal{E}, P_z)}{\omega - \frac{k_z P_z}{m\gamma} + \Omega}$$

$$\cdot \left[E^+ + \frac{iP_z}{m\gamma c} B^+ \left\{ 1 - \frac{k_z^2 P_z (\mathcal{E}, P_z)}{2 m^2 \gamma^2 \Omega \omega - \frac{k_z P_z}{m\gamma} + \Omega} \right\} \right]$$

$$\approx \frac{i\omega^2 p_z}{4\pi k_z c} \times A^+ \quad (H6)$$

$$J^- \approx \frac{i\omega^2 p_z A^-}{4\pi(\omega - k_z u_o - \Omega)} \quad (H7)$$

where

$$E^{\pm} = E_x \pm i E_y$$

$$B^{\pm} = B_x \pm i B_y$$

$$A^{\pm} = E^{\pm} \pm i \frac{u_o}{c} B^{\pm}$$

$$u_o = \frac{p_o}{m\gamma_o} = c \left(1 - \frac{m^2 c^4}{\mathcal{E}_o^2} \right)^{1/2}$$

$$\chi = 2 \frac{cu_o}{\hat{v}^2} \{ 1 - Q E_1(Q) \exp(Q) \}$$

$$\hat{v}^2 = \frac{\hat{p}^2}{m^2 \gamma_o^2}$$

$$Q = \frac{2 u_o}{k_z \hat{v}^2} (\omega - k_z u_o + \Omega)$$

and we have assumed that $\frac{\omega^2}{k_z^2 c^2} \ll 1$, $\frac{|\omega - k_z u_o + \Omega|}{\Omega} \ll 1$.

$E_1(Q)$ is the exponential integral

$$E_1(Q) = \int_Q^{\infty} \frac{d\theta \exp(-\theta)}{\theta} \quad (H8)$$

The perturbed parallel current J_z and the perturbed charge density ρ are

$$J_z \approx \rho u_0$$

$$\rho \approx \frac{i}{2(k_z u_0 - \omega)} \frac{\partial}{\partial x} J^+$$

From Maxwell's equations

$$\frac{\partial^2 A^{\pm}}{\partial x^2} = \left(k_z^2 - \frac{\omega^2}{c^2} \right) A^{\pm}$$

$$= 4\pi \frac{\partial}{\partial x} \left(p - \frac{u_0}{c^2} J_z \right) + \frac{4\pi i}{c^2} (k_z u_0 - \omega) J^{\pm}$$

$$\approx \frac{2\pi i}{(k_z u_0 - \omega) \gamma_0^2} \frac{\partial^2}{\partial x^2} J^+ + \frac{4\pi i}{c^2} (k_z u_0 - \omega) J^{\pm} \quad (H9)$$

Let

$$\Psi^{\pm} = A^{\pm} - \gamma_0^2 \frac{2\pi i J^{\pm}}{(k_z u_0 - \omega)}$$

Equation (H9) for the electron cyclotron beam mode then takes the form

$$\begin{aligned} \frac{\partial^2 \Psi^+}{\partial x^2} - (k_z^2 - \frac{\omega^2}{c^2}) \Psi^+ \\ = - \hat{\chi} \left\{ 1 + \frac{(k_z^2 c^2 - \omega^2)}{2\gamma_0^2 (k_z u_0 - \omega)^2} \right\} \Psi^+ \end{aligned} \quad (H10)$$

$$\begin{aligned} \frac{\partial^2 \Psi^-}{\partial x^2} - \xi^2 \Psi^- \\ = - \hat{\chi} \frac{\xi^2 c^2}{2\gamma_0^2 (k_z u_0 - \omega)^2} \Psi^+ \end{aligned} \quad (H11)$$

where

$$\hat{\chi} = \frac{\frac{\omega_p^2}{k_z c^3} (k_z u_o - \omega)}{1 + \frac{\frac{\omega_p^2}{k_z c^3} \chi}{2\gamma_o^2 k_z c (k_z u_o - \omega)}}$$

$$\xi^2 = k_z^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} \frac{(k_z u_o - \omega)}{(k_z u_o - \omega + \Omega)}$$

