PFC/RR-87-8

DOE/ET-51013-225

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in Rotating Tokamak Plasma, Part I**

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September 1987

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Evolution of Poloidal Variation of Impurity Density and Ambipolar Potential in Rotating Tokamak Plasma, Part I

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Abstract

When the (beam driven) toroidal plasma rotation ($V_{\varphi j}$) reaches thermal velocity (v_{thj}) levels such that

$$v_{thz} \ll V_{\varphi z} \sim V_{\varphi i} \lesssim v_{thi}$$

where i denotes the hydrogenic and z the impurity ions, the particle density n_j and the ambipolar potential Φ can develop strong poloidal variations in the flux surface. The rapid time evolution of $n_j(r, \theta)$, $\nabla \Phi(r, \theta)$, and the plasma flow velocity components in the flux surface $\vec{V}_{\parallel j}$, $\vec{V}_{\perp j}$ is calculated in Part I. In a forthcoming Part II these results will be evaluated numerically using a spectral code to describe the strong poloidal couplings. Results will be compared to data from the ASDEX tokamak.

I. Introduction

The connection between neutral beam induced plasma rotation and impurity transport has been the subject of numerous investigations [1,2,3], mainly focusing on the radial (across the flux surface) transport of impurities in experiments with neutral beam heating. In most of these works, the steady state beam-plasma equilibrium was analyzed describing the potential variations of all quantities A_j in the simplest form

$$A(r, \theta) = \bar{A} + A^c \cos \theta + A^s \sin \theta$$

where $A_j^{c,s}/\bar{A}_j$ was ordered $\sim \varepsilon = r/R_0$. However, using soft X-ray imaging techniques Smeulders [4] obtained results from the ASDEX tokamak of the form

$$n_z(r, \theta) = \bar{n}_z + \sum_{\ell=1}^2 (n_z^\ell \cos \ell\theta + n_z^\ell \sin \ell\theta)$$

with $0(1)$, not $0(\varepsilon)$ - Fourier amplitudes. This necessitates a theoretical approach different from that used by previous authors. This is the subject of the present paper which will be restricted to the Pfirsch-Schlüter regime of collisionality for all species, and will assume the impurity strength parameters $\alpha \equiv n_z Z^2/n_i$ to be of $0(1)$. The transient time evolution of the particle densities, flow-velocities and ambipolar potential is treated. The nonlinear couplings introduced by the inertia and other terms in the momentum balance equations are fully retained and contribute essentially to the final steady state poloidal impurity distribution. A complete description results of the fast time evolutions of the poloidal rotation and the ambipolar electric field (first described in a simplified model in Ref. [5]). On the slow time scale of the toroidal evolution, the neutral beam momentum source $\vec{M}_j$ is explicitly kept, balanced by the Braginskii viscous force modelled by a phenomenological momentum drag term - $m_j n_j \nu_{d,j} \vec{V}_j$ where $\nu_{d,j}$ is taken from experimental observations. (A first principle explanation of this drag as a gyroviscous force was given previously in Ref. [6]. Its fully selfconsistent theory is still under investigation [7].) The slow (diffusion) time scale evolution of toroidal rotation (driven by the radial flow) is not treated in this report.

In Section II, the underlying fluid equations are formulated. Section III contains a summary. Appendices A and B contain details on the parallel viscous force and inertia, Appendix C gives a simplified physical model for the time evolution of the radial ambipolar field, E_r , and Appendix D justifies the neglect of the radial flow velocity on the fast time scale of interest here.

II. Fluid Equations

We adopt the usual (ψ, θ, φ) flux coordinate system of an axisymmetric tokamak. The particle conservation equation for the ion species $j = i, z$ is

$$\frac{\partial n_j}{\partial t} + \nabla \cdot n_j \vec{V}_j = 0 \quad (1a)$$

For the purpose of this paper, namely to calculate the evolution in the flux surfaces where the slower radial diffusion flow is frozen in, the component $\vec{V} \cdot \nabla \psi$ is ordered out (cf. Appendix D) and (1a) becomes

$$\frac{\partial n_j}{\partial t} + \frac{1}{J} \frac{\partial}{\partial \theta} J n_j \vec{V}_j \cdot \nabla \theta = 0 \quad (1b)$$

where

$$J^{-1} = \vec{B} \cdot \nabla \theta$$

is the inverse Jacobian of the equilibrium. The momentum conservation equation is

