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The Theory of Alfvén Wave Heating in General Toroidal Geometry

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1. INTRODUCTION

The purpose of this paper is to present a general treatment of Alfvén wave heating based on the linearized equations of ideal magnetohydrodynamics (MHD). The conclusion of this study is that the geometry of the plasma equilibrium could play an important role on the effectiveness of this heating mechanism, and for certain geometries the fundamental equations may not possess solutions which satisfy prescribed boundary conditions.

In general terms, the prediction of rf energy absorption and eventual plasma heating is based on the idea that the Alfvén wave is a mode in the continuous spectrum of the linearized MHD equations, and consequently the amplitude of this wave is characterized by a spatial singularity which is not square integrable. One expects then that energy from an external source can be transferred to this mode, and hence to the plasma, without bound. Heating is predicted by introducing into the equations a mechanism which dissipates the absorbed energy without grossly effecting the overall MHD dynamics of the plasma.

The existence of a continuous spectrum and the nature of the associated spatial singularities has been established for equilibria possessing sufficient symmetry, in particular, for the planar sheet pinch¹ and the cylindrical screw pinch.² In these cases, the equilibria depend only on a single spatial coordinate permitting one to apply a Fourier transform with respect to the other coordinates. This reduces the equation which determines the continuum to an eigenvalue problem with an algebraic operator. The spatial singularities are found in general to be logarithmic. The continuum in axisymmetric toroidal geometry has also been considered.³ In this case one solves on each magnetic surface an eigenvalue problem with an ordinary differential operator. However, the nature of the singularity is not known. In ref. [3], a criterion is given which if satisfied implies that the singularity is logarithmic.

However, the criterion is not analyzed to determine if it can be satisfied. In regard to toroidal equilibria with no spatial symmetry, the existence of a continuum and the associated singular solution has not yet been established. In ref. [4], a formalism is presented which leads to equations which would characterize the continuum, but it is unclear whether these equations can be solved. In this paper we present further details concerning the nature of these equations and comment on possible methods of solution.

2. CONTINUOUS SPECTRUM

Our analysis is based on the linearized MHD equations which we write in the following form, assuming a time dependence of $\exp(i\omega t)$:

$$i\omega \rho \mathbf{v} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{b} + \frac{1}{\mu_0} (\mathbf{b} \cdot \nabla) \mathbf{B} - \nabla p^* \quad (1)$$

$$i\omega \mathbf{b} = (\mathbf{B} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{v} \quad (2)$$

$$i\omega p = - \mathbf{v} \cdot \nabla P - \gamma P \nabla \cdot \mathbf{v} \quad (3)$$

where ρ , $\mathbf{B}$ and P are respectively the equilibrium mass density, magnetic field and pressure; p , $\mathbf{b}$ and $\mathbf{v}$ are respectively the perturbed pressure, magnetic field and plasma velocity; μ_0 is the vacuum permeability, γ is ratio of specific heats and p^* designates the total linearized pressure, $p + \mathbf{B} \cdot \mathbf{b} / \mu_0$. The equilibrium is assumed to be static with closed toroidal magnetic surfaces, $\psi(\mathbf{r}) = \text{const}$.

The key in identifying the continuous spectrum of eqs. (1) - (3) and the associated singularities of the dependent variables is a transformation to the coordinates $\psi(\mathbf{r})$, $\theta(\mathbf{r})$ and $\phi(\mathbf{r})$, where θ and ϕ are angle-like variables which span the magnetic surfaces. The advantage of these coordinates is that the operator $\mathbf{B} \cdot \nabla$ does not contain derivatives with respect to ψ , $\mathbf{B} \cdot \nabla = (\mathbf{B} \cdot \nabla \theta) \partial / \partial \theta + (\mathbf{B} \cdot \nabla \phi) \partial / \partial \phi$. The only source of ψ derivatives in the system is found in the terms ∇p^* and $\nabla \cdot \mathbf{v}$.