For conducting walls at the beam surface $x = \pm a$,
the boundary conditions on $\psi^\pm$ are:

$$\left. \begin{aligned} \psi^+(a) - \psi^-(a) &= 0 \\ \frac{\partial}{\partial a} \psi^+(a) + \frac{\partial}{\partial a} \psi^-(a) &= 0 \end{aligned} \right\} \quad (H12)$$

The antisymmetric solutions of equations (H10) and
(H11) are:

$$\psi^+ = \alpha_0 \sin \kappa x$$

$$\psi^- = \beta_0 \sinh \xi x$$

$$+ \alpha_0 \delta \sin \kappa x$$

where

$$\kappa^2 = -k_z^2 + \frac{\omega^2}{c^2}$$

$$+ \left\{ 1 + \frac{(k_z^2 c^2 - \omega^2)}{2\gamma_0^2 (k_z u_0 - \omega)^2} \right\} \hat{\chi} \quad (H13)$$

$$\delta = \frac{\xi^2 c^2 \hat{\chi}}{2\gamma_0^2 (k_z u_0 - \omega)^2 (\kappa^2 + \xi^2)}$$

Substituting these solutions in equation (H12)

$$(1 - \delta) \xi a \coth \xi a + (1 + \delta) \kappa a \cot \kappa a = 0 \quad (H14)$$

Equation (H14) is the dispersion relation.

If

$$\xi^2 a^2 \approx k_z^2 a^2 + \frac{\omega^2 a^2}{2c^2} \ll 1$$

$$\delta \approx \frac{\xi^2 c^2}{2\gamma_o^2 \left\{ (k_z u_o - \omega)^2 + \frac{k_z^2 c^2}{2\gamma_o^2} \right\}} \frac{(\kappa^2 + k_z^2)}{(\kappa^2 + \xi^2)} \ll 1$$

equation (H14) may be approximated by

$$ka \cot ka \approx -1$$

and thus $ka \approx 2.03, 4.91, \dots$. The frequency ω of each eigenmode is determined by the solutions of equation (H13).

Let

$$k^2 \equiv \kappa^2 + k_z^2$$

Then from equation (H13)

$$k^2 = \frac{2\omega^2 u_o (k_z u_o - \omega)}{k_z c^2 \hat{v}^2} \left\{ 1 - \frac{\kappa^2 c^2}{2\gamma_o^2 (k_z u_o' - \omega)^2} \right\} \left\{ 1 - Q E_1(Q) \exp(Q) \right\} \quad (H15)$$

For Q real,

$$1 - Q E_1(Q) \exp(Q)$$

$$+ \begin{cases} \frac{1}{Q} - \frac{2}{Q^2} + \dots & , Q \gg 1 \\ \frac{1}{Q} - \frac{2}{Q^2} - 1 \pi |Q| \exp(-|Q|) & , Q \ll -1 \\ 1 + Q \ln Q & , |Q| \rightarrow 0 \end{cases}$$

a) $\frac{\kappa^2 c^2}{2\gamma_o^2 \Omega^2} < 1$

In the limit of a small thermal spread

$$1 \gg \epsilon_1 \equiv \frac{\omega_p^2}{k^2 c^2} \left(1 - \frac{\kappa^2 c^2}{2\gamma_o^2 \Omega^2} \right) \gg \frac{k_z \hat{v}^2}{2u_o \Omega}$$

where $Q \gg 1$, the frequency ω obtained from equation (H15) is

$$\omega = k_z u_o - \Omega + \epsilon_1 \Omega - \frac{k_z \hat{v}^2}{u_o} \quad (\text{H16})$$

A finite thermal spread introduces a correction $\Delta\omega \approx (k_z \hat{v}^2)/(u_o)$ to the frequency.

The frequency ω remains real as $\hat{v}^2$ increases until

$$\frac{k_z \hat{v}^2}{2u_o \Omega} \rightarrow \epsilon_1$$

when $Q \rightarrow 0$ and $\omega \rightarrow k_z u_o - \Omega$

b) $\frac{\kappa^2 c^2}{2\gamma_o^2 \Omega^2} > 1$

In the limit of a small thermal spread

$$1 \gg \epsilon_2 \equiv \frac{\omega_p^2}{k^2 c^2} \left(\frac{\kappa^2 c^2}{2\gamma_0^2 \Omega^2} - 1 \right) \gg \frac{k_z \hat{v}^2}{2u_0 \Omega}$$

where $Q \ll -1$, the frequency ω obtained from equation (H15) is

$$\omega \approx k_z u_0 - \Omega - \epsilon_2 \Omega \{1 + i\pi Q_0^2 \exp(-Q_0)\} - \frac{k_z \hat{v}^2}{u_0}$$

where

$$Q_0 = \frac{2 \epsilon_2 \Omega u_0}{k_z \hat{v}^2}$$

The eigenmode is weakly damped.

