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A GRAPH-THEORETIC ALGORITHM
FOR HIERARCHIAL
DECOMPOSITION OF DYNAMIC SYSTEMS

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MARCH 24, 1982

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A GRAPH-THEORETIC ALGORITHM FOR HIERARCHIAL
DECOMPOSITION OF DYNAMIC SYSTEMS

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ABSTRACT

A graph-theoretic scheme is proposed for partitioning of dynamic systems into hierarchially ordered subsystems having independent inputs and outputs. The resulting subsystems are input-output reachable as well as structurally controllable and observable, so that a "piece-by-piece" design of estimators and controllers can be accomplished for systems with large dimensions without excessive computer requirements.

1. INTRODUCTION

"Decomposition Principle" stands for a variety of more or less disparate problems and techniques which are held together by a common theme: A large system of equations is broken up into subsystems which are then solved separately and put together to produce a global solution (see, for example, [1-3]). The actual tearing of the system may be performed in any number of ways in order to achieve computational and conceptual simplifications of the overall problem. Most efficient tearing techniques so far has been those formulated in the graph-theoretic setting because of the convenience in computer implementations. The partitioning algorithms take advantage of the inherent structure of the system to decompose it into a number of irreducible hierarchially ordered subsystems, which can subsequently be solved from top to bottom independently of each other. This approach has been used to solve a system of algebraic equations with several thousand nodes without requiring excessive computation time [4].

In order to reproduce in the dynamic systems context some of the benefits of structural insights and simplifications achieved in solving large algebraic systems, directed graphs (digraphs) has been associated with large-scale dynamic systems [5], and various stability and decentralized control problems has been solved in this way [3]. This prompted a number of efforts to introduce a hierarchial decomposition of dynamic systems using graph-theoretic methods [6-9], and come up with simplified stability problems. The underlying idea has been to

mimic the approach used in the algebraic setting and identify the irreducible dynamic subsystems with *strong components* of the corresponding digraph. Since to test stability of the resulting hierarchially ordered system one needs to test only stability of the subsystems, the approach can bring about a considerable dimensionality reduction of stability problems, especially in sparse dynamic systems.

A decomposition of a dynamic system along the lines of strong components with respect to states assigns a secondary role to inputs and outputs. Thus, the strong component partitions are not particularly suitable for solving control and estimation problems. For this reason, in [10], *input reachability* has been utilized in an essential way to develop a more appropriate concept for such problems, whereby the strong components of the corresponding digraph, have been replaced by input reachable and structurally controllable subsystems. Once a hierarchial ordering of input reachable (structurally controllable) subsystems is obtained, stabilization of the overall system is accomplished by stabilizing a number of low-order subsystems.

Duals of irreducible partitions of dynamic systems into input reachable subsystems, can be used to obtain such partitions into output reachable subsystems for purposes of estimation and filtering. If dimensionality reduction of both the estimation and control problems is required, duality does not work in general, and input and output reachability of each subsystem has to be achieved simultaneously. This, however, requires a major reformulation of the available decomposition

scheme for input reachable irreducible partitions. The major objective of this work is to develop *input-output reachable acyclic decompositions* of dynamic systems, which can be used for efficient and numerically attractive schemes for "piece-by-piece" design of estimators and controllers for large systems.

The organization of the paper is as follows: In the next section, we associate directed graphs with continuous- and discrete-time dynamic systems, and introduce the necessary notation and terminology. In Section 3, we present a scheme for acyclic decomposition of dynamic systems into input-output reachable subsystems. The decomposition is achieved by permuting the variables of the system until the system matrices become simultaneously block-triangular matrices with blocks having compatible dimensions. We then describe in Section 4 the decomposition algorithm, and discuss various aspects of the algorithm, such as nonuniqueness of partitions, row and column alternatives, use of graph condensations, input and output reachable partitions, as well as structural controllability and observability of the subsystems. Section 5 is devoted to applications. Here, we show how the decomposition scheme can be used in "piece-by-piece" designs of filters, dynamic controllers, and observers for large dynamic systems.

2. DIRECTED GRAPHS AND DYNAMIC SYSTEMS

Before we associate directed graphs with dynamic systems, let us recall several standard definitions of graph theory (e.g., Harary et al. [11]). A *directed graph (digraph)* is a pair $D = (V, E)$, where V is a set of vertices and E is a set of edges (v_j, v_i) directed from the vertex v_j to the vertex v_i . If $(v_j, v_i) \in E$ then $v_j(v_i)$ is said to be adjacent to v_i (adjacent from v_j). Adjacency relation defines a square binary matrix $R = (r_{ij})$ where $r_{ij} = 1$ if and only if $(v_j, v_i) \in E$, which is called the adjacency or *interconnection matrix*. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)\}$ where all the vertices are distinct, is called a path from v_1 to v_k . When v_k coincides with v_1 , then the path is a cycle. A digraph D is said to be *acyclic* if it contains no cycles. If there is a path from v_j to v_i , then we say that v_i is *reachable* from v_j . Similarly, a subset $V_i \subset V$ is reachable from a subset $V_j \subset V$ if every vertex in V_i is reachable from some vertex in V_j . We say that $V_j \subset V$ reaches $V_i \subset V$ if every vertex in V_j reaches at least one vertex V_i . Like adjacency, the reachability relation on V can be represented by a binary matrix $\bar{R} = (\bar{r}_{ij})$ such that $\bar{r}_{ij} = 1$ if and only if v_i is reachable from v_j .

We consider a *subgraph* $D_k = (V_k, E_k)$ of D to be a digraph in which $V_k \subset V$ and $E_k \subset E$ such that for any $v_i, v_j \in V_k$, $(v_j, v_i) \in E_k$ if and only if $(v_j, v_i) \in E$. A maximal subgraph of D in which any two vertices are mutually reachable is called a *strong component* of D . We define a *partition* P of a digraph D as a set of subgraphs $D_k = (V_k, E_k)$, $k = 1, 2, \dots, s$, such that

$V_i \cap V_j = \emptyset$, $i \neq j$, and $V = \bigcup_{k=1}^S V_k$. A condensation of $\mathcal{D}$ with respect to the partition P is a digraph $\mathcal{D}^* = (V^*, E^*)$, where $V^* = \{v_1^*, v_2^*, \dots, v_S^*\}$, $(v_j^*, v_i^*) \in E^*$ if and only if $v_i^* \neq v_j^*$, and at least one vertex in V_j is reachable from some vertex in V_i . As in [10], our definitions of partitions and condensations are not standard because the subgraphs $\mathcal{D}_k$ are not required to be strong components of $\mathcal{D}$. This modification, however, is essential in the subsequent analysis of dynamic systems using directed graphs.