$$\begin{aligned} m_j n_j \left(\frac{\partial}{\partial t} + \vec{V}_j \cdot \nabla \right) \vec{V}_j + \nabla p_j + \nabla \cdot \vec{\Pi}_j = \\ e_j n_j \left(-\nabla \phi + \frac{\vec{V}_j}{c} \times \vec{B} \right) + \vec{R}_j + \vec{M}_j - m_j n_j \nu_d \vec{V}_j \end{aligned} \quad (2)$$

where $\vec{R}_j$ is the collisional friction, $\vec{M}_j$ the beam momentum source and $\vec{\Pi}_j$ the Braginskii viscous stress tensor. The last term models an effective drag force needed to describe those experimental observations where the observed rotation damping is faster than the classical perpendicular viscosity. (Cf. Ref. [6, 8] for a discussion.)

The system is closed invoking charge neutrality

$$\sum_{j=e,i,z} e_j n_j(\psi, \theta) = 0 \quad (3a)$$

and the $\nabla \psi$ component of Ampere's law

$$\frac{4\pi}{c} \vec{j} \cdot \nabla \psi + \frac{1}{c} \frac{\partial}{\partial t} \vec{E} \cdot \nabla \psi = \nabla \psi \cdot \nabla \times \vec{B} = 0$$

where the last step requires axisymmetry. Using

$$\vec{E} = -\nabla\phi = \left(-\frac{d}{d\psi}\phi^{(-1)}\right) \nabla\psi \equiv -\frac{\omega}{c} \nabla\psi$$

Ampere's law becomes

$$\frac{\partial}{\partial t}\omega|\nabla\psi|^2 = 4\pi c\vec{j} \cdot \nabla\psi \quad (3b)$$

$\Phi^{(-1)}$ is discussed below. From parallel electron momentum balance and assuming all temperatures are constant in the flux surface

$$T_j = T_j(\psi) \quad (4)$$

one obtains

$$0 = T_e \vec{B} \cdot \nabla n_e + e n_e \vec{B} \cdot \nabla\phi \quad (5a)$$

Combining Eqs (3) and (5) one can express $\vec{B} \cdot \nabla\Phi$ in terms of the ionic density variations in the flux surface:

$$e_j n_j \vec{B} \cdot \nabla\phi / T_e = Z_j n_j \vec{B} \cdot \nabla \ln(n_i + Z n_z) \quad (5b)$$

For the parallel friction we simply use

$$R_{\parallel i} = -m_i n_i \nu_{iz} (V_{\parallel i} - V_{\parallel z}), \quad R_{\parallel z} = -R_{\parallel i} \quad (6)$$

and the only stress tensor element needed explicitly is given in detail in App. A, Eq. A.4.

The required ordering $V_\varphi \sim v_{th}$ is incorporated by noting in Eq. (2) that to order δ^{-1} [$\delta \equiv \rho/a$, the Larmor radius expansion parameter,]

$$-\nabla\phi^{(-1)} + \vec{V}_\perp^{(0)} \times \vec{B}/c = 0 \quad (7a)$$

$$\vec{B} \cdot \nabla\phi^{(-1)} = 0 \quad \text{or} \quad \phi^{(-1)} = \phi^{(-1)}(\psi) \quad (7b)$$

Then,

$$\begin{aligned} \vec{V}_\perp^{(0)} &= c \frac{\partial}{\partial\psi} \phi^{(-1)} \frac{\vec{b} \times \nabla\psi}{B} \\ &= \omega \left(\frac{I}{B} \vec{b} - R^2 \nabla\varphi \right) \end{aligned} \quad (8)$$

with the rotation frequency

$$\omega \equiv c \frac{\partial}{\partial \psi} \phi^{(-1)} = \omega(\psi) \quad (9)$$

which can be assumed given at $t = 0$.

Here, $I = RB_\varphi = I(\psi)$, $\vec{b} \equiv \vec{B}/B$, $\nabla\varphi = \vec{e}_\varphi/R$, and the magnetic field is $\vec{B} = \nabla\varphi \times \nabla\psi + I\nabla\varphi$.