In conclusion, if $(\kappa^2 c^2)/(2\gamma_0^2 \Omega^2) < 1$ and the thermal spread is small enough so that

$$\frac{k_z \hat{v}^2}{2u_0 \Omega} < \frac{\omega_p^2}{k^2 c^2} \left(1 - \frac{\kappa^2 c^2}{2\gamma_0^2 \Omega^2} \right),$$

the electron cyclotron beam modes propagate undamped by thermal effects.

A P P E N D I X I

SUMMARY OF SURVEY CODE MODIFICATIONS

In this appendix we will summarize the previous SURVEY code, and discuss the modifications of this code that were required for the DC ARA parameter study.

A. Summary of the Previous SURVEY Code

Table II shows a sample of the output from the unmodified SURVEY code. The parameters chosen are those for the third stage of a fissile material breeder as discussed in the proposal for this design study (reference 1). The input parameters, are those listed at the top of the table. The user has the option of inputting FREQUENCY (MHZ) or WAVE PARAMETER ALPHA, rather than INITIAL MAGNETIC FIELD (KG), if desired.

1. W. E. Drummond, Proposal for "A Design Study for a High Current, Steady State Auto-Resonant Accelerator," December 1977. ARA Report #I-ARA-77-U-102.

TABLE II. Sample Output from Unmodified SURVEY Code

AUTO RESONANT PARAMETER SURVEY

INITIAL E-BEAM ENERGY (MV) =	0.7	INITIAL ION ENERGY (MV) =	100.00
INITIAL E-BEAM CURRENT (KAMPS) =	4.0	FINAL ION ENERGY (MV) =	1000.0
INITIAL BEAM RADIUS (CM) =	0.100	ION CURRENT (KAMPS) =	0.001
WALL TO BEAM RADIUS RATIO =	1.200	ION CHARGE NUMBER =	1.0
INITIAL MAGNETIC FIELD (KG) =	70.00	MASS NUMBER =	2.0
FREQUENCY (MHZ) =	40672.	WAVE PARAMETER ALPHA =	3.707
INITIAL KZ/KP VALUE =	0.911	INITIAL POTENTIAL (MV) =	0.01

LENGTH (METERS)	TIME (NS)	ELECTRON ENERGY (MV)	MAGNETIC FIELD (KG)	POTENTIAL (MV)	ION ENERGY (MV)	NUMBER OF E-FOLDS	WORST INEQUALITY
0.000	0.000	0.632	70.000	0.010	100.000	0.000	DELR = 0.28
2.713	26.033	0.618	52.160	0.013	150.001	0.000	DELR = 0.36
5.449	48.568	0.603	41.684	0.016	199.999	0.000	DELR = 0.42
8.256	69.310	0.588	34.661	0.019	249.998	0.000	DELR = 0.48
11.156	89.022	0.573	29.572	0.021	299.999	0.000	DELR = 0.53
14.160	108.120	0.558	25.690	0.023	349.998	0.000	DELR = 0.57
17.274	126.858	0.543	22.620	0.025	399.998	0.000	DELR = 0.61
20.505	145.408	0.528	20.126	0.026	450.000	0.000	DELR = 0.65
23.855	163.891	0.513	18.057	0.028	499.997	0.000	DELR = 0.69
27.327	182.389	0.499	16.312	0.029	549.995	0.000	DELR = 0.72
30.926	200.984	0.484	14.820	0.031	599.994	0.000	DELR = 0.76
34.652	219.726	0.469	13.529	0.032	649.995	0.000	DELR = 0.79
38.509	238.650	0.454	12.402	0.033	699.995	0.000	DELR = 0.82
42.498	257.806	0.439	11.409	0.034	749.992	0.000	DELR = 0.85
46.624	277.226	0.424	10.529	0.036	799.996	0.000	DELR = 0.88
50.887	296.935	0.410	9.744	0.037	849.992	0.000	DELR = 0.90
55.291	316.968	0.395	9.039	0.038	899.998	0.000	DELR = 0.93
59.838	337.339	0.380	8.404	0.038	949.995	0.000	DELR = 0.96
64.533	358.084	0.366	7.829	0.039	999.992	0.000	DELR = 0.98

