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MASTER

METHOD OF VIBRATIONAL CONTROL IN
THE PROBLEM OF STABILIZATION
OF CHEMICAL REACTORS

Progress Report
for Period July 1, 1981 - June 30, 1982

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Abstract - This annual progress report describes the results obtained during the second year of the three years study of vibrational control application to the problem of stabilization of chemical reactors. The work was performed at the Department of Electrical Engineering, Illinois Institute of Technology, and sponsored by the Division of Engineering, Mathematical and Geosciences, Office of Basic Energy Sciences of the U.S. Department of Energy. The main objective of the second year study was development of a nonlinear approach to analytical and numerical investigation of the properties of chemical reactors with oscillating parameters. This objective was achieved and the following results were obtained:

1. Nonlinear vibrational control theory, developed in the first year of the study, was applied to two types of chemical reactors --- Catalytic Reactor and Continuous Stirred Tank Reactor. It was shown that introduction of *sufficiently fast* oscillations results in a decrease of the negative slope part (unstable area) of the reactor steady state characteristic. Consequently, the steady states, corresponding to larger productivity, are stabilized.
2. A simulation program (based on the Gear's algorithm) was developed and the numerical investigation of reactors with oscillating parameters was carried out. Experiments with the program showed that:

- α). the averaged description yields an adequate precision if $\epsilon = 1/T_r \omega \approx 0.16$ (ω is the frequency of introduced vibrations and T_r is the rise time of the system without vibrations).
- β). admissible vibrations could result in an increase in productivity of the reactor by a factor of $2 \div 3$.

Also, on the basis of numerical simulations, the following phenomena were discovered:

- (1) Stabilization of unstable steady states by "small and slow" oscillations;
- (2) Enlargement of the attraction domain of stable steady states due to "medium" amplitude and frequency oscillations.
- (3) Destruction of limit cycles by "any" types of oscillations.

None of these phenomena has theoretical explanations.

- 3. A basis of vibrational control theory for distributed parameter systems was established. This opened the possibility for vibrational control applications to chemical reactors described by partial differential equations.

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I. INTRODUCTION

In this annual progress report, the results obtained during the second year of the three years study of vibrational control application to chemical reactors are presented. The goals of the second year, as formulated in the Renewal Proposal submitted in March of 1981, were:

1. Verification of linear vibrational control approach to the problem of stabilization of chemical reactors;
2. Nonlinear vibrational control approach to chemical reactors;
3. Experimental analysis of a vibrationally controlled reactor.

The goals 1 and 2 were accomplished and the results are presented below. The experiment could not have been carried out due to complications in setting up the experimental device. Consequently, the problem of vibrational control of distributed parameter systems (originally scheduled for the third year of the research) was solved during the second year, and the completion of the experiment was rescheduled for the third year.

In Section II below, the results obtained along the lines of nonlinear vibrational control approach to chemical reactors are briefly described (see also Appendix 1 where these results are presented in sufficient details). Section III describes the phenomena discovered during the numerical simulations of vibrationally controlled reactors. Explanation of these phenomena is unknown. (Appendix 2 contains the simulation program for CSTR with a vibrating parameter). In Section IV a summary of vibrational control theory for distributed parameter systems is given (see Appendix 3 for full presentation). In Section V the information is given on the papers published and submitted for publications during the second year as well as on the percentage of time spent by the principle investigator on this project.

II. NONLINEAR VIBRATIONAL CONTROL APPROACH TO CHEMICAL REACTORS

In order to describe the class of systems analyzed in this project, the following definition is introduced:

Definition 1. A system with n states $x = [x_1, \dots, x_n]'$ is referred to as a *system with Arrhenius dynamics* (or just an *Arrhenius system*) if its behavior is characterized by at least two positive parameters, A and B , in such a manner that the norm $\|x_s(A, B)\|$ of a steady state solution $\dot{x} = 0$ has the form shown in Fig. 1.

A simple example of a system with Arrhenius dynamics can be given as:

$$\dot{x} = -x + A(1-x)e^{Bx}, \quad 0 \leq x \leq 1. \quad (1)$$

Obviously, the steady state characteristic is defined here by

$$A = \frac{x_s}{1-x_s} e^{-Bx_s}, \quad 0 \leq x_s \leq 1,$$

which for small B 's has a form similar to curve 1 in Fig. 1, and for large B 's similar to curve 2.

A large number of equations of different mathematical nature can be recognized as systems with Arrhenius dynamics. These could be ordinary and partial differential equations or other types of dynamic operators. Arrhenius dynamics appears naturally in the systems where nonlinearities are of the form:

$$e^{-bx_i} \frac{\phi_1(x_j)}{\phi_2(x_j)}. \quad (2)$$

Here x_i and x_j are components of the state vector x (possibly $i=j$), $\phi_s(x)$,

FIG. 1.

$s = 1, 2$, are polynomials in x , and b is a constant.

Every chemical or biochemical process described by Arrhenius law is, due to (2), a system with Arrhenius dynamics. Therefore, majority of chemical reactors are Arrhenius systems. Here $||x_s(A,B)||$ has interpretation as the productivity of the reactor in a steady state $x_s(A,B)$, and maximization of this productivity in a stable steady state is an important practical goal. Since in most instances B happens to be larger than B^* and since the upper branch of curve 2 (Fig. 1) is not acceptable because of technological reasons, parameter A (which is usually identified with the inverse of the input flow rate or the external temperature) is chosen in such a manner that the operating point $||x_s^*||$ is close to m_1 (with a runaway margin R_m - see Fig. 1). Because of that, fluctuations in input flow rate or external temperature could remove the system in the instability area and cause a "runaway". To prevent this, and to keep the system in a desirable x_s^* , in many cases a feedback cooling system is utilized. However, the dynamics of this feedback is slower than that of possible runaways. That is why unstable behavior could be observed even if the feedback control is present, and that is why *new control techniques for chemical reactors are desirable*.

In this Section, the results of nonlinear vibrational control [1] application to Arrhenius systems are described.

Assume that vibrations of the form

$$A_1 = A_0 + Df(\omega t) \quad (3)$$

are introduced in parameter A of an Arrhenius system. Here A_0 and D are positive constants and $f(\omega t)$ is a periodic zero mean scalar function with frequency ω .

It is evident that if ω is small enough, stabilization of an unstable

steady state cannot be achieved since in the limit $\omega \rightarrow 0$ the system just stays in a (possibly, unstable) steady state, x_s , defined by $A_0 + \text{const.}$ Therefore, it is equally evident that to achieve the desired effect, it is necessary to assume that

$$\omega \gg \frac{1}{T_r}, \quad (4)$$

where T_r is the rise time of the system. The rise time is defined as the time necessary to reach a sufficiently small neighborhood of the equilibrium position, starting from an initial deviation of order 1.

Introducing relative time

$$\tau = \frac{t}{T_r},$$

from (3) we find that

$$A_1 = A_0 + Df\left(\frac{\tau}{\epsilon}\right),$$

where, as it follows from (4),

$$\epsilon = \frac{1}{T_r \omega} \ll 1. \quad (5)$$

Because of formal reasons, we assume that

$$D = \frac{\alpha}{\epsilon}, \quad 0 < \alpha < 1. \quad (6)$$

This does not necessarily imply that the amplitude of oscillations is large, since α can be sufficiently small. However, (6) describes also large amplitudes, for instance, the case where $\alpha/\epsilon f(t/\epsilon)$ is a sequence of large impulses added to a constant part of the form $A_0 - 1/T \int_0^T \delta(t-kT)dt$, $k = 0, 1, \dots$; here $\delta(\cdot)$ is a δ -function and T is a period of $f(t/\epsilon)$.

Thus, the oscillating parameter in an Arrhenius system we represent in the form:

$$A_1 = A_0 + \frac{\alpha}{\epsilon} f\left(\frac{t}{\epsilon}\right), \quad (7)$$

where t is time normalized to T_r .

Definition 2. A steady state x_s of an Arrhenius system is said to be vibrationally stabilizable if for every δ there exists $0 < \epsilon_0 \ll 1$ such that the Arrhenius system with vibrations of the form (5), (7) and $0 < \epsilon < \epsilon_0$ has an asymptotically stable periodic solution $x^S(t/\epsilon)$, $-\infty < t < \infty$, characterized by

$$||\bar{x}^S(t/\epsilon) - x_s|| < \delta,$$

where $\bar{u}(\tau)$ is the averaged value of $u(\tau)$.

During the second year of the study, sufficient conditions of vibrational stabilizability for two examples of Arrhenius systems (Catalytic Reactor and CSTR) were given. It was shown that the behavior of an Arrhenius system with vibrations (5), (7) is, in a sense, equivalent to the behavior of a *new* Arrhenius system (with constant A), the steady state characteristic of which, $||y_s(A,B)||$, has a smaller instability area and the upper branch located lower than that of the original system. The mentioned in Definition 1 periodic solution $x^S(t/\epsilon)$ is centered around the corresponding steady state, y_s , of the new Arrhenius system, and the asymptotic stability of $x^S(t/\epsilon)$ follows from the asymptotic stability of y_s . (For a full description of these results see Appendix 1).

The above mentioned work involved efforts of Dr. Richard Bellman (USC), Mr. Joseph Bentsman (Ph.D. student, IIT) and the principal investigator. At

the early stages of CSTR numerical simulation, Mr. K. Tokarz (IIT pre-doctoral research associate) took part.

The results presented in Appendix 1 show that *linear* vibrational control theory cannot be considered as an instrument of analysis and design of chemical reactor with oscillating parameters. Indeed, vibrational stabilization of catalytic reactor (Sections III and IV of Appendix 1) cannot be explained by the linear vibrational control theory which states that vibrational stabilization of a first order *linear* system is impossible. However, nonlinear approach as well as numerical simulations, reported in Appendix 1, show that this is indeed possible.

III. DISCOVERED PHENOMENA

Numerical simulations of CSTR were carried out for the following model:

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + Da(1-x_1)\exp\{x_2\}, \\ \frac{dx_2}{dt} &= -x_2 + BDa(1-x_1)\exp\{x_2\}, \quad 0 \leq x \leq 1.\end{aligned}\tag{8}$$

Reactor with vibrations was simulated as

$$\begin{aligned}\frac{dx_1^1}{dt} &= -(1 + C \sin \omega t)x_1^1 + Da(1-x_1^1)\exp\{x_2^1\}, \\ \frac{dx_2^1}{dt} &= -(1 + C \sin \omega t)x_2^1 + BDa(1-x_1^1)\exp\{x_2^1\}, \quad 0 \leq C \leq 1.\end{aligned}\tag{9}$$

The averaged equation, corresponding to (9), was modeled as

$$\begin{aligned}\frac{dy_1}{dt} &= -y_1 + Da\{1-y_1 + \gamma^2[1-y_2-y_1y_2 + y_2^2-y_1y_2^2]\}\exp\{y_2\}, \\ \frac{dy_2}{dt} &= -(1+\beta)y_2 + BDa\{1-y_1 + \gamma^2[1-y_1y_2 + y_2^2-y_1y_2^2]\}\exp\{y_2\},\end{aligned}\tag{10}$$

where

$$\gamma^2 = \frac{C^2}{4\omega^2}.$$

Numerical solutions of (8)-(10) for identical initial conditions and various values of parameters C and ω , discovered the following phenomena:

1. Stabilization of Unstable Steady States by Small and Slow Oscillations

$$(A = \alpha\epsilon, \omega \leq \epsilon, \quad 0 < \epsilon \ll 1, \alpha \sim 1).$$

It is possible to show that in a *linear system* small and slow vibrations

do not change the dynamics. This is not the case for system (8). Figs. 2-5 illustrate this point. As far as we know, this effect has not been known before and does not have theoretical explanation. This effect could be very important from the practical point of view since here the stabilization is achieved when $A \approx 0.08$ and $\omega \approx 0.1$. (Not much energy is spent on introduction of this oscillation!)

2. Enlargement of Attraction Domain of Stable Steady States due to Medium Oscillations ($A \sim 1, \omega \sim 1$)

This effect is illustrated by Figs. 6-11. They show that

- (1). For medium vibrations, averaged equation "does not work".
- (2). System (9) has larger attraction area than systems (8) and (10).
- (3). Oscillations convert the attraction area of (8) into a *connected* set and enlarge the area of this set (see Fig. 11).

None of these phenomena is explained theoretically. They could also be important practically since the enlargement of the attraction area of stable steady states decreases the pathological effects of disturbances.

3. Destruction of Limit Cycles. (Figs. 12-16.)

Any type of oscillations destroy the limit cycles converting their attraction domain into the attraction domain of the stable steady state. The explanation of this effect is also unknown.

The discovered effects are summarized in the Table attached.

