Nonlinear Excitation of Magnetostatic Fluctuations by
Kinetic Drift-Alfven Waves

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Abstract

We show that, for typical tokamak parameters, superthermal magnetostatic convective-cell fluctuations can be nonlinearly excited by a moderate level of kinetic drift-Alfven turbulence (e.g., $\delta n/n_0 \sim 1\%$). Effects on the plasma cross-field transport and the nonlinear evolution of drift-Alfven turbulence are also briefly discussed.

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The possibility that zero-frequency convective cells can make a substantial contribution to anomalous plasma transport across a magnetic field has attracted a great deal of attention.¹⁻⁴ Simple analytical estimates and computer simulations have shown that even a thermal equilibrium level of electrostatic and magnetostatic convective-cell fluctuations in a uniform plasma can be quite effective at plasma transport. Recently, Cheng and Okuda⁵ have carried out plasma simulations which show that electrostatic drift waves in a non-uniform plasma can nonlinearly excite a superthermal level of zero-frequency electrostatic convective-cell modes and thereby produce significantly enhanced plasma transport. In this paper, we analytically estimate the level of nonlinearly driven zero-frequency magnetostatic fluctuations in a plasma with a given level of kinetic drift-Alfvén turbulence and then make an estimate of plasma transport.

The starting point of our calculation is the nonlinear gyrokinetic equation (Frieman and Chen⁶) which describes the low-frequency response of a magnetized plasma. The drift-Alfvén waves in a low β plasma may be completely described in terms of the electrostatic potential, $\delta\phi$ and the parallel component of the vector potential, $\delta A_{\parallel}$. Here, $\beta \equiv$ plasma pressure/magnetic pressure. These potentials satisfy the quasi-neutrality condition

$$\sum_j e_j \int \delta f_j d^3v = 0, \quad (1)$$

and the parallel Ampere's law

$$\nabla^2 \delta A_{\parallel} = -\frac{4\pi}{c} \sum_j e_j \int v_{\parallel} \delta f_j d^3v, \quad (2)$$

where δf_j satisfies the nonlinear gyrokinetic equation. Fourier analysis in

space-time and the usual manipulations lead to the following expression for δf_{jk} :

$$\delta f_{jk} = -\left(\frac{e}{T}\right)_j \delta \phi_k F_{oj} + \delta h_{jk} \exp\left(\frac{i \mathbf{k} \cdot \mathbf{x} \times \mathbf{e}_{\parallel}}{\Omega_j}\right), \quad (3a)$$

$$\begin{aligned} (\omega_k - k_{\parallel} v_{\parallel}) \delta h_{jk} &= \left(\frac{e}{T}\right)_j (\omega + \omega_{*j})_k F_{oj} \delta L_k J_o(k_{\perp} \rho_j) \\ &+ i \frac{c}{2B} \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} (\mathbf{k}' \times \mathbf{k}'') \cdot \mathbf{e}_{\parallel} [\delta L_k J_o(k'_{\perp} \rho_j) \delta h_{jk''} \\ &- \delta L_{k''} J_o(k''_{\perp} \rho_j) \delta h_{jk'}] - i v_j (\delta h_{jk} - \widetilde{\delta n}_{jk} F_{oj}) \end{aligned} \quad (3b)$$

with $\delta L_k = \delta \phi_k - v_{\parallel} \delta A_{\parallel k} / c$, $\widetilde{\delta n}_{jk} \equiv \int d^3v \delta h_{jk}$, $F_{oj} = F_{mj}$ the Maxwell-Boltzman distribution, $\Omega_j = e_j B_o / m_j c$, $\omega_{*j} = (cT / e_j B_o L_n) k_y$, and $L_n^{-1} = n'_{oj} / n_{oj}$. We have included a density-conserving Krook collisional term; this will be neglected everywhere except for describing the parallel electron response for the $k_{\parallel} \approx 0$ magnetostatic modes (since $\delta v_{\parallel i} \approx 0$ for modes of interest, the inclusion of a momentum conserving Krook operator makes negligible additional contributions). We have neglected the nonlinearity $(e/m) \delta E_{\parallel} \partial \delta f / \partial v_{\parallel}$ associated with parallel electric fields since it is weaker than the retained nonlinear terms by the ratio $|\omega / \Omega_i| \ll 1$.