Let us consider now a linear dynamic system

$$\begin{aligned} S: z(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \quad (2.1)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, and $y(t) \in R^l$ is the output of S at time $t \in T$, and A, B, C are constant matrices of appropriate dimensions. In (2.1), $z(t) \in R^n$ represents the change of the state $x(t)$ in time. When $z(t) \equiv dx(t)/dt$, T is the nonnegative real line R_+ , then S is a continuous-time dynamic system. In case $z(t) \equiv x(t+1)$, T is a set of nonnegative integers $\{0, 1, 2, \dots\}$, then S is a discrete-time system.

With the dynamic system S we associate a digraph $\mathcal{D} = (U \cup X \cup Y, E)$ such that $U = \{u_1, u_2, \dots, u_m\}$, $X = \{x_1, x_2, \dots, x_n\}$, and $Y = \{y_1, y_2, \dots, y_l\}$ are the sets of input, state, and output vertices. For the digraph $\mathcal{D}$, (x_j, x_i) , (u_j, x_i) , $(x_j, y_i) \in E$ if and only if $a_{ij} \neq 0$, $b_{ij} \neq 0$, $c_{ij} \neq 0$, respectively. The corresponding $(n+m+l) \times (n+m+l)$ interconnection matrix R has the form

$$R = \begin{array}{ccc|c} & X & U & Y \\ \begin{bmatrix} E & F & 0 \\ 0 & 0 & 0 \\ G & 0 & 0 \end{bmatrix} & X & U & Y \end{array} \quad (2.2)$$

where E , F , and G are state, input, and output interconnection matrices of appropriate dimension. The reachability matrix $\bar{R}$, which has the same dimension and form as R above,

$$\bar{R} = \begin{array}{ccc|c} & X & U & Y \\ \begin{bmatrix} \bar{E} & \bar{F} & 0 \\ 0 & 0 & 0 \\ \bar{G} & \bar{H} & 0 \end{bmatrix} & X & U & Y \end{array} \quad (2.3)$$

can be computed directly from the matrix R as shown in [3].

We also recall from [3] that $\mathcal{D}$ is said to be *input reachable* if X is reachable from U , and *output reachable* if X reaches Y . From (2.3), $\mathcal{D}$ is input reachable if and only if $\bar{F}$ has no zero rows, and output reachable if and only if $\bar{G}$ has no zero columns. When $\mathcal{D}$ is both input and output reachable, we say it is *IO-reachable*.

In terms of the associated digraph $\mathcal{D} = (U \cup X \cup Y, E)$, the above decomposition of S corresponds to a partitioning of $\mathcal{D}$ into subgraphs $\mathcal{D}_i = (U_i \cup X_i \cup Y_i, E_i)$ identified by the triplets (E_{ii}, F_{ii}, G_{ii}) of (3.3). These subgraphs are the digraphs associated with the decoupled subsystems S_i of (3.2).

For purposes of control and estimation, it is necessary that all subsystems S_i are both controllable and observable. Since controllability and observability are not purely structural properties, they are not suitable for applications of graph-theoretic methods and must be replaced by input and output reachability of $\mathcal{D}_i$'s. An *a posteriori* analysis which is needed to ensure that IO-reachability of $\mathcal{D}_i$'s implies controllability and observability of S_i 's, is postponed until after a decomposition scheme is presented.

Following [10], we introduce:

(3.4) DEFINITION. A partition P_{IO} of a digraph $\mathcal{D} = (U \cup X \cup Y, E)$ is said to be an *acyclic IO-reachable partition* if:

- (i) the interconnection matrix R has form (3.3);
- (ii) each subgraph $\mathcal{D}_i$ is IO-reachable.

An obvious necessary condition for existence of IO-reachable partitions is that the digraph $\mathcal{D}$ itself be IO-reachable. If

3. ACYCLIC DECOMPOSITIONS

We consider the problem of partitioning the system S of (2.1) as

$$\begin{aligned} S: z_i(t) &= \sum_{j=1}^i A_{ij}x_j(t) + \sum_{j=1}^i B_{ij}u_j(t) \\ y_i(t) &= \sum_{j=1}^i C_{ij}x_j(t), \quad i = 1, 2, \dots, s \end{aligned} \quad (3.1)$$

which represents an interconnection of s hierarchially ordered subsystems

$$\begin{aligned} S_i: z_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) \\ y_i(t) &= C_{ii}x_i(t), \quad i = 1, 2, \dots, s \end{aligned} \quad (3.2)$$

where $x_i(t) \in R^{n_i}$, $u_i(t) \in R^{m_i}$, and $y_i(t) \in R^{\ell_i}$ are the state, input, and output of the subsystem S_i , and $R^n = R^{n_1} \times R^{n_2} \times \dots \times R^{n_s}$, $R^m = R^{m_1} \times R^{m_2} \times \dots \times R^{m_s}$, $R^\ell = R^{\ell_1} \times R^{\ell_2} \times \dots \times R^{\ell_s}$.

The decoupled subsystems S_i of (3.2) are obtained simply from (3.1) by setting $A_{ij} = B_{ij} = C_{ij} = 0$, $j \neq i$.

The decomposition (3.1) is achieved by a permutation of the variables of S , which brings the matrices A , B , C of (2.1) into a compatible block-triangular form. Accordingly, the interconnection matrix R of (2.2) becomes

$$R = \left[\begin{array}{cc|cc|cc} E_{11} & & & & F_{11} & & & \\ \vdots & & & & \vdots & & & \\ E_{s1} & \dots & E_{ss} & & F_{s1} & \dots & F_{ss} & \\ \hline & & & & & & & \\ & & & & & & & \\ G_{11} & & & & & & & \\ \vdots & & & & & & & \\ G_{s1} & \dots & G_{ss} & & & & & \end{array} \right] \quad (3.3)$$

this is not the case, the decomposition procedure can easily be modified to remove the unreachable parts of $\mathcal{D}$ before the decomposition is started. To avoid trivial cases, we assume that each input vertex of $\mathcal{D}$ is adjacent to at least one state vertex, and each output vertex is adjacent from at least one state vertex. These assumptions imply that $\mathcal{D}$ itself is a trivial partition P_{IO} with $s=1$. To consider nontrivial partitions of $\mathcal{D}$, we assume that $n > 1$, $m > 1$, and $\ell > 1$.

