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**TIME-INDEPENDENT ONE-SPEED
NEUTRON TRANSPORT EQUATION
WITH ANISOTROPIC SCATTERING IN ABSORBING MEDIA**

by

R. J. Hangelbroek

ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS

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Abstract

This report treats the time-independent, one-speed neutron transport equation with anisotropic scattering in absorbing media. For nuclear gain operators existence and uniqueness of solutions to the half-space and finite-slab problems are proved in L_2 -space. The formulas needed for explicit calculations are derived by the use of perturbation theory techniques.

Introduction

The study of the time-independent, one-speed neutron transport equation with anisotropic scattering acquired a new impetus by the publication in 1961 of J. R. Mika's work [1], which used the technique of expansions in terms of singular eigenfunctions. Of the many papers which followed Mika's only recently a few did investigate the mathematical structure of his approach. The importance of these investigations is not only the achievement of a greater mathematical rigour (which certainly was lacking in the earlier papers) but also the indication of points where the used techniques can be generalized to more complicated problems within the area of neutron transport or even to problems in other areas of mathematical physics. We mention E. W. Larsen's contribution [2] based on resolvent operator techniques and C. G. Lekkerkerker's approach [3] which uses the theory of generalized functions. These authors as well as Mika did restrict their studies to the case of anisotropic scattering of finite degree.

In a new development R. Beals proved the existence and uniqueness of solutions to the standard half-space and finite-slab problems associated with equations of a large class of integro-differential and partial differential equations many of which originate from transport and scattering theory [4]. One of the main features which these equations have in common is that the quintessential linear operator is of the form $T^{-1}A$ with T and A self-adjoint

in a certain L_2 -space, say H , of scalar valued functions. The existence proof requires a modification of the space H into a more suitable space, say $\hat{H}$. The correspondence between Beals' work and the other papers is that the operator $T^{-1}A$ is diagonalizable since it is self-adjoint relative to some inner product. Beals uses this fact to establish his results on boundary value problems whereas the other papers are mainly concerned with the derivation of generalized eigenfunction expansions. Though he refers to the time-independent neutron transport equation with isotropic scattering as an example, Beals' results also apply to the case of neutron transport when scattering is described by means of a L_1 -function, i.e., in (fairly) general cases of anisotropic scattering (see [5]).

The neutron transport equation with L_1 -scattering function has as a special property that the operator A is a compact perturbation of the identity operator with the Legendre polynomials as a complete set of eigenfunctions (Theorem 1.1 of this report). This property causes the neutron transport problems to be well suited to direct computation but is partially lost when the problems are considered in the space $\hat{H}$ needed in Beals' approach. In this report we avoid the drawback of modifying the space H by assuming the operator A to be a nuclear perturbation of the identity. Thus, we are able to present an existence proof for the solutions to boundary value problems in the space H itself as well as derivations of the relevant generalized eigenfunction expansions both by the use of perturbation techniques.

A few comments should be made on the half-range formulas presented in Section 3, formulas (3.17), (3.18), and on the associated theory developed in Section 4. The formulas are equivalent to those published by C. G. Lekkerkerker in [3] and are the most explicitly known today. Though the formulas can be obtained in a direct way ([3] and final remark in Section 4 of this report) that derivation is rather nontransparent and the result is less explicit than the comparable full-range formulas (3.12), (3.13). The analysis presented in Section 4 is intended to clarify the underlying mathematical structure. Based on this analysis we venture to state the following reasons for the less complete form of the half-range formulas:

- (i) In the case of finite degree of anisotropy no algorithm is known by which the polynomials r_j can be obtained as easily as the polynomials h_j which occur in the full-range formulas.
- (ii) In the case of infinite degree of anisotropy the half-range operator $\hat{S} = P_+ A^{-1} T P$ is not likely to have the polynomial functions as an invariant subspace.

Notations

All vector spaces considered are over the complex numbers.

The identity operator in any vector space is denoted by E .

For J an interval of the reals $\mathcal{P}(J)$ denotes the vector space of polynomial functions defined on J . $\mathcal{P}_n(J)$ is the subspace of $\mathcal{P}(J)$ of polynomial functions of degree at most n . For ϕ a polynomial we denote its degree by $\deg \phi$.

For H a Hilbert space the Banach algebra of bounded linear operators in H with the uniform operator norm is denoted by $\mathcal{B}(H)$.

The null space and range of a linear operator T are denoted by $\ker T$ and $\text{ran } T$, respectively.

The dimension of a vector space V is indicated by $\dim V$.

Definitions

For V a vector space and T a linear operator acting in V an element $v \in V$ is called a cyclic vector for T in V if V is the linear span of $\{T^j v: j=0,1,2,\dots\}$.

For H a Hilbert space and T a linear operator acting in H an element $v \in H$ is called a cyclic vector for T in H if H is the closure of the linear span of $\{T^j v: j=0,1,2,\dots\}$.

1. The Transport Equation

In this report we discuss some boundary value problems associated with the time-independent, one-speed transport equation,

$$(1.1) \quad \mu \frac{\partial \psi}{\partial x}(x, \mu) = -\psi(x, \mu) + \frac{c}{4\pi} \int_{-1}^{+1} \psi(x, \mu_1) \int_0^{2\pi} f(\mu\mu_1 + (1-\mu^2)^{1/2}(1-\mu_1^2)^{1/2} \cos \phi) d\phi d\mu_1$$

where x ranges over an open interval $J \subset \mathbb{R}$, $\mu \in I = [-1, +1]$, ψ is a non-negative real valued function, c is a non-negative real parameter. The scattering function f is a Lebesgue integrable function on I and satisfies the conditions

$$(i) \quad f \geq 0,$$

$$(ii) \quad \frac{1}{2} \int_{-1}^{+1} f(\mu) d\mu = 1.$$

(E.g., [7].)

The integral operator in (1.1) having a symmetric kernel we study the equation in the complex Hilbert space $H = L_2(I)$ of Lebesgue square integrable functions on I . Thus equation (1.1) is interpreted as a linear differential equation for functions on J with values in H ,

$$(1.2) \quad T \frac{d\psi}{dx}(x) = -A\psi(x), \quad x \in R,$$

where T is defined in H as the operator of multiplication by μ and A stands for the identity perturbed by the integral operator. The operator T is bounded, self-adjoint with unbounded inverse. Its spectrum is the interval I and purely continuous. The properties of the integral operator are enumerated in Theorem 1.1 for as far as we will need them. We denote the customary inner product and associated norm in H by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively.

Theorem 1.1. The operator C defined in H by

$$(Cv)(\mu) = \frac{c}{4\pi} \int_{-1}^{+1} v(\mu_1) \int_0^{2\pi} f(\mu\mu_1 + (1-\mu^2)^{1/2}(1-\mu_1^2)^{1/2} \cos \phi) d\phi d\mu_1,$$

$v \in H$, a.e. $\mu \in I$, is a compact self-adjoint operator with norm $\|C\| = c$. The Legendre polynomials P_j form an orthogonal basis of eigenvectors of C with corresponding eigenvalues

$$c_j = \frac{c}{2} \int_{-1}^{+1} f(\mu) P_j(\mu) d\mu, \quad j = 0, 1, 2, \dots,$$

which satisfy the inequality $|c_j| < c_0 = c$ ($j \neq 0$) and have the property $\lim_{j \rightarrow \infty} c_j = 0$.

Proof. The interval I being compact any element $v \in H$ is also Lebesgue integrable on I . In [8], 4.9 (note also 4.10, exercise 3) it is shown that for v and f Lebesgue integrable one may interchange the positions of v and f in the definition of Cv . In particular, taking $v = P_j$ we arrive at

$$CP_j = \frac{c}{4\pi} \int_{-1}^{+1} f(\mu_1) \int_0^{2\pi} P_j(\mu\mu_1 + (1-\mu^2)^{1/2}(1-\mu_1^2)^{1/2} \cos \phi) d\phi d\mu_1.$$

Application of the addition theorem for Legendre polynomials yields

$$CP_j = \frac{c}{2} \int_{-1}^{+1} f(\mu_1) P_j(\mu_1) d\mu_1 P_j, \quad j = 0, 1, 2, \dots,$$

showing that the numbers c_j as defined in the theorem actually are eigenvalues of the operator C with the P_j corresponding eigenvectors.

The property $\lim_{j \rightarrow \infty} c_j = 0$ can be proved in a way completely analogous to the Riemann-Lebesgue lemma for the Fourier transform on $L_1(\mathbb{R})$. This property combined with the fact that the P_j form an orthogonal basis of H then leads to the conclusion that C is a compact operator in H .

Finally, the estimate $|c_j| < c_0$ ($j \neq 0$) is a simple consequence of the conditions imposed on the scattering function f and the fact $|P_j(\mu)| \leq 1$, $\mu \in I$.

Remark. The proof given above is a slight modification of the proofs which one encounters in Transport Theory, where f is assumed to admit an expansion in terms of the Legendre polynomials (in particular, if f is square integrable; e.g., [9]).

In this report we will be interested in the special case that C is nuclear (i.e., $\sum_{j=0}^{\infty} |c_j| < \infty$). In Transport Theory many results on (1.1) have been obtained by direct calculations if C is finite dimensional. The number $n = \sup \{j: c_j \neq 0\}$ is called the degree of anisotropy.

Since C is self-adjoint with its largest eigenvalue being c we have the following estimates for $A = E - C$,

$$(1.3) \quad (1-c)(v,v) \leq (Av,v) < (1+c)(v,v), \quad v \in H.$$

Hence, A is a strictly positive operator if $0 \leq c < 1$, i.e., in terms of the physical meaning of c , if the medium is absorbing. We shall restrict our discussion to this range of c values further on.

Denoting the normalized Legendre polynomials by p_j , so $p_j = (j+1/2)^{1/2} P_j$, we deduce from Theorem 1.1,

$$(1.4) \quad Cv = \sum_{j=0}^{\infty} c_j (v, p_j) p_j, \quad v \in H$$

and hence

$$A^{-1}v = v + \sum_{j=0}^{\infty} \gamma_j (v, p_j) p_j, \quad v \in H, \quad \gamma_j = c_j (1 - c_j)^{-1}.$$

For later use it is convenient to define the nuclear operator B in H by

$$(1.5) \quad Bv = \sum_{j=0}^{\infty} \gamma_j (v, p_j) p_j, \quad v \in H.$$

Thus, $B = C(E - C)^{-1}$ and $A^{-1} = E + B$.

Setting $S = A^{-1}T$ we can write equation (1.2) in the equivalent ways $S\psi'(x) = -\psi(x)$ and $\psi'(x) = -S^{-1}\psi(x)$, with S bounded and S^{-1} densely defined, unbounded. The operators A^{-1} and T do not commute, hence S is not self-adjoint. However, S is self-adjoint relative to the inner product $(\cdot, \cdot)_A$ defined by

$$(v, w)_A = (Av, w), \quad v, w \in H.$$

Because of (1.3) the norm $\| \cdot \|_A$ associated with $(\cdot, \cdot)_A$ and the norm $\| \cdot \|$ are equivalent. For this reason we use the symbol H also for the Hilbert Space $L_2(I)$ with inner product $(\cdot, \cdot)_A$ and frequently the operator S will be called self-adjoint in H without explicit reference to the inner product $(\cdot, \cdot)_A$. A useful additional property of S is stated in Lemma 1.1.

Lemma 1.1. If ψ is a polynomial function on I of degree j , then $S\psi$ is a polynomial function of degree $j + 1$.

Proof. We prove the statement using the recurrence relation for the Legendre polynomials viewed as functions on I ,

$$(2j+1)T P_j = (j+1)P_{j+1} + j P_{j-1}, \quad j=0,1,\dots, \quad (P_{-1} \equiv 0, P_0 \equiv 1).$$

Application of the operator A^{-1} yields

$$(1.6) \quad (2j+1)S P_j = \frac{j+1}{1-c_{j+1}} P_{j+1} + \frac{j}{1-c_{j-1}} P_{j-1},$$

which suffices for the proof.