The linearized kinetic drift-Alfven waves of interest here satisfy the limits $|k_{\parallel} v_e| \gg |\omega_k| \gg v_e$ and, therefore, except for resonant-particle effects, may be described by the moments

$$\widetilde{\delta n}_{ek} = -\left(1 - \frac{\omega_{*e}}{\omega}\right)_k \overline{\delta A}_{\parallel k}, \quad (4a)$$

$$\widetilde{\delta n}_{ik} = \tau \left(1 + \frac{\omega_{*i}}{\omega}\right)_k \overline{\delta \phi}_k \Gamma_{oi}(k), \quad (4b)$$

$$\delta v_{\parallel e} = -\frac{(\omega - \omega_{*e})_k}{k_{\parallel}} (\delta \bar{\phi}_k - \bar{\delta A}_{\parallel k}), \quad (4c)$$

where $\tau = T_e/T_i$, $\Gamma_{oi}(k) = I_0(k_{\perp}^2 \rho_i^2) \exp(-k_{\perp}^2 \rho_i^2)$, $\Gamma_{oe}(k) \approx 1$, $\bar{\delta \phi}_k \equiv (e/T_e) \delta \phi_k$ and $\bar{\delta A}_{\parallel k} = (e/T_e) (\omega_k/k_{\parallel} c) \delta A_{\parallel k}$. But for finite Larmor-radius corrections, these equations could be directly obtained from the moment equations (e.g. see Kadomtsev⁷). Using Eqs. (1) and (2), one may obtain the well-known linear dispersion relation

$$\left(\tau + \frac{\omega_{*e}}{\omega}\right)_k (1 - \Gamma_{oi}(k)) + \left(1 - \frac{\omega_{*e}}{\omega}\right)_k \left[1 - \frac{\omega}{k_{\parallel}^2 v_A^2} \frac{1 - \Gamma_{oi}}{k_{\perp}^2 \rho_s^2}\right] \left(\tau + \frac{\omega_{*e}}{\omega}\right)_k = 0, \quad (5)$$

where $\rho_s^2 = (T_e/M\Omega_i^2)$, $v_A^2 = (B_0^2/4\pi m_o M)$. For low- β plasmas with $(\omega_{*e}/k_{\parallel} v_A)^2 \ll 1$ Eq. (5) has two branches, viz., the drift wave branch $\omega_{kd} = \omega_{*e} \Gamma_{oi}/(1 + \tau(1 - \Gamma_{oi}))$ and the kinetic shear-Alfvén branch $\omega_{kA} \approx k_{\parallel} v_A \{k_{\perp}^2 \rho_s^2 [1 + \tau(1 - \Gamma_{oi})]/\tau(1 - \Gamma_{oi})\}^{1/2}$. It is well-known that the drift wave branch may be driven unstable in a typical confined plasma by resonant wave-particle interactions, trapped-particle effects, temperature gradients, etc. Furthermore, note that the Γ_{oi} terms lead to dispersion effects, such that for $k_{\perp} \rho_i \sim O(1)$ we may have drift waves with the same frequency but different $k_{\perp}$; nonlinear interaction of such modes can readily lead to the excitation of zero-frequency modes in the plasma. Finally, the linear $\delta A_{\parallel} - \delta \phi$ relationships in the drift branch and Alfvén branch are, respectively, given by

$$\bar{\delta A}_{\parallel k} = (\omega_k^2/k_{\parallel}^2 v_A^2) [(1 - \Gamma_{oi})/k_{\perp}^2 \rho_s^2] (\tau + \omega_{*e}/\omega)_k \bar{\delta \phi}_k, \quad (6a)$$

and

$$\delta \bar{A}_{\parallel k} = [1 + \tau(1 - \Gamma_{01})]_k \delta \bar{\Phi}_k \quad (6b)$$