Let us assume that $\mathcal{D} = (U \cup X \cup Y, E)$ has an acyclic partition into two subgraphs $\mathcal{D}_i = (U_i \cup X_i \cup Y_i, E_i)$, $i=1, 2$. Then, the reachability matrix $\bar{R}$ of $\mathcal{D}$ can be obtained from the interconnection matrix R of (3.3) with $s=2$ by direct computation [3] as

$$\bar{R} = \begin{array}{c} \begin{array}{cc|cc|cc} X_1 & X_2 & U_1 & U_2 & Y_1 & Y_2 \\ \hline \bar{E}_{11} & 0 & \bar{F}_{11} & 0 & 0 & 0 \\ \bar{E}_{21} & \bar{E}_{22} & \bar{F}_{21} & \bar{F}_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \bar{G}_{11} & 0 & \bar{H}_{11} & 0 & 0 & 0 \\ \bar{G}_{21} & \bar{G}_{22} & \bar{H}_{21} & \bar{H}_{22} & 0 & 0 \end{array} \begin{array}{l} X_1 \\ X_2 \\ U_1 \\ U_2 \\ Y_1 \\ Y_2 \end{array} \end{array} \quad (3.5)$$

where, by assumption, $\bar{F}_{11}$ and $\bar{F}_{22}$ have no zero rows, $\bar{G}_{11}$ and $\bar{G}_{22}$ have no zero columns, and $\bar{H}_{11}$ and $\bar{H}_{22}$ each have at least one nonzero element. From (3.5), we observe that a necessary condition for the existence of an acyclic IO-reachable partition is that the input-output reachability submatrix $\bar{H}$ contains at least one zero element. In [10], such a simple

condition on $\bar{F}$ was necessary and sufficient for existence of acyclic input reachable partitions of $\mathcal{D}$. For IO-reachability, this is not even close to sufficiency, and we proceed to consider a Boolean "OR" operation on the rows and columns of $\bar{R}$ corresponding to U_1, U_2, Y_1 , and Y_2 . This yields a reduced reachability matrix

$$\bar{R} = \begin{array}{c} \begin{array}{cc|cc|cc} x_1 & x_2 & \tilde{u}_1 & \tilde{u}_2 & \tilde{y}_1 & \tilde{y}_2 \\ \hline X & 0 & e_1 & 0 & 0 & 0 \\ X & X & X & e_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline e_1^T & 0 & 1 & 0 & 0 & 0 \\ X & e_2^T & X & 1 & 0 & 0 \end{array} \begin{array}{l} x_1 \\ x_2 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{array} \end{array} = \begin{bmatrix} \bar{E} & \tilde{F} & 0 \\ 0 & 0 & 0 \\ \tilde{G} & \tilde{H} & 0 \end{bmatrix} \quad (3.6)$$

where $e_i = (1, 1, \dots, 1)^T$, $i = 1, 2$, and "X" denotes submatrices or elements that are of no concern at this point. Finally, taking the Boolean "AND" of $\tilde{F}$ and $\tilde{G}^T$ in (3.6), that is, $\tilde{S} = \tilde{F} \wedge \tilde{G}^T$, we can get the matrix

$$\tilde{S} = \begin{bmatrix} (\tilde{u}_1 \wedge \tilde{y}_1) & (\tilde{u}_2 \wedge \tilde{y}_2) \\ e_1 & 0 \\ 0 & e_2 \end{bmatrix}, \quad (3.7)$$

and state the following:

(3.8) THEOREM. A given digraph $\mathcal{D}$ has a P_{IO} partition if and only if U and Y can be partitioned into U_1, U_2 and Y_1, Y_2 such that the matrix $\tilde{H}$ has the form in (3.6), and the matrix $\tilde{S}$ contains exactly one 1 in each row.

PROOF. Necessity is obvious from the construction of the matrices $\tilde{R}$ and $\tilde{S}$. On the other hand, if the matrix $\tilde{S}$ has exactly one 1 in each row, defining $X_j = \{x_i \in X \mid \tilde{S}_{ij} = 1, i = 1, 2, \dots, n\}$, $j = 1, 2$, we obtain a partition of X into X_1 and X_2 . A permutation of X according to this partition puts $\tilde{S}$ into the form (3.7), which implies that X_j is reachable from U_j and reaches Y_j . Furthermore, the form of $\tilde{R}$ in (3.6) implies that neither U_2 reaches X_1 nor X_2 reaches X_1 or Y_1 . Thus, we obtain a P_{IO} partition of $\mathcal{D}$ into $\mathcal{D}_j = \{U_j \cup X_j \cup Y_j, E_j\}$, $j = 1, 2$. This completes the sufficiency and the proof.

To get the maximum benefit from the decomposition, we would like to obtain irreducible P_{IO} partitions of $\mathcal{D}$, in which no subgraph $\mathcal{D}_i$ can be further decomposed into acyclic IO-reachable subgraphs. For this purpose we introduce:

(3.9) DEFINITION. We say that in the input-output reachability $\ell \times m$ matrix $\bar{H} = (\bar{h}_{ij})$ of (2.3), the p -th row dominates q -th row if $\bar{h}_{pj} = 1$ whenever $\bar{h}_{qj} = 1$, and for at least one j , $\bar{h}_{pj} = 1$, but $\bar{h}_{qj} = 0$, $j = 1, 2, \dots, m$.

Now, the following result is automatic:

(3.10) THEOREM. Under the conditions of Theorem (3.7), the subgraph $\mathcal{D}_1$ of the P_{IO} partition induced by the matrix $\tilde{S}$ is irreducible if and only if no row of the matrix $\bar{H}$ corresponding to Y_1 dominates any other row corresponding to Y_2 .

In other words, if we start the partitioning of the digraph $\mathcal{D}$ from those rows of $\bar{H}$ which are not dominated by any other

row of $\bar{H}$, we guarantee irreducibility of the first digraph $\mathcal{D}_1$ of the resulting partition P_{IO} , provided such a partition exists. Once, such a partition is obtained, the algorithm can be applied successively to the remaining subgraph $\mathcal{D}_2$ to generate the irreducible partitions of the entire digraph $\mathcal{D}$.

4. DECOMPOSITION ALGORITHM

The following are the steps of an algorithm for generating all irreducible acyclic partitions of a given digraph $\mathcal{D}$:

Step 1: Compute the reachability matrix $\bar{R}$ of $\mathcal{D}$ using any standard method [12].