A consequence of the lemma is that if $\tilde{\phi}$ is a polynomial then $\tilde{\phi}(S)P_0$ is a polynomial function on I of the same degree as $\tilde{\phi}$. Consequently, P_0 is a cyclic vector in H for S . For this reason we denote P_0 viewed as an element of H also by e . Thus we may formulate the spectral theorem for S in the following convenient form.

Theorem 1.2. There exist a finite, positive Borel measure σ on the spectrum N of the operator S and a linear transformation F that maps H onto the space $L_2(N, \sigma)$ of complex valued functions on N which are square integrable with respect to σ , such that

$$(i) \quad (v, w)_A = \int_N (Fv)(v) \overline{(Fw)(v)} d\sigma(v), \quad v, w \in H;$$

$$(ii) \quad FSv = T_N Fv, \quad v \in H$$

where T_N is defined by $(T_N \tilde{v})(v) = v \tilde{v}(v)$, $\tilde{v} \in L_2(N, \sigma)$, $v \in N$;

- (iii) F and σ are uniquely determined by the condition $Fe = \tilde{e}$ where $\tilde{e}(v) = 1, v \in N$.

As a consequence of (ii) and (iii) one obtains that for $\tilde{\phi}$ a polynomial $F\tilde{\phi}(S)e$ is the same polynomial $\tilde{\phi}$ but now considered as an element of $L_2(N, \sigma)$, whence we write $F\tilde{\phi}(S)e = \tilde{\phi}$. On the other hand by Lemma 1.1 $\tilde{\phi}(S)e$ is a polynomial function on I of the same degree as $\tilde{\phi}$. Consequently, F is a bijective map from $P(I)$ onto $P(N)$, where $P(I)$ and $P(N)$ denote the spaces of polynomial functions on I and N , respectively. In particular, the basis $\{p_j\}$ in H is mapped onto a basis of polynomial functions $\{h_j\}$ in $L_2(N, \sigma)$ where $h_j = FP_j$. Since $\{p_j\}$ is orthogonal relative to $(\cdot, \cdot)$ as well as $(\cdot, \cdot)_A$ the basis $\{h_j\}$ is orthogonal (though not normalized) in $L_2(N, \sigma)$ by Theorem 1.2.(i). The functions h_j can be obtained from a recurrence relation and for this reason are considered as known. In fact, application of F to relation (1.6) yields

$$(1.7) \quad (2j+1)T_N H_j = \frac{j+1}{1-c_{j+1}} H_{j+1} + \frac{j}{1-c_{j-1}} H_{j-1}, \quad j=0,1,\dots \quad (H_{-1}=0, H_0=\tilde{e}),$$

where $H_j = FP_j$. Then, $h_j = (j+1/2)^{1/2} H_j$.

The F -transform of any $v \in H$ can be obtained by expansion of v relative to the basis $\{p_j\}$ in H and subsequent application of the continuous linear map F . An analogous procedure yields the F^{-1} -transform of any element in $L_2(N, \sigma)$.

If the degree of anisotropy n is finite one actually needs to know only the first $n+1$ functions h_j in order to represent the transformations F and F^{-1} on large classes of functions. This fact is the main feature of the expressions for F and F^{-1} as derived by K. M. Case for $n=0$ [10] and J. Mika for finite nonzero values of n [1]. These latter expressions are in terms of eigenfunctionals of the operator S ; they have been of great practical use in solving transport problems and have led to the present study of the mathematical structure of such problems. In a later paper we shall discuss how solutions for finite n approximate solutions of boundary value problems with n infinite [11].

The spectrum N of the operator S is located in the interval $[-(1-c)^{-1}, (1-c)^{-1}]$ since $\|A^{-1}T\| \leq \|A^{-1}\| \|T\| = (1-c)^{-1}$. Moreover,

$$(1.8) \quad A^{-1}T = T+BT,$$

B defined by (1.5). Thus S is obtained from the selfadjoint operator T by adding a nuclear perturbation. According to Weyl's lemma the spectrum N will be the union of I, being the spectrum of T, and a denumerable set of isolated eigenvalues (which have algebraic multiplicity one since S has a cyclic vector). Denoting the largest absolute value of the eigenvalues of S by ν_0 we have $\|S\|_A = \nu_0 \leq (1-c)^{-1}$. The eigenvalues of S coincide with the zeros of the perturbation determinant (or W-A determinant) for the pair of operators $\{T, S\}$. This determinant has been calculated for n finite and shown to be equal (up to a non-zero constant factor) to a holomorphic function on $\mathbb{C} \setminus I$ which in Transport Theory is called the dispersion function [12]. Using that result we infer from [13]IV,1.2 that for C nuclear the perturbation determinant for $\{T, S\}$ is given by

$$(1.9) \quad \det\{(\lambda-S)(\lambda-T)^{-1}\} = (1-c)\Lambda(\lambda)\det A^{-1}, \quad \lambda \notin I,$$

where the dispersion function Λ is defined by

$$(1.10) \quad \Lambda(\lambda) = 1 + \sum_{j=0}^{\infty} \gamma_j h_j(\lambda) \int_{-1}^{+1} \frac{\mu p_j(\mu)}{\mu - \lambda} d\mu, \quad \lambda \notin I.$$

(It should be noted that in most of the literature on Transport Theory the dispersion function Λ is defined to be a factor $1-c$ larger than here and in our earlier publications.)

Finally, for finite degree of anisotropy the measure σ has been obtained in terms of the boundary values and residues of the dispersion function (e.g., [12]). The measure σ for general C will be a subject of our study [11].

2. Statement of the Boundary Value Problems

With disregard of the Infinite-Medium problem where $J = \mathbb{R}$ the classical formulations of the boundary value problems associated with equation (1.1) are,

(i) Finite-Slab problem (F.S.): $J = (0, a)$, $a > 0$ and the conditions

$$\psi(0, \mu) = g_+(\mu), \mu \in I_+ = [0, 1],$$

$$\psi(a, \mu) = g_-(\mu), \mu \in I_- = [-1, 0];$$

and

(ii) Half-Space problem (H.S.): $J = (0, \infty)$ and the conditions

$$\psi(0, \mu) = g_+(\mu), \mu \in I_+,$$

$$\lim_{x \rightarrow \infty} \psi(x, \mu) = 0, \mu \in I,$$

where g_+ and g_- are given functions.

The nature of the boundary conditions as well as the change of sign of the coefficient of $\frac{\partial \psi}{\partial x}(x)$ in (1.1) as μ goes through zero reflects the physical situation of particles moving in the positive and negative x directions.

In order to reformulate the boundary conditions (i) and (ii) in a way consistent with our interpretation of equation (1.1), we consider the space H as the direct sum of the Hilbert Spaces $H_+ = L_2(I_+)$ and $H_- = L_2(I_-)$, $H = H_+ \oplus H_-$, which is orthogonal with respect to $(\cdot, \cdot)$. The projections on $H_{\pm}$ along $H_{\mp}$ are denoted by $P_{\pm}$. Thus, the boundary conditions are taken to be

$$\text{F.S.: } P_+ \psi(0) = g_+, g_+ \in H_+; P_- \psi(a) = g_-, g_- \in H_-;$$

and

$$\text{H.S.: } P_+ \psi(0) = g_+, g_+ \in H_+; \lim_{x \rightarrow \infty} \psi(x) = 0 \text{ in } H.$$

The spectral theorem (Th. 1.2) provides us with a second decomposition of H . In fact, let $L_2(N, \sigma) = L_2(N_+, \sigma) \oplus L_2(N_-, \sigma)$ with $N_+ = N \cap \mathbb{R}_+$, $N_- = N \cap \mathbb{R}_-$ be a decomposition of $L_2(N, \sigma)$ analogous to the above decomposition of $L_2(I)$. (Note that 0 is not an eigenvalue of S , hence σ is not discrete at $0 \in N$.) Under the transformation F^{-1} we then obtain

$$H = H_p \oplus H_m ,$$

which is orthogonal with respect to $(\cdot, \cdot)_A$ and whose components H_p and H_m are the maximal S -invariant subspaces on which S is positive and negative, respectively. The projections on H_p, H_m along H_m, H_p are denoted by P_p, P_m , respectively.

The boundary value problems will be attacked by the use of the following lemma.

Lemma 2.1. (i) A function ψ mapping $[0, a]$ with finite $a > 0$ into H is continuous on $[0, a]$ and a solution of

$$\frac{d\psi}{dx}(x) = -S^{-1}\psi(x) , \quad x \in (0, a)$$

if and only if there exists a $h \in H$ such that

$$(2.1) \quad \psi(x) = P_p e^{-x|S^{-1}|} h + P_m e^{-(a-x)|S^{-1}|} h , \quad x \in (0, a)$$

where $e^{-t|S^{-1}|}$, $t > 0$ is the holomorphic semigroup of selfadjoint bounded operators generated by $-|S^{-1}| = -S^{-1}(P_p - P_m)$. The semigroup is contractive with

$$(2.2) \quad \|e^{-t|S^{-1}|}\|_A = e^{-t/\nu_0} , \quad t > 0 ,$$

where $\nu_0 = \|S\|_A$.

(ii) A function ψ mapping $[0, \infty)$ into H is continuous with $\lim_{x \rightarrow \infty} \psi(x) = 0$ and solves

$$\frac{d\psi}{dx}(x) = -S^{-1}\psi(x) , \quad x > 0$$

if and only if there exists a $h \in H_p$ such that

$$(2.3) \quad \psi(x) = e^{-xS^{-1}} h , \quad x > 0 .$$

Proof. (see [6])

In view of the lemma the F.S. problem is reduced to the question whether a $h \in H$ can be determined such that ψ given by (2.1) satisfies the F.S.-boundary conditions, i.e., such that

$$(P_+P_p + P_+P_me^{-a|S^{-1}|})h = g_+; \quad (P_-P_pe^{-a|S^{-1}|} + P_-P_m)h = g_-.$$

Adding these two relations together and putting $g = g_+ + g_-$ we obtain equivalently,

$$(2.4) \quad (V + We^{-a|S^{-1}|})h = g,$$

where

$$V = P_+P_p + P_-P_m, \quad W = P_+P_m + P_-P_p.$$

The operator in the left member of (2.4) will also be denoted by V_a , thus

$$V_a = V + We^{-a|S^{-1}|}.$$

In a similar way the H.S. problem is reduced to the question whether there is a $h \in H_p$ such that

$$(2.5) \quad P_+h = g_+,$$

or equivalently

$$(2.6) \quad Vh = g_+.$$

Note that $V^{-1}H_+ \subset H_p$ if V^{-1} exists.

The following relations between V and W are easy to verify and lead to the injectivity of V in H ,

$$(2.7) \quad V + W = E$$

and

$$(2.8) \quad 2V - E = V - W = (P_+ - P_-)(P_p - P_m).$$

Theorem 2.1. The operators V and V_a are bijective in H . Hence, the F.S. and H.S. problems have unique solutions given by (2.1) when $h = V_a^{-1}g$ and by (2.3) when $h = V^{-1}g$, respectively.

Proof. First we show that V is injective. Let $Vv = 0$ for some $v \in H$ and define $|T| = T(P_+ - P_-)$ then by (2.8), $-(|T|v, v) = (T(P_p - P_m)v, v) = (S(P_p - P_m)v, v)_A = (|S|v, v)_A$. The operators $|T|$ and $|S|$ are injective and positive relative to $(\cdot, \cdot)$ and $(\cdot, \cdot)_A$, respectively. Hence, $-(|T|v, v) = (|S|v, v)_A$ implies $v = 0$.