The total lowest order flow velocity $\vec{V}^{(0)}$ can then be written as

$$\begin{aligned} \vec{V}^{(0)} &= V_{\parallel} \vec{b} + \omega \left(\frac{I}{B} \vec{b} - R^2 \nabla\varphi \right) \\ &= \left(V_{\parallel} + \omega \frac{I}{B} \right) \vec{b} - \omega R^2 \nabla\varphi \end{aligned} \quad (10a)$$

whence

$$\begin{aligned} R^2 \nabla\varphi \cdot \vec{V}^{(0)} &= V_{\parallel} \frac{I}{B} + \omega \left(\frac{I^2}{B^2} - R^2 \right) \\ &= V_{\parallel} \frac{I}{B} - \omega R^2 (\nabla\psi)^2 / R^2 B^2 \end{aligned} \quad (10b)$$

$$\frac{\vec{B}_p \cdot \vec{V}^{(0)}}{B_p} = \left(V_{\parallel} + \omega \frac{I}{B} \right) \left(\frac{B_p}{B} \right) \quad (10c)$$

Using (10), the continuity Eq. (1b) is rewritten as

$$\frac{\partial n_j}{\partial t} + \vec{B} \cdot \nabla \left(\frac{n_j}{B} V_{\parallel j} + \frac{n_j}{B^2} \omega I \right) = 0 \quad (11)$$

Using (9), also the parallel inertial term $(\vec{V} \cdot \nabla \vec{V}) \cdot \vec{b}$ can now be worked out and one finds the result given in App. B, Eq. (B. 1). Combining these equations (including Eq. A. 4) there results for the parallel momentum balance Eq. (2) of ion species j

$$\begin{aligned} m_j n_j \frac{\partial}{\partial t} V_{\parallel j} + \frac{m_j n_j}{2} \vec{b} \cdot \nabla \left[\left(V_{\parallel} + \omega \frac{I}{B} \right)^2 - \omega^2 R^2 \right] + T_j \vec{b} \cdot \nabla n_j + m_j n_j \nu_{dj} V_{\parallel j} \\ - \frac{2}{3} B^{3/2} (\vec{b} \cdot \nabla) \left\{ \frac{\eta_0}{B^{3/2}} \left[2 \vec{b} \cdot \nabla V_{\parallel j} + \frac{\vec{b} \cdot \nabla B}{B} V_{\parallel j} - \frac{I}{B^2} \omega \vec{b} \cdot \nabla B \right] \right\} \\ = -n_j Z_j T_e \vec{b} \cdot \nabla \ln(n_i + Z n_z) - m_j n_j \nu_{jj'} (V_{\parallel j} - V_{\parallel j'}) + M_{j\parallel} \end{aligned} \quad (12)$$

To obtain the evolution of ω , we take the $\frac{\nabla\psi}{B} \cdot \vec{b} \times$ projection of Eq. (2), i.e. the radial component of the perpendicular momentum balance, and sum over species which yields the radial current

$$\vec{j} \cdot \vec{\nabla}\psi = \nabla\psi \cdot \frac{c}{B} \vec{b} \times \sum \left\{ m_j n_j \left(\frac{\partial}{\partial t} + \vec{V}_j \cdot \vec{\nabla} \right) \vec{V}_j + \vec{\nabla} p_j + \vec{\nabla} \cdot \Pi_j + m_j n_j \nu_{dj} \vec{V}_j \right\} - \vec{M}_j \quad (13a)$$

Here, the lowest order $\vec{V}_j^{(0)}$ of Eq. (10a) can be used for V_j . Before summing over species, in tokamak flux coordinates, this equation is

$$\begin{aligned} e_j \frac{n_j}{c} \vec{V} \cdot \nabla\psi &= -m_j n_j \frac{|\nabla\psi|^2}{B^2} \frac{\partial\omega}{\partial t} \\ &- m_j n_j \left[\frac{I}{B} \left(V_{\parallel j} + \frac{I}{B} \omega \right)^2 \frac{\vec{b} \cdot \nabla B}{B} + \left(V_{\parallel j} \omega + \frac{I}{2B} \omega^2 \right) \vec{b} \cdot \nabla R^2 \right] \\ &- \frac{I}{B} \left(T_j \vec{b} \cdot \nabla n_j + e_j n_j \vec{b} \cdot \nabla \phi \right) + \frac{IB^2}{2} \vec{b} \cdot \nabla (\Pi_{\parallel j} / B^3) \\ &- m_j n_j \nu_{dj} \omega \frac{|\nabla\psi|^2}{B^2} - R \left[\left(M_{\perp}^j \right)_{\varphi} + \left(R_{jj'}^{(1)} \right)_{\perp \varphi} \right] \end{aligned} \quad (13b)$$