$B=6.30$, $BT=0.50$, $DA=0.1220$, $AM=0.08$, $OM=0.1000$, $AV=0.1600$,

FIG. 2

$B=6.30$, $BT=0.50$, $DA=0.1220$, $AM=0.16$, $OM=0.2000$, $AV=0.1600$,
 $X_{10}=0.156$, $X_{20}=0.65$, $T=98.99$, $DT=0.02000$

FIG. 3

$B=6.30$, $BT=0.50$, $DA=0.1220$, $AM=0.08$, $OM=0.1000$, $AV=0.1600$,
 $X10=0.156$, $X20=0.65$, $T=580.0$, $DT=0.04000$

FIG. 4

$B=6.30$, $BT=0.50$, $DA=0.1220$, $AM=0.08$, $OM=0.1000$, $AV=0.1600$,
 $X10=0.156$, $X20=0.65$, $T=580.0$, $DT=0.04000$

FIG. 5

B=16.00, BT=2.00, DA=0.0830, AM=0.00,
X10=0.350, X20=2.09, T=4.00, DT=0.00100

FIG. 6

$B=16.00$, $BT=2.00$, $DA=0.0830$, $AM=0.00$,
 $X10=0.350$, $X2C=2.09$, $T=4.00$, $DT=0.00100$

FIG. 7

B=16.00, BT=2.00, DA=0.0830, AM=1.50, OM=4.0000, AV=0.0351

FIG. 8

$B=16.00$, $BT=2.00$, $DA=0.0830$, $AM=1.50$, $OM=4.0000$, $AV=0.0351$
 $X_{10}=0.350$, $X_{20}=2.09$, $T=14.00$, $DT=0.00100$

FIG. 9

FIG. 10

$B=16.00$, $BT=2.00$, $DA=0.0830$, $AM=0.00$,
 $X10=0.350$, $X20=2.09$, $T=4.00$, $DT=0.00100$

① - Lower eq. position
 ② - Upper eq. position

FIG. 11

$B=16.00000$
 $BETA=2.00000$
 $DA=0.08300$
 $ACTIV. ENERGY=INFIN.$
 $X2C=0.00000$

$X10=0.91200$
 $X20=4.80000$
 $AMPL = 0.0$

FIG. 14

FIG. 15

During the third year of the study we plan to work on the explanation of the above mentioned effects. Evidently, this work will require a substantial theoretical development. That is why participation of a highly qualified consultant will be extremely beneficial for the project.

IV. VIBRATIONAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

Consider an equation of the form:

$$\frac{\partial^2}{\partial t^2} A\{u(x,t)\} + \frac{\partial}{\partial t} B\{u(x,t)\} + C\{u(x,t)\} = 0, \quad (11)$$

where $u(x,t)$ is an m -dimensional vector function of $n+1$ variables $\{x_1, \dots, x_n, t\}$ and A , B , and C are linear partial differential matrix operators; their domain defines the boundary conditions for (11).

Equation (11) describes a large number of linear distributed parameter systems under static "forces" of different nature.

Assume now that a periodic zero mean "force" of the form $\alpha/\epsilon f(t/\epsilon)$, $0 < \epsilon \ll 1$, is introduced. Assume also that it would result in an additional term $\alpha/\epsilon f(t/\epsilon) \mathcal{D}\{u(x,t)\}$ in equation (11) where $\mathcal{D}$ is again a linear partial differential matrix operator (in many cases, $\mathcal{D} \equiv C$). Thus, the dynamics defined by periodic forces is described as

$$\frac{\partial^2}{\partial t^2} A\{u\} + \frac{\partial}{\partial t} B\{u\} + [C + \frac{\alpha}{\epsilon} f(\frac{t}{\epsilon}) \mathcal{D}]\{u\} = 0. \quad (12)$$

Under which condition there exists $\frac{\alpha}{\epsilon} f(\frac{t}{\epsilon})$ such that system (12) exhibits a stable behavior whereas system (11) might be unstable? To answer this question, assume that

$$u(x,t) = \sum_{k=1}^N q_k(t) \psi_k(x), \quad (13)$$

where $\psi_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $k = 1, \dots, N$, is a sufficiently representative set of vector-valued functions which satisfy the boundary condition for $u(x,t)$, and q_k is a scalar. In this case the following set of ordinary differential equa-

tions is obtained:

$$A\ddot{q} + B\dot{q} + Cq = 0, \quad (14)$$

where $q = [q_1, \dots, q_N]^T$, and A , B and C are constant matrices with the following elements, consequently,

$$a_{ij} = (A\psi_j, \psi_i) ; \quad b_{ij} = (B\psi_j, \psi_i) ; \quad c_{ij} = (C\psi_j, \psi_i). \quad (15)$$

For equation (12) we obtain:

$$A\ddot{q} + B\dot{q} + [C + \frac{\alpha}{\epsilon} f(\frac{t}{\epsilon})D]q = 0, \quad (16)$$

where A , B , and C are as in (15) and

$$D = ||d_{ij}||_{i,j=1}^n = ||D\psi_j, \psi_i||.$$

Definition 3. Equation (11) is said to be stable if the equilibrium position of (14) is asymptotically stable.

Definition 4. Equation (12) is said to be stable if (16) has an asymptotically stable periodic solution $q^*(t/\epsilon)$, $-\infty < t < \infty$, such that $||q^*(t/\epsilon)|| < \delta(\epsilon)$, where $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Definition 5. System (11) is vibrationally stabilizable if there exists a periodic zero mean function $\alpha/\epsilon f(t/\epsilon)$ such that equation (12) is stable.

The definition of vibrational stabilizability of partial differential equations differs from that for ordinary differential equations (see [1] and [2]). The reason is in the following: In the later case vibrations are intro-

duced in the form $B(t)$ or $\alpha/\epsilon B(t/\epsilon)$ where B is a periodic *matrix* chosen arbitrarily. In the former case D cannot be chosen at will and only a *scalar* $\alpha/\epsilon f(t/\epsilon)$ can be chosen arbitrarily. That is why general theorems of [1] and [2] cannot be applied to the problem under consideration.

In Appendix 3, a sufficient condition of vibrational stabilizability of linear distributed parameter systems is given. Also an example of stabilization of a nonlinear partial differential equation is described.

The results presented in Appendix 3 lay a ground for vibrational control of chemical reactors described by partial differential equations. Ammonia production will be considered during the third year of the study as an example of this situation.

V. PRESENTATION OF THE RESULTS OBTAINED
DURING THE SECOND YEAR OF THE STUDY

During the second year of the study (July 1, 1981 - June 30, 1982)

1. A paper entitled "Condition of Vibrational Stabilizability for a class of Nonlinear Systems" appeared in *IEEE Transactions on Automatic Control*, vol. AC-27, pp. 485-487, April 1982.
2. A paper entitled "Vibrational Control: Nonlinear Theory" was presented at the 20th IEEE Conference on Decision and Control (San Diego, December 1981).
3. A paper entitled "Vibrational Control - An Alternative Method of Control for Distributed Parameter Systems" was accepted for presentation at the 3rd IFAC Symposium on Control of Distributed Parameter Systems (Toulouse, France, June 29 - July 2, 1982).
4. A paper entitled "Vibrational Control of a Nonlinear System" was accepted for presentation at the IFAC Workshop on Singular Perturbations and Robustness of Control Systems (Ohrid, Yugoslavia, July 13 - 16, 1982).
5. A paper entitled "Vibrational Control of Systems with Arrhenius Dynamics" (with R. Bellman and J. Bentsman) was submitted for presentation at the 21st IEEE Conference on Decision and Control (December, 1982) and for publication in the *Journal of Mathematical Analysis and Applications*.
6. A paper entitled "Vibrational Control of Distributed Parameter Systems" was submitted for presentation at the 21st IEEE Conference on Decision and Control (December 1982) and for publication in *IEEE Transactions on Automatic Control*.

During the period of July - August, 1981, the principal investigator spent 100% of time on this project; September - November, 1981 - 20% of time; December 1981 - January 1982 - 50% of time; February - March, 1982 - 20% of time. Projected activity in April - May 1981 - 20% of time; June - 100% of time.

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A P P E N D I C E S

VIBRATIONAL CONTROL OF SYSTEMS

WITH ARRHENIUS DYNAMICS

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ABSTRACT

A nonlinear system with S-shape steady state characteristic is referred to as a system with Arrhenius dynamics. The negative slope part of the S-shape curve represents a set of unstable steady states. Using two examples of Arrhenius systems (catalytic reactor and continuous stirred tank reactor), it is shown that introduction of sufficiently fast oscillations in system's parameters generates a new Arrhenius system, the steady state characteristic of which has a smaller negative slope part. Results of analytical investigation as well as numerical simulation are presented. It is shown that vibrational stabilization of Arrhenius systems gives an increase in productivity of the plants.

I. INTRODUCTION

Definition 0. A system with n states $x = [x_1, \dots, x_n]'$ is referred to as a *system with Arrhenius dynamics* (or just an *Arrhenius system*) if its behavior is characterized by at least two positive parameters, A and B , in such a manner that the norm $\|x_s(A, B)\|$ of a steady state solution $\dot{x} = 0$ has the form shown in Fig. 1.

A simple example of a system with Arrhenius dynamics can be given as:

$$\dot{x} = -x + A(1-x)e^{Bx}, \quad 0 \leq x \leq 1. \quad (1)$$

Obviously, the steady state characteristic is defined here by

$$A = \frac{x_s}{1-x_s} e^{-Bx_s}, \quad 0 \leq x_s \leq 1,$$

which for small B 's has a form similar to curve 1 in Fig. 1, and for large B 's similar to curve 2.

A large number of equations of different mathematical nature can be recognized as systems with Arrhenius dynamics. These could be ordinary and partial differential equations or other types of dynamic operators. Arrhenius dynamics appears naturally in the systems where nonlinearities are of the form:

$$e^{-bx_i} \frac{\phi_1(x_j)}{\phi_2(x_j)}. \quad (2)$$

Here x_i and x_j are components of the state vector x (possibly $i=j$), $\phi_s(x)$, $s = 1, 2$, are polynomials in x , and b is a constant.

Obviously, due to Definition 0 and (2), systems with Arrhenius dynamics can be recognized as a specific subclass of nonlinear systems [1].

Arrhenius systems are characterized by rich dynamics. Indeed, if $B > B^*$, the system has multiple (stable and/or unstable) steady states; hysteresis behavior is observed if parameter A is slowly varied; limit cycles are known to

exist in the vicinity of some steady states. Therefore, Arrhenius systems are of a *considerable system-theoretic interest*.

On the other hand, these systems are important from the practical stand point as well. Indeed, every chemical or biochemical process described by Arrhenius law is, due to (2), a system with Arrhenius dynamics. Therefore, majority of chemical reactors are Arrhenius systems. Here $||x_s(A,B)||$ has interpretation as the productivity of the reactor in a steady state $x_s(A,B)$, and maximization of this productivity in a stable steady state is an important practical goal. Since in most instances B happens to be larger than B^* and since the upper branch of curve 2 (Fig. 1) is not acceptable because of technological reasons, parameter A (which is usually identified with the inverse of the input flow rate or the external temperature) is chosen in such a manner that the operating point $||x_s^*||$ is close to m_1 (with a runaway margin R_m - see Fig. 1). Because of that, fluctuations in input flow rate or external temperature could remove the system in the instability area and cause a "runaway". To prevent this, and to keep the system in the desirable x_s^* , in many cases a feedback cooling system is utilized. However, the dynamics of this feedback is slower than that of possible runaways. That is why unstable behavior could be observed even if the feedback control is present, and that is why *new control techniques for chemical reactors are desirable*.

In the present paper, vibrational control approach [2], [3] is applied to the problem of stabilization of systems with Arrhenius dynamics. It is said [3] that equilibrium position x_s is vibrationally stabilized if introduction of zero mean periodic oscillations in system's parameters causes a bifurcation of x_s into an asymptotically stable limit cycle $x^s(t)$ having the averaged value close to x_s . In case of Arrhenius systems, A and B are the parameters where zero mean oscillations are introduced. The theoretical aim of this paper is to

give conditions under which an unstable steady state of an Arrhenius system can be vibrationally stabilized. The practical goals of this paper are: 1). to show that some of the unstable steady states, where productivity is large, are vibrationally stabilizable; 2). to give recommendations on the choice of amplitudes and frequencies of the vibrations which ensure the desired properties of a vibrationally stabilized system; 3). to show that nonlinear vibrational control technique permits to predict the benefits and drawbacks that might be expected after introduction of vibrations.

All the material of this paper is based on two examples of systems with Arrhenius dynamics, one being a Catalytic Reactor, which is considered in Sections III and IV, and the second - Continuous Stirred Tank Reactor discussed in Sections V and VI.

Remark 1. Dynamics of chemical reactors with Arrhenius kinetics was a subject of analysis of numerous publications, starting from classical papers [4]-[5] followed by [6]-[9]. We base our development on the models derived in papers [10], [11] (catalytic reactor) and extremely complete and detailed papers [12], [13] (continuous stirred tank reactor).

Remark 2. There exists also vast literature on oscillations as a tool of control of chemical reactors (see a review [14]). Conditionally, this literature can be divided into three groups: α). Periodic Optimization; β). Push-pull control; and γ). Asynchronous Quenching.

In group α), a use is made of the fact that in a nonlinear system with non-convex admissible velocity set, introduction of proper oscillations generates a new "effective" velocity set which is a convex hull of the original one. Therefore, the optimized function could assume "better" values being defined on the enlarged velocity set. In other words, in the works of group α), a usual

statement of optimal control problem is complemented by a condition:

$$x(0) = x(T),$$

where T is an unknown period, and the problem is to find

$$\max_{u \in U; T} \frac{1}{T} \int_0^T f_0(x, u) dt.$$

This approach was discussed, for instance, in [15]-[19]. The main difference between periodic optimization and vibrational control is that the former one is a feedback control whereas the latter one is not. Note that in chemical reactors, some of the states, required for feedback, are hardly measurable (for example, the rate of conversion). The other difference is that periodic optimization is a substantially nonlinear phenomena whereas vibrational control is not (see, for example [2]).