Step 2: If the input-output reachability matrix $\bar{H}$ contains no zero elements, STOP ($\mathcal{D}$ has no acyclic partitions). Otherwise, SET $k \leftarrow 0$, $j \leftarrow 0$ (k is the number of distinct acyclic partitions of $\mathcal{D}$).

Step 3: (a) If $j = \ell$, STOP. Otherwise, SET $i \leftarrow j + 1$.
 (b) If row i of $\bar{H}$ is labeled, SET $j \leftarrow j + 1$ and GO TO 3(a).
 (c) If row i of $\bar{H}$ dominates any other row labeled (+), LABEL row i (+), SET $i \leftarrow j + 1$ and GO TO 3(b).
 (d) If row i of $\bar{H}$ dominates an unlabeled row p , SET $i \leftarrow p$, and GO TO 3(c).

Step 4: (a) Define the index sets: $I = \{p = 1, 2, \dots, \ell \mid \text{row } j \text{ of } \bar{H} \text{ dominates row } p\}$; $J = \{j = 1, 2, \dots, m \mid \bar{h}_{ij} = 1\}$.
 (b) Define the partitions of U and Y
 $U_1 = \{u_j \in U \mid j \in J\}$, $U_2 = U - U_1$;
 $Y_1 = \{y_j \in Y \mid j \in I\}$, $Y_2 = Y - Y_1$.

Step 5: Construct the reduced reachability matrix $\tilde{R}$ of (3.6) based on the above partitions of U and Y , and compute the matrix $\tilde{S} = \tilde{F} \wedge \tilde{G}^T$.

Step 6: If the matrix $\tilde{S}$ has one and only one nonzero element in each row, then:

- (a) Define $X_1 = \{x_p \mid x_p \text{ first column of } \tilde{S} \text{ contains a nonzero element in } p\text{-th row}\}$,
 $X_2 = X - X_1$.
- (b) SET $k \leftarrow k + 1$, and OUTPUT $\mathcal{D}_1^k = \{U_1 \cup X_1, U_1 \cup Y_1, E_1\}$.
- (c) LABEL row i of $\bar{H} (+)$, and GO TO 3(a).

Otherwise, LABEL row i of $\bar{H} (-)$, and GO TO 3(a).

Now, several remarks are in order:

(i) Nonuniqueness of Partitions.

From the description of the algorithm, we observe that neither the irreducible acyclic partitions of $\mathcal{D}$ nor the number of subgraphs in such partitions need be unique. For example, the digraph of Fig. 1 has two distinct decompositions, one having two and the other three components, as illustrated in Fig. 1a and b.

(ii) Row and Column Alternatives.

Although the algorithm operates on the rows of the input-output reachability matrix $\bar{H}$, it is also possible to generate all irreducible acyclic partitions by applying the dual procedure to the columns of $\bar{H}$. The only difference between the two approaches is that in the former case, those subgraphs that are at the top of the hierarchy (namely, $\mathcal{D}_1$'s) are first determined, while in the latter case, those subgraphs that are at the bottom are first selected. Depending on the number of input and output

Fig. 1. Two distinct decompositions

vertices of $\mathcal{D}$ one approach may be preferred to the other.

(iii) Condensation.

As is the case in the input reachability considerations [], the computations involved in the above algorithm can be considerably reduced by considering a condensation of $\mathcal{D}$ instead of $\mathcal{D}$ itself. For this purpose, we partition the vertex sets of $\mathcal{D}$ as

$$X = \bigcup_{i=1}^p X_i^*, \quad U = \bigcup_{i=1}^q U_i^*, \quad Y = \bigcup_{i=1}^r Y_i^* \quad (4.1)$$

where X_i^* 's are the vertex sets of the strong components of the state truncation subgraph $\mathcal{D}_x = (X, E_x)$ [], and each $U_i^*(Y_i^*)$ contains all the input (output) vertices that are adjacent to (from) those state vertices that occur in the same strong component of $\mathcal{D}_x$, but no others. If we denote the condensation of $\mathcal{D}$ with respect to the above partition by $\mathcal{D}^* = (U^* \cup X^* \cup Y^*, E^*)$, then it is not difficult to show that to any irreducible acyclic partition of $\mathcal{D}$ into s subgraphs, there corresponds a unique irreducible acyclic partition of $\mathcal{D}^*$ into the same number of subgraphs, and conversely. Intuitively, this follows from the fact that any two state vertices of $\mathcal{D}$, that belong to the same strong component of the truncation of $\mathcal{D}_x$, should occur in the same subgraph of any irreducible acyclic decomposition of $\mathcal{D}$, as well as any two input (output) vertices that are adjacent exactly to (from) the same strong component of $\mathcal{D}_x$. A complete proof of a similar result can be found in [10].

The fact outlined above simply states that the decomposition problems for $\mathcal{D}$ and $\mathcal{D}^*$ are interchangeable. However, since $\mathcal{D}^*$

Fig. 2. A digraph without IO-reachable decomposition

is considerably smaller than $\mathcal{D}$, irreducible acyclic partitions of $\mathcal{D}$ can be generated much more easily by first obtaining $\mathcal{D}^*$ and then applying the decomposition scheme to $\mathcal{D}^*$. An additional advantage of using such an approach is that the strong components of each subgraph $\mathcal{D}_k$ of the decoupled subsystems would also be identified as a by-product of the decomposition scheme, allowing for a further simplification of the subsequent design procedure.

(iv) Input and Output Reachable Partitions.

In certain cases, we may not be interested in the outputs of the system S , and only a decomposition of S into input reachable subsystems is desired. Such decompositions can easily be generated by appending an output equation $y = x$ to the first equation of (2.1), and applying the algorithm to the digraph of the resulting system. It is obvious that our decomposition algorithm would produce an acyclic input reachable partition which has the form of (3.1) with $C_{ii} = I_i$ and $C_{ij} = 0$, $i \neq j$, and I_i is the $n_i \times n_i$ identity matrix. Similar arguments can be made for output reachable partitions.

In general, decompositions of a digraph into IO-reachable subgraphs are more restrictive than decompositions into only input (output) reachable subgraphs without considerations of outputs (inputs). For example, the digraph of Fig. 2a has no IO-reachable partitions, but it does have an input reachable partition as shown in Fig. 2b, as well as an output reachable partition as illustrated by Fig. 2c. This observation suggests that each subgraph $\mathcal{D}_k$ of an IO-reachable partition can be further decomposed into input (output) reachable subgraphs. This way

we obtain a multilevel partition of $\mathcal{D}$ as illustrated by the corresponding interconnection matrix of Fig. 3.