SURVEY, as an interactive computer code used to obtain design parameters for particular accelerator configurations, has two main subdivisions. The code first checks that the input design parameters satisfy certain minimal requirements which can be evaluated without a detailed calculation of accelerator parameters along the length of the accelerator. If these minimal requirements are not met, a message indicating appropriate changes is issued, and control is returned to the user to input new parameters. When all of the preliminary requirements are met, the code then begins calculating the axial magnetic field, instantaneous ion energy, potential well depth and residual electron energy as a function of distance through the accelerator. It is in this section of the code that the quantities shown in the lower part of Table II are computed and printed. The calculation is completed and control returned to the user for new input when the desired final ion energy is reached or when one of several operating restrictions is violated.

B. Detailed Modifications to SURVEY

The analysis of equilibrium profiles for the electron beam and the conditions for the existence of such equilibria are given in Appendix E. One result of this analysis, equation (E33), is that the space charge of the electron beam reduces the electron energy from the value, given by γ_0 , attained during preacceleration. With these results for the beam profiles in the equilibrium, the necessary conditions for the existence of such equilibria may be evaluated as given in equations (E25) and (E26). The unmodified SURVEY code uses these conditions with $\beta_z = 1$, which is valid for the higher energy beams employed in the pulsed ARA. For the DC ARA this approximation is no longer valid, and the modified SURVEY code contains this correction.

A previous analysis of the stability of wide class of equilibria to arbitrary electromagnetic perturbations can be used to give as a sufficient condition for stability that

$$I_e < \frac{5 B_0 a^2 \gamma \beta_z}{b} , \quad (11)$$

where I_e is the electron current in kiloamperes, B_0 is the vacuum guide field in kilogauss, and a and b are the beam radius and wall radius, respectively, in centimeters. Again, the fact that $\beta_z < 1$ for the DC ARA is an important modification.

In addition to the assumption that the flare in the accelerator section is adiabatic, with $Ba^2 = \text{constant}$ and $b/a = \text{constant}$, evaluation of further requirements in SURVEY necessitates some knowledge of the eigenfunctions for the wave fields in the DC ARA. These eigenfunctions may be obtained from the analysis given in Appendix F.

For example, it is required that the cyclotron wave, which holds and accelerates the ions, not become so large as to selftrap the electron beam. In the wave frame, moving at the wave phase velocity relative to the laboratory frame, the quantity $\gamma - |e|\phi/mc^2$ is a constant of the motion. The total potential, φ , is the sum of the space charge potential, ϕ , and the wave potential, Φ . The requirement that electron selftrapping be avoided is that $\gamma > 1$ in the wave frame. This requirement is most severe at the

wave potential minima where $\phi = -|\hat{\phi}|$. With the definition of $\bar{\gamma}$ as the value of the constant of the motion evaluated when $\hat{\phi} = 0$, the selftrapping condition can be written

$$\bar{\gamma} - \left| \frac{e\hat{\phi}}{mc^2} \right| \geq 1 \quad (I2)$$

as evaluated in the wave frame. In the unmodified SURVEY code this requirement is approximated in terms of laboratory frame quantities as

$$\gamma - \gamma_i > \left| \frac{c\hat{\phi}}{mc^2} \right| \quad (I3)$$

The approximation made is, roughly, that the ion energy is not too large, $\gamma_i \sim 1$. For some of the DC ARA applications $\gamma_i \gtrsim 2$, and this approximation is not made. Instead the requirement is that, in the laboratory frame,

$$\gamma_i \gamma (1 - \beta\beta_i) - 1 > \left| \frac{eE_z}{mc\omega} \right| \beta_i \gamma_i \quad (I4)$$

This condition is evaluated at the center of the beam because there the wave field E_z is largest, and γ has a minimum due to its reduction by beam space charge. It must be fulfilled at every point in the accelerator.

As one moves down the accelerator, SURVEY must evaluate the ion energy and related field energy and residual electron beam energy.