The surjectivity of V is a consequence of Theorem 3.1 where W is shown to be Hilbert-Schmidt. Assuming the validity of that theorem and since $V = E - W$ by (2.7), we infer from Fredholm's alternative that V is bijective in H . Then the same argument proves that $V_a = V + We^{-a|S|^{-1}} = V(E + WV^{-1}e^{-a|S|^{-1}})$ is bijective. (That V_a is injective is a consequence of the uniqueness theorem for the F.S. problem, see [7]2.3. A direct proof is given in [5],[6].)

The main subject of section 3 is the derivation of expressions for the transformations F and F^{-1} and the projection operator P which is directly related to the operator V^{-1} . We proceed to define the operator P and to present some of its relevant properties. Most of the proofs will be deleted since they appeared in [14]. It may be remarked that practical expressions for the operator V_a^{-1} are unknown, even in the case of a finite degree of anisotropy.

From its definition given above it will be clear that V restricted to H_p, H_m coincides with P_+, P_- , respectively. Hence, V^{-1} restricted to H_+, H_- maps onto H_p, H_m , respectively. This observation leads to the introduction of the operators P and Q defined by

$$P_+ = V^{-1}P_+, \quad Q = V^{-1}P_-$$

or, equivalently,

$$P = P_p V^{-1}, \quad Q = P_m V^{-1}.$$

The operators P and Q are projections. P projects onto H_p along H_- and Q projects onto H_m along H_+ . The following properties are simple consequences.

$$(2.9) \quad P_+P = P_+ , \quad PP_+ = P , \quad P_-Q = P_- , \quad QP_- = Q ,$$

$$(2.10) \quad P_P P = P , \quad P_P Q = 0 , \quad P_m P = 0 , \quad P_m Q = Q .$$

From the fact that P_+, P_P commute with T, S , respectively, it then follows that

$$(2.11) \quad P_+TP = P_+T , \quad PT = PTP ,$$

$$(2.12) \quad P_P SP = SP , \quad SP = PSP .$$

The operator VSV^{-1} is reduced by the same pair of subspaces $\{H_+, H_-\}$ as T because

$$VSV^{-1} = P_+P_P SP + P_-P_m SQ = P_+SP + P_-SQ .$$

Moreover, substitution according to (1.8) in the right member yields

$$(2.13) \quad \begin{aligned} VSV^{-1} &= P_+TP + P_-TQ + P_+BTP + P_-BTQ \\ &= T + P_+BTP + P_-BTQ , \end{aligned}$$

hence, VSV^{-1} , like S , is obtained from T by adding a nuclear perturbation.

These properties of VSV^{-1} lead to the factorization

$$(2.14) \quad (\lambda - VSV^{-1})(\lambda - T)^{-1} = \Omega_+(\lambda)\Omega_-(\lambda) , \quad \lambda \notin I ,$$

where $\Omega_{\pm}(\lambda)$ are bounded linear operators defined by

$$(2.15) \quad \begin{aligned} \Omega_+(\lambda) &= (\lambda - P_+SP)(\lambda - P_+T)^{-1} , \quad \lambda \notin I_+ , \\ \Omega_-(\lambda) &= (\lambda - P_-SQ)(\lambda - P_-T)^{-1} , \quad \lambda \notin I_- . \end{aligned}$$

The operators $\Omega_{\pm}(\lambda)$ coincide with the identity E on $H_{\pm}$ and are analytic functions of λ in $\mathbb{C} \setminus I_{\pm}$ with analytic inverses on the half planes $\mathbb{C}_{\mp} = \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq 0\}$. These operators are also of the type, identity perturbed by a nuclear operator. Consequently,

$$\det\{(\lambda - VSV^{-1})(\lambda - T)^{-1}\} = \det\Omega_+(\lambda) \cdot \det\Omega_-(\lambda), \quad \lambda \notin I.$$

Since $V = E - W$ with W a Hilbert-Schmidt operator,

$$\det\{(\lambda - VSV^{-1})(\lambda - T)^{-1}\} = \det\{(\lambda - S)(\lambda - T)^{-1}\}, \quad \lambda \notin I.$$

These last results together with (1.9) lead to the factorization of the dispersion function Λ ,

$$\Lambda(\lambda) = X_+(\lambda)X_-(\lambda), \quad \lambda \notin I,$$

where $X_{\pm}(\lambda) = \{(1-c)^{-1}\det A\}^{1/2} \det\Omega_{\pm}(\lambda)$.

Finally, $X_-(\lambda) = X_+(-\lambda)$ because of the symmetry of the spectral properties of T and S relative to $\lambda = 0$. To be precise, let the transformation U in H be defined by $(Uv)(\mu) = v(-\mu)$, $v \in H$, $\mu \in I$ then $U^2 = E$, $UT = -TU$ and $US = -SU$. Moreover, $UP_+ = P_-U$, $UP_p = P_mU$ and $UP = QU$. These relations imply that $\Omega_-(\lambda) = U\Omega_+(-\lambda)U$. Consequently,

$$(2.16) \quad \Lambda(\lambda) = X(\lambda)X(-\lambda), \quad \lambda \notin I,$$

where $X(\lambda) = X_+(\lambda)$. From the analyticity properties of Ω_+ we infer that X is analytic in $\mathbb{E} \setminus I_+$ with an analytic inverse in the half-plane $\mathbb{E}_-$. The function X has simple zeros at the eigenvalues of the operator P_+SP . These eigenvalues coincide with the positive eigenvalues of the operator S .

3. Representations for F , F^{-1} and P

Since $S = A^{-1}T$ is selfadjoint with no eigenvalue at $\lambda = 0$ the projection P_p can be represented by the contour integral

$$(3.1) \quad P_p v = \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - S)^{-1} v d\lambda, \quad v \in H,$$

where Γ_+ is a simple, positively oriented contour in $\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \geq 0\}$, enclosing N_+ and approaching N perpendicularly at and only at $\lambda = 0$. The integral is defined as a Cauchy principal value at $\lambda = 0$ in the strong sense (e.g., [15]VI, 5.2).

More generally, if ϕ_+ is a complex valued function, which is locally holomorphic in an open set O_+ containing N_+ , then we define the bounded linear operator $\phi_+(S)$ by

$$(3.2) \quad \phi_+(S)v = \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda)(\lambda - S)^{-1} v d\lambda, \quad v \in H,$$

where the definition of the integral is the same as in the previous paragraph except that here necessarily $\Gamma_+ \subset O_+$. The convergence of the integral follows from the convergence of the integral in (3.1), the analyticity of ϕ_+ at $\lambda = 0$ and the estimate $\|\lambda(\lambda - S)^{-1}\|_A \leq 1$ for imaginary values of λ . Considering the cases $v \in H_p$ and $v \in H_m$ separately one easily verifies via contour deformation that $\phi_+(S) = \phi_+(S')P_p$, where S' denotes the part of S in H_p and $\phi_+(S')$ is defined in H_p by means of a Dunford-Taylor integral of standard type.

We denote by $\mathcal{K}_+$ the linear set of complex valued functions on N_+ which admit locally holomorphic extensions to open neighborhoods of N_+ . Evidently, there is a one to one linear correspondence between functions $\phi_+ \in \mathcal{K}_+$ and operators $\phi_+(S) \in \mathcal{B}(H)$ defined by (3.2) via holomorphic extension of ϕ_+ .

Similary, if $\mathcal{K}_-$ denotes the linear set of complex valued functions on N_- which admit locally holomorphic extensions to neighborhoods of N_- then we may associate to $\phi_- \in \mathcal{K}_-$ a bounded linear operator $\phi_-(S)$ using an integral over a contour Γ_- in $\{\lambda \in \mathbb{C}: \operatorname{Re} \lambda \leq 0\}$ which encloses N_- .

The direct sum of $\mathcal{K}_+$ and $\mathcal{K}_-$ is denoted by $\mathcal{K}$ (hence, $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$) and for $\phi = \phi_+ + \phi_- \in \mathcal{K}$ we define

$$\phi(S) = \phi_+(S) + \phi_-(S)$$

and denote

$$(3.3) \quad \phi(S) = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda)(\lambda-S)^{-1} d\lambda$$

with $\Gamma = \Gamma_+ \cup \Gamma_-$ and subject to the above described interpretation of the integrals over Γ_+ and Γ_- .

In the preceding paragraphs we used of S only the properties that S is selfadjoint, bounded and injective. Hence, in the same way as $\phi(S)$ in (3.3) we may define

$$(3.4) \quad \phi(T) = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda)(\lambda-T)^{-1} d\lambda$$

for $\phi \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_{\pm}$ denote the linear sets of complex valued functions on $I_{\pm}$ that admit locally holomorphic extensions to neighborhoods of $I_{\pm}$.

The derivations and proofs presented in this section will be based mainly on the following lemma and its counterpart Lemma 3.2.

Lemma 3.1. Let K and L be bounded linear operators in H , L^+ the adjoint of L relative to the inner product $(\cdot, \cdot)_A$, and $\phi \in \mathcal{H}$. Let the operator M be defined by

$$M = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda)(\lambda-T)^{-1} KCL^+S(\lambda-S)^{-1} d\lambda$$

in the sense of (3.3). That is, if $\phi = \phi_+ + \phi_-$, $\phi_{\pm} \in \mathcal{H}_{\pm}$ then the operator M is decomposed into $M = M_+ + M_-$ with

$$M_{\pm} = \frac{1}{2\pi i} \int_{\Gamma_{\pm}} \phi_{\pm}(\lambda)(\lambda-T)^{-1} KCL^+S(\lambda-S)^{-1} d\lambda,$$

where the latter integrals are defined as discussed above. The convergence of the Cauchy principle value integrals at $\lambda = 0$ in the strong sense is assumed.

Then for each $v \in H$,

$$(3.5) \quad Mv(\mu) = \sum_{j=0}^{\infty} \gamma_j p_j'(\mu) \int_N \frac{\phi(v) - \phi(\mu)}{v - \mu} h_j'(v) \tilde{v}(v) d\sigma(v), \quad \text{a.e. } \mu \in I,$$

where $\tilde{v} = Fv$, $p_j' = Kp_j$ and $h_j' = FLp_j$.

Proof. We study the contour integral for M_+ with T as an ancillary smoothing factor in the integrand,

$$TM_+ = \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda) T(\lambda - T)^{-1} KCL^+ S(\lambda - S)^{-1} d\lambda.$$

The mappings $\lambda \mapsto T(\lambda - T)^{-1}$ and $\lambda \mapsto S(\lambda - S)^{-1}$, $\lambda \in \Gamma_+$ are continuous into $\mathcal{B}(H)$. Moreover, these mappings are bounded on Γ_+ . Hence, the latter integral exists as a Riemann integral in $\mathcal{B}(H)$. We substitute $C = \sum_{j=0}^{\infty} C_j$ where $C_j v = c_j(v, p_j) p_j$, $v \in H$ (cf. (1.4)). Since $\sum_{j=0}^{\infty} \|C_j\| = \sum_{j=0}^{\infty} |c_j| < \infty$ the integrand becomes an absolutely convergent series, the convergence being uniform in $\lambda \in \Gamma_+$. Consequently, the integration and summation can be interchanged yielding,

$$TM_+ = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda) T(\lambda - T)^{-1} KC_j L^+ S(\lambda - S)^{-1} d\lambda.$$

Due to the fact that the latter series converges absolutely in $\mathcal{B}(H)$ we have for each $v \in H$,

$$(3.6) \quad TM_+ v(\mu) = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \left\{ \int_{\Gamma_+} \phi_+(\lambda) T(\lambda - T)^{-1} KC_j L^+ S(\lambda - S)^{-1} v d\lambda \right\}(\mu)$$

with the series convergent for a.e. $\mu \in I$.

