Gorter-Mellink pulsed-source problem in cylindrical and spherical geometry*

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Abstract

An exact solution to the Gorter-Mellink pulsed-source problem is known in plane geometry [Dresner, L. *Advances in Cryogenic Engineering* **29** (1984) 323]. According to this solution, the central temperature (i.e., the temperature at the source location) falls as $t^{-3/2}$, where t is the elapsed time after an instantaneous, plane pulse. No such exact solutions are known to the pulsed-source problem in cylindrical and spherical geometry. But in cylindrical geometry, it can be shown that if the initial condition is an instantaneous temperature rise ΔT inside a cylinder of radius R , the central temperature is bounded from above by a decreasing exponential function of time. The relaxation time of this exponential is related to ΔT and R . In spherical geometry, it can be shown that if the initial condition is an instantaneous temperature rise ΔT inside a sphere of radius R , the central temperature is bounded from above by a function proportional to $(t_0 - t)^{9/2}$. The extinction time t_0 is related to ΔT and R . These predictions have not been tested by experiment, and the author recommends such experiments.

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Introduction

The pulsed-source problem is a problem of heat diffusion in which a plane, line, or point source in an infinite medium is pulsed at $t = 0$ and the resulting temperature distribution studied as a function of elapsed time. Depending on which source is used, the pulsed-source problem is said to refer to plane, cylindrical, or spherical geometry or to one, two, or three dimensions, respectively.

If the diffusion is described by the ordinary diffusion equation, $\partial T/\partial t = \nabla^2 T$, the pulsed-source problem has well-known analytic solutions in which the central temperature falls with elapsed time t as $t^{-n/2}$, where n is the dimensionality mentioned above.

Heat transport in turbulent He-II (superfluid helium) does not obey the ordinary diffusion equation. The latter equation is based on Fourier's law of conduction, which says that the heat flux is proportional to the gradient of the temperature. Turbulent superfluid helium instead obeys the Gorter-Mellink relation, which says that the heat flux is proportional to the cube root of the gradient of the temperature. For the Gorter-Mellink relation, the diffusion equation takes the form

$$S \frac{\partial T}{\partial t} = \nabla \cdot [K(\nabla T)^{1/3}] \quad (1)$$

where S is the heat capacity per unit volume and K is the Gorter-Mellink heat conductance. Strictly speaking, S and K are functions of temperature, but if they are taken constant, the pulsed-source problem has an analytic solution in plane geometry ($n = 1$).¹ For this solution, the central temperature falls with elapsed time t as $t^{-3/2}$.

The purpose of this paper is to explore the behavior of the central temperature in cylindrical ($n = 2$) and spherical ($n = 3$) geometries. To simplify the appearance of the equations, we work in special units in which $K = S = 1$.

Pulsed-source problem in spherical geometry

In spherical geometry, Eq. (1) takes the form

$$\frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(\frac{\partial T}{\partial r} \right)^{1/3} \right] \quad (2)$$

The boundary and initial conditions corresponding to the pulsed-source problem are

$$T(r, 0) = 0 \quad (r > 0) \quad (3a)$$

$$T(\infty, t) = 0 \quad (3b)$$

$$\int_0^\infty 4\pi r^2 T \, dr = Q \quad (t > 0) \quad (3c)$$

The partial differential equation (2) is invariant to the family of affine groups

$$T' = \lambda^\alpha T \quad (4a)$$

$$t' = \lambda^\beta t \quad (4b)$$

$$r' = \lambda r \quad (4c)$$

subject to the subsidiary relation

$$2\alpha - 3\beta = -4 \quad (4d)$$

The boundary condition (3c) requires that $\alpha = -3$, so that $\beta = -2/3$. With these values, the similarity solution of Eq. (2) becomes²

$$T = t^{9/2} y(rt^{3/2}) \quad (5)$$

This last form is unsatisfactory because it represents a temperature distribution that peaks up as time goes on rather than one that spreads out. This defect is fundamental and means that the pulsed-source problem in spherical geometry has

no solution of the form (5). It does, however, have solutions of a very closely related form, as we shall see next.