In group β) (see [20]-[21]) it was assumed that a state, which is to be stabilized, is located between two unstable steady states. In this case a feedback procedure can be specified under which the operating point is being "push-pulled" from one unstable steady state to another. As a result, a system is oscillating around the desired state. The difference between group β) and vibrational control is the same as between group α) and vibrational control.

Group γ) (see [22]-[25]) is the closest to vibrational control. In [22]-[24] time dependent vibrations were introduced in system's parameters in order to modify the dynamics. However, the conditions under which vibrational stabilization can be achieved as well as the properties of necessary vibrations were not derived. In [25], a condition of vibrational stabilization of runaway in catalytic reactor was given. The relationship between this condition and the one derived in this paper is discussed in Sections III and IV.

II. NOTATIONS AND DEFINITION

Assume that vibrations of the form

$$A_1 = A_0 + Df(\omega t) \quad (3)$$

are introduced in parameter A of an Arrhenius system. Here A_0 and D are positive constants and $f(\omega t)$ is a periodic zero mean scalar function with frequency ω .

It is evident that if ω is small enough, stabilization of an unstable steady state cannot be achieved since in the limit $\omega \rightarrow 0$ the system just stays in a (possibly, unstable) steady state, x_s , defined by $A_0 + \text{const}$ (see Section IV). Therefore, it is equally evident that to achieve the desired effect, it is necessary to assume that

$$\omega \gg \frac{1}{T_r}, \quad (4)$$

where T_r is the rise time of the system. The rise time is defined as the time necessary to reach a sufficiently small neighborhood of the equilibrium position, starting from an initial deviation of order 1.

Introducing relative time

$$\tau = \frac{t}{T_r},$$

from (3) we find that

$$A_1 = A_0 + Df\left(\frac{\tau}{\epsilon}\right),$$

where, as it follows from (4),

$$\epsilon = \frac{1}{T_r \omega} \ll 1. \quad (5)$$

Because of formal reasons, we assume that

$$D = \frac{\alpha}{\epsilon}, \quad 0 < \alpha < 1. \quad (6)$$

This does not necessarily imply that the amplitude of oscillations is large, since α can be sufficiently small. However, (6) describes also large amplitudes, for instance, the case where $\alpha/\epsilon f(t/\epsilon)$ is a sequence of large impulses added to a constant part of the form $A_0 = 1/T \int_0^T \delta(t-kT)dt$, $k = 0, 1, \dots$; here $\delta(\cdot)$ is a δ -function and T is a period of $f(t/\epsilon)$ (see Section VII where it is indicated that δ -function is the most efficient shape of vibrations).

Thus, the oscillating parameter in an Arrhenius system we represent in the form:

$$A_1 = A_0 + \frac{\alpha}{\epsilon} f\left(\frac{t}{\epsilon}\right), \quad (7)$$

where t is time normalized to T_r .

Definition 1. A steady state x_s of an Arrhenius system is said to be vibrationally stabilizable if for every δ there exists $0 < \epsilon_0 \ll 1$ such that the Arrhenius system with vibrations of the form (5), (7) and $0 < \epsilon < \epsilon_0$ has an asymptotically stable periodic solution $x^s(t/\epsilon)$, $-\infty < t < \infty$, characterized by

$$||\bar{x}^s(t/\epsilon) - x_s|| < \delta,$$

where $\bar{u}(\tau)$ is the averaged value of $u(\tau)$.

In the following Sections, sufficient conditions of vibrational stabilizability for two examples of Arrhenius systems are given. It is shown that the behavior of an Arrhenius system with vibrations (5), (7) is, in a sense, equivalent to the behavior of a *new* Arrhenius system (with constant A), the steady state characteristic of which, $||y_s(A,B)||$, has a smaller instability area. The mentioned in Definition 1 periodic solution $x^s(t/\epsilon)$ is centered around the corresponding steady state, y_s , of the new Arrhenius system, and the asymptotic stability of $x^s(t/\epsilon)$ follows from the asymptotic stability of y_s .

III. VIBRATIONAL CONTROL OF CATALYTIC REACTOR: THEORY

In [10], [11] and [25], a set of nonlinear partial differential equations, which describe heat and mass transfer in a spherically shaped catalytic pellet, was reduced to a dimensionless first order ordinary differential equation which we represent in the form:

$$\frac{dx}{dt} = A - g(x, B), \quad B > B^*, \quad 0 \leq x \leq 1, \quad (8)$$

where x is the temperature in the reactor, t is time expressed in units of T_r , A is external temperature, B is parameter defined by the reacting substances, and $g(x, B)$, $B \geq B^*$, is of the form shown in Fig. 2.

Since in this application, parameter B remains constant and is not being vibrated, we omit B , denoting $g(x, B) = g(x)$.

Obviously, (8) is a system with Arrhenius dynamics since $x_s(A)$ has a form as in Fig. 1.

Stability analysis in the first order system (8) is trivial: the steady states on the negative slope part of the curve in Fig. 3 are unstable (including points K_0 and R_0), whereas all other points x_s are asymptotically stable. Consequently, $x_s \in [m_1, m_2]$ are not accessible in a steady state operation. Since the productivity of the reactor is a monotonically increasing function of x and since the upper branch of $g(x)$ is not acceptable due to technological reasons, stabilization of $x_s \in [m_1, m_2]$ is an important practical problem. To address this problem we assume, following [25], that parameter A is oscillating according to the law given in (7). As a result, the following periodic equation is obtained:

$$\dot{x} = (A + \frac{\alpha}{\epsilon} f(t/\epsilon)) - g(x). \quad (9)$$

Theorem 1. Assume that $g(x)$ is an analytic function on $x \in [0, 1)$. In this case x_s is vibrationally stabilizable if there exists A such that the equation

$$\dot{y} = A - [g(y) + \gamma^2 \frac{d^2 g}{dy^2}], \quad (10)$$

$$\gamma^2 = \frac{\alpha^2}{2} [\phi(t/\epsilon)]^2, \quad (11)$$

$$\phi(\tau) = \int_0^\tau f(\tau) d\tau - \sqrt{\int_0^\tau f(\tau) d\tau} \quad (12)$$

has a steady state $y_s = x_s$ which is asymptotically stable.

Remark 3. Equation (10) is a new system with Arrhenius dynamics, the steady state characteristic of which is defined by

$$g_1(x) = g(x) + \gamma^2 \frac{d^2 g}{dx^2}. \quad (13)$$

Since $d^2 g/dx^2$ is negative in the vicinity of the maximum of $g(x)$ and positive in the vicinity of the minimum of $g(x)$, the relative positions of $g(x)$ and $g_1(x)$ are as it is shown in Fig. 4.

Therefore, introduction of fast oscillations results in modified steady state characteristic $g_1(x)$, lower branch of which yields larger x_s (productivity) than the lower branch of $g(x)$.

Remark 4. If a specific analytic form of $g(x)$ is assumed, $g_1(x)$ can be analyzed in more details. Fig. 5 shows $g(x)$ and $g_1(x, \gamma^2)$ for

$$g(x) = \frac{x}{1-x} e^{-5.5x}.$$

Note that the area of instability (negative slope part of the curves) is decreasing as γ^2 is increasing.

Remark 5. Assume that system (8) has $A = A_K$ which results in an unstable steady state x_s^* . As it follows from Fig. 5, with $\gamma^2 = 0.044$, x_s^* is asymptotically stable, and therefore system (9) will oscillate around x_s^* in asymptotically stable manner, if $A = A_1 < A_K$. This means that in order to keep the system in x_s^* , parameter A should be decreased to the value of A_1 along with introduction of vibrations. Otherwise after introduction of vibrations the system will jump onto the upper branch of $g_1(x)$. This conclusion differs from the result of [25] which states that introduction of vibrations creates an asymptotically stable steady state in the vicinity of point K when A_K is unchanged. The numerical experiments reported in Section IV support our conclusion.

Proof of Theorem 1. Introducing $\tau = t/\epsilon$ and a substitution

$$x(\tau) = z(\tau) + \alpha\phi(\tau), \quad (14)$$

where $\phi(\tau)$ is defined by (12), we rewrite (9) as

$$\frac{dz}{d\tau} = \epsilon[A - g(z + \alpha\phi(\tau))]. \quad (15)$$

This is an equation in so called standard form. Since $g(x)$ is analytic, the averaging principle [26] is applicable to (15). Application of this principle yields (with accuracy to $O(\alpha^4)$):

$$\frac{dy}{d\tau} = \epsilon[A - (g(y) + \gamma^2 \frac{d^2 g}{dy^2})], \quad (16)$$

where γ^2 is defined in (11).

If the equilibrium position, y_s , of equation (16) is asymptotically stable, the second theorem of the averaging principle [26, p. 497] guarantees that for every $\delta > 0$ there exists ϵ_0 such that equation (15) with $0 < \epsilon < \epsilon_0$ has an asymptotically stable periodic solution, $z^s(\tau)$, $-\infty < \tau < \infty$, characterized by

$$|z^s(\tau) - y_s| < \delta, \quad -\infty < \tau < \infty. \quad (17)$$

If there exists A such that (16) has an asymptotically stable equilibrium position $y_s = x_s$, from (14) and (17) follows:

$$|\bar{x}^s(\tau) - y_s| = |\bar{x}^s(\tau) - x_s| = |\bar{z}(\tau) - y_s| < \delta, \quad -\infty < \tau < \infty.$$

This, according to Definition 1, means that equilibrium position x_s is vibrationally stabilizable. Theorem is proven.

IV. VIBRATIONAL CONTROL OF CATALYTIC REACTOR:

NUMERICAL SIMULATIONS

Numerical simulations were carried out for a model of the form:

$$\frac{dx}{dt} = A_0 - \frac{x}{1-x} e^{-Bx}. \quad (18)$$

The reactor with introduced vibrations was described as:

$$\frac{dx_1}{dt} = A_0(1 + C \sin \omega t) - \frac{x_1}{1-x_1} e^{-Bx_1}. \quad (19)$$

Note that in (19) vibrations are introduced in a form showing the relative amplitude C.

The averaged equation (10) in this case is of the form:

$$\frac{dy}{dt} = A_0 - e^{-By} \left\{ \frac{y}{1-y} + \gamma^2 \left[\frac{2}{(1-y)^3} - \frac{2B}{(1-y)^2} + \frac{B^3 y}{1-y} \right] \right\}, \quad (20)$$

where

$$\gamma^2 = \frac{A_0^2 C^2}{4\omega^2}. \quad (21)$$

In the course of simulations, the properties of (18), (19) and (20) were comparatively analyzed. Solutions of (18)-(20) were obtained for various

values of parameters A_0 , B , C and ω under different initial conditions using the Gear's algorithm. The results are described below.

Estimation of ϵ_{max}

In Section II, small parameter ϵ was introduced as

$$\epsilon = \frac{1}{T_r \omega}.$$

For vibrations of (19)

$$\epsilon = \frac{T_0}{2\pi T_r}, \quad (22)$$

where T_0 is the period of $\sin \omega t$.

Fig. 6 represents the steady state characteristics, $x_s(A)$ and $y_s(A)$, of equations (18) and (20) (denoted as 1 and 2, respectively) and time solutions, $x(t)$, $x_1(t)$ and $y(t)$, of (18)-(20) (denoted as 3, 4 and 5, respectively*) for various T_0 , C and A . The value of ϵ in the experiment presented in Fig. 6A is

$$\epsilon = \frac{1}{\pi} \approx 0.32.$$

Evidently, this ϵ does not ensure an acceptable precision of averaged description since $\bar{x}^s$ does not approximate y_s .

In Fig. 6B, corresponding to $\omega = 0.25$,

$$\epsilon = \frac{1}{2\pi} \approx 0.16,$$

and $\bar{x}^s$ approximates $y(t)$ with 10% precision. We conclude that for the system under consideration

*) The same notations are used in Figs. 7-9. T in Figs. 6-9 denotes the simulation time.

$$\epsilon_{\max} \leq 0.16.$$

Figs. 6C, D illustrate that $\bar{x}^s \rightarrow y_s$ when $\epsilon \rightarrow 0$. Consequently, given δ we can find $\epsilon_0 \leq 0.16$ (or $\omega \geq 0.25$) such that

$$|\bar{x}^s - y_s| < \delta \quad \text{for} \quad t \geq T_r.$$

The discrepancy between $\bar{x}^s$ and y_s for $\epsilon = 0.32$ (Fig. 6A) is due to the fact that here solution $x_1(t)$ reaches the steady state corresponding to $A_0(1-C) = A_{\min}$. This means that under small frequencies, the dynamics is "partially eliminated" and the system is partially drifting along the steady state characteristic. Under these conditions vibrational control effect does not take place. For $\epsilon = 0.16$ (Fig. 6B) $A_{\min}$ is not reached and the averaged description is acceptable.

Increase in Productivity

Consider the original system (18) with $A = A_0$ (Fig. 7A). Denote the runaway margin, corresponding to the resulting operating point, x_s , as R_m . Along with (18), consider (19) and (20) with $C = C_1$ and $\omega = 0.25$ (which ensures $\epsilon = \epsilon_{\max}$). The steady state characteristic of (20) is shown in Fig. 7A for $C_1 = 0.5$; 1; and 1.5. Choose operating points on each of these characteristics in such a manner that the runaway margin, R_m , remains unchanged. Denote these operating points as $y_s(C_1)$. The ratio

$$e(C_1) = y_s(C_1)/x_s \tag{23}$$

describes the increase in productivity. Fig. 7A shows that $e(0.5) = 1.25$; $e(1) = 1.75$; $e(1.5) = 2.3$. Fig. 7B shows the time solution of (19) for $C_1 = 1.25$.