The summand in the latter relation is evaluated as follows. Denoting $Fv = \tilde{v}$, $Kp_j = p_j'$, $FLp_j = h_j'$ we obtain from Theorem 1.2 (i) for a.e. $\mu \in I$,

$$\begin{aligned} (3.7) \quad & \frac{1}{2\pi i} c_j \int_{\Gamma_+} \phi_+(\lambda) \frac{\mu}{\lambda - \mu} p_j'(\mu) (L^+ S(\lambda - S)^{-1} v, p_j) d\lambda \\ &= \frac{1}{2\pi i} \gamma_j p_j'(\mu) \int_{\Gamma_+} \phi_+(\lambda) \frac{\mu}{\lambda - \mu} \int_N \frac{v}{\lambda - v} h_j'(v) \tilde{v}(v) d\sigma(v) d\lambda \\ &= \frac{1}{2\pi i} \gamma_j p_j'(\mu) \int_N h_j'(v) \tilde{v}(v) \int_N \frac{\mu v \phi_+(\lambda)}{(\lambda - \mu)(\lambda - v)} d\lambda d\sigma(v). \end{aligned}$$

The first member represents the summand in (3.6) as can be verified by taking inner products with an arbitrary $w \in H$. The step to the third member is based on Fubini's theorem. The contour integral in the third member is evaluated by taking residues for a.e. $\mu \in I$ and the factor μ is taken out subsequently. We obtain

$$M_+ v(\mu) = \sum_{j=0}^{\infty} \gamma_j p_j^+(\mu) \left\{ \int_{N_+} v \frac{\phi_+(v) - \phi_+(\mu)}{v - \mu} h_j^+(v) \tilde{v}(v) d\sigma(v) \right. \\ \left. - \int_{N_-} v \frac{\phi_+(\mu)}{v - \mu} h_j^+(v) \tilde{v}(v) d\sigma(v) \right\}, \text{ a.e. } \mu \in I_+,$$

$$M_+ v(\mu) = \sum_{j=0}^{\infty} \gamma_j p_j^+(\mu) \int_{N_+} v \frac{\phi_+(v)}{v - \mu} h_j^+(v) \tilde{v}(v) d\sigma(v), \text{ a.e. } \mu \in I_-.$$

Replacing $+$ by $-$ and $-$ by $+$ in these formulas one obtains expressions for $M_- v(\mu)$, a.e. $\mu \in I$. Addition of the formulas for $M_+ v(\mu)$ and $M_- v(\mu)$ then yields (3.5).

As a first application of the lemma we prove that the operator W is Hilbert-Schmidt and hence that the F.S. and H.S. problems have a unique solution (cf. Theorem 2.1).

Theorem 3.1. The operators $P_- P_+$ and $W = P_- P_+ + P_+ P_-$ are Hilbert-Schmidt and for each $v \in H$

$$Wv(\mu) = \sum_{j=0}^{\infty} \gamma_j p_j(\mu) \int_{N_-} \chi(\mu, v) \frac{v}{v - \mu} h_j(v) \tilde{v}(v) d\sigma(v), \text{ a.e. } \mu \in I,$$

where $\tilde{v} = Fv$ and χ is the characteristic function of the set $I_+ \times N_- \cup I_- \times N_+$.

Proof. For any $v \in H$ we have

$$P_- v - P_+ v = \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - S)^{-1} v d\lambda - \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - T)^{-1} v d\lambda \\ = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda) (\lambda - T)^{-1} C S (\lambda - S)^{-1} v d\lambda,$$

where $\phi \in \mathcal{K}$, $\phi = \phi_+ + \phi_-$ with ϕ_+, ϕ_- identical equal to one, zero, respectively. We may apply the lemma with $K = L = E$ to obtain

$$(3.8) \quad (P_p - P_+)v(\mu) = \sum_{j=0}^{\infty} \gamma_j \phi_j(\mu) \int_N \chi(\mu, v) \frac{|v|}{v-\mu} h_j(v) \tilde{v}(v) d\sigma(v), \text{ a.e. } \mu \in I.$$

The function $(\mu, v) \mapsto \chi(\mu, v) \frac{|v|}{v-\mu} p_j(\mu) h_j(v)$, $(\mu, v) \in I \times N$ is square integrable relative to the product measure $d\mu \times d\sigma(v)$ with

$$\begin{aligned} & \int_I \int_N \left| \chi(\mu, v) \frac{|v|}{v-\mu} p_j(\mu) h_j(v) \right|^2 d\sigma(v) d\mu \\ &= \left\{ \int_{I_+} \int_{N_-} + \int_{I_-} \int_{N_+} \right\} \left| \frac{v}{v-\mu} p_j(\mu) h_j(v) \right|^2 d\sigma(v) d\mu \\ &\leq 2 \int_I |p_j(\mu)|^2 d\mu \int_N |h_j(v)|^2 d\sigma(v) = 2(1-c_j). \end{aligned}$$

Since $F: H \rightarrow L_2(N, \sigma)$ is continuous we deduce that the operator R_j in H defined by

$$R_j v(\mu) = p_j(\mu) \int_N \chi(\mu, v) \frac{|v|}{v-\mu} h_j(v) \tilde{v}(v) d\sigma(v), \quad \mu \in I, \quad \tilde{v} = Fv,$$

is a Hilbert-Schmidt operator with Hilbert-Schmidt norm independent of j . Since $\sum_{j=0}^{\infty} |\gamma_j| < \infty$ the series $\sum_{j=0}^{\infty} \gamma_j R_j$ is absolutely convergent in the Hilbert-Schmidt norm. Thus, $P_p - P_+ = \sum_{j=0}^{\infty} \gamma_j R_j$ is Hilbert-Schmidt.

Left multiplication of the operator $P_p - P_+$ by the bounded operator $P_- - P_+$ yields that

$$(P_- - P_+)(P_p - P_+) = P_- P_p - P_+ P_p + P_+ = P_- P_p + P_+ P_m = W,$$

is Hilbert Schmidt. The formula for Wv then follows from (3.8).

Remarks.

1. The author has been informed that for finite degree of anisotropy the operators $P_p - P_+$ and W are shown to be nuclear in [16].

2. It may be noted that $P_m - P_- = -(P_p - P_+)$ since $E = P_+ + P_- = P_p + P_m$.

The next theorem presents a representation of F^{-1} restricted to $\mathcal{K}$ considered as a linear subspace of $L_2(N, \sigma)$. This representation corresponds with the singular integral equation which in Transport Theory is called the Full-range Completeness Theorem (e.g., [7]). Formula (3.9) restricted to

$\tilde{\phi} \in \mathcal{K}_+ \subset L_2(N_+, \sigma)$ and $\mu \in I_+$ corresponds with the Half-Range Completeness Theorem. The expansion coefficients of these Completeness Theorems are given by the formulas in the Theorems 3.3 and 3.4, respectively.

Theorem 3.2. Let $\tilde{\phi} \in \mathcal{K} \subset L_2(N, \sigma)$ then

$$(3.9) \quad F^{-1}\tilde{\phi}(\mu) = \tilde{\phi}(\mu) + \sum_{j=0}^{\infty} \gamma_j p_j(\mu) \int_N \frac{\tilde{\phi}(\nu) - \tilde{\phi}(\mu)}{\nu - \mu} h_j(\nu) d\sigma(\nu), \text{ a.e. } \mu \in I.$$

Proof. Since $\tilde{\phi} = \frac{1}{2\pi i} \int_{\Gamma} \tilde{\phi}(\lambda)(\lambda - T_N)^{-1} e d\lambda$ and $F T_N F^{-1} = S$ we derive

$$\begin{aligned} F^{-1}\tilde{\phi} &= \frac{1}{2\pi i} \int_{\Gamma} \tilde{\phi}(\lambda)(\lambda - S)^{-1} e d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} \tilde{\phi}(\lambda)(\lambda - T)^{-1} e d\lambda + \frac{1}{2\pi i} \int_{\Gamma} \tilde{\phi}(\lambda)(\lambda - T)^{-1} C S (\lambda - S)^{-1} e d\lambda. \end{aligned}$$

Application of Lemma 3.1 with $K = L = E$, $v = e$ and $\phi = \tilde{\phi}$ yields formula (3.9).

Remarks.

1. Formula (3.9) yields $F^{-1}\tilde{\phi}$ subject to the knowledge of the measure σ due to the fact that the polynomials $h_j = F p_j$ are considered to be known from the recurrence relation (1.7). As regards to the measure σ see the last paragraph of Section 1.

2. The series in the formula is finite with j running from 0 through n if $\tilde{\phi}$ is a polynomial of degree n or also if the degree of anisotropy is n . In the latter case the right member of (3.9) is continuous in μ except for a possible finite jump at $\mu = 0$. Moreover, then the formula can be extended to larger classes of functions $\tilde{\phi}$ (e.g., see [3],[17] for extensions to functions that are Hölder continuous on N except for a jump at $\nu = 0$).

Corollary. For $\nu' \in N \setminus I$ (i.e. ν' is an isolated eigenvalue of S), the corresponding eigenvector such that $(e, \phi_{\nu'})_A = 1$ is given by

$$(3.10) \quad \phi_{\nu'}(\mu) = \frac{\nu'}{\nu' - \mu} \sum_{j=0}^{\infty} \gamma_j p_j(\mu) h_j(\nu'), \text{ a.e. } \mu \in I.$$

Proof. The function $\tilde{\phi} \in \mathcal{K} \subset L_2(N, \sigma)$ such that $\tilde{\phi}(\nu) = 0$, $\nu \in N \setminus \{\nu'\}$ and $\tilde{\phi}(\nu') = (\sigma(\nu'))^{-1}$ is an eigenvector of T_N corresponding to ν' . Formula (3.9) with

this choice of $\tilde{\phi}$ yields the result stated in the corollary. Moreover, $(e, \phi_v)_A = \int_N \tilde{\phi}(v) d\sigma(v) = 1$.

In order to obtain a representation for F and to derive half-range formulas we need another lemma.

Lemma 3.2. Let K and L be bounded linear operators in H , K^* be the adjoint of K relative to $(\cdot, \cdot)$ and $\phi \in \mathcal{K}$. Let the operator R be defined by

$$R = \frac{1}{2\pi i} \int_{\Gamma} \phi(\lambda) (\lambda - S)^{-1} L B K^* T (\lambda - T)^{-1} d\lambda,$$

with $\Gamma = \Gamma_+ \cup \Gamma_-$ in the sense of (3.3). The convergence of the Cauchy principle value integrals over Γ_+ and Γ_- at $\lambda = 0$ in the strong sense is assumed. Then for each $v \in H$,

$$(3.11) \quad F R v(v) = \sum_{j=0}^{\infty} \gamma_j h_j^!(v) \int_{\Gamma} \mu \frac{\phi(\mu) - \phi(v)}{\mu - v} p_j^!(\mu) v(\mu) d\mu, \sigma - a.e. v \in N,$$

where $p_j^! = K p_j$ and $h_j^! = F L p_j$.