The equation governing the spatial change in ion energy is

$$\frac{\partial}{\partial z} \gamma_i = \frac{Ze}{Mc^2} E_z \quad . \quad (I5)$$

In the unmodified SURVEY code the approximation is made the $E_z = F(k_z - \omega/c)\phi$, where $0 \leq f = \sin \psi \leq 1$ as determined by the phase angle, ψ , at which the ions are loaded into the wave potential. In the new SURVEY, E_z is more accurately calculated.

The maximum acceleration is obtained with $f = 1$, but a margin of safety from spilling the ions is allowed by taking $f = 1/2$.

Two further conservation laws must be satisfied by the electrons, ions, and waves, and are used to relate field energy and residual electron beam energy to the instantaneous ion energy:

(1) Conservation of wave energy flux, which may be written as

$$\vec{\nabla} \cdot [\omega \vec{Q} + Mn_i c^2 (\gamma_i - 1) \vec{v}_i \hat{z}] = 0 \quad (\text{I6})$$

where

$$\vec{Q} = \frac{E^2 - B^2}{8\pi} \frac{\frac{\partial \epsilon}{\partial \vec{k}}}{\omega_p^2 \frac{\partial \epsilon}{\partial \omega_p^2}}, \text{ and} \quad (\text{I7})$$

(2) The corresponding expression of total energy conservation:

$$\vec{\nabla} \cdot \left[\frac{c}{4\pi} \vec{E} \times \vec{B} + Mn_c (\gamma - 1) \vec{v}_e + Mn_i c^2 (\gamma_i - 1) \vec{v}_i \hat{z} \right] = 0 \quad (\text{I8})$$

In the wave energy conservation law, the expression for $\vec{Q}$ given in equation (I7) is particularly convenient because ϵ may be chosen to be any function of $\vec{k}$ and ω provided only that $\epsilon = 0$ gives the correct eigenvalue $\omega = \omega(k)$. In particular, $\epsilon = \omega - \omega(\vec{k})$ is allowed.

In the unmodified SURVEY code equation (I6) is evaluated in the high- γ limit evaluated to give

$$\Delta \left[\frac{M}{Zm} I_i (\gamma_i - 1) - 4.25 \alpha \frac{a^2 \left(k^2 - \frac{\omega^2}{c^2} \right)^2}{k_i^2} \left(\frac{\vec{E} \Phi}{mc^2} \right)^2 \right] = 0 \quad (I9)$$

where Δ indicates total change in the quantity in brackets as a function of distance down the accelerator. Here I_i is the ion current in kiloamperes, and $\alpha \equiv \omega \Omega / \omega_p^2$ is an approximately constant wave parameter.

The negative energy character of the cyclotron wave is evident in equation (I9) in that an increase in the ion energy requires an increase in the wave amplitude.

Equation (I8) is similarly evaluated in the high- γ limit to yield:

$$I_e \Delta \gamma_e = - \Delta \left[4.25 \frac{\left(k^2 - \frac{\omega^2}{c^2} \right)^2}{k_{\perp}^2} a^2 \left(\frac{e\phi}{mc^2} \right)^2 \right. \\ \left. \times \left(\alpha + \frac{k_{\perp}^2}{k^2 - \frac{\omega^2}{c^2}} \right) \right] \quad (I10)$$

which states that, for every unit increase of field energy, $1 + \alpha [k^2 - (\omega^2/c^2)]/k_{\perp}^2$ units of energy are extracted from the electrons with the remaining $\alpha[k^2 - (\omega^2/c^2)]/k_{\perp}^2$ units of energy going to the ions as given by equation (I9).

In the new SURVEY code, the following procedure is used to evaluate equations (I6) and (I8): We first integrate over the volume enclosed by the waveguide between two axial locations. Then in the corresponding surface integrations only the axial fluxes contribute. For the ions we assume that n_i , γ_i , and v_i are independent of r , so that the wave energy flux equation becomes

$$\frac{M}{Zm} I_i \Delta \gamma_i = - \Delta \left[\frac{17}{8} \left| \frac{e}{mc^2} \right|^2 \frac{\frac{\omega}{c} \frac{\partial \epsilon}{\partial k_z}}{\omega_p^2 \frac{\partial \epsilon}{\partial \omega_p^2}} \int_0^a r dr (|\tilde{E}|^2 - |\tilde{B}|^2) \right] \quad (111)$$