(v) Structural Controllability and Observability.

We recall that our main objective has been to obtain an acyclic decomposition of S into controllable and observable subsystems. In the context of graph theory, it is natural to replace these properties by their structural counterparts: Structural controllability and observability [13]. A possible loss of controllability or observability of a structurally controllable and observable system can occur only in pathological cases when there is an exact matching of system parameters. We further note that once input and output reachability of the subsystems are established by our decomposition algorithm, structural controllability and observability of the subsystems follow provided their associated subgraphs contain no dilation [14]. Without going into details, it suffices to mention that an input (output) reachable digraph contains a dilation only when it does not have sufficient number of edges. Thus, if a subgraph turns out to contain a dilation causing structural uncontrollability, it can be eliminated by adding the subgraph with one or more subgraphs of the upper level of the hierarchy, thus making use of the additional edges gained by the combination process. Similarly, a dilation causing structural unobservability can be eliminated by combining the subgraph with the lower level subgraphs in the hierarchy. Provided the overall system is structurally controllable and observable, this process leads eventually to a partition with structurally controllable and observable subsystems.

Fig. 3. Interconnection matrix

In the case of input reachability and structural controllability, such an approach is described in detail in [10].

5. APPLICATIONS

In this section, we outline a couple of possible applications of the acyclic partition scheme:

(i) Filter Design. Let us consider the state estimation of the discrete dynamic system

$$\begin{aligned} S: \quad x(t+1) &= Ax(t) + Bw(t) \\ y(t) &= Cx(t) + v(t) \end{aligned} \quad (5.1)$$

where $w(t) \in R^m$ and $v(t) \in R^l$ are independent Gaussian processes with zero mean and covariances R_w and R_v , respectively, and the initial state $x(0) = x_0$ a Gaussian random variable with mean $\bar{x}_0$ and covariance $P(0) = P_0$, which is independent of $v(t)$ and $w(t)$. The standard Kalman filter [15] requires the solution of a discrete-time Riccati equation of the same order as the state vector of the system S . In systems of large dimensions, this fact may cause considerable numerical problems. Decomposing the system into acyclic components, a suboptimal design can be achieved by solving a number of low-order Riccati equations in a sequential manner.

To illustrate the idea of sequential filter design, suppose that the system S of (5.1) is decomposed into two structurally observable subsystems using the proposed scheme,

$$\begin{aligned} S_1: \quad x_1(t+1) &= A_{11}x_1(t) + B_1w(t) \\ y_1(t) &= C_{11}x_1(t) + D_1v(t) \end{aligned} \quad (5.2a)$$

$$\begin{aligned} S_2: \quad x_2(t+1) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2w(t) \\ y_2(t) &= C_{21}x_1(t) + C_{22}x_2(t) + D_2v(t) \end{aligned} \quad (5.2b)$$

where the matrices D_1 and D_2 are blocks of the matrix $D = \begin{bmatrix} D_1^T & D_2^T \end{bmatrix}^T$ which is nothing but the identity matrix of appropriate dimension with rows permuted.

As usual, the filter design consists of a prediction step

$$\begin{aligned}\hat{x}_1(t+1|t) &= A_{11}\hat{x}_1(t|t) \\ \hat{x}_2(t+1|t) &= A_{21}\hat{x}_1(t|t) + A_{22}\hat{x}_2(t|t)\end{aligned}\quad (5.3)$$

and an updating step

$$\begin{aligned}\hat{x}_1(t|t) &= \hat{x}_1(t|t-1) + K_1(t)[y_1(t) - C_{11}\hat{x}_1(t|t-1)] \\ \hat{x}_2(t|t) &= \hat{x}_2(t|t-1) + K_2(t)[y_2(t) - C_{21}\hat{x}_1(t|t) - C_{22}\hat{x}_2(t|t-1)]\end{aligned}\quad (5.4)$$

where $\hat{x}_i(t|t-1)$ and $\hat{x}_i(t|t)$ are the predicted and updated estimates of $x_i(t)$ given the output measurements up to $t-1$ and t , respectively. The filter gains $K_1(t)$ and $K_2(t)$ in (5.4) are computed sequentially as follows: First, $K_1(t)$ is determined to minimize the trace of the covariance of the error $e_1(t|t) = x_1(t) - \hat{x}_1(t|t)$, as in the standard Kalman filter. Then, with the updated estimate $\hat{x}_1(t|t)$ of $x_1(t)$ substituted in the expression of $\hat{x}_2(t|t)$ in (5.4), $K_2(t)$ is computed to minimize the trace of the covariance of the error $e_2(t|t) = x_2(t) - \hat{x}_2(t|t)$.

Defining the error covariance matrices as $E[e_1(t|t)e_1^T(t|t)] = P_{11}(t|t)$, $E[e_2(t|t)e_1^T(t|t)] = P_{21}(t|t)$, etc., the filter gains can be computed as

$$K_1(t) = L_1(t)M_1^{-1}(t), \quad K_2(t) = L_2(t)M_2^{-1}(t) \quad (5.5)$$

where

$$\begin{aligned}
L_1(t) &= P_{11}(t|t-1)C_{11}^T \\
M_1(t) &= C_{11}P_{11}(t|t-1)C_{11}^T + D_1R_vD_1^T \\
L_2(t) &= P_{22}(t|t-1)C_{22}^T + P_{21}^T(t|t-1)[I-K_1(t)C_{11}]^TC_{21}^T \\
M_2(t) &= C_{22}P_{22}(t|t-1)C_{22}^T + C_{22}P_{21}^T(t|t-1)[I-K_1(t)C_{11}]^TC_{21}^T \\
&\quad + C_{21}[I-K_1(t)C_{11}]P_{21}(t|t-1)C_{22}^T \\
&\quad + C_{21}[I-K_1(t)C_{11}]P_{11}(t|t-1)[I-K_1(t)C_{11}]^TC_{21}^T \\
&\quad + [D_2 - C_{21}K_1(t)D_1]R_v[D_2 - C_{21}K_1(t)D_1]^T.
\end{aligned} \tag{5.6}$$