We next look at the linearized magnetostatic mode ($k \approx q$) near zero frequency. It is found in the limit $q_{\perp} \rho_i \sim 0(1)$, $|\omega_i| \sim v_e \gg |q_{\parallel} v_e|$ and is described by the moment

$$\delta v_{\parallel e} = i \left(\frac{\omega_{*e} - \omega}{v_e - i\omega} \right)_q \frac{q_{\parallel}^2 v_{eth}^2}{\omega^2} \delta \bar{A}_{\parallel q} \quad (7)$$

which, coupled with the parallel Ampere's law, Eq. (2), leads to the dispersion relation

$$\omega_q = (\omega_{*eq} - i v_e) / (1 + \omega_{pe}^2 / c^2 q_{\perp}^2) \quad (8)$$

In contrast to the uniform plasma result,³ the magnetostatic convective-cell mode in an inhomogeneous plasma has a real frequency $\omega_{*eq} / (1 + \omega_{pe}^2 / c^2 q_{\perp}^2)$. In a moment equation approach, this ω_{*e} effect enters through a parallel force term of the form $(T_e / n_0) (\hat{b}_x / B_0) (dn_0 / dx)$, in the electron equation of motion. The real frequency of the convective cell is, however, typically much less than ω_{*e} because for $q_{\perp} \rho_i \sim 0(1)$ and $\beta \gg m_e / M$, one has $(\omega_{pe} / c q_{\perp})^2 \gg 1$. Since the convective cell mode may also be strongly damped by a shear electron viscosity that it creates itself,³ one can introduce a phenomenological collective electron viscosity coefficient μ in the electron equations; this is readily incorporated by replacing v_e with $v_e + \mu q_{\perp}^2 \equiv \tilde{v}_e$.

We now consider the nonlinear excitation of this low-frequency magnetostatic mode by two drift-Alfven waves with nearly the same frequency

but different $\underline{k}_\perp$. Thus we take $\omega_q = \omega_k - \omega_{k'}$ and $\underline{q} = \underline{k} - \underline{k}'$ such that $|\omega_q| \ll |\omega_k|, |\omega_{k'}|, q_\parallel \approx 0$ but $q_\perp \rho_s \sim 0(1)$. Using Eq. (3b) to calculate a nonlinear parallel electron current at the beat frequency and substituting in the parallel Ampere's law for the convective-cell modes, we get

$$\begin{aligned} \delta A_{\parallel q} \left[q_\perp^2 + i \left(\frac{\omega_{*e} - \omega}{v_e - i\omega} \right) q \frac{\omega_{pe}^2}{c} \right] &= \left(\frac{2\pi m_e}{c} \right) \left(\frac{\Omega_i}{v_e - i\omega_q} \right) \frac{\omega_{*eq}}{k_\parallel} (\underline{k} \times \underline{k}') \cdot \hat{e}_\parallel \rho_s^2 \\ &\times \left[\overline{\delta \phi_k} \overline{\delta \phi_{k'}}^* + \frac{k_\parallel^2 v_{eth}^2}{2 \omega_k} \overline{\delta A_{\parallel k}} \overline{\delta A_{\parallel k'}}^* \right. \\ &\left. - \overline{\delta \phi_k} \overline{\delta A_{\parallel k'}}^* - \overline{\delta \phi_{k'}} \overline{\delta A_{\parallel k}}^* \right]. \end{aligned} \quad (9)$$

The nonlinear effects contributing to convective cell excitation can be interpreted as coming from $m(\vec{v} \cdot \nabla) \vec{v}_\parallel$, $(T/n_0)(\vec{b}_x/B_0)(d\hat{n}/dx)$ and $(\vec{v}_{E \times B} \times \vec{b}) \cdot \hat{e}_\parallel$ terms in the parallel equation of motion for electrons. In the absence of ω_* effects, the right side of Eq (9) is dominated by weaker resonant particle effects and electron polarization drift nonlinearities.