Proof. We consider the bounded linear operator

$$S R_+ = \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda) S (\lambda - S)^{-1} L B K^* T (\lambda - T)^{-1} d\lambda,$$

where ϕ_+ is the component of ϕ in $\mathcal{K}_+$. Proceeding in a similar way as in the proof of Lemma 3.1 we substitute $B = \sum_{j=0}^{\infty} B_j$ where $B_j v = \gamma_j(v_j, p_j) p_j$, $v \in H$ (cf. (1.5)) and change the summation and integration to obtain

$$S R_+ = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda) S (\lambda - S)^{-1} L B_j K^* T (\lambda - T)^{-1} d\lambda$$

or, equivalently, in $L_2(N, \sigma)$,

$$F S R_+ F^{-1} = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_+} \phi_+(\lambda) T_N (\lambda - T_N)^{-1} F L B_j K^* T (\lambda - T)^{-1} F^{-1} d\lambda,$$

with the series for $S R_+$ and $F S R_+ F^{-1}$ both converging in the respective operator norms. Hence, if $\tilde{v} = F v$, $v \in H$ we arrive at

$$FSR_+ v(v) = \sum_{j=0}^{\infty} \frac{1}{2\pi i} \left\{ \int_{\Gamma_+} \phi_+(\lambda) T_N(\lambda - T_N)^{-1} FLB_j K^* T(\lambda - T)^{-1} v d\lambda \right\} (v),$$

with the series converging for σ -a.e. $v \in N$. Then the summand in the right member is evaluated in a similar way as in the proof of Lemma 3.1 (see (3.7)) yielding

$$\begin{aligned} FR_+ v(v) &= \sum_{j=0}^{\infty} \gamma_j h_j^+(v) \left\{ \int_{\Gamma_+} \mu \frac{\phi_+(\mu) - \phi_-(v)}{\mu - v} p_j^+(\mu) v(\mu) d\mu \right. \\ &\quad \left. - \int_{\Gamma_-} \mu \frac{\phi_+(v)}{\mu - v} p_j^+(\mu) v(\mu) d\mu \right\}, \quad \sigma\text{-a.e. } v \in N_+, \\ FR_+ v(v) &= \sum_{j=0}^{\infty} \gamma_j h_j^+(v) \int_{\Gamma_+} \mu \frac{\phi_+(\mu)}{\mu - v} p_j^+(\mu) v(\mu) d\mu, \quad \sigma\text{-a.e. } v \in N_-, \end{aligned}$$

where $p_j^+ = K p_j$ and $h_j^+ = FL p_j$.

Replacing $+$ by $-$ and $-$ by $+$ in these formulas one obtains the expression for $FR_- v(v)$, σ -a.e. $v \in N$. Addition of the formulas for $FR_+ v(v)$ and $FR_- v(v)$ then yields (3.11).

A representation of F on $\mathcal{A}$ considered as a linear subspace of H is given in the following theorem.

Theorem 3.3. Let $\phi \in \mathcal{A} \subset H$ then

$$(3.12) \quad F\phi(v) = \phi(v) - \sum_{j=0}^{\infty} \gamma_j h_j(v) \int_{\Gamma} \mu \frac{\phi(\mu) - \phi(v)}{\mu - v} p_j(\mu) d\mu, \quad \sigma\text{-a.e. } v \in I$$

and

$$(3.13) \quad F\phi(v) = - \sum_{j=0}^{\infty} \gamma_j h_j(v) \int_{\Gamma} \frac{\mu \phi(\mu)}{\mu - v} p_j(\mu) d\mu, \quad v \in N \setminus I.$$

Proof. Let $\phi_e \in \mathcal{K}$ be an extension of $\phi \in \mathcal{A}$ then with $\Gamma = \Gamma_+ \cup \Gamma_-$ as in (3.3),

$$\begin{aligned} \phi &= \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) (\lambda - T)^{-1} e d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) (\lambda - S)^{-1} e d\lambda - \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) (\lambda - S)^{-1} B T (\lambda - T)^{-1} e d\lambda. \end{aligned}$$

Application of Lemma 3.2 with $L = K = E$ and $v = e$ yields formula (3.12) for $v \in I$ and

$$\begin{aligned} F\phi(v) = & \left(1 + \sum_{j=0}^{\infty} \gamma_j h_j(v) \int_I \frac{\mu p_j(\mu)}{\mu - v} d\mu\right) \phi_e(v) \\ & - \sum_{j=0}^{\infty} \gamma_j h_j(v) \int_I \frac{\mu \phi(\mu)}{\mu - v} p_j(\mu) d\mu, \quad v \in N \setminus I. \end{aligned}$$

The extension $\phi \mapsto \phi_e$ being not unique we conclude to (3.13). This conclusion is in accordance with the observation $\Lambda(v) = \det A \det\{(v-S)(v-T)^{-1}\} = 0$ for $v \in N \setminus I$ (cf. (1.9), (1.10)).

Remarks analogous to the remarks following Theorem 3.2 can be made regarding formula (3.12). Formula (3.13) actually holds for $\phi \in H$ and can be obtained from the Corollary to Theorem 3.2 as follows. The linear eigenfunctional $\phi \mapsto F\phi(v)$ is defined and H -continuous for $\phi \in H$ if $v \in N \setminus I$. Hence, $F\phi(v) = (\phi, \phi_v)_A$, $\phi \in H$ for some eigenvector ϕ_v of $S = A^{-1}T$ corresponding to $v \in N \setminus I$. This eigenvector ϕ_v is uniquely determined by the condition $Fe(v) = (e, \phi_v)_A = 1$. Therefore,

$$F\phi(v) = (\phi, \phi_v)_A = \frac{1}{v}(\phi, T\phi_v)$$

for $\phi \in H$ and ϕ_v given by (3.10).

The last two theorems of this section present half range results in terms of the functions Fp_j , $(E-Q)p_j$ and $(E-P)p_j$. For convenience we denote

$$(3.14) \quad \eta_j = Fp_j, \quad \zeta_j = (E-Q)p_j, \quad j=0,1,2,\dots$$

with $\eta_j \in L_2(N_+, \sigma)$, $\zeta_j \in H_+$.

Applying the symmetry operator U , defined in the last paragraph of Section 2, to ζ_j one obtains

$$U\zeta_j = (E-P)Up_j = (-1)^j(E-P)p_j$$

since $p_j(-\mu) = (-1)^j p_j(\mu)$ and, hence,

$$(3.15) \quad (E-P)p_j(\mu) = (-1)^j \zeta_j(-\mu), \quad \mu \in I_-.$$

We also need the following relation between the operators P and $E-Q$,

$$(3.16) \quad TP = (E-Q)^* T,$$

where $(E-Q)^*$ is the $(\cdot, \cdot)_A$ -adjoint of $E-Q$. Relation (3.16) is proved as follows. Using the definitions of S and $(\cdot, \cdot)_A$ one has

$$(TPv, w) = (SPv, w)_A, \quad v, w \in H.$$

The operators P and Q project on the subspaces H_p and H_m , respectively, which are orthogonal relative to $(\cdot, \cdot)_A$ and invariant under S . Hence, $(SPv, Qw)_A = 0$ and

$$(TPv, w) = (SPv, (E-Q)w)_A = (TPv, (E-Q)w).$$

The operators $E-P$ and $E-Q$ project on the subspaces H_- and H_+ (see (2.9)), which are orthogonal relative to $(\cdot, \cdot)$ and invariant under T . Hence, $(TPv, (E-Q)w) = (Tv, (E-Q)w)$ and

$$(TPv, w) = (Tv, (E-Q)w),$$

which proves (3.16).

Theorem 3.4. Let $\phi \in H$ such that $P_+\phi \in \mathcal{H}_+$, then

$$(3.17) \quad FP\phi(v) = \sqrt{2} \phi(v) \eta_0(v) - \sum_{j=0}^{\infty} \gamma_j \eta_j(v) \int_{I_+} \mu \frac{\phi(\mu) - \phi(v)}{\mu - v} \zeta_j(\mu) d\mu, \\ \sigma\text{-a.e. } v \in I_+,$$

$$(3.18) \quad FP\phi(v) = - \sum_{j=0}^{\infty} \gamma_j \eta_j(v) \int_{I_+} \frac{\mu \phi(\mu)}{\mu - v} \zeta_j(\mu) d\mu, \quad v \in N_+ \setminus I_+,$$

and

$$FP\phi(v) = 0, \quad v \in N_- ,$$

where η_j and ζ_j are defined by (3.14).

Proof. Since $\ker P = H_-$ we may take $\phi \in \mathcal{H}_+$. Let ϕ_e be an extension of $\phi \in \mathcal{H}_+$ to $\mathcal{H}_+$ then with $\Gamma = \Gamma_+ \cup \Gamma_-$ as in (3.3),

$$P\phi = P\phi(T)e = \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) P(\lambda-T)^{-1} e d\lambda.$$

We have $P(\lambda-T)^{-1} - (\lambda-S)^{-1}P = -(\lambda-S)^{-1}\{P(\lambda-T) - (\lambda-S)P\}(\lambda-T)^{-1}$. Since T commutes with P_+ and $PP_+ = P$ one has $PT = PTP$. Moreover, P projects on H_p which is invariant under S , so $SP = PSP$. Therefore,

$$\begin{aligned} P(\lambda-T)^{-1} - (\lambda-S)^{-1}P &= -(\lambda-S)^{-1}P(S-T)P(\lambda-T)^{-1} \\ &= -(\lambda-S)^{-1}PBTP(\lambda-T)^{-1} \\ &= -(\lambda-S)^{-1}PB(E-Q)^*T(\lambda-T)^{-1}, \end{aligned}$$

with the last step a consequence of (3.16). Thus,

$$\begin{aligned} P\phi &= \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) (\lambda-S)^{-1} P e d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda) (\lambda-S)^{-1} PB(E-Q)^*T(\lambda-T)^{-1} e d\lambda \end{aligned}$$

and (3.17) then follows from Lemma 3.2 with $L = P$, $K = E-Q$ and $v = e$. (Note that $e = \sqrt{2} p_0$ and $\eta_j \in L_2(N_+, \sigma)$, $\zeta_j \in H_+$). For $v \in N_+ \setminus I_+$ we obtain

$$\begin{aligned} FP\phi(v) &= (\sqrt{2} \eta_0(v) + \sum_{j=0}^{\infty} \gamma_j \eta_j(v) \int_{I_+} \frac{\mu}{\mu-v} \zeta_j(\mu) d\mu) \phi_e(v) \\ &\quad - \sum_{j=0}^{\infty} \gamma_j \eta_j(v) \int_{I_+} \frac{\mu \phi(\mu)}{\mu-v} \zeta_j(\mu) d\mu. \end{aligned}$$

The extension $\phi \mapsto \phi_e$ being not unique we conclude to (3.18). For finite degree of anisotropy this conclusion also follows from formula (4.18) which shows that the coefficient of $\phi_e(v)$ in the equation above equals $\Lambda(v)$ and hence has to be zero.

Remark. Formulas (3.17) and (3.18) yield $FP\phi$ subject to the knowledge of the functions $\eta_j = FPp_j$ and $\zeta_j = (E-Q)p_j$. The formulas may be compared with the representation of F by (3.12), (3.13). The latter equations are expressed in terms of the functions $h_j = FPj$, which are known because of a recurrence

relation via which they can be calculated. Not having such a relation for η_j and ζ_j one is forced to obtain these functions as solutions of a set of singular integral equations derived from (3.9). For the functions η_j one derives these equations as follows. Take $P_+ = P_+P$ (cf. (2.9)) and write this relation as $P_+ = P_+P F^{-1}FP$ then from (3.9)

$$p_j(\mu) = \left(1 - \sum_{\ell=0}^{\infty} \gamma_{\ell} p_{\ell}(\mu) \int_N \frac{v h_{\ell}(v)}{v-\mu} d\sigma(v)\right) \eta_j(\mu) + \sum_{\ell=0}^{\infty} \gamma_{\ell} p_{\ell}(\mu) \int_{N_+} \frac{v h_{\ell}(v)}{v-\mu} \eta_j(v) d\sigma(v), \quad \mu \in I_+.$$

If the degree n of anisotropy is finite one has $n+1$ linear equations in the unknowns $\eta_0, \eta_1, \dots, \eta_n$. The actual solution can be simplified by an a prior determination of the form of the functions η_j . It is known and in Section 4 we shall show by different means that $\eta_j(v) = r_j(v)X(-v)$, $v \in N_+$ with r_j a polynomial of degree at most n for $j=0,1,\dots,n$, and $X(-v)$ the factor in the factorization (2.16) of the dispersion function Λ . It turns out that the functions ζ_j can also be expressed in terms of the same polynomials r_j and the function X .

Finally, we derive a representation for P_-P on H . The operator P_-P corresponds with Larsen's extension operator E (cf. [2]). Note that $P = P_+ + P_-P$.