In addition to its invariance to the groups (4), Eq. (2) is invariant to the group of translations in time: $T' = T$, $t' = t + \lambda$, $r' = r$. Solutions invariant to this group must be functions of the time difference $t - t_0$, where t_0 is some fiducial time that is determined by the boundary and initial conditions. These considerations suggest that we look for similarity solutions of the form

$$T = (t_0 - t)^{9/2} y \left[r(t_0 - t)^{3/2} \right] \quad (6)$$

If we substitute this form into Eq. (2), we obtain for y the following ordinary differential equation:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \dot{y}^{1/3} \right) + \frac{3}{2} \xi \dot{y} + \frac{9}{2} y = 0 \quad , \quad \xi = r(t_0 - t)^{3/2} \quad (7)$$

After multiplication by ξ^2 , this can be integrated once to give

$$\xi^2 \dot{y}^{1/3} + \frac{3}{2} \xi^3 y = C \quad (8)$$

where C is a constant of integration. If y is regular at the origin $\xi = 0$, then $C = 0$, in which case Eq. (8) can be integrated a second time to give

$$y = \left(a + \frac{27}{16} \xi^4 \right)^{-1/2} \quad (9)$$

where a is a second constant of integration.

The solution (9) is an exact solution of the superfluid diffusion equation. Its overall form $c = (t_0 - t)^{9/2} y[r(t_0 - t)^{3/2}]$ was chosen so that the total heat $Q = 4\pi \int_0^\infty r^2 c \, dr$ would be conserved. But with the form (9) for y , the Q -integral diverges. Thus the solution (9) cannot fulfill the initial condition of a finite source pulse, and so, strictly speaking, the solution to the pulsed-source problem is not a similarity solution of the form (6).

Suppose we consider a pulsed source in which the finite heat deposition is such that the initial temperature distribution is given by the similarity solution (9) out to a radius $r = R$, but is zero for $r > R$. If R is large enough, the initial effect of the region $r > R$ on the temperature distribution near $r = 0$ should be small. So the central temperature should begin falling like $(t_0 - t)^{9/2}$. Since the region $r > R$ is initially colder than in the similarity solution (9), it seems plausible for the central temperature to continue falling faster than $(t_0 - t)^{9/2}$. Indeed, we might expect that any temperature distribution that is initially smaller than some similarity solution of the family (9) will always remain smaller. Thus the central temperature should be bounded from above by $a^{-1/2}(t_0 - t)^{9/2}$, which means it should vanish at some finite time after the pulse! The proof of these last assertions, given next, involves some complicated but rather standard analysis involving maximum and minimum principles.

Proof of the foregoing assertions

Let us consider the infinitesimal difference δc between two neighboring solutions of the superfluid diffusion equation $c_t = r^{-2}(r^2 c_r^{1/3})_r$. It obeys the *linear* partial differential equation $(\delta c)_t = r^{-2}[r^2(c_r^{-2/3}/3)(\delta c)_r]_r$, which has the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 k \frac{\partial \phi}{\partial r} \right) = \frac{\partial \phi}{\partial t} \quad , \quad k > 0 \quad (10)$$

Equation (10) describes heat conduction in spherical geometry with a thermal conductivity k . The boundary and initial conditions that interest us are

$$\phi(r, 0) > 0 \quad (11a)$$

$$\phi(\infty, t) = 0 \quad (11b)$$

$$\phi \text{ regular at } r = 0 \quad (11c)$$

What we should like to show is that $\phi(r, t) > 0$ for all $t > 0$. In that case, two solutions c_1 and c_2 of the superfluid diffusion equation that obey the boundary conditions (11b) and (11c) and the initial condition $c_1 > c_2$ are always ordered so that $c_1 > c_2$.

We begin first by considering a function ψ that obeys the strict differential inequality

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 k \frac{\partial \psi}{\partial r} \right) - \frac{\partial \psi}{\partial t} < 0 \quad (12)$$

and the boundary condition (11c) in the rectangle Σ in the (r, t) -plane (see Fig. 1). The minimum value of ψ cannot lie in the interior of Σ , for if it did, say at a point P , then $\psi_t(P) = 0$, $\psi_r(P) = 0$, and $\psi_{rr}(P) \geq 0$. These conditions contradict the strict inequality (12). Moreover, the minimum value of ψ cannot lie at an interior point Q of S_4 , for if it did, then $\psi_r(Q) = 0$ and $\psi_{rr}(Q) \geq 0$. Then from Eq. (12) it follows that $\psi_t(Q) > 0$, which means that there must be yet smaller values of ψ in the neighborhood of Q inside Σ , contrary to hypothesis.