Note that $C_1 = 1.25$ is achieved if $f(\omega t)$ is a rectangular waveform with relative amplitude 1 (see Section VII).

The results presented in Figs. 7A, B are obtained for a system with $B = 5.5$. For other B 's conclusions might be different. For instance, if $B = 4.5$, oscil-

lations with $C_1 = 0.9$ and $\omega = 0.3$ (Fig. 7C) result in an Arrhenius system without instability area at all. Therefore, the increase in productivity is by a factor of 2÷3. Note that in Fig. 7C stabilization was achieved for a new steady state defined by the same A as in system (18) without vibrations. Fig. 7D shows the time solution for this situation.

Comparison with a Conclusion of [25]

Consider (18) with $A = A_{K_0}$ (see Fig. 8A where $x_s(A)$ and $x(t)$ defined by initial condition x_{K_0} are shown). Obviously if $x(t_0) > x_{K_0}$, $x(t) \rightarrow x_{M_0}$ as $t \rightarrow \infty$. In [25] was concluded that if vibrations $\mu \sin \omega t$ are added to A_{K_0} :

$$A_1 = A_{K_0} + \mu \sin \omega t, \quad (24)$$

where ω is large enough and $\mu \ll A_{K_0}$, an asymptotically stable periodic solution appears in the vicinity of K_0 . Since our results contradict this conclusion (see Fig. 8B which illustrates that introduction of vibrations moves the steady state characteristic to the left of K_0 and, therefore, $x(t) \rightarrow x_{M_1}$), an experiment was carried out for small μ and large ω . Fig. 8C shows the solution of (19) for $A = A_{K_0}$, $\omega = 2$, $C = 0.15$, from which we conclude that introduction of vibrations of the form (24) destabilizes point K_0 (in the sense that $x_1(x_{K_0}, t) \rightarrow x_{M_0}$), and the asymptotically stable periodic solution appears not in the vicinity of point K_0 but in the vicinity of point M_0 .

Analysis of the Domains of Attractions of $\bar{x}^s(t)$ and y_s

Theorem 1 gives a sufficient condition under which asymptotic stability of the equilibrium position, y_s , of (20) ensures the existence and asymptotic stability of the periodic solution, $\bar{x}^s(t)$, of (19) such that $|\bar{x}^s(t) - y_s|$ is small. However no formal results are known concerning the relationship between the attraction areas of y_s and $\bar{x}^s(t)$. Note that the attraction area V_1 , of y_s is easily established (see Fig. 9A): $V_1 = [0, y_0]$; attraction area, V_2 , of y_s ,

is: $V_2 = (y_0, 1]$.

Experiments show that if $x_1(t) \in V_1$ for every t , attraction areas of y_s and $x^s(t)$ coincide (Figs. 6). This is not necessarily true if $x_1(t) \in V_1$ for $t \in [t_0, t_1]$ and $x_1(t) \in V_2$ for $t \in [t_1, t_2]$. Fig. 9A illustrates this statement and shows that opposite phases of the introduced oscillations yield different time solutions for the same initial condition. Averaged system describes adequately only one solution. This calls for introduction of the "averaged" initial condition $\bar{y}_0$. Fig. 9B shows solution of (20) defined by $\bar{y}_0$ (computed from Fig. 9A as an average of the first two oscillations of $x_1(x_0, t)$). Fig. 9C gives another example. It looks plausible that the domains of attraction of $x^s(t)$ and y_s coincide if the initial condition for (20) is chosen not as $y(0) = x_1(0)$ but rather $y(0) = \bar{y}_0$. However, since $\bar{y}_0$ is calculated through $x_1(t)$, this statement is not constructive, and analytical investigation is required to obtain a condition with a predictive power.

V. VIBRATIONAL CONTROL OF CONTINUOUS

STIRRED TANK REACTOR (CSTR): THEORY

1. Behavior of CSTR Under Constant Input Flow [12]

Following [12], consider a first order exothermic, irreversible reaction $A \rightarrow B$ in a well stirred tank reactor under constant fresh feed rate. To simplify the calculations, we assume, without loss of generality, that the recycling as well as the cooling are absent. In this case, the kinetic equations of the reaction are:

$$\begin{aligned} V \frac{dc_A}{dt} &= F(c_{Af} - c_A) - V k_0 \exp\{-E/RT\} c_A, \\ V \rho C_p \frac{dT}{dt} &= \rho C_p F(T_f - T) + V(-\Delta H) k_0 \exp\{-E/RT\} c_A - hAT, \end{aligned} \quad (25)$$

where

A heat transfer area,
 c concentration,
 C_p specific heat,
 E activation energy,
 h heat transfer coefficient,
 ΔH heat of reaction,
 k_0 reaction rate constant,
 R universal gas constant,
 T temperature,
 V reactor volume,
 ρ density.

The subscripts denote:

A species A,
 f feed state.

As in [12], (25) can be reduced to a dimensionless form using the following notations:

$$\begin{aligned} x_1 &= \frac{c_{Af} - c_A}{c_{Af}} ; \quad x_2 = \frac{T - T_f}{T_f} \left(\frac{E}{RT_f} \right) ; \quad \tau = \frac{V}{F} ; \\ t &= \frac{t' F}{V} = \frac{t'}{\tau} ; \quad Da = \frac{k_0 e^{-\gamma} V}{F} = k_0 e^{-\gamma} \tau ; \\ B &= \frac{(-\Delta H) c_{Af}}{\rho C_p T_f} \left(\frac{E}{RT_f} \right) ; \quad \beta = \frac{hA}{F \rho C_p} = \frac{hA \tau}{V \rho C_p} ; \quad \gamma = \frac{E}{RT_f} . \end{aligned} \quad (26)$$

Dimensionless equations corresponding to (25) are:

$$\frac{dx_1}{dt} = -x_1 + Da(1-x_1) \exp\left[\frac{x_2}{1 + x_2/\gamma}\right] ,$$

$$\frac{dx_2}{dt} = -x_2 + BDa(1-x_1)\exp\left\{\frac{x_2}{1+x_2/\gamma}\right\} - \beta x_2, \quad 0 \leq x \leq 1.$$

To further simplify calculations, we assume (without loss of generality) that γ is large ($\gamma \rightarrow \infty$). In this case,

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 + Da(1-x_1)\exp\{x_2\}, \\ \frac{dx_2}{dt} &= -(1+\beta)x_2 + BDa(1-x_1)\exp\{x_2\}. \end{aligned} \tag{27}$$

The steady states, x_{1s} , x_{2s} , of (27) are defined by

$$\begin{aligned} Da &= \frac{x_{1s}}{1-x_{1s}} \exp\left\{-\frac{B}{1+\beta} x_{1s}\right\}, \\ x_{2s} &= \frac{B}{1+\beta} x_{1s}, \quad 0 < x < 1. \end{aligned} \tag{28}$$

Therefore,

$$||x_s|| = x_{1s} \left(1 + \frac{B^2}{(1+\beta)^2}\right)^{1/2},$$

which results in a steady state characteristic shown in Fig. 10. Identifying Da (Damköhler number) with A and $B/(1+\beta)$ with B from Definition 0, we conclude that (27) is a system with Arrhenius dynamics.

To discuss the stability properties of the equilibrium positions of (27) as a function of Da , B and β , introduce a linear approximation of (27) around a steady state x_{1s} , x_{2s} :

$$\begin{aligned} \frac{d\delta x_1}{dt} &= -\frac{1}{1-x_{1s}} \delta x_1 + x_{1s} \delta x_2, \\ \frac{d\delta x_2}{dt} &= -\frac{Bx_{1s}}{1-x_{1s}} \delta x_1 - [(1+\beta) - Bx_{1s}] \delta x_2, \end{aligned} \tag{29}$$

where $\delta x_i = x_i - x_{is}$, $i = 1, 2$. The necessary and sufficient conditions for (29) to be asymptotically stable are:

$$\det A > 0$$

$$\text{tr } A < 0$$

where A is a matrix of the coefficients of system (29).

The set $\mathcal{D} = \{x_{1s} : \det A(x_{1s}) \leq 0\}$ we shall refer to as *det-instability area* in the space of steady states. The set $\mathcal{T} = \{x_{1s} : \text{tr } A(x,s) \geq 0\}$ we shall refer to as *tr-instability area* in the space of steady states. In [12] it is shown that the det-instability area coincides with the negative slope part of the steady state characteristic $||x_s(Da, B/(1+\beta))||$. Also it is shown that the necessary condition for existence of the tr-instability area is

$$B \geq 3 + \beta + 2\sqrt{2+\beta}.$$

Denote

$$\mathcal{D} = \{x_{1s} \in [m_1, m_2]\},$$

$$\mathcal{T} = \{x_{1s} \in [s_1, s_2]\}.$$

The relative positions of m_1, s_1, m_2, s_2 depend upon the relationship between B and β and assume all possible combinations (for details, see [12], Fig. 2).

2. Equations of CSTR Under Oscillating Input Flow

Assume now that the fresh feed F is oscillating in time according to the law:

$$F(t') = F_0 + \alpha_1 f(\omega t'),$$

where F_0 and α_1 are constants and $f(\omega t')$ is a periodic zero mean function. The material and energy balances in this case are of the form:

$$V \frac{dc_A}{dt} = (F_0 + \alpha_1 f(\omega t'))(c_{Af} - c_A) - V k_0 \exp\{-E/RT\} c_A,$$

(30)

$$V \rho C_p \frac{dT}{dt} = \rho C_p (F_0 + \alpha_1 f(\omega t'))(T_f - T) + V(-\Delta H) k_0 \exp\{-E/RT\} c_A - hAT,$$

where all notations mean the same as in (25).

Introduce a parameter

$$\tau = \frac{V}{F_0}, \quad (31)$$

which describes the *average* residence time of the reactor, and consider a dimensionless time

$$t = \frac{t'}{\tau}.$$

In time t system (30) has the form:

$$\frac{dc_A}{dt} = (1 + \frac{\alpha_1}{F_0} f(\frac{\omega V}{F_0} t)) (c_{Af} - c_A) - \tau k_0 \exp\{-E/RT\} c_A,$$

$$\rho C_p \frac{dT}{dt} = \rho C_p (1 + \frac{\alpha_1}{F_0} f(\frac{\omega V}{F_0} t)) (T_f - T) + \tau (-\Delta H) k_0 \exp\{-E/RT\} - \frac{hA}{F_0} T.$$

Using notation (26), this system can be rewritten as

$$\frac{dx_1}{dt} = -(1 + \frac{\alpha_1}{F_0} f(\omega \tau t)) x_1 + Da(1-x_1) \exp\{x_2\}, \quad (32)$$

$$\frac{dx_2}{dt} = -(1 + \beta + \frac{\alpha_1}{F_0} f(\omega \tau t)) x_2 + BDa(1-x_1) \exp\{x_2\}.$$

Let T_r be the rise time (in units τ) of the reactor (27) without vibrations. To satisfy requirement (5), we assume that

$$\frac{1}{\omega \tau T_r} = \varepsilon \ll 1.$$

To satisfy (6), we denote

$$\frac{\alpha_1}{F_0} = \frac{\alpha}{\varepsilon T_r}.$$

In this case, we obtain the following dimensionless kinetic equations of the reactor with oscillating input flow rate:

$$\frac{dx_1}{dt} = -(1 + \frac{\alpha}{\epsilon T_r} f(\frac{t}{\epsilon T_r}))x_1 + Da(1-x_1)\exp\{x_2\}, \quad (33)$$

$$\frac{dx_2}{dt} = -(1 + \beta + \frac{\alpha}{\epsilon T_r} f(\frac{t}{\epsilon T_r}))x_2 + BDa(1-x_1)\exp\{x_2\},$$

where the vibrations are of the form (7).

3. Behavior of CSTR under Oscillating Input Flow

Theorem 2. The equilibrium position x_s of system (27) is vibrationally stabilizable if there exists Da such that the system

$$\frac{dy_1}{dt} = -y_1 + Da\{1 - y_1 + \gamma^2[1-y_2 - y_1y_2 + y_2^2 - y_1y_2^2]\}\exp\{y_2\}, \quad (34)$$

$$\frac{dy_2}{dt} = -(1+\beta)y_2 + BDa\{1-y_1 + \gamma^2[1-y_2 - y_1y_2 + y_2^2 - y_1y_2^2]\}\exp\{y_2\},$$

$$\gamma^2 = \frac{\alpha^2}{2} [\phi^2(t/\epsilon)], \quad (35)$$

$$\phi(\tau) = \int_0^\tau f(\tau)d\tau - \left[\int_0^\tau f(t)d\tau \right], \quad (36)$$

has a steady state $y_s = x_s$ which is asymptotically stable.

Remark 6. Equations (34) represents a new system with Arrhenius dynamics, steady state characteristic of which is defined by

$$Da = \frac{y_{1s}}{1 - y_1 + \gamma^2(1-y_{2s}-y_{1s}y_{2s} + y_{2s}^2 - y_{1s}y_{2s}^2)} \exp\{-y_{2s}\},$$

$$y_{2s} = \frac{B}{1+\beta} y_{1s},$$

$$||y_s|| = y_{1s} \left(1 + \frac{B^2}{(1+\beta)^2}\right)^{1/2},$$

and is shown in Fig. 11 (compared with the steady state characteristics of (27)).