One can similarly show that the nonlinear excitation of the electrostatic convective-cell mode by kinetic drift-Alfven waves is governed by the equation

$$\begin{aligned} \left[1 + \tau + \frac{\omega_{*eq}}{\omega_q} (1 - \Gamma_{oi}(q)) \right] \overline{\delta \phi_q} &= \frac{iQ}{2\omega_k} \left(\frac{\omega_{*eq}}{\omega_q} \right) \rho_s^2 (\underline{k} \times \underline{k}') \cdot \hat{e}_\parallel \\ &\times \left[\langle J_0 J_0 J_0 \rangle \overline{\delta \phi_k} \overline{\delta \phi_{k'}}^* + \overline{\delta A_{\parallel k}} \overline{\delta A_{\parallel k'}}^* - \overline{\delta \phi_k} \overline{\delta A_{\parallel k'}}^* - \overline{\delta \phi_{k'}} \overline{\delta A_{\parallel k}}^* \right] \end{aligned} \quad (10)$$

where $\langle J_0 J_0 J_0 \rangle$ corresponds to the usual gyrokinetic average of the triple product $J_0(k_\perp v_\perp / \Omega_i) J_0(k'_\perp v_\perp / \Omega_i) J_0(q_\perp v_\perp / \Omega_i)$. Equation (10) is a more accurate description of the physical processes considered by Cheng and Okuda.⁵

We now concentrate on implications of Eq. (9), viz., the level of driven magnetostatic convective cells in a non-uniform plasma with a given ambient drift-Alfvén turbulence in the background. For typically relevant plasmas and maximum nonlinear coupling $|\omega_{*eq}| \gg |\omega_q|$ and $|\omega_{*eq}| \frac{\omega_{pe}^2}{c^2 q_{\perp}^2 (\bar{v}_e - i\omega_q)} \gg 1$, so that

$$\begin{aligned} \frac{\delta b_{\perp q}}{B_0} &= \frac{q_{\perp}}{B_0} \frac{\delta A_{\parallel q}}{\rho_s} = \frac{1}{2} (\hat{k} \times \hat{k}') \cdot \hat{e}_{\parallel} \rho_s^2 (q_{\perp} \rho_s) \frac{\Omega_i}{k_{\parallel} v_e} \left(\frac{m_e}{m_i}\right)^{1/2} \\ &\times \left[\overline{\delta \phi_k} \overline{\delta \phi_k^*} - \frac{k_{\parallel}^2 v_{eth}^2}{2 \omega_k} \overline{\delta A_{\parallel k}} \overline{\delta A_{\parallel k}^*} - \overline{\delta \phi_k} \overline{\delta A_{\parallel k}^*} - \overline{\delta \phi_k^*} \overline{\delta A_{\parallel k}} \right]. \end{aligned} \quad (11)$$

Using Eq. (6a) we may estimate that for the drift branch $\overline{\delta A_{\parallel k}} \sim (\omega_{*}/k_{\parallel} v_A)^2 \overline{\delta \phi_k}$. Taking $k_{\perp} \rho_s \sim O(1)$, the first two terms in (11) may be estimated as

$$\left(\frac{\delta b_{\perp q}}{B_0}\right) \sim \frac{m_e}{m_i} \left(\frac{1}{k_{\parallel} \rho_s}\right) \left[1 + \beta \frac{m_i}{m_e} \frac{\beta}{(k_{\parallel} L_n)^2}\right] |\overline{\delta \phi_k}|^2. \quad (12)$$

The magnitude of driven magnetostatic mode strongly depends on the $k_{\parallel}$ of the exciting drift waves. Thus, in a sheared magnetic field, there will be an intense excitation of these modes near the mode-rational surfaces. Taking a typical value of $k_{\parallel}$ be $L_s^{-1} \approx (qR)^{-1}$, we may now estimate $\delta b_{\perp q}/B_0$. For low β plasmas with the second term negligible, we find $(\delta b_{\perp q}/B_0) \sim (m_e/m_i) (L_s/\rho_s) |\overline{\delta \phi_k}|^2$. For a $(L_s/\rho_s) \sim 10^3$, $\overline{\delta \phi_k} = 1$ percent, we find $\delta b_{\perp q}/B_0 \approx 10^{-4}$. Thus, with a rather modest level of drift wave turbulence (~ 1 per cent) one can drive up very high levels of the magnetostatic mode. The above levels may be compared with the thermal equilibrium level of these modes in a uniform plasma, given by³