The covariance equations are

$$\begin{aligned}
P_{11}(t+1|t) &= A_{11}P_{11}(t|t)A_{11}^T + B_1R_wB_1^T \\
P_{21}(t+1|t) &= A_{11}P_{11}(t|t)A_{21}^T + A_{11}P_{21}(t|t)A_{22}^T + B_1R_wB_2^T \\
P_{22}(t+1|t) &= A_{21}P_{11}(t|t)A_{21}^T + A_{21}P_{21}(t|t)A_{22}^T \\
&\quad + A_{22}P_{21}(t|t)A_{21}^T + A_{22}P_{22}(t|t)A_{22}^T + B_2R_wB_2^T
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
P_{11}(t|t) &= P_{11}(t|t-1) - L_1(t)M_1^{-1}(t)L_1^T(t) \\
P_{21}(t|t) &= [I-K_1(t)C_{11}]P_{21}(t|t-1)[I-K_2(t)C_{22}]^T \\
&\quad - [I-K_1(t)C_{11}]P_{11}(t|t-1)[I-K_1(t)C_{11}]^TC_{21}^TK_2^T(t) \\
&\quad + K_1(t)D_1R_v[D_2 - C_{21}K_1(t)D_1]^TK_2^T(t) \\
P_{22}(t|t) &= P_{22}(t|t-1) - L_2(t)M_2^{-1}(t)L_2^T(t).
\end{aligned} \tag{5.8}$$

Equations (5.3)-(5.8) together with the initial conditions $x(0|-1) = \bar{x}_0$, $P(0|-1) = P_0$, where $P = (P_{ij})_{2 \times 2}$, constitute a *sequentially Designed Kalman Filter* for the system S . It can be shown using the results of [16] that the filter is stable and suboptimal in the

sense of [17], that is, there exists a number $\mu (0 < \mu < 1)$ such that for sufficiently large t ,

$$\text{tr}(P_{11}) + \text{tr}(P_{22}) \leq \mu^{-1} \text{tr}(P^0), \quad (5.9)$$

where P^0 is the error covariance of the optimal filter for S .

(ii) *Decentralized Stabilization.* It is well-known that all eigenvalues of a controllable and observable system can be assigned arbitrarily using dynamic output feedback. Suppose that for each decoupled subsystem S_i of (3.2) a dynamic controller is designed, which has the form

$$\begin{aligned} C_i: \dot{\bar{x}}_i(t) &= \bar{A}_i \bar{x}_i(t) + \bar{B}_i \bar{y}_i(t) \\ \bar{u}_i(t) &= \bar{C}_i \bar{x}_i(t) + \bar{D}_i y_i(t) \end{aligned} \quad (5.10)$$

so that the eigenvalues are placed at any desired locations. In (5.10) $\bar{x}_i(t) \in R^{\bar{n}_i}$ is the state of the i -th controller C_i . The closed-loop system $\bar{S}$ consisting of the interconnected subsystems S_i of (3.1) and the local controllers C_i of (5.10), can be described as

$$\begin{aligned} \bar{S}: \begin{bmatrix} \dot{x}_i(t) \\ \dot{\bar{x}}_i(t) \end{bmatrix} &= \begin{bmatrix} A_{11} + B_{11} \bar{D}_1 C_{11} & B_{11} \bar{C}_1 \\ \bar{B}_1 C_{11} & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x_i(t) \\ \bar{x}_i(t) \end{bmatrix} \\ &+ \sum_{j=1}^{i-1} \begin{bmatrix} A_{ij} + \sum_{k=j}^i B_{ik} \bar{D}_k C_{kj} & B_{ij} \bar{C}_j \\ \bar{B}_i C_{ij} & \bigcirc \end{bmatrix} \begin{bmatrix} x_j(t) \\ \bar{x}_j(t) \end{bmatrix}. \end{aligned} \quad (5.11)$$

From (5.11), we observe that the closed-loop system also has a block triangular structure, the diagonal blocks corresponding to the closed-loop subsystems consisting of S_i 's and the attached C_i 's. The eigenvalues of the overall system $\bar{S}$ is the union of the eigenvalues of the individual closed-loop subsystems and, hence, can be assigned arbitrarily.

(iii) Observer Design. Decomposition into acyclic components can also be very useful in reducing the amount of computations involved in Luenberger observer design for large systems, by carrying out the design sequentially. Consider the acyclic decomposition of (3.1) rewritten as

$$\begin{aligned} S: \dot{x}_i(t) &= A_{ii}x_i(t) + \sum_{j=1}^{i-1} A_{ij}x_j(t) + \sum_{j=1}^i B_{ij}u_j(t) \\ y_i(t) &= C_{ii}x_i(t) + \sum_{j=1}^{i-1} C_{ij}x_j(t), \quad i = 1, 2, \dots, s \end{aligned} \quad (5.12)$$

and assume that the pairs (A_{ii}, C_{ii}) are observable. Treating all the summation terms in (5.12) as inputs, a reduced order observer can be designed sequentially as

$$\begin{aligned} 0: \dot{\bar{x}}_i(t) &= \bar{A}_{ii}\bar{x}_i(t) + \sum_{j=1}^{i-1} \bar{L}_{ij}\hat{x}_j(t) + \sum_{j=1}^i \bar{M}_{ij}u_j(t) \\ &\quad + \bar{K}_{ii} \left[y_i(t) - \sum_{j=1}^{i-1} C_{ij}\hat{x}_j(t) \right], \\ \hat{x}_i(t) &= \bar{P}_{ii}\bar{x}_i(t) + \bar{Q}_{ii} \left[y_i(t) - \sum_{j=1}^{i-1} C_{ij}x_j(t) \right], \quad i = 1, 2, \dots, s \end{aligned} \quad (5.13)$$

where $\bar{x}_i(t) \in \mathbb{R}^{n_i - \ell_i}$ is the state of the i -th component of the

observer, and $\hat{x}_i(t)$ is the estimate of $x_i(t)$. The matrices appearing in (5.13) are computed in a standard way: Matrices $\bar{T}_{ii}$ and $\bar{K}_{ii}$ are chosen such that $[\bar{T}_{ii}^T \bar{C}_{ii}^T]$ is nonsingular, and the equation

$$\bar{A}_{ii}\bar{T}_{ii} - \bar{T}_{ii}A_{ii} + \bar{K}_{ii}C_{ii} = 0, \quad i = 1, 2, \dots, s \quad (5.14)$$