Finally, the minimum value of ψ cannot lie at an interior point of S_1 , but in order to prove this, we shall have to consider in detail the consequences of boundary condition (11c). First of all, any solution c of the superfluid diffusion equation (2) that is regular at the origin must behave near $r = 0$ like $a + br^4 + \dots$, for only then can the right-hand side of Eq. (2) have a finite, nonzero limit as $r \rightarrow 0$. In that case $c_r^{-2/3} \sim r^{-2}$ near $r = 0$ and thus so does k in Eq. (10). Furthermore, ψ , being the infinitesimal difference between two solutions of the superfluid diffusion equation, also has the form $a + br^4 + \dots$ near $r = 0$. If we substitute this form for the first term on the left-hand side of Eq. (12) we find that it equals $12b \cdot \lim_{r \rightarrow 0} (r^2 k)$ at $r = 0$. If the minimum of ψ occurs at an interior point T of S_1 , then $\psi_t(T) = 0$. Therefore $b(T) < 0$, which means that there are yet smaller values of ψ in the neighborhood of T inside Σ , a contradiction. The minimum of ψ must therefore lie on S_2 or S_3 .

When ϕ satisfies the equality (10), we introduce the auxiliary function

$$\psi = \phi + \epsilon t \quad , \quad \epsilon > 0 \quad , \quad (13)$$

which satisfies the differential inequality (12). Furthermore, ψ behaves like $a + br^4 + \dots$ near $r = 0$, as it should. Therefore, its minimum value in Σ must lie on S_2 or S_3 ; call it m_ψ and let it be attained at point P . Let m_ϕ be the smallest value ϕ attains on S_2 and S_3 (not yet proved to be its minimum value in Σ !). Then from Eq. (13), for any point Q in Σ ,

$$\phi(Q) + \epsilon t_Q = \psi(Q) > m_\psi \quad (14)$$

and

$$m_\psi = \psi(P) = \phi(P) + \epsilon t_P \geq m_\phi + \epsilon t_P \quad . \quad (15)$$

Thus, combining Eqs. (14) and (15), we find

$$\phi(Q) > m_\phi + \epsilon(t_P - t_Q) \quad . \quad (16)$$

Finally, if we let $\epsilon \rightarrow 0$, we see that ϕ is always $\geq$ the smallest value it attains on S_2 and S_3 , so that its minimum must lie on S_2 or S_3 .

If $\phi \geq 0$ on S_2 and $\phi = 0$ on S_3 , then its minimum value must be zero. Therefore, $\phi \geq 0$ everywhere in Σ , as was to be proved. The last step is to let $R \rightarrow \infty$.

Discussion

Suppose we establish an initial temperature rise T_1 inside a sphere of radius R . We can use the foregoing results to obtain an upper limit to the time at which the temperature rise disappears. To do so, we need to find a similarity solution whose initial form at $t = 0$, namely, $T = t_0^{9/2}(a + 27r^4t_0^6/16)^{-1/2}$, exceeds T_1 for $r \leq R$. The best (smallest) upper limit t_0 will correspond to an initial form T that just

equals T_1 at $r = R$ (see Fig. 2). The choices of a and t_0 are therefore constrained by the relation

$$T_1 = t_0^{9/2} (a + 27R^4 t_0^6 / 16)^{-1/2} \quad (17)$$

or

$$t_0^9 - \frac{27R^4 T_1^2}{16} t_0^6 - a T_1^2 = 0 . \quad (18)$$

Equation (18) has one real root, which is always larger than

$$t_{0*} = \frac{3}{2} \left(\frac{R^4 T_1^2}{2} \right)^{1/3} \quad (19)$$

and approaches t_{0*} as a becomes smaller (see Fig. 3).

The radius R , the temperature rise T_1 , and the source strength Q are related by

$$Q = \frac{4\pi}{3} T_1 R^3 , \quad (20)$$

and if we use Eq. (20) to eliminate R in Eq. (19), we obtain

$$t_{0*} = \frac{3}{2^{4/3}} \left(\frac{3}{4\pi} \right)^{4/9} \cdot Q^{4/9} T_1^{2/9} \quad (21)$$

The numerical constant in (21) equals 0.629889... .

Equation (21) is correct in special units in which K and S are numerically equal to 1. In ordinary units, Eq. (21) must be written

$$t_{0*} = 0.629889 (Q^4 T_1^2 S^5)^{1/9} / K \quad (22)$$

When the bath temperature is 1.8 K (and the pressure is 1 atm), $S = 0.410 \text{ J}\cdot\text{cm}^{-3}\cdot\text{K}^{-1}$ and $K = 10.4 \text{ W}\cdot\text{cm}^{-5/3}\cdot\text{K}^{-1/3}$. In the one-dimensional (1-D) experiment of Lottin and van Sciver,² a heat pulse of 0.92 J was deposited at a point in a 6-mm tube. Taking $T_1 = T_\lambda - T_b = 0.37 \text{ K}$, we find, from Eq. (22), $t_{0*} = 28.5 \text{ ms}$. So we expect the temperature rise to be extinguished in less than 30 ms; this is quite different from the 1-D experiment of Lottin and van Sciver

in which a 0.050 K temperature rise was still perceptible after one full second. This rather remarkable prediction has not yet been tested experimentally and is a tempting target for further study.