On summing over species the $\vec{b} \cdot \nabla \phi$ term and the friction term $R_{jj'}^{(1)}$ will annihilate. Equation (13b) can also be obtained by subtracting the toroidal ($R^2 \nabla \varphi$) component of the momentum balance from the parallel component given in Eq. (12) as can be seen from the identity (used in Eq. (8) before)

$$\vec{A} \times \vec{b} \cdot \frac{\nabla\psi}{B} = - \left(R^2 \nabla \varphi - \frac{I}{B} \vec{b} \right) \cdot \vec{A}$$

where $\vec{A}$ represents the momentum balance equation. This reveals the meaning of $(M_{\perp}^j)_{\varphi}$ as the perpendicular component (in the $-\vec{b} \times \nabla\psi$ direction) of the momentum source term, projected onto the toroidal (φ) direction. We then utilize Ampere's law Eq. (3b) and take the flux average of Eq. (13b) to obtain

$$\begin{aligned} &\left(\sum \langle mn \rangle + \frac{\langle B^2 \rangle}{4\pi c^2} \right) \frac{\partial\omega}{\partial t} + \frac{I}{|\nabla\psi|^2} \sum \langle mn V_{\parallel}^2 \vec{b} \cdot \nabla B \rangle \\ &+ 2 \sum \langle mn V_{\parallel} \frac{\vec{b} \cdot \nabla R}{R} \rangle \omega + \sum \langle mn \nu_d \rangle \omega \end{aligned}$$

$$\begin{aligned}
& + \frac{I}{BR} \langle mn \vec{b} \cdot \nabla R \rangle \omega^2 + \frac{3}{2} \frac{I}{|\nabla \psi|^2} \langle (\vec{b} \cdot \nabla B) \sum \Pi_{\parallel} \rangle \\
& = - \frac{B^2 R^2}{|\nabla \psi|^2} \left\langle \frac{1}{R} \sum M_{\perp \varphi} \right\rangle
\end{aligned} \tag{13c}$$

where we dropped the species index for brevity and $\sum = \sum_j$. Here $R^2 B^2 = I^2 + |\nabla \psi|^2$ was used (and for a low β -equilibrium $|\nabla \psi|$ is a flux surface quantity.) We note that using $B^2/4\pi m n c^2 = c_A^2/c^2 \ll 1$ the term $\langle B^2 \rangle/4\pi c^2$ is negligible. (13c) is an equation for the time evolution of $\omega(\psi) \propto E_r$ of the form

$$\frac{\partial \omega}{\partial t} + C_1 \omega^2 + C_2 \omega + C_3 = 0 \tag{14a}$$

$$C_1 = \frac{1}{D} \frac{B_T}{B} \left\langle \sum mn \vec{b} \cdot \nabla R \right\rangle \tag{14b}$$

$$C_2 = \frac{1}{D} \left\{ \left\langle \sum mn \nu_d \right\rangle - 2 \left\langle \sum mn V_{\parallel} \frac{\vec{b} \cdot \nabla B}{B} \right\rangle \right. \tag{14c}$$

$$\left. + \frac{B_T^2}{B_p^2} \sum \left\langle \eta_0 (\vec{b} \cdot \nabla B) (\vec{b} \cdot \nabla B) / B^2 \right\rangle \right\}$$

$$C_3 = \frac{1}{D} \left\{ \frac{B_T}{B_p^2 R} \sum \left\langle mn V_{\parallel}^2 \vec{b} \cdot \nabla B \right\rangle + \frac{B^2}{B_p^2} \left\langle \sum M_{\perp \varphi} / R \right\rangle \right. \tag{14d}$$

$$\left. - \sum \left\langle \eta_0 \left((\vec{b} \cdot \nabla B) (2\vec{b} \cdot \nabla V_{\parallel} + V_{\parallel} \frac{\vec{b} \cdot \nabla B}{B}) \right) \right\rangle \frac{B_T}{B_p^2 R} \right\}$$

$$D = \sum \langle mn \rangle + \langle B^2 \rangle / 4\pi c^2 \tag{14e}$$

III. Summary

We have obtained a closed set of Eqs. (11) for $\frac{\partial}{\partial t} n_j(\psi, \theta)$, (12) for $\frac{\partial}{\partial t} V_{\parallel j}(\psi, \theta)$ and (14) for $\frac{\partial}{\partial t} \omega(\psi) (j = i, Z)$ which describes the evolution of n_j and $V_{\parallel j}$ in the domain $0 \leq \theta \leq 2\pi$, on a fixed surface ψ , and the evolution of $\omega \propto E_r$, as an initial value problem.