As shown in ref. [4], one can then write eqs. (1) - (3) in a convenient matrix form:

$$\frac{\partial \tilde{X}}{\partial \psi} = \tilde{A} \cdot \tilde{X} + \tilde{C} \cdot \tilde{Y} \quad (4)$$

$$\tilde{\chi} \cdot \tilde{Y} = \tilde{K} \cdot \tilde{X} \quad (5)$$

where $\tilde{X}$ and $\tilde{Y}$ are vectors defined by,

$$\tilde{X} = \begin{bmatrix} v^1 \\ p^* \end{bmatrix} \quad \tilde{Y} = \begin{bmatrix} v^2 \\ v^3 \\ b^2 \\ b^3 \end{bmatrix} \quad (6)$$

and $\tilde{A}$, $\tilde{C}$, $\tilde{\chi}$ and $\tilde{K}$ are matrices whose elements are linear combinations of first derivatives with respect to θ and ϕ only. In eq. (6) the terms v^i ($i = 1, 2, 3$) and b^i ($i = 1, 2$) are respectively contravariant components of the vectors $\tilde{v}$ and $\tilde{b}$ ($v^1 = \tilde{v} \cdot \nabla \psi$, $v^2 = \tilde{v} \cdot \nabla \theta$, $v^3 = \tilde{v} \cdot \nabla \phi$ and similarly for $\tilde{b}$). The component b^1 is obtained from v^1 with $i\omega b^1 = \tilde{E} \cdot \nabla v^1$.

In eq. (5) the coordinate ψ is a parameter. Therefore, one can solve eqs. (4) and (5) formally by using eq. (5) to obtain $\tilde{Y}$ as a function of $\tilde{X}$ on a particular magnetic surface, $\tilde{Y} = \tilde{\chi}^{-1} \cdot \tilde{K} \cdot \tilde{X}$ where $\tilde{\chi}^{-1}$ is the inverse of the operator $\tilde{\chi}$.

Eliminating $\tilde{Y}$ in eq. (4) with this expression yields an equation which one solves for $\tilde{X}$. The boundary conditions that must be satisfied are periodicity with respect to θ and ϕ (since the magnetic surfaces are nested tori), regularity about the magnetic axis, and some form of continuity condition at the plasma-vacuum interface. It is clear however that this

formal procedure will break down if the inverse of $\tilde{\chi}$ does not exist. The condition for non-existence of $\tilde{\chi}^{-1}$ is that the equation,

$$\tilde{\chi} \cdot \underline{g} = 0 \quad (7)$$

possesses a non-trivial solution $\underline{g}$ which is single valued on a toroidal surface $\psi \equiv \psi_0 = \text{const.}$ If eq. (7) can be satisfied, the surface corresponding to ψ_0 is said to be singular.

To determine if a null vector of $\tilde{\chi}$ exists, we consider an explicit form of eq. (7) obtained by G. Spies⁵:

$$\frac{1}{\mu_0} \underline{B} \cdot \nabla (\underline{M} \underline{B} \cdot \nabla \phi) + \rho \omega^2 \underline{N} \phi = 0 \quad (8)$$

where ϕ is a two component column vector and $\underline{M}$ and $\underline{N}$ are 2×2 matrices defined as,

$$\underline{M} = \begin{bmatrix} \frac{\gamma P}{B_*} \underline{B} \cdot \underline{B} & \frac{\gamma P}{B_*} \underline{B} \cdot \underline{J} \\ \frac{\gamma P}{B_*} \underline{B} \cdot \underline{J} & \underline{J} \cdot \underline{J} - \frac{(\underline{J} \cdot \underline{B})^2}{\mu_0 B_*} \end{bmatrix} \quad (9)$$

$$\underline{N} = \begin{bmatrix} \underline{B} \cdot \underline{B} & \underline{B} \cdot \underline{J} \\ \underline{B} \cdot \underline{J} & \underline{J} \cdot \underline{J} \end{bmatrix} \quad (10)$$

In eqs. (9) and (10), B_* has been written in place of $\gamma P + B^2/\mu_0$, and $\tilde{J}$ is the equilibrium current density. Equation (8) is a system of hyperbolic partial differential equations which is to be solved over a closed toroidal surface. The boundary condition to be satisfied is that ϕ be single valued on the torus, i.e., ϕ is to be periodic in θ and ϕ , and this condition will in general be satisfied only for certain values of ω . The set of all such ω 's forms the continuous spectrum of eqs. (1) - (3).