is satisfied for some matrix $\bar{A}_{ii}$ with specified eigenvalues. Other matrices in (5.13) are then computed as $\bar{L}_{ij} = T_{ii}A_{ij}$, $\bar{M}_{ij} = \bar{T}_{ii}B_{ij}$ and $[\bar{T}_{ii}^T \bar{C}_{ii}^T]^{-1} = [\bar{P}_{ii} \bar{Q}_{ii}]^T$. With this choice of the matrices and defining $\bar{e}_i = \bar{x}_i = \bar{T}_{ii}x_i$, it takes a little matrix algebra to show that

$$\dot{\bar{e}}_i = \bar{A}_{ii}\bar{e}_i + \sum_{j=1}^{i-1} \bar{A}_{ij}\bar{e}_j, \quad (5.15a)$$

and

$$x_i - \hat{x}_i = -P_{ii}e_i - Q_{ii} \sum_{j=1}^{i-1} C_{ij}(x_j - \hat{x}_j), \quad i = 1, 2, \dots, s \quad (5.15b)$$

where $\bar{A}_{ij}$ are defined in terms of $\bar{K}_{ii}$, $\bar{P}_{jj}$, C_{pq} , $j < p \leq i$, $j \leq q < i$, and $\bar{Q}_{rr}$, $j < r < i$. From (5.15) it is clear that $\hat{x}_i(t) \rightarrow x_i(t)$ as $t \rightarrow +\infty$.

Thus, the overall design is achieved by designing small individual observers for the acyclic components of the system. Equations (5.13) indicate that the observer has the same hierarchial structure as the system itself.

6. AN EXAMPLE

To illustrate the decomposition algorithm of Section 4, let us consider the system S of (2.1) given as

$$\begin{aligned}
 S: \quad \dot{x}(t) = & \begin{bmatrix} & & a_{15} & & & & & & & \\ & a_{32} & & & & & & & & \\ & & & & & & a_{3,10} & & & \\ & & & & a_{48} & a_{49} & & & & \\ & & & & & a_{59} & & & & \\ & & & & & a_{69} & & & & \\ & a_{72} & & & a_{76} & & & & & \\ & & & & & & & & & \\ & & & & a_{95} & & & & & \\ & & a_{10,3} & & & a_{10,7} & & & & \\ a_{11,1} & & & & & & & & & \\ & & & & & & a_{12,8} & & & \end{bmatrix} x(t) \\
 & + \begin{bmatrix} & & & & & b_{15} & & & & \\ & & & & b_{24} & & & & & \\ & b_{31} & & & & & & & & \\ & & & & & & b_{65} & & & \\ & & b_{62} & & & & & & & \\ & & & & b_{83} & & & & & \\ & & & & & & b_{95} & & & \\ & & & & & & & & & \\ & & & & & & & & & b_{12,1} \end{bmatrix} u(t) \\
 y(t) = & \begin{bmatrix} & & & & & & c_{1,12} & & & \\ & & c_{24} & & & & & & & \\ & c_{42} & & & & & & c_{3,11} & & \\ & & & & c_{47} & & & & & \\ & & & & & & & & c_{5,10} & \end{bmatrix} x(t)
 \end{aligned}
 \tag{6.1}$$

where the zero elements are omitted for simplicity.

We first compute the reachability matrix $\bar{R}$ of (2.3), and

get the submatrix $\bar{H}$ as

$$\bar{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}. \quad (6.2)$$

Following the steps of the decomposition, we set $i = 1$. Since the first row of $\bar{H}$ does not dominate any other row, we define the sets $I = 1$, $J = 1, 3$. The matrix $\tilde{S}$ can be obtained as

$$\tilde{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}^T, \quad (6.3)$$

which indicates that the partitions

$$\begin{aligned} U_1 &= \{u_1, u_3\}, & U_2 &= \{u_2, u_4, u_5\} \\ X_1 &= \{x_8, x_{12}\}, & X_2 &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\} \\ Y_1 &= \{y_1\}, & Y_2 &= \{y_2, y_3, y_4, y_5\} \end{aligned} \quad (6.4)$$

yield an acyclic IO-reachable decomposition of $\mathcal{D}$ into $\mathcal{D}_i^1 = \{U_i \cup X_i \cup Y_i, E_i\}$, $i = 1, 2$, where $\mathcal{D}_1^1$ is irreducible.

Having achieved a decomposition, we label the first row of $\bar{H}$ by (+), and check the second row. Since the second row dominates the unlabeled third row, which does not dominate any other row, we set $i = 3$, and define $I = \{3\}$, $J = \{5\}$. This gives

$$\tilde{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T, \quad (6.5)$$

indicating that the partitions

$$\begin{aligned}
 U_1 &= \{u_5\} & U_2 &= \{u_1, u_2, u_3, u_4\} \\
 X_1 &= \{x_1, x_5, x_9, x_{11}\}, & X_2 &= \{x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{12}\} \\
 Y_1 &= \{y_3\} & Y_2 &= \{y_1, y_2, y_4, y_5\}
 \end{aligned} \tag{6.6}$$

yield another acyclic decomposition of $\mathcal{D}$ into $\mathcal{D}_1^2$ and $\mathcal{D}_2^2$, where $\mathcal{D}_1^2$ is irreducible. We label the third row (+).

Now, we go back to the second row of $\bar{H}$. Since it dominates the (+) labeled third row, we label it (+), too. The fourth and fifth row of $\bar{H}$ are also labeled (+) since they both dominate the third row, and the algorithm stops.

This first round of the algorithm produces two distinct acyclic IO-reachable decompositions, $\mathcal{D} = \mathcal{D}_1^1 \cup \mathcal{D}_2^1$ and $\mathcal{D} = \mathcal{D}_1^2 \cup \mathcal{D}_2^2$, where $\mathcal{D}_1^k$, $k = 1, 2$, are irreducible. In the second round, we apply the algorithm to $\mathcal{D}_2^1$ and $\mathcal{D}_2^2$ separately, to decompose them further into acyclic IO-reachable components. We continue this procedure until the final decompositions are reached. For the system of (6.1), possible P_{IO} partitions are summarized in Table 1.