Pulsed-source problem in cylindrical geometry

In cylindrical geometry, Eq. (1) takes the form

$$\frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{\partial T}{\partial r} \right)^{1/3} \right] \quad (23)$$

We wish to solve it subject to the boundary and initial conditions

$$T(r, 0) = 0 \quad (r > 0), \quad (24a)$$

$$T(\infty, t) = 0 \quad (24b)$$

$$\int_0^\infty 2\pi r T \, dr = Q \quad (t > 0) \quad (24c)$$

Equation (23) is invariant to the mixed stretching-translation group

$$T' = \lambda^{-2} T \quad (25a)$$

$$t' = t + \frac{1}{\beta} \ln \lambda \quad (0 < \lambda < \infty) \quad (25b)$$

$$r' = \lambda r \quad (25c)$$

where β is an arbitrary constant not yet determined. The most general form for $T(r, t)$ invariant to a group of the family (25) is

$$T = e^{-2\beta t} y(re^{-\beta t}) \quad (26)$$

where y is an arbitrary function of the single argument $\xi = re^{-\beta t}$. Furthermore, Eq. (26) has the right form to satisfy the boundary condition (24c):

$$2\pi \int_0^\infty y(\xi) \xi d\xi = Q \quad (27)$$

If we substitute Eq. (26) into Eq. (23), we get

$$\frac{1}{\xi} \frac{d}{d\xi} (\xi \dot{y}^{1/3}) + \beta (2y + \xi \dot{y}) = 0 \quad , \quad \xi = re^{-\beta t} \quad (28)$$

After multiplication by ξ , this can be integrated to give

$$\xi \dot{y}^{1/3} + \beta \xi^2 y = C \quad (29)$$

where C is a constant of integration. If y is regular at the origin $\xi = 0$, then $C = 0$.

Then Eq. (29) can be integrated again to give

$$y = \left(a + \frac{\beta^3}{2} \xi^4 \right)^{-1/2} \quad (30)$$

where a is another constant of integration.

Again the Q -integral diverges, so the solution (30) cannot fulfill the initial condition of a finite heat pulse. But again, by a repetition of the argument of the last section, we expect it to exceed for all time any solution that it exceeds at $t = 0$. Thus the central temperature should be bounded from above by $a^{-1/2}e^{-2\beta t}$, which means it should fall exponentially with time.

As before, the choices of a and β in Eq. (30) are constrained by the initial condition

$$T_1 = \left(a + \frac{\beta^3}{2} R^4 \right)^{-1/2} \quad (31)$$

from which it follows that the largest possible value of β is

$$\beta_* = \left(\frac{2}{R^4 T_1^2} \right)^{1/3} \quad (32)$$

If we now define $t_{0*} = 1/2\beta_*$, the relaxation time of the central temperature, we find

$$t_{0*} = \left(\frac{R^4 T_1^2}{16} \right)^{1/3} \quad (33)$$

The source strength Q is related to R and T_1 by

$$Q = \pi R^2 T_1 \quad (34)$$

so that

$$t_{0*} = \left(\frac{Q^2}{16\pi^2} \right)^{1/3} \quad (35)$$

Whereas in the spherical case Q has the dimensions of J, in the cylindrical case it has the dimensions of J·m⁻¹. So in ordinary units,

$$t_{0*} = \frac{1}{K} \left(\frac{Q^2 S}{16\pi^2} \right)^{1/3} \quad (36)$$

An energy deposition Q of 1 J·cm⁻¹ results in a relaxation time of 13.2 ms.

References

1. Dresner, L., *Advances in Cryogenic Engineering* **29** (1984) 323.
2. Lottin, J.C., Van Sciver, S.W. Proc 9th Intrn Cryo Engr Conf, Butterworths (1982) 269.

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Figure captions

Fig. 1 The rectangle in the (r, t) plane used in proving the minimum principle for the differential inequality (12).

Fig. 2 A uniform initial temperature rise in the sphere of radius R compared with the initial temperature distribution of a similarity solution.

Fig. 3 An auxiliary sketch depicting the relation of the real root of Eq. (18) to the coefficient p .

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