The characteristics of eq. (8) are the magnetic field lines along which the operator $\underline{B} \cdot \nabla$ is an ordinary derivative d/dt where t is a parameter following $\underline{B}$. Therefore, by integrating eq. (8) along the characteristics, the problem is reduced to a system of ordinary differential equations. Since $\tilde{M}$ and $\tilde{N}$ are periodic functions of θ and ϕ , the coefficients of this equation depend on t , and in particular they are quasi-periodic functions of t if Hamada coordinates are assumed. The boundary condition that must be satisfied with respect to t depends on whether the surface is rational or irrational. If it is irrational, then the field line ergotically fills the surface, and ϕ would be obtained over the entire surface. The boundary condition would be that $\phi(t)$ is a quasi-periodic function of t with the same periods as the coefficients of eq. (8) (quasi-periodicity in t corresponds to periodicity in θ and ϕ). If the surface is rational, the field lines close over a finite length. The boundary condition is then periodicity over all closed lines. This implies that ω may then depend on the field line.

One can solve eq. (8) on the characteristics explicitly in two cases: ρ , $\tilde{M}$ and $\tilde{N}$ are either constant or are periodic functions of one coordinate, θ for example. If they are constant, then the solution is $\phi = \phi_0 \exp(i\alpha t)$ where ϕ_0 is a constant vector and α^2 is an eigenvalue of $\mu_0 \rho \omega^2 \tilde{M}^{-1} \tilde{N}$. For example, for the screw pinch, there are two eigenvalues:

$$\alpha^2 = \mu_0 \rho \omega^2 \quad (\text{Alfvén continuum}); \quad (11)$$

$$\alpha^2 = \mu_0 \rho \omega^2 B_*/\gamma P \quad (\text{cusp continuum}). \quad (12)$$

Let the coordinates of the field lines be written as $\theta(t) = B^2 t + \theta_0$, and $\phi(t) = B^3 t + \phi_0$ where $B^2 = \underline{B} \cdot \nabla \theta$ and $B^3 = \underline{B} \cdot \nabla \phi$. Then, the solution will be quasi-periodic in t and hence periodic in θ and ϕ if integers m and n exist such that the following decomposition can be made,

$$\alpha = mB^2 + nB^3 \quad (13)$$

If the coefficients of eq. (8) are periodic in θ , as would occur if the equilibrium is an axisymmetric torus, then solutions of the Floquet form exist, $\phi = \exp(i\alpha t) P(\theta)$, where P is periodic in θ . Again, quasi-periodicity requires that α be real and have the decomposition of eq. (13). There exists an extensive literature on the solution of ordinary differential equations with periodic coefficients, and in particular one can find conditions which guaranty that the Floquet coefficient α be real. However, the theory of equations with quasi-periodic coefficients does not appear to be extensively developed, and it is unclear whether such equations possess solutions which are also quasi-periodic. Therefore, the existence of single-valued solutions of eq. (8) over a toroidal surface must be regarded as questionable at this time.

It should be noted that if quasi-periodic solutions of eq. (8) exist, then ω^2 is real and greater or equal to zero. This was first demonstrated in ref. [4]. It can also be shown by vector multiplying eq. (8) by the complex conjugate of the transpose of ϕ and then integrating over the toroidal surface. One finds,

$$\omega^2 \oint ds \rho (\phi^+)^* \cdot \underline{N} \cdot \phi = - \frac{1}{\mu_0} \oint ds (\phi^+)^* \underline{B} \cdot \nabla (\underline{M} \cdot \underline{B} \cdot \nabla \phi) \quad (14)$$

where ds is an appropriate surface element and ϕ^+ designates the transpose of ϕ . It can be shown by integrating by parts that the right hand side is real and positive semi-definite while the integral on the left hand side is real and positive definite.

3. SINGULAR SOLUTIONS

The crucial question at this point concerns the behavior of $\underline{X}$ and $\underline{Y}$ about the singular surface. Absorption of energy would require that $\underline{X}$ and $\underline{Y}$ possess singularities which are not

square integrable, but there is no guaranty that such singularities exist even if the surface is singular. To see this, let us look for solutions of eqs. (4) and (5) which have the following expansion about ψ_0 :

$$\begin{aligned} \tilde{X}(\psi, \theta, \phi) &= (\psi - \psi_0)^\nu [\tilde{f}_0(\theta, \phi) + (\psi - \psi_0)\tilde{f}_1(\theta, \phi) + \dots] \\ \tilde{Y}(\psi, \theta, \phi) &= (\psi - \psi_0)^{\nu-1} [\tilde{g}_0(\theta, \phi) + (\psi - \psi_0)\tilde{g}_1(\theta, \phi) + \dots] \end{aligned} \quad (15)$$