Similarly, we obtain the following expression for total energy conservation:

$$I_e \Delta \gamma_e = \Delta \left[\frac{17}{8} \left| \frac{e}{mc^2} \right|^2 \frac{\frac{\omega}{c} \frac{\partial \epsilon}{\partial k_z}}{\omega_p^2 \frac{\partial \epsilon}{\partial \omega_p^2}} \int_0^a r dr (|\tilde{E}|^2 - |\tilde{B}|^2) \right] \\ - \Delta \left[\frac{17}{4} \left| \frac{e}{mc^2} \right|^2 \int_0^a r dr (\tilde{E}_r \tilde{B}_\phi^* - \tilde{E}_\phi \tilde{B}_r^*) \right] \quad (112)$$

The dispersion function, $\epsilon = 0$, is basically that given in equation (F61):

$$\epsilon = \omega - k_z \beta_z c + \frac{\Omega (k^2 c^2 - \omega^2)}{(k^2 c^2 - \omega^2 + \omega_p^2)} \left(1 - \frac{\omega_p^2}{\gamma^2 \Omega^2} \right) + \frac{\omega_p^2 k_\perp^2 c^2 (k^2 c^2 - \omega^2 + \omega_p^2)}{2 \gamma^4 \Omega (k^2 c^2 - \omega^2)^2} . \quad (I13)$$

Note that in equations (I11) and (I12), the $\partial/\partial\omega_p^2$ does not act upon the underlined term in equation (I13) because this term is from the equilibrium orbits of the electrons rather than their coupling to the wave.

The remaining class of conditions used in the SURVEY code consists of limitations on perturbations of the electron beam. The first of these is the requirement that the beam radial excursion, δr , due to the wave, not exceed either a or $b - a$.

The unmodified SURVEY code uses

$$\frac{|\delta r|}{a} = \frac{2.24}{I_e \text{ (kA)}} \frac{[k^2 - (\omega^2/c^2)] a}{k_\perp} \left(\frac{e |\Phi|}{mc^2} \right) . \quad (I14)$$

In the present SURVEY code, the more exact expression given in Appendix F for $\tilde{\beta}_r$ is used in the relation

$$\left| \frac{\delta r}{c} \right| = \frac{|\tilde{\beta}_r|}{|k_z v_e - \omega|} \quad (\text{I15})$$

Note that equation (I9) allows the $|\delta r|/a$ requirement at the end of the accelerator to be checked, given the desired final ion energy. This check is therefore made in the first part of the SURVEY code.

Both the unmodified SURVEY code and the new SURVEY code also include the obvious restriction $\tilde{\beta}_r < 1$.

C. The Landau Resonance

The possible occurrence of Landau spatial resonances has been examined in several previous analyses of Auto-Resonant Accelerators where the ultrarelativistic approximation was made. In all of these treatments it was found that such resonances could never occur precisely, but could only be approached up to a certain positive limit. However, during the development of the new SURVEY code, it became apparent that such a resonance may occur precisely due to certain finite- γ effects.

The source of the possible Landau resonance is the longitudinal compression term which appears as the correction $+ \omega_p^2 k_{\perp}^2 / 2k^2 \gamma_o^2 \Omega^2$ in equation (F62). The condition of resonance in the transverse electron motion is given by $(\omega - k_z \beta_o c) / \bar{\Omega} = -1$. The dispersion relation shows that this condition is met when

$$\frac{k_{\perp}^2 c^2}{\gamma_o^2 \bar{\Omega}^2} \approx 2 \quad . \quad (I16)$$

If this condition is satisfied, then one must examine the possibility that finite field perturbations can imply, e.g., very large $\tilde{\beta}_r$. The earliest versions of the new SURVEY code did encounter large $\tilde{\beta}_r$ and δ_r perturbations.

However, an analysis of the resonance point showed that there should be no divergent behavior in the perturbed quantities at the resonance (see Appendix G) and indicated that the difficulties encountered by SURVEY near the resonance might be removed by a different computational procedure.

Subsequent modification to alleviate this problem resulted in code results that showed little change from results obtained from the previous version of the code for cases where the resonance is not encountered. Moreover, the presence of the resonance does not appear to introduce dramatic changes in the axial profiles of the physical quantities which are computed. In fact, we suspect that the particle resonance, as treated in the fluid perturbation theory forming the basis of the calculations, is not especially important in determining the relevant axial profiles and is problematical only as an artifact of the numerical treatment.