Remark 7. To compare the stability properties of (34) and (27), consider linear approximation of (34) around a steady state y_{1s}, y_{2s} :

$$\begin{bmatrix} \frac{d\delta y_1}{dt} \\ \frac{d\delta y_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 - Da e^{y_{2s}} [1 + \gamma^2 (y_{2s} + y_{2s}^2)] & y_{1s} - Da e^{y_{2s}} \gamma^2 (1 + y_{1s} - 2y_{2s} + 2y_{1s}y_{2s}) \\ -B Da e^{y_{2s}} [1 + \gamma^2 (y_{2s} + y_{2s}^2)] & -1 - \beta + B y_{1s} - B Da e^{y_{2s}} \gamma^2 (1 + y_{1s} - 2y_{2s} + 2y_{1s}y_{2s}) \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix}, \quad (37)$$

$$\delta y_i = y_i - y_{is}, \quad i = 1, 2, \quad (37)$$

Since $Da = \frac{x_{1s}}{1-x_{1s}} e^{-x_{2s}}$, linear approximations (29) and (37) coincide when $\gamma^2 = 0$.

Denoting the square matrix in (37) through A_1 , we calculate:

$$\begin{aligned} \det A_1 = & 1 + \beta - B y_{1s} + Da e^{\frac{B}{1+\beta} y_{1s}} \{ 1 + \beta + \gamma^2 [\frac{B^3 + 2B(1+\beta)^2}{(1+\beta)^2} y_{1s}^2 + \\ & + \frac{[B - (1+\beta)]^2}{1+\beta} y_{1s} + (1+\beta)] \} , \end{aligned} \quad (38)$$

$$\begin{aligned} \text{tr } A_1 = & -2 - \beta + B y_{1s} - Da e^{\frac{B}{1+\beta} y_{1s}} \{ 1 + \gamma^2 B [\frac{B(3+2\beta)}{(1+\beta)^2} y_{1s}^2 + \\ & + \frac{2+\beta-2B}{1+\beta} y_{1s} + 1] \} . \end{aligned} \quad (39)$$

Again, $\det A_1 = \det A$, $\text{tr } A_1 = \text{tr } A$ when $\gamma^2 = 0$.

For every steady state, y_{1s}, y_{2s} , and all B and β , the quantity in square brackets in (38) is always positive. This means that the det-instability area $\mathcal{D}_1$ of (37) is smaller than $\mathcal{D}$ of (29):

$$\mathcal{D}_1 \subset \mathcal{D}.$$

Therefore, *vibrational control always improves the det-stability property of every steady state of the Arrhenius system (27)*. That is why the steady state characteristics of (34) has a smaller negative slope part, compared to that of (27). When γ^2 is large enough, the negative slope part of (34) disappears.

As far as the trace of A_1 is concerned, the improvement takes place only when the quantity in the square brackets in (39) is positive. This, obviously, occurs for all y_{1s}, y_{2s} only in a bounded domain of the plane (β, B) . To find this domain, consider the inequality

$$\frac{B(3+\beta)}{(1+\beta)^2} u^2 + \frac{2+\beta-2B}{1+\beta} u + 1 > 0. \quad (40)$$

Evidently (40) holds for all real u 's if and only if

$$(2+\beta-2B)^2 - 4B(3+2\beta) < 0,$$

which can be rewritten as

$$\beta^2 + (4-12B)\beta - 20B + 4B^2 + 4 < 0. \quad (41)$$

Therefore, *vibrational control improves tr-stability property of every steady state of Arrhenius system (27) only when β and B satisfy (41)*. Fig. 12 shows the area of (β, B) where (41) is met.

Of course, the last statement does not mean that the value of the trace is always increased by vibrations when B and β do not satisfy (41): there are steady states where trace is improved as well as steady states where the trace becomes worse. Fig. 13 illustrates this situation for $\beta = 0.5$.

Proof of Theorem 2. System (27) satisfies the conditions of the Theorem of paper [3] with

$$X(x) = \begin{bmatrix} -x_1 + Da(1-x_1)\exp\{x_2\} \\ -(1+\beta)x_2 + BDa(1-x_1)\exp\{x_2\} \end{bmatrix}. \quad (42)$$

Therefore, the property of vibrational stabilizability of x_s follows from the

property of asymptotic stability of the equilibrium position $y_s = x_s$ of the system

$$\frac{dy}{dt} = \frac{1}{T} \int_0^T [\exp\{\frac{-\alpha}{T} \int_0^\tau B(\frac{\tau}{T}) d\tau\} X(\exp\{\frac{\alpha}{T} \int_0^\tau B(\frac{\tau}{T}) d\tau\} y)] d\tau, \quad (43)$$

where, as it follows from (27) and (33),

$$y = [y_1, y_2]^T, \quad B(\tau) = -I f(\tau), \quad (44)$$

and $X(\cdot)$ is defined by (42).

Expanding matrix exponents of (43) into power series and taking into account (44), with accuracy to $O(\alpha^4)$ we obtain:

$$\frac{dy}{dt} = X(y) + \gamma^2 [X(y) - \frac{\partial X}{\partial y} y + (\frac{\partial X}{\partial y} y) \otimes (\frac{\partial X}{\partial y} y)], \quad (45)$$

where γ^2 is defined by (35) and (36), and $z \otimes z$ is used to denote the element by element multiplication of vectors:

$$z \otimes z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \otimes \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1^2 \\ z_2^2 \end{bmatrix}.$$

Calculating the partial derivatives of (45), we have:

$$\frac{dy}{dt} = X(y) + \gamma^2 Da Y(y), \quad (46)$$

where

$$Y(y) = \begin{bmatrix} \exp\{y_2\}(1-y_2-y_1y_2 + y_2^2 - y_1y_2^2) \\ B \exp\{y_2\}(1-y_2-y_1y_2 + y_2^2 - y_1y_2^2) \end{bmatrix}. \quad (47)$$

Equations (46), (47) can be rewritten in the form (34). If there exists Da such that (46), (47) and, therefore, (34) has an asymptotically stable equilibrium position $y_s = x_s$, the steady state x_s is vibrationally stabilizable. Theorem 2 is proven.

VI. VIBRATIONAL CONTROL OF CONTINUOUS STIRRED TANK REACTOR:
NUMERICAL SIMULATIONS

The following model of CSTR was analyzed numerically:

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + Da(1-x_1)\exp\{x_2\}, \\ \frac{dx_2}{dt} &= -x_2 + BDa(1-x_1)\exp\{x_2\}.\end{aligned}\tag{48}$$

Reactor with vibrations was simulated as

$$\begin{aligned}\frac{dx_1^1}{dt} &= -(1+C \sin \omega t)x_1^1 + Da(1-x_1^1)\exp\{x_2^1\}, \\ \frac{dx_2^1}{dt} &= -(1+C \sin \omega t)x_2^1 + BDa(1-x_1^1)\exp\{x_2^1\}.\end{aligned}\tag{49}$$

Note that vibrations in (49) are characterized by a relative amplitude C only, since equations (48) are normalized with respect to average input flow F_0 .

Finally, the averaged equation was modeled as

$$\begin{aligned}\frac{dy_1}{dt} &= -y_1 + Da\{1-y_1+\gamma^2[1-y_2-y_1y_2 + y_2^2-y_1y_2^2]\}\exp\{y_2\}, \\ \frac{dy_2}{dt} &= -(1+\beta)y_2 + BDa\{1-y_1+\gamma^2[1-y_2-y_1y_2 + y_2^2-y_1y_2^2]\}\exp\{y_2\},\end{aligned}\tag{50}$$

where

$$\gamma^2 = \frac{C^2}{4\omega^2}.$$

Estimation of $\epsilon_{\max}$

In Section VI, small parameter ϵ was introduced in the following form (ω and T_r are in units $1/\tau$ and τ , respectively):

$$\epsilon = \frac{1}{\omega T_r}.$$

For vibrations shown in (49),

$$\epsilon = \frac{T_0}{2\pi T_r},$$

where T_0 is the period of vibrations (in τ).

Fig. 14 presents justification for the choice of $\epsilon_{\max}$. In Fig. 14A, T_r is estimated to be 6 (1 and 3 in this figure represent the steady state characteristic x_{1s} (Da) and time solution $x_1(t)$ of (48), respectively). In Fig. 14B results of simulation for $\omega = 0.25$ ($\epsilon = 0.64$) are presented. Here 2 is the steady state characteristic, y_{1s} (Da), of (50), 4 denotes the time solution, $x_1^1(t)$, of (49), and 5 represents the time solution, $y_1(t)$, of (50)*). As before, all time solutions are calculated for identical initial conditions. As it follows from Fig. 14B, y_{1s} does not approximate $\bar{x}_1^s$ adequately (precision is 45%). In Fig. 14C, $\epsilon = 0.08$, and the approximation is sufficiently good (10%). This is also supported by the simulation of Fig. 14D. The experiments with different ϵ 's indicated that $\epsilon \approx 0.16$ guarantees acceptable precision (of the order 10-25%). Therefore, we assume that

$$\epsilon_{\max} = 0.16.$$

Note that this $\epsilon_{\max}$ is the same as for catalytic reactor of Section IV. Additional examples of systems with Arrhenius dynamics have to be analyzed in order

*) The same notations are used in all Figs. 14-17.

to be sure that $\epsilon_{\max} \approx 0.16$ is acceptable for the averaged description of any Arrhenius system.

Increase in Productivity

Fig. 15 represents data concerning this aspect of CSTR vibrational control. As it follows from Figs. 15A, B, for $C = 1.25$, $\omega = 0.8$, an increase in productivity (defined in Section IV) by a factor of 2 can be achieved. Note that unlike catalytic reactor, in case of CSTR, the increase in productivity is due to the fact that the upper branch of $y_{1s}(\text{Da})$ is much lower than that of $x_{1s}(\text{Da})$ (Fig. 15A); this could ensure practical feasibility of the operating points on the upper branch of $y_{1s}(\text{Da})$. Figs. 15C, D give another example of this effect.

Vibrational Control of a tr-Unstable System

In Fig. 16 a system with a tr-instability area is considered. The operating point x_s , chosen for analysis, is an unstable focus (Fig. 16A). Time solution, $x_1(t)$, of (48) is shown in Fig. 16B. After introduction of vibration a stable limit cycle appears, averaged value of which is close to x_{1s} .

A Note on the Domains of Attractions of $x^s(t)$ and y_s

A Comparison made in Section IV for the catalytic reactor is, probably, true for CSTR as well. Fig. 17 illustrates this point. Additional theoretical and numerical work is necessary to clarify this phenomenon.

VII. CONCLUDING REMARKS

1. Theorems 1 and 2 show that vibrational control affects the behavior of an Arrhenius system proportionally to

$$\gamma^2 = \frac{\alpha^2}{2} \zeta^2,$$

where

$$\zeta^2 = \overline{\phi^2(t)}.$$

If the vibrations are described as

$$f_1(t) = A_0 f(\omega t), \quad (51)$$

$$f(\omega t) = \sum_{i=1}^{\infty} a_i \cos(i\omega t + \phi_i), \quad (52)$$

we find that

$$\gamma^2 = \frac{A_0^2}{2\omega^2} \zeta^2, \quad (53)$$

where

$$\zeta^2 = \frac{1}{2} \sum_{i=1}^{\infty} \frac{a_i^2}{i^2}. \quad (54)$$

Maximization of γ^2 is an important part in applications of vibrational control. The following recommendations can be given in this respect:

From (51) follows that

$$\inf_t f(\omega t) \geq -1.$$

On the other hand, the supremum of $f(\omega t)$ can be significantly larger than 1. Therefore, the problem of optimization of γ^2 can be understood as maximization of a functional, specified by (54), in a class of piece-wise continuous functions characterized by

$$-1 \leq f(\omega t) \leq M. \quad (55)$$

Using dynamic programming [27], it is possible to show that the solution of this optimization problem is given by a *rectangular waveform satisfying (55)*. Note that in case $M = 1$, the first Fourier coefficient of $f(\omega t)$ is ≈ 1.27 . If $M \rightarrow \infty$, $\sup \zeta^2$ is achieved:

$$\sup \zeta^2 = \frac{2}{3} \pi^2$$

Therefore, from (53) we conclude that

$$\sup \gamma^2 = \frac{A_0^2}{2\omega^2} \frac{2}{3} \pi^2 \approx 3.3 \left(\frac{A_0}{\omega} \right)^2.$$

2. Analytical and numerical investigations presented in this paper show that for both catalytic reactor and CSTR,

- α) $\varepsilon < \varepsilon_{\max} \approx 0.16$ ensures an acceptable precision of the averaged description;
- β) the introduction of vibrations results in a new Arrhenius system with a smaller negative slope part of the steady state characteristic; in other words, the det-instability area is decreased;
- γ) the new Arrhenius system has the upper branch located lower than that of the original system; this makes it feasible to have an averaged operating point located on the upper branch; resulting increase of productivity might be by a factor of 2÷3.

3. On the basis of the above analysis, the following hypothesis can be formulated:

Conjecture. For any Arrhenius system with oscillating parameter A,

- 1). the averaged description has an acceptable precision if $\frac{1}{T_r \omega} \leq 0.16$;
- 2). the averaged system is also a system with Arrhenius dynamics;
- 3). the steady state characteristic of this new system has a smaller negative slope part and the upper branch located lower than that of the original system.