Theorem 3.5. Let $v \in H$, then

$$(3.19) \quad P_-Pv(\mu) = \sum_{j=0}^{\infty} \gamma_j (-1)^j \zeta_j(-\mu) \int_{I_+} \frac{\mu_1}{\mu_1 - \mu} \zeta_j(\mu_1) v(\mu_1) d\mu_1, \text{ a.e. } \mu \in I_+,$$

where ζ_j is defined by (3.14).

Proof. For $v \in H$ and the integral interpreted as in (3.4) we have

$$P_-Pv = \frac{1}{2\pi i} \int_{\Gamma_-} (\lambda - T)^{-1} Pvd\lambda.$$

Since $PP_- = 0$ we also have

$$PP_-v = \frac{1}{2\pi i} \int_{\Gamma_-} P(\lambda - T)^{-1} vd\lambda = 0.$$

Hence,

$$(3.20) \quad P_- P v = \frac{1}{2\pi i} \int_{\Gamma_-} \{(\lambda-T)^{-1} P - P(\lambda-T)^{-1}\} v d\lambda.$$

From (2.11) we obtain

$$\begin{aligned} (\lambda-T)^{-1} P - P(\lambda-T)^{-1} &= (\lambda-T)^{-1} \{TP - PT\} (\lambda-T)^{-1} \\ &= (\lambda-T)^{-1} (E-P) TP (\lambda-T)^{-1}. \end{aligned}$$

On the other hand $(E-P)SP = 0$ according to (2.12). Substitution of $S = T+BT$ in this relation yields

$$(E-P)TP = -(E-P)BTP.$$

Thus, using equation (3.16) we get

$$\begin{aligned} (\lambda-T)^{-1} P - P(\lambda-T)^{-1} &= -(\lambda-T)^{-1} (E-P)BTP (\lambda-T)^{-1} \\ &= -(\lambda-T)^{-1} (E-P)B(E-Q)^* T (\lambda-T)^{-1}, \end{aligned}$$

and from (3.20)

$$P_- P v = -\frac{1}{2\pi i} \int_{\Gamma_-} (\lambda-T)^{-1} (E-P)B(E-Q)^* T (\lambda-T)^{-1} v d\lambda.$$

We substitute $B = \sum_{j=0}^{\infty} \gamma_j B_j$ (cf. the proof of Lemma 3.2) and obtain

$$TP_- P v(\mu) = -\sum_{j=0}^{\infty} \frac{1}{2\pi i} \left\{ \int_{\Gamma_-} T(\lambda-T)^{-1} (E-P)B_j(E-Q)^* T(\lambda-T)^{-1} v d\lambda \right\}(\mu)$$

with the series converging for a.e. $\mu \in I$. The summand of the series is represented by

$$\frac{1}{2\pi i} \gamma_j (-1)^j \zeta_j(-\mu) \int_{\Gamma_-} \frac{\mu}{\lambda-\mu} \int_{\Gamma_+} \frac{\mu_1}{\lambda-\mu_1} \zeta_j(\mu_1) v(\mu_1) d\mu_1 d\lambda$$

because of (3.14), (3.15). Interchanging the integrals and taking residues like in (3.7) we then arrive at (3.19).

4. The Functions η_j and ζ_j for Finite Degree of Anisotropy

In the case of a finite degree n of anisotropy additional information about the operator P can be obtained by studying the properties of the operator P_+SP when restricted to the subspace H_+ of H . We denote this restriction by $\hat{S}$ and the restriction of T to H_+ by T_+ . From (2.13) we obtain

$$(4.1) \quad \hat{S} = T_+ + P_+BTP,$$

i.e., $\hat{S}$ is obtained from T_+ by adding the perturbation P_+BTP . This perturbation is finite dimensional since $\text{ran}(P_+BTP) \subset \mathcal{P}_n(I_+)$. More precisely we infer from (4.1),

Lemma 4.1. The linear space $\mathcal{P}(I_+)$ of polynomial functions on I_+ is invariant under $\hat{S}$. If $\phi \in \mathcal{P}(I_+)$ and $\deg \phi = j$ then $\deg(\hat{S}\phi) = j+1$ if $j \geq n$ and $\deg(\hat{S}\phi) \leq n$ if $j=0,1,\dots,n-1$. In particular, $\hat{S}^{n-1}\mathcal{P}_{n-1}(I_+) \subset \mathcal{P}_{2n-2}(I_+)$ for $n=1,2,3,\dots$.

The operator $\hat{S}$ is diagonalized by the transform $FP: H_+ \rightarrow L_2(N_+, \sigma)$ since

$$(4.2) \quad FPP_+SP = FPSP = FSP = T_NFP$$

(cf. Theorem 1.2 and Section 2). An immediate consequence of (4.1) and (4.2) is that formulas (3.17) and (3.18) for FP can be obtained in a similar way as the formulas (3.12) and (3.13) for F . In fact, if $\phi \in \mathcal{A}_+$ with $\mathcal{A}_+$ considered as a subspace of H_+ , ϕ_e an extension of ϕ to $\mathcal{K}_+$ and for $e_+ = P_+e$,

$$\begin{aligned} \phi &= \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda)(\lambda - T_+)^{-1} e_+ d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda)(\lambda - \hat{S})^{-1} e_+ d\lambda \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda)(\lambda - \hat{S})^{-1} P_+BTP(\lambda - T_+)^{-1} e_+ d\lambda, \end{aligned}$$

where Γ is a simple closed contour around the spectrum N_+ of $\hat{S}$. Application of the operator P to the right- and left-hand sides of the latter equation yields

$$P\phi = \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda)(\lambda - S)^{-1} P e d\lambda - \frac{1}{2\pi i} \int_{\Gamma} \phi_e(\lambda)(\lambda - S)^{-1} P B(E-Q)^* T(\lambda - T)^{-1} e d\lambda,$$

since $\ker P = H_-$. This formula may be compared with the second equation for $P\phi$ occurring in the proof of Theorem 3.4.

Before proving lemma 4.2 we want to remark that an equivalent statement is true for the operator S in $\mathcal{P}(I)$, viz., that S has no eigenvectors in $\mathcal{P}(I)$. This property of S is a consequence of S having a cyclic vector (i.e., the function e) in $\mathcal{P}(I)$. For if we assume the existence of an eigenvector in $\mathcal{P}(I)$ corresponding to an eigenvalue λ then there would exist a nonzero polynomial $\tilde{\phi}$ such that $(\lambda - S)\tilde{\phi}(S)e = 0$. Since e separates the elements of the algebra of polynomials in S this would imply $(\lambda - S)\tilde{\phi}(S) = 0$ or, in other words, the existence of a minimal polynomial for S in the vector space $\mathcal{P}(I)$, which contradicts either the infinite dimensionality of $\mathcal{P}(I)$ or the existence of the cyclic vector e .

Lemma 4.2. The operator $\hat{S}$ has no eigenvectors in $\mathcal{P}(I_+)$.

Proof. The projection P_+ is a linear isomorphism from H_p onto H_+ and the operator P restricted to H_+ is its inverse. Hence, $\phi_+ \in H_+$ is an eigenvector of $\hat{S}$ if and only if S has an eigenvector $\phi \in H_p$ such that $\phi_+ = P_+\phi$. Moreover, we have shown in [12] that for finite degree of anisotropy the measure σ is nowhere discrete on I . Hence, I contains no eigenvalues of S and the eigenvectors of S in H_p are obtained from (3.10) to be constant multiples of

$$\phi_v(\mu) = \frac{1}{v-\mu} \sum_{j=0}^n \gamma_j p_j(\mu) h_j(v), \quad \mu \in I, \quad v \in N \setminus I_+.$$

This means that $P_+\phi_v \in \mathcal{P}(I_+)$ if and only if $\phi_v \in \mathcal{P}(I)$, which cannot be the case according to the paragraph preceding the lemma.

Remark. The fact that S has no polynomial eigenvectors together with formula (3.10) reaffirms that $M(\lambda) = \lambda \sum_{j=0}^n \gamma_j p_j(\lambda) h_j(\lambda)$ has no zeros in $N \setminus I$ (cf. [12], lemma 4).

Lemma 4.3. The operator $\hat{S}$ has a cyclic vector in $\mathcal{P}(I_+)$. The cyclic vector is unique up to a nonzero constant factor. It is in $\mathcal{P}_0(I_+)$ if $n=0$ and in $\mathcal{P}_{n-1}(I_+)$ if $n \geq 1$.

Proof. For $n=0$ the statement is a direct consequence of Lemma 4.1. Thus we may assume $n \neq 0$. For any nonzero $x \in \mathcal{P}(I_+)$ and any nonnegative integer m the set $\{x, \hat{S}x, \dots, \hat{S}^m x\}$ is linearly independent. If not, a nontrivial polynomial ϕ would exist such that $\phi(\hat{S})x = 0$ and, hence, $\hat{S}$ would have an eigenvector in $\mathcal{P}(I_+)$ contradicting Lemma 4.2.

We claim the existence of a $x \in \mathcal{P}_{n-1}(I_+)$ such that $\{x, \hat{S}x, \dots, \hat{S}^{n-1}x\}$ is a basis of $\mathcal{P}_{n-1}(I_+)$. If such an x did not exist then

$$\hat{S}^{n-1} \mathcal{P}_{n-1}(I_+) \cap \mathcal{P}_{n-1}(I_+) = \emptyset.$$

Hence, $\dim(\mathcal{P}_{n-1}(I_+) + \hat{S}^{n-1} \mathcal{P}_{n-1}(I_+)) = 2 \dim \mathcal{P}_{n-1}(I_+) = 2n$. But $\mathcal{P}_{n-1}(I_+) + \hat{S}^{n-1} \mathcal{P}_{n-1}(I_+) \subset \mathcal{P}_{2n-2}(I_+)$ by Lemma 4.1. Since $\dim \mathcal{P}_{2n-2}(I_+) = 2n-2$ a contradiction has been obtained.

With these results the remainder of the proof is an immediate consequence of Lemma 4.1.

Corollary. If x is a cyclic vector of $\hat{S}$ in $\mathcal{P}(I_+)$ and $\{d_0, d_1, \dots, d_{n-1}\}$ is a basis of $\mathcal{P}_{n-1}(I_+)$ then for any $\lambda \in \mathbb{C}$, $\{x, (\lambda - \hat{S})d_0, (\lambda - \hat{S})d_1, \dots, (\lambda - \hat{S})d_{n-1}\}$ is a basis of $\mathcal{P}_n(I_+)$.

We note that $e_+ = P_+ e$ is a cyclic vector of T_+ in $\mathcal{P}(I_+)$ and, analogously to the result stated in the corollary, $\{e_+, (\lambda - T_+)d_0, \dots, (\lambda - T_+)d_{n-1}\}$ is a basis of $\mathcal{P}_n(I_+)$ for any $\lambda \in \mathbb{C}$. We shall use these two bases in deriving formula (4.6) by direct calculation of the perturbation determinant for $\{\hat{S}, T_+\}$. As a preliminary we remark that for each $\phi \in \mathcal{P}_n(I_+)$

$$(4.3) \quad (\lambda - T_+)(\lambda - \hat{S})^{-1}\phi = \phi + P_+ BTP(\lambda - \hat{S})^{-1}\phi, \quad \lambda \notin N_+$$

is an element of $\mathcal{P}_n(I_+)$. Moreover, on H_+

$$(4.4) \quad (\lambda - T_+)(\lambda - \hat{S})^{-1} = \Omega_+^{-1}(\lambda), \quad \lambda \notin N_+$$

with $\Omega_+(\lambda)$ as defined by (2.15). Consequently,

$$(4.5) \quad (\lambda - T_+)(\lambda - \hat{S})^{-1}\phi = \phi - \sum_{j=0}^n \gamma_j p_{j+} \int_{N_+} \frac{v}{v-\lambda} (FP\phi)(v) h_j(v) d\sigma(v), \quad \lambda \notin N_+$$

for any $\phi \in H_+$ and $p_{j+} = P_+ p_j$.