The evolution of $n_j(\psi, \theta)$ is driven by the poloidal flow $V_p = (V_{\parallel} + \frac{I}{B} \omega) \frac{B_p}{B}$ and will relax to steady state such that $\frac{n V_p}{B_p}$ becomes a flux function. The evolution of $V_{\parallel}$ and thus V_p is naturally driven by the parallel component of the momentum source $M_{\parallel}$, which

simultaneously couples to the poloidal variation of the density via terms involving $\vec{b} \cdot \vec{\nabla} n_j$ in Eq. (12). For $\frac{\partial}{\partial t} \omega$, it is interesting to note that in addition to the explicit drive by the usually small toroidal projection of $M_\perp$, ω is driven indirectly by $M_\parallel$ through $V_\parallel$ as shown in the first term of Eq. (14d). That is, even for purely parallel injection, a large radial electric field can be built up when $V_\parallel$ reaches a level $V_\parallel \sim v_{thi}$.

Concerning the damping mechanisms of the system, a simplified physical discussion of the poloidal damping of V_p is given in Appendix C. Here we only remark that the parallel viscosity involving η_0 provides a damping mechanism for both ω and $V_\parallel$ on the time scale τ_p , while the momentum drag term provides the toroidal damping mechanism on the time scale τ_T . However, it is worth mentioning that in most tokamak experiments the time scales for τ_p and τ_T are not as widely separated as is usually assumed theoretically, involving the small classical perpendicular viscosity [9] $\eta_1 \sim \eta_0(\Omega_i \tau_i)^{-2}$. As mentioned after Eq. (2), τ_T^{-1} (experimental) $\sim (50 - 100)(\eta_1/a^2)$ where a is the plasma radius, cf. Ref. [8]. After the transients $\frac{\partial}{\partial t} \{n_j, V_{\parallel j}, \omega\}$ have vanished, Eqs. (11), (12) and (14) yield a steady state system with nontrivial solutions for $V_{\parallel j}$ and $V_{pj} = B_p K_j(\psi)/n_j$ maintained by the finite momentum sources. Note that the quadratic form of Eq. (14a) implies two possible roots for $\omega \sim E_r$ thus permitting a bifurcation of the steady state plasma.

Appendix A on $\vec{B} \cdot \nabla \cdot \vec{\Pi}$

From Braginskii [9] the $0(\delta^\circ)$ - part of the viscous stress tensor is

$$\vec{\Pi}^{(0)} = \frac{3}{2} \Pi_{\parallel} \left(\vec{b}\vec{b} - \frac{1}{3} \vec{I} \right)$$

with

$$\Pi_{\parallel} = -\eta_0 W_{\parallel} ,$$

and

$$\begin{aligned} W_{\parallel} = \vec{b}\vec{b} : \left[\nabla \vec{V} + (\nabla \vec{V})^T - \frac{2}{3} \vec{I} (\nabla \cdot \vec{V}) \right] \\ = 2\vec{b} \cdot \nabla V_{\parallel} - 2\vec{b} \cdot \nabla \vec{b} \cdot \vec{V}_{\perp} - \frac{2}{3} \nabla \cdot \vec{V} \end{aligned} \quad (A1)$$

where η_0 corresponds to μ_1 in Eq. (C5) such that $\eta_0 = \frac{\mu_1}{2}$. In the Pfirsch-Schlüter regime, $\eta_0 = .96p/\nu_{jj}$. Using the lowest order Eq. (10) for $\vec{V}$ gives either

$$W_{\parallel} = 2 \left[\frac{2}{3} \vec{b} \cdot \nabla V_{\parallel} + \frac{1}{3} B^{-1} (\vec{b} \cdot \nabla B) V_{\parallel} - \frac{I}{3B^2} \omega \vec{b} \cdot \nabla B \right] \quad (A2a)$$

or

$$W_{\parallel} \simeq 2 \left[\frac{I}{B} \vec{b} \cdot \nabla \frac{V_{\varphi}}{R} - \frac{1}{3B^2} (\vec{b} \cdot \nabla B^3) \frac{V_p}{B_p} \right] \quad (A2b)$$

depending on the decomposition

$$\vec{V} = V_{\parallel} \vec{b} + \vec{V}_{\perp} = V_p \frac{\vec{B}_p}{B} + V_{\varphi} R \nabla \varphi , \quad V_{\parallel} = V_{\varphi} \frac{I}{RB} + V_p \frac{B_p}{B}$$