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Fig. 1,

FIG. 2

FIG. 3

FIG. 4

FIG. 5

Fig. 6

Fig. 7

Fig. 8

(A)

(B)

(C)

Fig. 9

FIG. 10

FIG. 11

FIG. 12

FIG. 13

Fig. 14

(a)

(b)

(c)

(d)

Fig. 15 .

Fig. 16

Fig. 17

Not found. (std\$cp)

!SLIST CC.FOR

C*** THIS PROGRAM SIMULATES CSTR DYNAMICS USING GEAR'S METHOD ****

C*** PLOTTING FILE FOR UNIVAC IS NOT CREATED *****

C** EVALUATING SUBROUTINES DO NOT CONSIDER ACTIV. ENERGY AND COOLING **

```
INTEGER*4 LB,LC,LK,NN,M,I
INTEGER N,METH,MITER,INDEX,IWK(2),IER
INTEGER IWKZ(2),INDEXZ,METHZ
INTEGER K,KX,KZ,KTR,JA
REAL Y(2),WK(26),X,TOL,H,XX(160000),Y1(160000),Y2(160000)
REAL TIME,B,BETA,DA,A,OMEGA,GAM,X2C
REAL Z1(160000),Z2(160000),Z1S(3),DETZ(3),TRACEZ(3)
REAL WKZ(26),ZA,ZB,XZ,TOLZ,TIMEZ,HZ
REAL XST1(2501),DAX(2501),ZST1(2501),DAZ(2501),BBB(2501)
REAL PAR,DALIM,X1S,F
REAL S1,S2,M1,M2,DA1,DA2,DA3,DA4,DM,DS
EXTERNAL FCN,FCNJ,FCNE,FCND
COMMON /GEAR/DUMMY(48),SDUMMY(4),IDUMMY(38)
COMMON TIME,B,BETA,DA,A,OMEGA,GAM,X2C,WER
COMMON /SPACE1/ XX
COMMON /SPACE2/ Y1,Y2
COMMON /SPACE3/ Z1,Z2
COMMON /SPACE4/ XST1,DAX,ZST1,DAZ,BBB
```

C ***** CSTR PARAMETERS SETTING *****

```
READ(5,*) B,BETA,DA,T,Y(1),Y(2),A,OMEGA,DELT
GAM=999999999.
X2C=0.0
DALIM=0.14
X=0.0
XZ=0.0
WER=A**2/(OMEGA**2*4)
```

C***** DET AND TRACE INSTABILITY RANGE *****

```
DM=B*(B*(2.*BETA*X2C/GAM**2+(2./GAM-1.)*(1.+BETA))**2-4.*(1.+BETA+
*BETA*X2C/GAM)**2*(B/GAM**2+(1.+BETA)))
IF(DM.LT.0.) GO TO 1111
M1=(-B*((2./GAM)*(1.+BETA+BETA*X2C/GAM)-(1.+BETA))-SQRT(DM))/(2*(B
**2/GAM**2+B*(1.+BETA)))
M2=(-B*((2./GAM)*(1.+BETA+BETA*X2C/GAM)-(1.+BETA))+SQRT(DM))/(2*(B
**2/GAM**2+B*(1.+BETA)))
DA1=M1/(1.-M1)*EXP(-(B*M1+BETA*X2C)/((1.+BETA)+(1./GAM)*(B*M1+BETA
**X2C)))
DA2=M2/(1.-M2)*EXP(-(B*M2+BETA*X2C)/((1.+BETA)+(1./GAM)*(B*M2+BETA
**X2C)))
```

127 CONTINUE

```
DS=(B+1.+BETA)**2-4.*B*(2.+BETA)
```

```
IF(DS.LT.0.) GO TO 9898
```

```
S1=(B+1.+BETA-SQRT(DS))/(2.*B)
```

```
S2=(B+1.+BETA+SQRT(DS))/(2.*B)
```

```
DA3=S1/(1.-S1)*EXP(-(B*S1+BETA*X2C)/((1.+BETA)+(1./GAM)*(B*S1+BETA
**X2C)))
```

```
DA4=S2/(1.-S2)*EXP(-(B*S2+BETA*X2C)/((1.+BETA)+(1./GAM)*(B*S2+BETA
**X2C)))
```

```
GO TO 3333
```

9898 DO 3232 K=1,10

3232 WRITE(1,987)

```
WRITE(1,8989)
```

8989 FORMAT('S1,S2 ARE COMPLEX')

```

      GO TO 3333
2222 FORMAT('M1,M2 ARE COMPLEX')
1111 WRITE(1,2222)
      GO TO 127
3333 DO 4675 K=1,5
4675 WRITE(1,987)
7575 FORMAT('DET INSTABILITY M1= ',F15.8,6X,'M2= ',F15.8)
      WRITE(1,7575) M1,M2
7676 FORMAT('DET INSTABILITY DA1= ',F15.8,6X,'DA2= ',F15.8)
      WRITE(1,7676) DA1,DA2
8484 FORMAT('TRACE INSTABILITY S1= ',F15.8,6X,'S2= ',F15.8)
      WRITE(1,8484) S1,S2
8686 FORMAT('TRACE INSTABILITY DA3= ',F15.8,6X,'DA4= ',F15.8)
      WRITE(1,8686) DA3,DA4
C*****END*****

C***** CHOOSING OF EQUIL.POINT RELATIVELY TO M1 *****
      X1S=TAIL+M1
      X2S=(B*X1S+BETA*X2C)/((1.+BETA)
      DA5=X1S/((1.-X1S)*EXP(-(B*X1S+BETA*X2C)/((1.+BETA)+(1./GAM)*(B*X1S+
      *BETA*X2C)))
796 FORMAT('DEVIATION FROM M1 UP = ',F12.6)
      WRITE(1,796)TAIL
787 FORMAT('SUGGESTED EQUILIBRIUM: ',1X,'X1= ',F10.5,4X,'X2= ',F10.5,4X,
      *'DA= ',F10.5)
      WRITE(1,787)X1S,X2S,DA5
C*****END *****
      DO 4375 K=1,8
4375 WRITE(1,987)
C*****DGEAR PARAMETERS *****
      TOL=0.0001
      TOLZ=0.0001
      N=2
      H=0.0000000000000001
      HZ=0.0000000000000001
      METH=2
      METHZ=2
      MITER=2
      INDEX=1
      INDEXZ=1
C*****END *****

C***** PARAMETERS OF SIMULATION FOR OBSERVATION *****
3 FORMAT('PARAMETERS OF SIMULATION:')
      WRITE(1,3)
      WRITE(1,987)
12 FORMAT('B= ',F6.3,6X,'BETA= ',F6.3,6X,'DAMKOHLER NO.= ',F11.9)
      WRITE(1,12)B,BETA,DA
5 FORMAT('X1= ',F6.3,6X,'X2= ',F6.3,15X,'FOR T= ',F6.3)
      WRITE(1,5)Y(1),Y(2),X
78 FORMAT('DIM-LESS ACTIVATION ENERGY = ',F20.1,6X,'X2C= ',F6.3)
      WRITE(1,78)GAM,X2C
6 FORMAT('SIN VIBRATION PARAMETERS: ',15X,'A= ',F8.4,3X,'OMEGA= '
      *,F8.4)
      WRITE(1,6)A,OMEGA
545 FORMAT('SIMULATION TIME= ',F10.3,10X,'STEP= ',F12.9)
      WRITE(1,545)T,DELT
432 FORMAT('AVERAGED SYSTEM CO.= ',F12.6)
      WRITE(1,432)WER
C*****END *****
      DO 53 K=1,150
53 WRITE(1,987)
      ZA=Y(1)

```

```

      ZB=Y(2)
C***** ORIG. SYST. SIMULATION *****
      Y1(1)=ZA
      Y2(1)=ZB
      XX(1)=X
23  FORMAT('ERRORS OF SIMULATION:')
      WRITE(1,23)
      LB=INTL(T/DELT)
      DO 10 M=1,LB
      TIME=DELT*FLOAT(M)
      CALL DGEAR(N,FCN,FCNJ,X,H,Y,TIME,TOL,METH,MITER,INDEX,IWK,WK,IER)
      IF(IER.GT.128) GO TO 20
      XX(M+1)=TIME
      Y1(M+1)=Y(1)
      Y2(M+1)=Y(2)
      LC=M+1
      IF(A.NE.0.) GO TO 10
      IF(Y(1).GE.0.99) GO TO 20
10  CONTINUE
20  CONTINUE
21  FORMAT('ORIG. SYST. ERROR=',I3)
      WRITE(1,21)IER
C***** END *****
987  FORMAT(' ')
C***** AVER. SYST. SIMULATION *****
      Y(1)=ZA
      Y(2)=ZB
      Z1(1)=ZA
      Z2(1)=ZB
      DO 63 NN=1,LB
      TIMEZ=DELT*FLOAT(NN)
      CALL DGEAR(N,FCNE,FCND,XZ,HZ,Y,TIMEZ,TOLZ,METHZ,MITER,INDEXZ,IWKZ,
*WKZ,IER)
      IF(IER.GT.128) GO TO 26
      Z1(NN+1)=Y(1)
      Z2(NN+1)=Y(2)
      LK=NN+1
      IF(A.NE.0.) GO TO 63
      IF(Y(1).GE.0.99) GO TO 26
63  CONTINUE
26  CONTINUE
24  FORMAT('AVER. SYST. ERROR=',I3)
      WRITE(1,24)IER
C***** END *****
      DO 54 K=1,150
54  WRITE(1,987)
C***** PLOTS X1S AND Z1S VS DA, AND Z1S VS B *****
      XST1(1)=0.
      DAX(1)=0.
      DO 17 K=1,2499
      PAR=FLOAT(K)/2501.
      XST1(K+1)=PAR
      DAX(K+1)=PAR/(1.-PAR)*EXP(-(B*PAR+BETA*X2C)/((1.+BETA)+(1./GAM)*(B
**PAR+BETA*X2C)))
      KX=K+1
      IF(DAX(K+1).GE.DALIM) GO TO 18
17  CONTINUE
18  ZST1(1)=0.
      DAZ(1)=0.
      DO 71 K=1,2499
      PAR=FLOAT(K)/2501.
      ZST1(K+1)=PAR
      F=B/(1.+BETA)

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      IF(DAZ(K+1).GE.(DALIM+0.05)) GO TO 52
      DAZ(K+1)=PAR/(1.-PAR+WER*(1.-F*PAR-F*PAR**2+F**2*PAR**2-F**2*PAR**
*3))*EXP(-F*PAR)
      KZ=K+1
52    DAZ(K+2)=DAZ(K+1)
      BBB(K+1)=-((2*PAR+PAR*BETA+1.+BETA)*(1.+BETA))/((3.+2.*BETA)*PAR**
*2-(2*PAR*(1.+BETA)))
      IF(BBB(K).LT.32..AND.BBB(K).GT.30.) KB=K
71    CONTINUE
C***** END *****

C*****AVER SYST. TRACE,DET,AND EQUIL. Z1S *****
      DO 801 K=1,2500
      IF(DA.LE.DAZ(K)) GO TO 802
801   CONTINUE
802   Z1S(1)=ZST1(K)
      JA=K+1
      KTR=1
      DO 803 K=JA,2500
      IF(DA.GE.DAZ(K)) GO TO 804
      IF(DAZ(K).GE.DALIM) GO TO 807
803   CONTINUE
804   Z1S(2)=ZST1(K)
      JA=K+1
      DO 805 K=JA,2500
      IF(DA.LE.DAZ(K)) GO TO 806
805   CONTINUE
806   Z1S(3)=ZST1(K)
      KTR=3
807   DO 808 K=1,KTR
      TRACEZ(K)=-2.-BETA+B*Z1S(K)-DA*EXP((B/(1.+BETA))*Z1S(K))*(1.+WER*B
**((B*(3.+2.*BETA)/(1.+BETA)**2*Z1S(K)**2+(2.+BETA-2.*B)/(1.+BETA)*Z
*1S(K)+1.))
      DETZ(K)=1.+BETA-B*Z1S(K)+DA*EXP(B/(1.+BETA)*Z1S(K))*(1.+BETA+WER*(
**((B**3+2.*B*(1.+BETA)**2)/(1.+BETA)**2)*Z1S(K)**2+((B-(1.+BETA))*
*2/(1.+BETA))*Z1S(K)+1.+BETA))
808   CONTINUE
      DO 809 K=1,10
809   WRITE(1,987)
      DO 810 K=1,KTR
810   WRITE(1,811)K,Z1S(K),TRACEZ(K),DEZ(K)
811   FORMAT(1,'Z1S',I2,1X,F14.8,6X,'TRACE',1X,F14.8,6X,'DET',1X,F14.8)
C***** END *****

      DO 812 K=1,150
812   WRITE(1,987)
C***** DRAWING OF PLOTS *****
      CALL INITT(96)
      CALL OPEN$( 'PLOT',4,CODE)
C***** PLOT X2 VS X1 AND Z2 VS Z1 *****
      CALL DWINDO(-0.2,1.,-2.8,14.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(0.,14.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(1.,0.)
      CALL MOVEA(Y1(1),Y2(1))
      DO 779 I=1,LC
      CALL DRAWA(Y1(I),Y2(I))
779   CONTINUE
      CALL MOVEA(Z1(1),Z2(1))
      DO 778 I=1,LK
      CALL DASHA(Z1(I),Z2(I),1)
778   CONTINUE
C***** PLOT X1 AND Z1 VS DA *****

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      CALL NEWPAG
      DAL=DALIM/5.
      CALL DWINDO(-DAL,DALIM,-0.2,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(0.,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(DALIM,0.)
      CALL MOVEA(DAX(1),XST1(1))
      DO 45 K=1,KX
      CALL DRAWA(DAX(K),XST1(K))
45    CONTINUE
      CALL MOVEA(DA,0.)
      CALL DRAWA(DA,1.)
      CALL MOVEA(DAZ(1),ZST1(1))
      DO 43 K=1,KZ
      CALL DASHA(DAZ(K),ZST1(K),1)
43    CONTINUE
C***** PLOT Z1S VS B *****
      CALL NEWPAG
      CALL DWINDO(-6.,30.,-0.2,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(0.,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(30.,0.)
      CALL MOVEA(B,0.)
      CALL DRAWA(B,1.)
      CALL MOVEA(0.,Z1S(1))
      CALL DRAWA(30.,Z1S(1))
      CALL MOVEA(BBB(2),ZST1(2))
      DO 46 K=1,KB
      CALL DRAWA(BBB(K),ZST1(K))
46    CONTINUE
C***** PLOT X1 AND Z1 VS T *****
      CALL NEWPAG
      TL=T/5.
      CALL DWINDO(-TL,T,-0.2,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(0.,1.)
      CALL MOVEA(0.,0.)
      CALL DRAWA(T,0.)
      CALL MOVEA(XX(1),Y1(1))
      DO 4 I=1,LC
      CALL DRAWA(XX(I),Y1(I))
4    CONTINUE
      CALL MOVEA(XX(1),Z1(1))
      DO 113 I=1,LK
      CALL DASHA(XX(I),Z1(I),1)
113  CONTINUE
      CALL FINITT(1,1)
      CALL CLOS$G(CODE)
C***** END OF DRAWING *****
      STOP
      END
C***** SUBROUTINES *****
      SUBROUTINE FCN(N,X,Y,YPRIME)
      INTEGER N
      REAL Y(2),YPRIME(2),X
      COMMON TIME,B,BETA,DA,A,OMEGA,GAM,XZC,WER
      YPRIME(1)=-Y(1)*(1.+A*SIN(OMEGA*TIME))+DA*(1.-Y(1))*EXP(Y(2))
      YPRIME(2)=-Y(2)*(1.+BETA+A*SIN(OMEGA*TIME))+B*DA*(1.-Y(1))*EXP(Y(2)
*)
      RETURN
      END