$$\frac{\langle \delta b^2 \rangle_q}{8\pi} = \frac{T}{2} \left(\frac{\omega_{pe}^2}{\omega_{pe}^2 + q^2 c^2} \right), \quad (13)$$

which on integrating over the spectrum gives $\delta b/B_0 = (\omega_{pe}/\Omega_e)(T/m_e c^2)^{1/2} [(e^2/m_e c^2)/\langle c/\omega_{pe} \rangle]^{1/2} = 10^{-8}$ for a typical tokamak plasma. Thus the driven levels of the magnetostatic mode are superthermal by several orders of magnitude, even for rather modest levels of drift wave turbulence. It is also interesting to compare the driven magnetostatic fluctuation level with the magnetic field perturbation associated with the drift wave itself. The latter may be readily estimated as $\delta b_k/B_0 \approx (\beta/k_{\parallel} L_n) \overline{\delta \phi}_k$. Thus for low β values, when the drift waves are primarily electrostatic, $\delta b_q/\delta b_k \gg 1$; even for modest β values δb_q may exceed δb_k . Thus the level of transport associated with the driven magnetostatic mode may well exceed that due to the drift wave itself.⁸

Following standard methods we may estimate the perpendicular transport of electrons as

$$\begin{aligned} D &= \int_0^\infty \langle \ddot{v}_1(t) \cdot \dot{v}_1(t+\tau) \rangle d\tau \\ &= (v_o^2/B_o^2) \int_0^\infty \langle \ddot{b}_1(t) \cdot \dot{b}_1(t+\tau) \rangle d\tau. \end{aligned} \quad (14)$$

If various mode-rational surfaces are packed close to each other so that there is an overlap of the driven magnetostatic mode on different surfaces, we may use a 2-d approximation where the correlation time in Eq. (14) is $\sim (q_{\perp}^2 D)^{-1}$. (Note that the driven mode has zero frequency and we assume that the diffusive process provides a collective electron shear viscosity $\mu \sim D$.) Taking $q_{\perp} \rho_s \sim 0(1)$, $v_o^2 \sim v_{the}^2$, we find

$$D \sim v_{the} \rho_s (\delta b / B_0) \quad . \quad (15)$$

This diffusion coefficient could be relevant in the outer part of usual tokamak discharges (or in such cold devices as Macrotor where B-field fluctuation measurements have actually been made⁹) where $v_e > (\Delta k_{\parallel}) v_{eth} \sim (v_{eth}/qR)$. If, on the other hand, magnetic surfaces open up because of overlap of associated magnetic island structures, and if particles freely stream along diffusing field lines, we find the usual expression¹⁰

$$D \sim v_{the} L_s (\delta b / B_0)^2 \quad . \quad (16)$$

We see that the two estimates are comparable.

Finally, we note that the zero-frequency convective cells can also couple back to the kinetic drift-Alfvén waves and hence trigger parametric decay instabilities.¹¹ There one finds that the wave energy of the kinetic drift-Alfvén waves cascades toward long wavelengths. Meanwhile, calculations with adiabatic electrons also predict a similar wave-energy cascading process.¹² While the latter interaction tends to occur over a larger phase-space volume, it becomes rather inefficient for $k_{\perp} \rho_i < 1$ due to the near cancellation between the bare-particle and shielding-cloud effects. In fact, one finds that the decay rate via convective cells is $O[(k_{\perp} \rho_i)^{-3}]$ larger than that predicted with adiabatic electrons. It therefore suggests that, in addition to causing anomalous cross-field transport, convective cells should also play a major role in determining the nonlinear evolution of long-wavelength ($k_{\perp} \rho_i < 1$) kinetic drift-Alfvén waves.

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