Using

$$\nabla \cdot \vec{\Pi}^{(0)} = \frac{3}{2} \left(\vec{B} \cdot \nabla \frac{\Pi_{\parallel}}{B} \right) \vec{b} + \frac{3}{2} (\vec{b} \cdot \nabla \vec{b}) \Pi_{\parallel} - \frac{1}{2} \nabla \Pi_{\parallel}$$

one obtains

$$\vec{B} \cdot \nabla \cdot \vec{\Pi}^{(0)} = B^{3/2} \vec{B} \cdot \nabla (\Pi_{\parallel} / B^{3/2}) \quad (A)$$

and using $W_{\parallel}$ from (A.1) in the (V_{φ}, V_p) representation one obtains Eq. (17) of Ref. [6] for $\vec{B} \cdot \nabla \cdot \vec{\Pi}^{(0)}$. Using (A.2a) gives

$$\vec{B} \cdot \nabla \cdot \vec{\Pi}^{(0)} = -\frac{2}{3} B^{3/2} (\vec{B} \cdot \nabla) \left[\frac{\eta_0}{B^{3/2}} \left(2\vec{b} \cdot \nabla V_{\parallel} + \frac{\vec{b} \cdot \nabla B}{B} V_{\parallel} - \frac{I\omega}{B^2} \vec{b} \cdot \nabla B \right) \right] \quad (A4)$$

Appendix B on the inertia term

Using Eq. (10) for the lowest order $\vec{V}$ one has

$$\begin{aligned}
 (\vec{V} \cdot \nabla \vec{V}) \vec{b} &= \left(V_{\parallel} + \omega \frac{I}{B} \right) \left(\vec{b} \cdot \nabla V_{\parallel} - \vec{b} \cdot \nabla \vec{b} \cdot \vec{V}_{\perp} - \omega^2 R^2 \nabla \varphi \cdot \nabla \vec{b} \cdot R^2 \nabla \varphi \right) \\
 &= \left(V_{\parallel} + \omega \frac{I}{B} \right) \vec{b} \cdot \nabla \left(V_{\parallel} + \omega \frac{I}{B} \right) - \frac{1}{2} \omega^2 \vec{b} \cdot \nabla R^2 \\
 &= \frac{1}{2} \vec{b} \cdot \nabla \left[\left(V_{\parallel} + \omega \frac{I}{B} \right)^2 - \omega^2 R^2 \right]
 \end{aligned} \tag{B1}$$

For the flux averaged equation determining $E_r \sim \omega$ (see Eq. (9)) the expression

$$G_{\varphi} \equiv \langle nm R^2 \nabla \varphi \cdot (\vec{V} \cdot \nabla) \vec{V} \rangle$$

is needed. It is worth mentioning that if $\frac{\partial \omega}{\partial t} \neq 0$ in Eq. (1a), G_{φ} does not vanish, even for axisymmetric systems. This is readily seen if one uses the relation

$$\frac{\partial}{\partial t} (n \vec{V}) + \nabla \cdot (n \vec{V} \vec{V}) = n \frac{\partial \vec{V}}{\partial t} + n (\vec{V} \cdot \nabla) \vec{V}$$

which is obtained invoking

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{V}) = 0$$

Then,

$$\langle R^2 \nabla \varphi \cdot n (\vec{V} \cdot \nabla) \vec{V} \rangle = \langle (\vec{V} \cdot R^2 \nabla \varphi) \frac{\partial n}{\partial t} \rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \cdot \langle R^2 \nabla \varphi \cdot n \vec{V} \vec{V} \cdot \nabla \psi \rangle \tag{B2}$$

The last term was obtained from the general flux averaged divergence formula

$$\langle \nabla \cdot \vec{A} \rangle = \frac{1}{V'} \frac{\partial}{\partial \psi} V' \langle \vec{A} \cdot \nabla \psi \rangle$$

and from,

$$n \vec{V} \vec{V} : \nabla R^2 \nabla \varphi = 0$$

since $\vec{V} \vec{V}$ is a symmetric and $\nabla R^2 \nabla \varphi$ an antisymmetric tensor (but only if $\frac{\partial}{\partial \varphi} = 0$). The last term in (B.2) vanished since $\vec{V} \cdot \nabla \psi = 0$ (see Eq. (10)). Thus, from (B.2),

$$\langle R^2 \nabla \varphi \cdot n (\vec{V} \cdot \nabla) \vec{V} \rangle \neq 0$$

if

$$\frac{\partial n}{\partial t} \neq 0$$

Appendix C on the evolution of E_r for a simplified model

In an early paper focusing on a physical explanation, Hirshman [5] constructed a simplified model problem to show the evolution of E_r , V_p and V_φ , leaving out inertia, variations of n_j in the flux surface and assuming a phenomenological toroidal damping time τ_T such that $\tau_p \ll \tau_T$. Here τ_p is the poloidal rotation damping time and the mechanism for τ_T can be a perpendicular viscous force.