```

```

SUBROUTINE FCNJ(N,X,Y,PD)
  INTEGER N
  REAL Y(2),PD(2,2),X
  RETURN
END
SUBROUTINE FCNE(N,XZ,Y,YPRIME)
  INTEGER N
  REAL Y(2),YPRIME(2),XZ
  COMMON TIME,B,BETA,DA,A,OMEGA,GAM,X2C,WER
  YPRIME(1)=-Y(1)+DA*EXP(Y(2))*(1.-Y(1)+WER*(1.-Y(2)-Y(1)*Y(2)+Y(2)*
**2-Y(1)*Y(2)**2))
  YPRIME(2)=- (1.+BETA)*Y(2)+B*DA*EXP(Y(2))*(1.-Y(1)+WER*(1.-Y(2)-Y(1)
*)*Y(2)+Y(2)**2-Y(1)*Y(2)**2))
  RETURN
END
SUBROUTINE FCND(N,XZ,Y,PDD)
  INTEGER N
  REAL Y(2),PDD(2,2),XZ
  RETURN
END

```

VIBRATIONAL CONTROL - AN ALTERNATIVE METHOD
OF CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS

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Abstract. A problem of vibrational stabilization of systems described by partial differential equations is considered. After reviewing the vibrational control theory for lumped parameter systems, a few cases of vibrational stabilization of distributed parameter systems are discussed. A condition of vibrational stabilizability for a class of linear partial differential equations is given and an example of vibrational stabilization in a nonlinear situation (Benard instability) is presented.

Keywords. Partial differential equations; vibrational stabilizability; parameter oscillations; averaging theory; Benard instability.

I. INTRODUCTION

Vibrational control is a method of stabilization of unstable systems on the basis of enforcing mean zero oscillations on the parameters of the system. A simple example of vibrational control is an inverted pendulum which can be stabilized by sufficiently frequent oscillations of its suspension point (see, for instance, Bogoliubov, 1961). Unlike conventional control techniques, vibrational control does not require measurements of either deviations or disturbances. A theory of vibrational control for lumped parameter systems was developed in Meerkov (1980) (linear case) and in Meerkov (1982) (nonlinear case). Section II below reviews the basics of this theory. There are, however, a few cases of vibrational stabilization of distributed parameter systems (Berge, 1972; Chelomey, 1956; Meerkov and Shapiro, 1976; Osovets, 1974; and Wolf, 1969); these cases are discussed in Section III. The power of vibrational control approach to distributed parameter systems is based on the fact that fast time-oscillations of system's parameters could stabilize *all* or "*almost all*" modes of spacial oscillation of the system simultaneously. Section IV presents a condition of vibrational stabilizability for a class of linear partial differential equations. In Section V an example of vibrational stabilization of a nonlinear distributed system (arising from the Benard problem) is considered.

II. VIBRATIONAL STABILIZABILITY OF LUMPED PARAMETER SYSTEMS

Linear Theory

Consider a plant described by the following system of differential equations:

$$\dot{x} = Ax, \quad x \in R^n, \quad A \text{ is a constant matrix.} \quad (1)$$

Introduce periodic oscillations in matrix A elements according to the law $B(t)$, where $B(t)$ is a periodic zero mean matrix. As a result, the following system is obtained:

$$\dot{x} = (A + B(t))x. \quad (2)$$

Definition 1. System (1) is vibrationally stabilizable if there exists a periodic matrix $B(t)$ (with zero mean) such that system (2) is asymptotically stable.

Theorem 1. (Meerkov, 1980). Assume that there exists a vector-row c such that the pair (c, A) is observable. Then, for (1) to be vibrationally stabilizable, it is necessary and sufficient that

$$\text{Tr } A < 0.$$

A set of theorems, which allow one to find a stabilizing matrix $B(t)$, was also given.

The linear vibrational control approach was applied to a number of situations; for example, to cyclic particle accelerators for betatron oscillations suppression (Livingston, 1954). In this application, matrix $B(t)$ was generated by a vibrating magnetic field.

Nonlinear Theory

Consider a nonlinear equation of the form:

$$\dot{x} = X(x), \quad x \in \mathbb{R}^N. \quad (3)$$

Introduce in this equation vibrations of the form:

$$\dot{x} = X(x) + \frac{\alpha}{\varepsilon} B\left(\frac{t}{\varepsilon}\right)x, \quad (4)$$

where $0 < \varepsilon \ll 1$, $\alpha > 0$, and $B(t/\varepsilon)$ is a periodic, zero mean matrix.

Vibrations of the form (4) naturally appear in equations of chemical and biochemical processes under periodic oscillations of input flow rates (Meerkov, 1981; 1982).

Let x_s be an equilibrium point of equation (3) and $x^s(t/\varepsilon)$ be a periodic solution of equation (4).

Definition 2. Equilibrium position x_s of equation (3) is vibrationally stabilizable if there exists a periodic, zero mean matrix $B(t/\varepsilon)$ such that equation (4) has an asymptotically stable periodic solution $x^s(t/\varepsilon)$ of the form:

$$x^s(\tau) = \exp\left(\alpha \int_0^\tau B(\tau) d\tau\right) y^*(\tau), \\ ||x_s - y^*(\tau)|| < \varepsilon, \quad \tau \in (-\infty, \infty), \quad \tau = \frac{t}{\varepsilon},$$

where $y^*(\tau)$ is a periodic vector-valued function of the same period T as $B(\tau)$.

Thus, if x_s is vibrationally stabilizable, introduction of vibrations causes a bifurcation of this equilibrium position into an asymptotically stable periodic solution $x^s(\tau)$ with the averaged value close (if α is small) to x_s .

Introduce a nonlinear *time-invariant* equation of the form:

$$\dot{z} = \frac{1}{T} \int_0^T \left[\exp\left(-\alpha \int_0^\tau B(\tau) d\tau\right) X\left(\exp\left(\alpha \int_0^\tau B(\tau) d\tau\right) z\right) \right] d\tau. \quad (5)$$

Theorem 2. (Meerkov, 1982). Assume that there exists a neighborhood D of the equilibrium position x_s of equation (3) such that

$$||X(x)|| \leq M_1 < \infty, \quad ||X(x') - X(x'')|| \leq M_2 < \infty$$

for all x, x' and $x'' \in D$. In this case, x_s is vibrationally stabilizable if (5) has equilibrium position $z_s = x_s$ which is asymptotically stable.

The nonlinear vibrational control theory was applied to stabilization of continuous stirred tank chemical reactors (Meerkov, 1981; 1982).

III. EXAMPLES OF VIBRATIONAL STABILIZATION OF DISTRIBUTED PARAMETER SYSTEMS

Vibrational Stabilization of Euler

Instability in Elastic Systems (Chelomey, 1956).

Consider an elastic rod under a periodic axial force of the form:

$$F(\omega t) = F_0 + f(\omega t),$$

where $F_0 = \text{const}$ and $f(\omega t)$ is a periodic zero mean function with sufficiently large frequency ω . The dynamics of the rod under this periodic force can be described in linear approximation as

$$\frac{\partial^2}{\partial t^2} A(u) + \frac{\partial}{\partial t} B(u) + C(u) + F(\omega t) D(u) = 0,$$

where A, B, C , and D are linear partial differential operators and $u(x, t)$ is the deviation from the equilibrium position in point x at time t . The solution of this equation can be represented as

$$u(x, t) = q(t)\psi(x), \quad (6)$$

where the "amplitude" $q(t)$ is described by the following differential equation:

$$\ddot{q} + 2n\dot{q} + \Omega^2 \left[\alpha - \frac{f(\omega t)}{F_c} \right] q = 0. \quad (7)$$

Here Ω is the eigenfrequency of the rod, F_c is the critical force for Euler instability, n is a coefficient of linear dissipation, and $\alpha = 1 - (F_0/F_c)$.

With $f(\omega t) \equiv 0$, Euler instability takes place for $F_0 > F_c$. With $f(\omega t) \neq 0$, equation (7) could be understood as a pendulum with a vibrating suspension point, stability of which is ensured even for $F_0 > F_c$ if ω is large enough. The calculations of Chelomey (1956) show that the rod remains stable for such F_0 which satisfy the inequality:

$$\frac{1}{2F_c} \left(\frac{\Omega}{\omega} \right)^2 \sum_{m \neq 0} \frac{\alpha_m^2 + \beta_m^2}{m^2 + \mu^2} > \frac{F_0}{F_c} - 1,$$

where α_m and β_m are Fourier coefficients of $f(\omega t)$ and $\mu = 2n/\omega$.

This result is presently used in the aerospace industry.

Vibrational Stabilization of Raleigh-Taylor Instability in Fluid Systems (Wolf, 1969).

This type of instability takes place for the equilibrium of superposed fluids with a horizontal boundary if the density, ρ_h , of the upper fluid is greater than the density, ρ_l , of the lower one. To stabilize this equilibrium position, harmonic oscillations, $a \cos \omega t$, were enforced in the vertical direction on the cylinder with superposed fluids. As a result, the stability of the system was achieved if

$$\omega^2 \approx \frac{Lg}{\pi a^2} \left(\frac{\rho_h + \rho_\ell}{\rho_h - \rho_\ell} \right),$$

where L is the wave length of the perturbation, and g is acceleration due to gravity. This theoretical result was found to be in a good agreement with experimental data.

Results of Wolf (1969) could be useful for stabilization of MHD instabilities in plasmas.

Vibrational Stabilization of Plasma Pinches (Berge, 1972; Osovets, 1974).

In this method, the stationary current (which stabilizes short wave perturbations) is complemented by a high frequency oscillating current. It was shown that this fast oscillations stabilize long wave perturbations. The method of analysis used by Berge and Osovets is based on the reduction of the partial differential equation describing the displacement of the pinch to a second order ordinary differential equation. In this case, a pinch with additional oscillating current is described by the equation of the inverted pendulum with a vibrating base.

Numerous experiments reported by Osovets (1974) show the feasibility of this method of plasma stabilization.

Vibrational Stabilization of Ionization-Thermal Instability of Glow Discharge in Powerful, Continuous CO₂ Lasers (Meerkov and Shapiro, 1976).