An investigation of Table 1 indicates that at least one subsystem of all decompositions except the one induced by P_{IO}^2 is either structurally uncontrollable or structurally unobservable. This information about the subsystems is provided in Table 1, where a (✓) in the column SC (SO) indicates that the corresponding subsystem is structurally controllable (observable) and a (X) indicates that it is not. Although structural controllability and observability can be restored by combining several components (for example, by combining $\mathcal{D}_2^5$, $\mathcal{D}_3^5$ and $\mathcal{D}_4^5$ in P_{IO}^5),

Table 1. Distinct partitions

p_{IO}^k	p_i^k	u_i	x_i	y_i	SO	SC
p_{IO}^1	p_1^1	$\{u_1, u_3\}$	$\{x_8, x_{12}\}$	$\{y_1\}$	✓	✓
	p_2^1	$\{u_5\}$	$\{x_1, x_4, x_5, x_9, x_{11}\}$	$\{y_2, y_3\}$	X	✓
	p_3^1	$\{u_2, u_4\}$	$\{x_2, x_3, x_6, x_7, x_{10}\}$	$\{y_4, y_5\}$	✓	✓
p_{IO}^2	p_1^2	$\{u_5\}$	$\{x_1, x_5, x_9, x_{11}\}$	$\{y_3\}$	✓	✓
	p_2^2	$\{u_3\}$	$\{x_4, x_8\}$	$\{y_2\}$	✓	✓
	p_3^2	$\{u_1\}$	$\{x_{12}\}$	$\{y_1\}$	✓	✓
	p_4^2	$\{u_2, u_4\}$	$\{x_2, x_3, x_6, x_7, x_{10}\}$	$\{y_4, y_5\}$	✓	✓
p_{IO}^3	p_1^3	$\{u_5\}$	$\{x_1, x_5, x_9, x_{11}\}$	$\{y_3\}$	✓	✓
	p_2^3	$\{u_3\}$	$\{x_4, x_8\}$	$\{y_2\}$	✓	✓
	p_3^3	$\{u_2, u_4\}$	$\{x_2, x_6, x_7\}$	$\{y_4\}$	✓	X
	p_4^3	$\{u_1\}$	$\{x_3, x_{10}, x_{12}\}$	$\{y_1, y_5\}$	✓	✓
p_{IO}^4	p_1^4	$\{u_5\}$	$\{x_1, x_5, x_9, x_{11}\}$	$\{y_3\}$	✓	✓
	p_2^4	$\{u_2, u_4\}$	$\{x_2, x_6, x_7\}$	$\{y_4\}$	✓	X
	p_3^4	$\{u_3\}$	$\{x_4, x_8\}$	$\{y_2\}$	✓	✓
	p_4^4	$\{u_1\}$	$\{x_3, x_{10}, x_{12}\}$	$\{y_1, y_5\}$	✓	✓
p_{IO}^5	p_1^5	$\{u_5\}$	$\{x_1, x_5, x_9, x_{11}\}$	$\{y_3\}$	✓	✓
	p_2^5	$\{u_2, u_4\}$	$\{x_2, x_6, x_7\}$	$\{y_4\}$	✓	X
	p_3^5	$\{u_i\}$	$\{x_3, x_{10}\}$	$\{y_5\}$	✓	✓
	p_4^5	$\{u_3\}$	$\{x_4, x_8, x_{12}\}$	$\{y_{1,2}\}$	X	✓

Fig. 4. Irreducible partition p_{10}^2

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 \\ 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} x(t) \quad (6.7)$$

$$\begin{aligned}
0_4: \dot{\bar{x}}_4 &= \begin{bmatrix} -3/2 & 0 & -1/4 \\ -1 & -1 & 0 \\ 1 & 0 & -1/2 \end{bmatrix} \bar{x}_4 + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} x_9 + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\
&+ u_5 + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} u_1 + \begin{bmatrix} 1 & 0 \\ -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_4 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1/2 & 3/4 \\ 1/4 & -1 \\ 2 & 3/2 \end{bmatrix} \begin{bmatrix} y_5 \\ y_4 \end{bmatrix} \\
\begin{bmatrix} \hat{x}_2 \\ \hat{x}_6 \\ \hat{x}_7 \\ \hat{x}_3 \\ \hat{x}_{10} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \bar{x}_4 + \begin{bmatrix} -1/4 & 0 \\ 0 & 1 \\ 1/2 & 1 \\ -1/2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_5 \\ y_4 \end{bmatrix} \quad (6.8)
\end{aligned}$$

With a little manipulation an overall representation of the dynamic part of the observer can be obtained as

$$\begin{aligned}
0: \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_4 \end{bmatrix} &= \left[\begin{array}{ccc|c|ccc} 0 & -1 & -2/3 & & & & \\ 1 & 0 & -2/3 & & & & \\ -3 & 0 & -3 & & & & \\ \hline 0 & 1 & 0 & -1 & & & \\ \hline 0 & 0 & 0 & & -3/2 & 0 & -1/4 \\ 0 & 3/2 & 0 & & -1 & -1 & 0 \\ 0 & 0 & 0 & & 1 & 0 & -1/2 \end{array} \right] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_4 \end{bmatrix} + \\
&\left[\begin{array}{c|ccc} 0 & & & \\ 2 & & & \\ 4 & & & \\ \hline 0 & -1 & & \\ \hline 0 & & 0 & 1 & 0 \\ 2 & & 0 & -2 & -1 \\ 0 & & -2 & 0 & 0 \end{array} \right] u(t)
\end{aligned}$$

$$+ \left[\begin{array}{c|c|c|c|c} -8/3 & & & & \\ -4/3 & & & & \\ -11 & & & & \\ \hline 2/3 & 1/2 & & & \\ \hline 0 & & & 1/2 & 3/4 \\ 1 & & & 1/4 & -1 \\ 0 & & & 2 & 3/2 \end{array} \right] y(t) , \quad (6.9)$$

which has the same hierarchical structure as S , save that the expression for $\dot{\bar{x}}_3$ is omitted as it is not necessary.

7. CONCLUSIONS

Standard estimation and control schemes may become numerically unattractive for large systems because of excessive computer time and memory requirements. By mimicking the graph-theoretic methods used for solving large systems of algebraic equations in the context of Decomposition Principle, a "piece-by-piece" design of estimation and control schemes for large dynamic systems has been outlined. By permuting the states of the system, the scheme can take advantage of the special structural features of the system and produce irreducible partitions consisting of hierarchially ordered input-output reachable and structurally controllable-observable subsystems with independent inputs and outputs. Low-order estimators and controllers can then be built from top to bottom sequentially for each subsystem separately. The application of this approach has significantly reduced the computer cost involved in regulator design for sparse dynamic systems of large dimensions. Further improvements of the algorithm are carried out presently in the area of reachability and structural controllability-observability calculations, as well as construction of reduced-order observers and controllers. These new research efforts should bring about numerical simplification which can increase our capability to design regulators for systems with dimensions of another order of magnitude.

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