where the coefficients $f_i(\theta, \phi)$ and $g_i(\theta, \phi)$ are periodic in θ and ϕ , and ν is a constant. Substituting eq. (15) into eqs. (4) and (5), expanding the operators $\tilde{A}, \tilde{C}, \tilde{\chi}$ and $\tilde{K}$ in a power series about ψ_0 ($\tilde{\chi} = \tilde{\chi}_0 + (\psi - \psi_0)\tilde{\chi}_1 + \dots$, etc.), and setting the coefficients of $(\psi - \psi_0)^{\nu-1}$ and $(\psi - \psi_0)^\nu$ equal to zero yield,

$$\tilde{\chi}_0 \cdot \tilde{g}_0 = 0 \quad (16)$$

$$\nu \tilde{f}_0 = \tilde{C}_0 \cdot \tilde{g}_0 \quad (17)$$

$$\tilde{\chi}_0 \cdot \tilde{g}_1 = (\tilde{K}_0 \cdot \tilde{C}_0 / \nu - \tilde{\chi}_1) \cdot \tilde{g}_0 \quad (18)$$

Equation (16) implies that $\tilde{g}_0$ belongs to the null space of $\tilde{\chi}_0$.

If the surface ψ_0 is indeed singular, then a non-trivial solution for $\tilde{g}_0$ exists. Since $\tilde{\chi}_0$ is a singular operator, eq. (18) can be solved for $\tilde{g}_1$ only if the right hand side satisfies a compatibility condition, namely,

$$\langle \tilde{h}^{(i)} | (\tilde{K}_0 \cdot \tilde{C}_0 - \nu \tilde{\chi}_1) \cdot \tilde{g}_0 \rangle = 0 \quad (19)$$

where $\langle \tilde{a} | \tilde{b} \rangle$ designates an inner product between the vectors $\tilde{a}$ and $\tilde{b}$ over the singular surface, $\tilde{h}^{(i)}$ is a null vector of the adjoint $\tilde{\chi}_0^+$ of $\tilde{\chi}_0$. Equation (19) can be used to obtain

the possible v 's. If two or more of the v 's are equal, then the expansion about ψ_0 may differ. For example, if all v are zero, i.e., if for all $\tilde{h}^{(i)}$, one obtains,

$$\left\langle \tilde{h}^{(i)} \left| \tilde{k}_0 \cdot \tilde{c}_0 \cdot \tilde{g}_0 \right. \right\rangle = 0 \quad (20)$$

then in place of eq. (15), it can be shown that the form of $\tilde{x}$ and $\tilde{y}$ about ψ_0 is,

$$\tilde{x} = [\tilde{f}_0 + (\psi - \psi_0)\tilde{f}_1 + \dots] \ln(\psi - \psi_0) + [\tilde{g}_0 + (\psi - \psi_0)\tilde{g}_1 + \dots]$$

$$\tilde{y} = (\psi - \psi_0)^{-1} [\tilde{h}_0 + (\psi - \psi_0)\tilde{h}_1 + \dots]$$

$$+ [\tilde{s}_0 + \tilde{s}_1(\psi - \psi_0) + \dots] \ln(\psi - \psi_0)$$

where $\tilde{x}_0 \cdot \tilde{h}_0 = 0$ and $\tilde{f}_0 = \tilde{c}_0 \tilde{h}_0$. With the exception of equilibria

which possess spatial symmetry it would be difficult to determine if eq. (13) is satisfied for all $\tilde{h}^{(i)}$. However, this condition is satisfied for the sheet pinch and for the screw pinch with compressible motion ($\nabla \cdot \tilde{v} \neq 0$). In both these cases, the singu-

larities are given by eq. (21). If the plasma motion is incompressible ($\nabla \cdot \tilde{v} = 0$), then eq. (20) is not satisfied for the

screw pinch, and v can be shown to be purely imaginary, implying that the singularity is of the type given by eq. (15). In either case, however, it is clear that $\tilde{y}$ is not square integrable. For most other equilibria, one cannot say too much more about v and hence about the singularity without further analysis.

4. CONCLUSION

In this paper we have presented a formalism which describes the continuous spectrum of ideal MHD. A continuum is said to exist if eq. (8) possesses solutions which are single-valued over a closed toroidal magnetic surface. However, in the absence of any spatial symmetry, it is questionable whether such solutions exist. The non-existence of a continuum would then bring into question the presence of an rf energy absorption mechanism based

on the continuous spectrum. However, the importance of this geometrical effect should be determined in comparison with other non-ideal mechanisms such as those arising from a kinetic description of the plasma.⁶

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