From the toroidal component of (2), and summing over species and using Ampere's law (3b), Hirshman obtained rigorously

$$\frac{\partial}{\partial t} P_\varphi \equiv \frac{\partial}{\partial t} \left\langle \sum_j m_j n_j R^2 \nabla \varphi \cdot \vec{V}_j + (\vec{E} \cdot \nabla \psi) / 4\pi c \right\rangle = 0 + 0(\tau_T^{-1}) \quad (C1)$$

Thus, P_φ is a constant of the evolution on the fast τ_p time scale. Using Eq. (10a) for $\vec{V}_j$, Eq. (C.1) can be seen to link the time evolution of $V_\parallel$ (or V_p) with E_r , in accordance with our more general result, Eq. (13c).

It is instructive to develop this link in more detail. From Eqs. (C.1), (10a) and $c\vec{E} \cdot \nabla \psi = -\omega(\nabla \psi)^2$ one gets

$$\frac{\partial}{\partial t} \sum \left\langle \frac{4\pi m n c^2}{B^2} [IBV_\parallel - \omega R^2 B_p^2] - \omega |\nabla \psi|^2 \right\rangle = 0 \quad (C2)$$

where the last term can be neglected since

$$4\pi m n c^2 / B^2 = c^2 / c_A^2 \gg 1$$

Equation (C2) implies that

$$\sum \frac{\partial}{\partial t} \left\langle m_j n_j \frac{I}{B} V_{\parallel j} \right\rangle = \sum \frac{B_p^2}{B^2} \frac{\partial}{\partial t} \omega \langle m_j n_j R^2 \rangle \quad (C3)$$

Next, from parallel momentum balance

$$\sum \langle n_j m_j B \dot{V}_{\parallel j} \rangle = - \sum \langle \vec{B} \cdot \nabla \cdot \vec{\Pi}_j \rangle = - \sum \frac{3}{2} \left(\frac{\epsilon}{Rq} \right)^2 B_0^2 \mu_{j1} \frac{V_{pj}}{B_p} \quad (C4)$$

The last step was taken from Ref. [5]:

$$\langle \vec{B} \cdot \nabla \cdot \vec{\Pi} \rangle = 3 \langle (\vec{b} \cdot \nabla B)^2 \rangle \mu_1 (V_p / B_p)$$

$$\mu_1 = Rqnmv_{th}\nu_*/[(1 + \nu_*)(1 + \epsilon^{3/2}\nu_*)] \quad (C5)$$

where for concentric circular flux surfaces

$$\langle (\vec{b} \cdot \nabla B)^2 \rangle / \langle B^2 \rangle = \frac{\epsilon^2}{2q^2 R^2}$$

Thus, in the Pfirsch-Schlüter regime ($\epsilon^{3/2}\nu_* > 1$) $\mu_j = n_j m_j v_{thj}^2 / \nu_j$. Then, from Eqs. (10b),
(C3)

$$\sum \frac{B}{B_p} \dot{V}_p = \sum \left(\dot{V}_\parallel + \frac{I}{B} \dot{\omega} \right) \simeq \sum \frac{B^2}{B_p^2} \dot{V}_\parallel \quad (C6)$$

Eq. (C4) thus becomes

$$\frac{\partial}{\partial t} \sum m_j n_j V_{pj} \simeq - \sum \frac{1}{\tau_p} \sum m_j n_j V_{pj}$$

with the poloidal damping frequency given by

$$\frac{1}{\tau_p} = \frac{3}{2} q^2 \sum \frac{\omega_{tj}^2}{\nu_j}$$

Finally, owing to Eq. (C6) the radial electric field, $E_r = RB_p \omega / c$ will relax to a quasisteady state at the same rate τ_p^{-1} as the poloidal flow.