Lemma 4.4. The operator $\hat{S}$ in H_+ has a cyclic vector $\eta \in \mathcal{P}(I_+)$ such that

$$(4.6) \quad (\lambda - T_+)(\lambda - \hat{S})^{-1}\eta = \frac{1}{X(\lambda)} e_+ + (\lambda - T_+) \sum_{i=0}^{n-1} \alpha_i(\lambda) d_i, \quad \lambda \notin N_+$$

where $X(\lambda)$ is the factor in the factorization of $\Lambda(\lambda)$ as given in (2.16) and $1/X(\lambda), \alpha_0(\lambda), \dots, \alpha_{n-1}(\lambda)$ are the coefficients in the expansion of the left member relative to the basis $\{e_+, (\lambda - T_+)d_0, \dots, (\lambda - T_+)d_{n-1}\}$ of $\mathcal{P}_n(I_+)$.

Proof. Since $\hat{S} - T_+$ is finite dimensional with range in $\mathcal{P}_n(I_+)$ the perturbation determinant for the pair $\{\hat{S}, T_+\}$ can be defined by

$$\det\{(\lambda - T_+)(\lambda - \hat{S})^{-1}\} = \det\{(\lambda - T_+)(\lambda - \hat{S})^{-1}| \mathcal{P}_n(I_+)\}, \quad \lambda \notin N_+,$$

where the right member indicates that one has to take the determinant of $(\lambda - T_+)(\lambda - \hat{S})^{-1}$ restricted to $\mathcal{P}_n(I_+)$. To evaluate the determinant on the right one can use the techniques of linear algebra. We use the definition of a determinant in terms of a multilinear functional. In fact, let Φ be a nonzero, alternating, $(n+1)$ -linear functional on $\mathcal{P}_n(I_+)$ and x be a cyclic vector for $\hat{S}$ in $\mathcal{P}(I_+)$ then from the corollary to Lemma 4.3 and (4.4),

$$(4.7) \quad \det\{(\lambda - T_+)(\lambda - \hat{S})^{-1}| \mathcal{P}_n(I_+)\} = \frac{\Phi\{\Omega_+^{-1}(\lambda)x, (\lambda - T_+)d_0, \dots, (\lambda - T_+)d_{n-1}\}}{\Phi\{x, (\lambda - \hat{S})d_0, \dots, (\lambda - \hat{S})d_{n-1}\}}.$$

Let the expansion of $\Omega_+^{-1}(\lambda)x$ relative to the basis $\{e_+, (\lambda - T_+)d_0, \dots, (\lambda - T_+)d_{n-1}\}$ of $\mathcal{P}_n(I_+)$ be given as

$$(4.8) \quad \Omega_+^{-1}(\lambda)x = \alpha(\lambda)e_+ + \sum_{i=0}^{n-1} \alpha_i(\lambda)(\lambda - T_+)d_i.$$

The sum over i in the right member is a linear combination of the last n

entries of Φ in the numerator in (4.7). Hence, the value of the numerator will not change if this sum is deleted when $\Omega_+^{-1}(\lambda)x$ is substituted according to (4.8). Thus, we obtain

$$(4.9) \quad \det\{(\lambda - T_+)(\lambda - S_+)^{-1}\} = \alpha(\lambda) \frac{\Phi\{e_+, (\lambda - T_+)d_0, \dots, (\lambda - T_+)d_{n-1}\}}{\Phi\{x, (\lambda - S_+)d_0, \dots, (\lambda - S_+)d_{n-1}\}}$$

The values of Φ occurring in this last relation are polynomials in λ of degree at most n . But since the arguments of Φ are bases of $\mathcal{P}_n(I_+)$ for any $\lambda \in \mathbb{E}$ these polynomials actually have to be nonzero constants. Because of (4.4) and since $\Omega_+^{-1}(\lambda) = E$ on H_- (cf. Section 2) we can replace the left member of (4.9) by $\det\{\Omega_+^{-1}(\lambda)\}$, i.e., by $(\det A)^{1/2} / X(\lambda)$ according to the definition of the function X given in Section 2. Hence,

$$\frac{(\det A)^{1/2}}{X(\lambda)} = \alpha(\lambda)k, \quad \lambda \notin N_+$$

with k some nonzero constant. Finally, the cyclic vector x being unique up to a constant factor we can choose $x = \eta$ in such a way that formula (4.6) holds.

Corollary.

$$(4.10) \quad \frac{1}{X(\lambda)} = \eta(\lambda) - \sum_{j=0}^n \gamma_j p_j(\lambda) \int_{N_+} \frac{v}{v-\lambda} \tilde{q}(v) h_j(v) d\sigma(v), \quad \lambda \notin N_+,$$

where $\tilde{q} = F P \eta \in L_2(N_+, \sigma)$.

Proof. Evaluate the left member of (4.6) according to (4.5) and take μ -values of the resulting polynomial functions on both sides of (4.6). Formula (4.10) follows by subsequent polynomial extension to $\mu = \lambda$.

Formula (4.10) enables us to determine $F P \eta$ by a technique which is more or less standard. We need three results from [12], section 4, viz., (i)

$$(4.11) \quad \frac{1}{A(\lambda)} = 1 - \sum_{j=0}^n \gamma_j p_j(\lambda) \int_N \frac{v}{v-\lambda} h_j(v) d\sigma(v), \quad \lambda \notin N,$$

(ii) σ is nowhere discrete on I , (iii) the polynomial $M(\lambda) = \lambda \sum_{j=0}^n \gamma_j p_j(\lambda) h_j(\lambda)$ has no zeros in $N \setminus I$ (cf. the Remark following Lemma 4.2).

Lemma 4.5. Let η be the cyclic vector for $\hat{S}$ in H_+ such that (4.6) holds, then

$$(4.12) \quad F\eta(v) = \begin{cases} X(-v), & \sigma\text{-a.e. } v \in N_+ \\ 0, & \sigma\text{-a.e. } v \in N_- \end{cases}.$$

Proof. We replace $1/X(\lambda)$ in the left member of (4.10) by $X(-\lambda)/\Lambda(\lambda)$ according to (2.16) and substitute for $\Lambda(\lambda)$ from (4.11) to obtain

$$\begin{aligned} X(-\lambda) \left(1 - \sum_{j=0}^n \gamma_j p_j(\lambda) \int_N \frac{v}{v-\lambda} h_j(v) d\sigma(v) \right) \\ = \eta(\lambda) + \sum_{j=0}^n \gamma_j p_j(\lambda) \int_{N_+} \frac{v}{v-\lambda} \tilde{q}(v) h_j(v) d\sigma(v), \quad \lambda \notin N. \end{aligned}$$

Let $[a, b]$ be an arbitrary closed interval on the positive part of the real axis. For $\varepsilon > 0$ let $\Gamma_{\pm\varepsilon} = \{s \pm i\varepsilon : s \in [a, b]\}$. Taking the integrals of both sides of (4.12) over $\Gamma_{\pm\varepsilon}$ and subsequently the limit $\varepsilon \rightarrow 0$ of the difference of these integrals one obtains

$$\sum_{j=0}^n \gamma_j \int_a^b v p_j(v) h_j(v) X(-v) d\sigma(v) = \sum_{j=0}^n \gamma_j \int_a^b v p_j(v) h_j(v) \tilde{q}(v) d\sigma(v)$$

since $X(-\lambda)$ is analytic in λ for $\lambda \notin I_-$. Hence,

$$M(v)(X(-v) - \tilde{q}(v)) = 0, \quad \sigma\text{-a.e. } v \in N_+ \setminus \{0\},$$

where M is the polynomial defined in the paragraph preceding the lemma. The properties (ii) and (iii) cited in that paragraph then yield $X(-v) = \tilde{q}(v)$, σ -a.e. $v \in N_+$, which is the nontrivial part of (4.12).

Theorem 4.1. Let $\eta_j = F\eta p_j$ and $\zeta_j = (E-Q)p_j$ for $j=0, 1, 2, \dots$, then

$$(4.13) \quad \eta_j(v) = \begin{cases} r_j(v)X(-v), & \sigma\text{-a.e. } v \in N_+ \\ 0, & \sigma\text{-a.e. } v \in N_- \end{cases}$$

and

$$(4.14) \quad \zeta_j(\mu) = \begin{cases} (-1)^j r_j(-\mu)/X(-\mu), & \text{a.e. } \mu \in I_+ \\ 0, & \text{a.e. } \mu \in I_- \end{cases}$$

where r_j is a polynomial with $\deg(r_j) \leq n-1$ if $j \leq n-1$ and $\deg(r_j) = j$ if $j \geq n$.

Proof. Formula (4.13) and the degree specifications of the polynomial r_j are immediate consequences of the Lemmas 4.3, 4.5 and 4.1. For $\lambda \notin N_+$ one has $P_+ p_j = r_j(\hat{S})\eta = (r_j(\hat{S}) - r_j(\lambda))\eta + r_j(\lambda)\eta$ with $(\lambda - \hat{S})^{-1}(r_j(\hat{S}) - r_j(\lambda))\eta \in \mathcal{P}_{m-1}(I_+)$, $m = \max\{j, n\}$. Thus, one obtains from (4.6),

$$(4.15) \quad (\lambda - T_+)(\lambda - \hat{S})^{-1} P_+ p_j = \frac{r_j(\lambda)}{X(\lambda)} e_+ + (\lambda - T_+) \phi_\lambda, \quad \lambda \notin N_+$$

with $\phi_\lambda \in \mathcal{P}_{m-1}(I_+)$. Evaluation of the left member according to (4.5) and subsequent polynomial extension to $\mu = \lambda$ like in the proof of the corollary to Lemma 4.4 yield

$$\frac{r_j(\lambda)}{X(\lambda)} = p_j(\lambda) - \sum_{\ell=0}^n \gamma_\ell p_\ell(\lambda) \int_{N_+} \frac{v}{v-\lambda} r_j(v) \tilde{q}(v) h_\ell(v) d\sigma(v), \quad \lambda \notin N_+.$$

Taking $\lambda = \mu \in [-1, 0)$ and using Theorem 3.1 with $v = Pp_j$ we deduce

$$\begin{aligned} \frac{r_j(\mu)}{X(\mu)} &= p_j(\mu) - P_- Pp_j(\mu) \\ &= P_-(E-P)p_j(\mu) = (E-P)p_j(\mu), \quad \text{a.e. } \mu \in I_- . \end{aligned}$$

The nontrivial part of (4.14) is then obtained from (3.15).

As a final result we derive an expression for $X(\lambda)$ in terms of the polynomials r_j .

Corollary.

$$(4.16) \quad X(\lambda) = \sqrt{2} r_0(\lambda) + \sum_{j=0}^n \gamma_j (-1)^j r_j(\lambda) \int_{I_+} \frac{\mu r_j(-\mu)}{\mu-\lambda} \frac{d\mu}{X(-\mu)}, \quad \lambda \notin I_+.$$

Proof. We write (4.6) in the form

$$(4.17) \quad (\lambda - S^{\wedge})(\lambda - T_+)^{-1} e_+ = X(\lambda) \eta \\ - (\lambda - S^{\wedge}) \sum_{i=0}^{n-1} X(\lambda) \alpha_i(\lambda) d_i, \quad \lambda \notin I_+,$$

with the validity at $\lambda \in N_+ \setminus I_+$ obtained by analytic extension. The left hand side of (4.17) is evaluated by the use of (4.1) and (3.16),

$$(\lambda - S^{\wedge})(\lambda - T_+)^{-1} e_+ = e_+ - P_+ B P (\lambda - T_+)^{-1} e_+ \\ = e_+ + \sum_{j=0}^n \gamma_j p_{j+} \int_{I_+} \frac{\mu \zeta_j(\mu)}{\mu - \lambda} d\mu.$$

Substitution of this result in (4.17) and subsequent application of FP yields for $\lambda \notin I_+$,

$$\sqrt{2} \eta_0(\nu) + \sum_{j=0}^n \gamma_j \eta_j(\nu) \int_{I_+} \frac{\mu \zeta_j(\mu)}{\mu - \lambda} d\mu \\ = X(\lambda) X(-\nu) - (\lambda - \nu) \sum_{i=0}^{n-1} X(\lambda) \alpha_i(\lambda) \text{FPd}_i(\nu), \quad \nu \in N_+.$$

The functions η_j and FPd_i being of the form $X(-\nu)$ times a polynomial in ν the validity of the last relation extends to $\nu \in \mathbb{C} \setminus I_-$. Taking $\nu = \lambda \in \mathbb{C} \setminus I$ we obtain

$$(4.18) \quad \Lambda(\lambda) = \sqrt{2} \eta_0(\lambda) + \sum_{j=0}^n \gamma_j \eta_j(\lambda) \int_{I_+} \frac{\mu \zeta_j(\mu)}{\mu - \lambda} d\mu, \quad \lambda \notin I,$$

and (4.16) follows from Theorem 4.1.

Remark. The results of this section as presented in Theorem 4.1 can be obtained in a more direct way if one considers the contour integral

$$(4.19) \quad \frac{1}{2\pi i} \int_{\Gamma_+} \tilde{\phi}(\lambda) X(-\lambda) (\lambda - S)^{-1} e d\lambda$$

with Γ_+ a contour as described in the first page of Section 3 and $\tilde{\phi}(\lambda)$ a polynomial in λ . Although the method has been presented by C. G. Lekkerkerker in [3] it is worthwhile to describe it briefly in this report.

The techniques used in Section 3 to obtain (3.9) can be adapted to show that (4.19) represents $F^{-1}\tilde{\phi}\tilde{q}$ where $\tilde{q}(v) = X(-v)$, $v \in N_+$; $\tilde{q}(v) = 0$, $v \in N_-$. On the other hand, one may replace the factor $(\lambda-S)^{-1}e$ in the integrand according to the formula

$$(\lambda-S)^{-1}e = \frac{1}{\Lambda(\lambda)} (\lambda-T)^{-1}e + \phi_\lambda$$

where $\phi_\lambda \in \mathcal{P}_{n-1}(I)$ (cf. [12], Lemma 1). Thus from (2.16) we obtain

$$(4.20) \quad F^{-1}\tilde{\phi}\tilde{q} = \frac{1}{2\pi i} \int_{\Gamma_+} \frac{\tilde{\phi}(\lambda)}{X(\lambda)} (\lambda-T)^{-1}e d\lambda + p(\tilde{\phi})$$

with $p(\tilde{\phi}) \in \mathcal{P}_{n-1}(I)$.

Taking point evaluations at $\mu \in I \setminus \{0\}$ and using the residue calculus in the region outside Γ_+ one obtains for the right member of (4.20) a polynomial ϕ in μ if $\mu \in (0,1]$ and $\phi(\mu) = \tilde{\phi}(\mu)/X(\mu)$ if $\mu \in [-1,0)$. Moreover, if $\deg(\tilde{\phi}) = j$ then $\deg(\phi) = j$ for $j \geq n$ and $\deg(\phi) \leq n-1$ for $j \leq n-1$. The correspondence $\tilde{\phi} \mapsto \phi$ is an isomorphism of the linear space of polynomials onto itself. Hence, for $\phi = p_j$ there does exist a polynomial $\tilde{\phi} = r_j$ such that

$$p_j(\mu) = (F^{-1}r_j\tilde{q})(\mu), \quad \mu \in (0,1]$$

and

$$p_j(\mu) - \frac{r_j(\mu)}{X(\mu)} = (F^{-1}r_j\tilde{q})(\mu), \quad \mu \in [-1,0).$$

These latter results immediately lead to the statement of Theorem 4.1.

Although the above described method immediately yields the desired results, it lacks a satisfactory a priori motivation why the integral in (4.19) is considered.

5. Addendum: Renormalization

The presence of the factor $1-c$ in formulas of time-independent transport theory may be compared with the occurrence of the factor 2π in the theory of the classical Fourier transform. Renormalization of certain basic quantities in both theories will lead these factors to disappear from some formulas but to turn up in others. Nevertheless, there is a difference between the roles of these two factors in the fact that c is a non-negative real parameter. The occurrence of $1-c$ serves as a reminder that the three cases, $0 \leq c < 1$, $c = 1$ and $c > 1$, are essentially different from a mathematical as well as a physical point of view. If one restricts oneself to the study of transport in absorbing media (i.e., the case $0 \leq c < 1$) as we did in this report then the normalization of the transform F , the functions h_j and the dispersion function Λ is a matter of taste and of no intrinsic importance. In this report we defined these quantities in a way which is consistent with our earlier publications and which we believe to be natural in the development of the theory. However, the consequence of the procedure is a rather annoying difference between our results and those published elsewhere. For this reason we present here the following conversion table. In this table the renormalized quantities are indicated by $^\circ$.

$$\text{Main quantities: } F^\circ = (1-c)F, \quad h_j^\circ = \frac{1-c}{1-c_j} h_j ,$$

$$\Lambda^\circ = (1-c)\Lambda , \quad \eta^\circ = (1-c)^{1/2} \eta .$$

$$\text{Secondary quantities: } X_\pm^\circ = (1-c)^{1/2} X_\pm , \quad \eta_j^\circ = \frac{1-c}{(1-c_j)^{1/2}} \eta_j ,$$

$$\zeta_j^\circ = (1-c_j)^{1/2} \zeta_j .$$

$$\text{Derived quantities: } \sigma^\circ = (1-c)^{-2} \sigma , \quad \tilde{q}^\circ = (1-c)^{1/2} \tilde{q} ,$$

$$\text{and } r_j^\circ = \frac{(1-c)^{1/2}}{(1-c_j)^{1/2}} r_j .$$

After renormalization and with omission of the indicator $^\circ$ the main formulas derived in this report and in [12] read as follows.

Full-range theory

$$Fe = (1-c)\tilde{e}, \quad h_j = FA^{-1}p_j.$$

$$\text{For } H_j = (j+1/2)^{-1/2} h_j,$$

$$(1.7) \quad (2j+1)(1-c_j)T_N H_j = (j+1)H_{j+1} + jH_{j-1}, \quad H_{-1} = 0, \quad H_0 = \tilde{e}.$$

(This recurrence relation is the same as [7], 4.10(9) for $\phi_{vj} = H_j(v)$.)

$$(1.9) \quad \det\{(\lambda-S)(\lambda-T)^{-1}\} = \Lambda(\lambda)\det A^{-1}$$

with

$$\begin{aligned} \Lambda(\lambda) &= 1-c + \sum_{j=0}^{\infty} c_j h_j(\lambda) \int_{-1}^{+1} \frac{\mu p_j(\mu)}{\mu-\lambda} d\mu \\ &= 1 + \lambda \sum_{j=0}^{\infty} c_j h_j(\lambda) \int_{-1}^{+1} \frac{p_j(\mu)}{\mu-\lambda} d\mu. \end{aligned}$$

(This dispersion function is the same as defined by [7], 4.10(18).)

$$\begin{aligned} [12], (14) \quad \frac{1}{\Lambda(\lambda)} &= \frac{1}{1-c} - \sum_{j=0}^{\infty} c_j p_j(\lambda) \int_N \frac{v h_j(v)}{v-\lambda} d\sigma(v) \\ &= 1 - \lambda \sum_{j=0}^{\infty} c_j p_j(\lambda) \int_N \frac{h_j(v)}{v-\lambda} d\sigma(v). \end{aligned}$$

$$(3.9) \quad F^{-1}\tilde{\phi}(\mu) = \frac{\tilde{\phi}(\mu)}{1-c} + \sum_{j=0}^{\infty} c_j p_j(\mu) \int_N v \frac{\tilde{\phi}(v) - \tilde{\phi}(\mu)}{v-\mu} h_j(v) d\sigma(v)$$

And rather easily from this since $A = E-C$,

$$AF^{-1}\tilde{\phi}(\mu) = \tilde{\phi}(\mu) + \mu \sum_{j=0}^{\infty} c_j p_j(\mu) \int_N \frac{\tilde{\phi}(v) - \tilde{\phi}(\mu)}{v-\mu} h_j(v) d\sigma(v).$$

$$(3.12) \quad F\phi(v) = (1-c)\phi(v) - \sum_{j=0}^{\infty} c_j h_j(v) \int_I \mu \frac{\phi(\mu) - \phi(v)}{\mu-v} p_j(\mu) d\mu$$

and from this with $A^{-1} = E+B$ (cf. (1.5))

$$FA^{-1}\phi(v) = \phi(v) - v \sum_{j=0}^{\infty} c_j h_j(v) \int_I \frac{\phi(\mu) - \phi(v)}{\mu-v} p_j(\mu) d\mu.$$

Moreover, for anisotropic scattering of finite degree

$$[12], \text{Th.3} \quad d\sigma(v) = \frac{dv}{\Lambda^+(v)\Lambda^-(v)}, \quad v \in I \quad \text{and}$$

$$\sigma(v_i) = \frac{1}{M(v_i)} \text{Res}\left[\frac{1}{\Lambda(\lambda)}\right]_{\lambda=v_i}, \quad v_i \in N \setminus I,$$

$$\text{where} \quad \Lambda^\pm(v) = \lim_{\epsilon \rightarrow 0} \Lambda(v \pm i\epsilon)$$

$$\text{and} \quad M(\lambda) = \lambda \sum_{j=0}^n c_j p_j(\lambda) h_j(\lambda).$$

(For anisotropic scattering of finite degree and fixed $v \in I$ the number $(\Lambda^+(v)\Lambda^-(v))^{-1} F\phi(v)$ equals the expansion coefficient $A(v) = \frac{1}{N(v)} \int \mu \phi(\mu) \phi_v(\mu) d\mu$ in [7], 4.6(17) with ϕ_v and $N(v)$ given in section [7], 4.10.)

Half-range theory

$$X_\pm(\lambda) = \{\det A\}^{1/2} \det \Omega_\pm(\lambda).$$

$$(3.17) \quad F\phi(v) = 2^{1/2} (1-c)^{1/2} \phi(v) \eta_0(v) - \sum_{j=0}^{\infty} c_j \eta_j(v) \int_{I_+} \mu \frac{\phi(\mu) - \phi(v)}{\mu - v} \zeta_j(\mu) d\mu.$$

$$(3.19) \quad P_- P v(\mu) = \sum_{j=0}^{\infty} c_j (-1)^j \zeta_j(-\mu) \int_{I_+} \frac{\mu_1}{\mu_1 - \mu} \zeta_j(\mu_1) v(\mu_1) d\mu_1.$$

For anisotropic scattering of finite degree,

$$(4.10) \quad \frac{1}{X(\lambda)} = \eta(\lambda) - \sum_{j=0}^n c_j p_j(\lambda) \int_{N_+} \frac{v}{v - \lambda} \tilde{q}(v) h_j(v) d\sigma(v),$$

where $\tilde{q} = F P \eta$ and

$$(4.12) \quad \tilde{q}(v) = \begin{cases} X(-v), & v \in N_+ \\ 0, & v \in N_- \end{cases}.$$

$$(4.13) \quad \eta_j(v) = \begin{cases} r_j(v) X(-v) & , \quad v \in N_+ \\ 0 & , \quad v \in N_- \end{cases} \quad \text{and}$$

$$(4.14) \quad \zeta_j(\mu) = \begin{cases} (-1)^j r_j(-\mu) / X(-\mu), & \mu \in I_+ \\ 0 & , \quad \mu \in I_- \end{cases}$$

$$(4.16) \quad X(\lambda) = 2^{1/2} (1-c)^{1/2} r_0(\lambda) + \sum_{j=0}^n c_j (-1)^j r_j(\lambda) \int_{I_+} \frac{\mu r_j(-\mu)}{\mu - \lambda} \frac{d\mu}{X(-\mu)}.$$

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