Appendix D on neglecting $\vec{V}^{(1)} \cdot \nabla \psi$

We neglected this term in the continuity equation (1b). Scaling $\frac{\vec{V}^{(1)} \cdot \nabla \psi}{|\nabla \psi|}$ through the inertial term $\sim V_{\parallel}^2 \sim v_{thi}^2$ in Eq. (13b), and $V_{\theta} = V_p$ from Eq. (10c), one obtains readily $V_r^{(1)}/V_p \sim \delta_p$, the Larmor radius parameters $\frac{\rho_p}{a}$ in the poloidal field. The diffusion driven radial flow $V_r^{(2)}$ is yet smaller than $V_r^{(1)}$ by the factor $(\Omega_i \tau_{iI})$ in an impure plasma, where τ_{iI}^{-1} is the collision frequency.

Appendix E on alternate form for ω evolution

The purpose of this appendix is to obtain an alternative form of Eq. (13c) which explicitly contains information of toroidal angular momentum conservation which is useful for numerical computation.

The flux-averaged equation, Eq. (13c) resulted from forcing the constraint $\omega = \omega(\psi)$ (defined in Eq. (9)) on the unaveraged equation (13b). However, one always has the choice to multiply by any θ -varying function before flux-averaging and thus to arrive at different forms of the resulting equation. In Eq. (13c), we chose to annihilate the term involving $T\vec{b} \cdot \nabla n$ by determining $\sum \langle B^2 \cdot \text{Eq. (13b)} \rangle$.

By taking $\sum \langle R^2 \vec{\nabla} \zeta \cdot \text{Eq. (2)} \rangle$, one finds the toroidal momentum conservation equation

$$\begin{aligned} & \sum \left\langle \frac{\partial}{\partial t} \left[mn \left(V_{\parallel} \frac{I}{B} - \frac{|\nabla \psi|^2}{B^2} \omega \right) \right] \right\rangle \\ & + \sum \langle mn \nu_d \left(V_{\parallel} \frac{I}{B} - \frac{|\nabla \psi|^2}{B^2} \omega \right) \rangle \\ & = \left\langle \frac{\vec{J}}{c} \cdot \nabla \psi \right\rangle + \sum \langle \vec{M} \cdot R^2 \nabla \zeta \rangle \end{aligned} \quad (E1)$$

Therefore, instead of annihilating $T\vec{b} \cdot \nabla n$, we preserve the form $\sum \frac{\partial}{\partial t} \langle mn \frac{|\nabla \psi|^2}{B^2} \omega \rangle$ by determining $\sum \langle \text{Eq. (13b)} \rangle$ which yields

$$\frac{\partial}{\partial t} \omega + C'_1 \omega^2 + C'_2 \omega + C'_3 = 0 \quad (E2)$$

where

$$\begin{aligned} C'_1 &= \frac{1}{D'} \sum m \left\langle \frac{n}{B^2} \vec{B} \cdot \nabla \frac{|\nabla \psi|^2}{B^2} \right\rangle I \\ C'_2 &= \frac{1}{D'} \sum \left[m \left\langle \frac{n}{B} V_{\parallel} \vec{B} \cdot \nabla \frac{|\nabla \psi|^2}{B^2} \right\rangle + m \langle n \nu_d \frac{|\nabla \psi|^2}{B^2} \rangle \right] \\ & \quad + \frac{2}{3} I^2 \langle \eta_0 \frac{(\vec{b} \cdot \nabla B)^2}{B^4} \rangle \\ C'_3 &= \frac{1}{D'} \sum \left[m \langle n V_{\parallel}^2 \frac{\vec{b} \cdot \nabla B}{B^2} \rangle I - \frac{2}{3} I \langle \eta_0 \left(\frac{\vec{b} \cdot \nabla B}{B^2} \right) \left(2\vec{b} \cdot \nabla V_{\parallel} + \frac{\vec{b} \cdot \nabla B}{B} V_{\parallel} \right) \rangle \right] \end{aligned}$$

$$+T\left\langle\frac{\vec{B}\cdot\nabla n}{B^2}\right\rangle I+\left\langle RM_{\perp\varphi}\right\rangle\Bigg]$$

$$D'=\frac{\langle|\nabla\psi|^2\rangle}{4\pi c^2}+\sum\left\langle\frac{mn}{B^2}|\nabla\psi|^2\right\rangle$$

By straightforwardly calculating D' · Eq. (E2), $\sum\langle\frac{|\nabla\psi|^2}{B^2}m\omega$ Eq. (11) $\rangle$, $\sum\langle\frac{I}{B}$ Eq. (12) $\rangle$, $\sum\langle V_{\parallel}\frac{I}{B}m$ Eq. (11) $\rangle$, one can reconstitute the toroidal momentum conservation equation (E1).

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