The glow discharge can be described by a set of partial differential equations representing energy, electrons and heavy particles balances. Under certain conditions, this set of equations has an equilibrium position corresponding to the uniform distribution of plasma parameters in the discharge chamber. In order to analyze stability of this equilibrium position, the balance equations were linearized in its vicinity, under the assumption that the deviations are proportional to $\exp\{-ikr\}$, $|\vec{k}| = k = 2\pi/\Lambda$, where Λ is the wave length of the perturbation. For k sufficiently large (small frequencies are known to be stable or easily stabilizable by a feedback), the initial system of partial differential equations can be approximated by the following infinite system of ordinary differential equations:

$$\frac{d}{dt} \delta n = -D_a k^2 \delta n + (k_i - k_a - \hat{k}) n \delta N, \quad (8)$$

$$\frac{d}{dt} \delta N = -\frac{h\nu}{\beta T} k_* N \delta n - \kappa k^2 \delta N, \quad k \geq L,$$

where δn and δN are the deviations of the free electron and heavy particle concentrations, respectively; D_a is the coefficient of ambipolar diffusion, k_i and k_a are the coefficients of ionization and dissociative adhesion of an electron to a molecule of CO₂, respectively; $h\nu$ is the energy of an oscillatory quantum; T is the temperature; $\beta = \gamma/(\gamma-1)$ and γ is the adiabatic constant; k_* is the constant of oscillatory excitation by means of electron collision; κ is the coefficient of thermal conductivity, and

tory quantum; T is the temperature; $\beta = \gamma/(\gamma-1)$ and γ is the adiabatic constant; k_* is the constant of oscillatory excitation by means of electron collision; κ is the coefficient of thermal conductivity, and

$$\hat{k} = k_i \hat{k}_i - k_a \hat{k}_a, \quad \hat{k}_i = \frac{d \ln k_i}{d \ln(E/N)},$$

$$\hat{k}_a = \frac{d \ln k_a}{d \ln(E/N)}.$$

In derivation of equations (8) it was taken into account that vector $\vec{k}$ orthogonal to $\vec{E}$ defines the direction in which instability has the greatest increment.

The conditions of stability of all spacial frequencies $k \geq L$, as it follows from (8), are:

$$D_a k^2 + \kappa k^2 > 0,$$

$$D_a \kappa k^4 + (k_i - k_a - \hat{k}) n \frac{h\nu}{\beta T} k_* N > 0.$$

The first of these inequalities holds for all k . Since under usual conditions $k_i \gg 1$, the following is true: $k_i - k_a - \hat{k} < 0$. Therefore, the second of the above inequalities holds for $k \geq L$ only if $n < n_0$, where n_0 is defined by the equation:

$$D_a \kappa k^4 + (k_i - k_a - \hat{k}) n \frac{h\nu}{\beta T} k_* N = 0.$$

Instability of the discharge with $n > n_0$ is called ionization-thermal instability. Since the power of the laser is proportional to the free electron concentration, ionization-thermal instability limits the achievable power of gas lasers. Thus, there is a necessity to stabilize equations (8) for $n > n_0$, $k \geq L$. Theorem 1 guarantees that system (8) is vibrationally stabilizable for all $k \geq L$. As it is proven by Meerkov and Shapiro (1976), vibrational stabilization of system (8) for $n_1 > n > n_0$, $k \geq L$, can be achieved on the basis of introduction of high frequency oscillations of the free electron concentration, n . These oscillations can be generated either by additional high frequency electric field or by injection of an external electron beam. The optimal shape of vibrations (which guarantees the largest increase in the stability area) was shown to be a δ -function. It was estimated that the described method increases the power of the laser by a factor of 5. Experimental evidences are in a good agreement with this estimate.

Concluding this Section, we would like to point out that the Euler, Rayleigh-Taylor, MHD and ionization-thermal instabilities cannot be stabilized by a feedback control system, unless the measurements and controls are distributed continuously as well, which is an impossible proposition. Therefore, since parameters oscillations affect all spacial modes of the system's dynamics simultaneously, vibrational control could be understood as a "distributed" control for a distributed parameter system.

IV. A CONDITION OF VIBRATIONAL STABILIZABILITY FOR A CLASS OF LINEAR DISTRIBUTED PARAMETER SYSTEMS

Consider a partial differential equation of the form:

$$A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial u}{\partial t} + Cu = 0 \quad (9)$$

where $u(x, t)$ is an m -dimensional vector function of $n+1$ variables and A , B , and C are linear partial differential matrix operators; their domain defines the boundary condition for (9). Examples of A , B , and C are:

$$Av = \alpha v, \quad \alpha = \text{const}, \quad \text{or} \quad Av = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix};$$

$$B = \lambda A, \quad \text{or} \quad B = \frac{\sigma \frac{d}{dt}}{1 + \sigma \frac{d}{dt}};$$

$$Cv = -\frac{\partial}{\partial x} \left(N \frac{\partial v}{\partial x} \right) \quad \text{or} \quad Cv = \begin{bmatrix} \frac{\partial^2}{\partial x^2} (a_1 \frac{\partial^2}{\partial x^2}) & 0 \\ 0 & \frac{\partial^2}{\partial x^2} (a_2 \frac{\partial^2}{\partial x^2}) - \frac{\partial}{\partial x} (a_3 \frac{\partial}{\partial x}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Equation (9) describes a large number of linear distributed parameter systems under static forces of different nature.

Assume now that there is a possibility to introduce a zero mean, periodic force $\alpha/\epsilon f(t/\epsilon)$ in the system; assume also that formally it would result in additional term $\alpha/\epsilon f(t/\epsilon) \mathcal{D}u$ in equation (9), where $\mathcal{D}$ is a linear partial differential matrix operator. (In the above example of Euler instability, $\alpha/\epsilon f(t/\epsilon)$ was generated by a fast periodic axial force.) With the vibrations introduced, dynamics of the system is described as:

$$A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial u}{\partial t} + [C + \alpha/\epsilon f(t/\epsilon) \mathcal{D}]u = 0. \quad (10)$$

Under which conditions there exists $\alpha/\epsilon f(t/\epsilon)$ such that system (10) exhibits a stable behavior, whereas system (9) might be unstable? To answer this question, let us represent equations (9) and (10) in the form of a system of ordinary differential equations. Assume that

$$u(x, t) = \sum_{k=1}^N q_k(t) \psi_k(x) \quad (11)$$

where $\psi_k: R^n \rightarrow R^m$, $k = 1, 2, \dots, N$, is a sufficiently representative set of vector-valued functions which satisfy the boundary condition for $u(x, t)$ and q_k is a scalar. For the above example of Euler instability $N = 1$ (see equation (7)). Based on assumption (11),

Galerkin method could be applied to equation (9) in order to obtain the following system of ordinary differential equations:

$$A\ddot{q} + B\dot{q} + Cq = 0 \quad (12)$$

where $q = \{q_1, \dots, q_N\}$ and A , B , and C are constant matrices with the following elements:

$$a_{ij} = (A\psi_j, \psi_i); \quad b_{ij} = (B\psi_j, \psi_i); \quad c_{ij} = (C\psi_j, \psi_i). \quad (13)$$

For equation (10) we have:

$$A\ddot{q} + B\dot{q} + [C + \alpha/\epsilon f(t/\epsilon) \mathcal{D}]q = 0, \quad (14)$$

where A , B , and C are the same as in (13) and

$$D = ||d_{ij}|| = ||(\mathcal{D}\psi_j, \psi_i)||.$$

Definition 3. System (9) is said to vibrationally stabilizable if there exists a periodic zero mean function $\alpha/\epsilon f(t/\epsilon)$ such that equation (14) has an asymptotically stable periodic solution $q^*(t/\epsilon)$, $-\infty < t < \infty$ such that $||q^*(t/\epsilon)|| < \delta(\epsilon)$, where $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Theorem 3. Assume that A is a nonsingular matrix. In this case (9) is vibrationally stabilizable if there exists γ^2 such that

$$N = \begin{bmatrix} 0 & I \\ -A^{-1}C - \gamma^2(A^{-1}D)^2 & -A^{-1} \end{bmatrix}$$

is a Hurwitz matrix.

Proof. Under assumption of the Theorem 3, equation (14) can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix} + \frac{\alpha}{\epsilon} f\left(\frac{t}{\epsilon}\right) \begin{bmatrix} 0 & 0 \\ -A^{-1}D & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (15)$$

where $x_1 = q$; $x_2 = \dot{q}$. Denoting

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ -A^{-1}D & 0 \end{bmatrix}$$

and introducing $s = t/\epsilon$, from (15) we obtain:

$$\frac{dx}{ds} = (\epsilon M + \alpha f(s)L)x. \quad (16)$$

Let

$$x = \exp\{\alpha\phi(s)L\}z, \quad (17)$$

where $\phi(s)$ is a zero mean periodic function defined as

$$\phi(s) = \int_0^s f(s)ds - \frac{1}{T} \int_0^T \int_0^s f(s)ds.$$

Here T is the period of $f(s)$. Substituting (17) into (16) we have:

$$\frac{dz}{ds} = \varepsilon \exp\{-\alpha\phi(s)L\} M \exp\{\alpha\phi(s)L\} z.$$

Expanding the matrix exponent in Taylor series and taking into account that $L^1 = 0$ for $i \geq 2$, we obtain:

$$\frac{dz}{ds} = \varepsilon [M + \alpha\phi(s)ML - \alpha\phi(s)LM - \alpha^2\phi^2(s)LML]z. \quad (18)$$

This is an equation in so called standard form (Bogoliubov, 1961). The averaging principle of Bogoliubov can be applied to this equation in order to obtain

$$\frac{d\bar{z}}{ds} = \varepsilon (M - \gamma^2 LML) \bar{z},$$

$$\text{where } \gamma^2 = \alpha^2 \frac{1}{T} \int_0^T \phi^2(s)ds.$$

Obviously, matrix $M - \gamma^2 LML$ coincides with matrix N of Theorem 3.

If there exists γ^2 (and, therefore, α/ε $f(t/\varepsilon)$) such that N is a Hurwitz matrix, from the second theorem of averaging principle (Bogoliubov, 1961) directly follows that there exist ε_0 such that for every $0 < \varepsilon < \varepsilon_0$ equation (18) has a periodic solution $z^*(s) = z^*(t/\varepsilon)$, $-\infty < t < \infty$, with

$$||z^*(t/\varepsilon)|| < \delta(\varepsilon), \quad \delta(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

This statement along with equality

$$[z_1, \dots, z_N]^T = x_1,$$

which follows from (17), proves that existence of such γ^2 , which makes N Hurwitz, is a sufficient condition of vibrational stabilizability of (9).

With some minor technical changes a theorem analogous to Theorem 3 can be proven in case when instead of (11) the following is assumed:

$$u(x, t) = \sum_{k=1}^{\infty} q_k(t) \psi_k(x).$$

V. VIBRATIONAL STABILIZATION OF BENARD INSTABILITY

One of the most celebrated cases of nonlinear hydrodynamic instability is so called Benard problem (see, for instance, Chandrasekhar, 1968), which can be described as follows:

Consider a layer of fluid between rigid horizontal surfaces. If this layer is heated

homogeneously from below and if the temperature gradient reaches certain threshold, an instability in the form of convection cells of regular structure takes place (Benard instability). If the layer is heated from above, the instability does not occur. In this Section, it is shown that if sufficiently fast oscillations with a small amplitude in vertical direction are enforced on the layer heated from below, the instability does not occur and the behavior is as in the case of a layer without vibrations being heated from above.

Consider the hydrodynamic equations for the Benard problem in Boussinesq approximation (Chandrasekhar, 1968):

$$\frac{\partial u_j}{\partial x_j} = 0, \quad (19)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + [1 - \alpha(T - T_0)] X_i + \nu \nabla^2 u_i, \quad (20)$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T, \quad (21)$$

where $u_i(x, t)$, $i = 1, 2, 3$, is a component of the velocity vector; ρ_0 is the density of the fluid at the temperature T_0 ; p is the pressure; α is the coefficient of volume expansion; $T(x, t)$ is the temperature; ν is the kinetic viscosity; κ is the coefficient of thermodynamic conductivity, and

$$X_i = \begin{cases} 0 & \text{for } i = 1, 2, \\ -g & \text{for } i = 3, \end{cases} \quad (22)$$

g is the acceleration of gravitation.

Assume that fast oscillations $\mu \sin \omega t$, $\omega \gg 1$, $\mu \sim 1/\omega$, are enforced on the layer in vertical direction. In this case the hydrodynamics of the Boussinesq approximation is defined by (19)-(21) along with

$$X_i = \begin{cases} 0 & , i = 1, 2, \\ -(g - \omega^2 \mu \sin \omega t) & , i = 3. \end{cases} \quad (23)$$

System (19)-(21) and (23) has fast and slow components. Therefore, it is clear that solutions $u_i(t, x)$ and $T(t, x)$ have "smooth" and "fast oscillating" components. Introduce notations:

$$u_i(x, t) = v_i(x, t) + \xi_i(x, \omega t), \quad i=1,2,3, \quad (24)$$

$$T(x, t) = \tau(x, t) + \zeta(x, \omega t),$$

where $v_i(x, t)$ and $\tau(x, t)$ are smooth functions of time, $\xi_i(x, \omega t)$ and $\zeta(x, \omega t)$ are fast oscillating functions of time with zero mean such that $|\xi_i(x, \omega t)|$, $i = 1, 2, 3$, are of the order 1 whereas $|\zeta(x, \omega t)|$ is of the order $1/\omega$.

Substituting (24) into (20) and (21), we obtain:

$$\frac{\partial(v_i + \xi_i)}{\partial t} + (v_j + \xi_j) \frac{\partial(v_i + \xi_i)}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + [1 - \alpha(\tau + \zeta - T_0)] \lambda_i (\omega^2 \mu \sin \omega t - g) + \nu \nabla^2 (v_i + \xi_i), \quad (25)$$

$$\frac{\partial(\tau + \zeta)}{\partial t} + (v_j + \xi_j) \frac{\partial(\tau + \zeta)}{\partial x_j} = \kappa \nabla^2 (\tau + \zeta), \quad (26)$$

$$\lambda_i = \begin{cases} 0 & \text{for } i=1,2, \\ 1 & \text{for } i=3. \end{cases}$$

In these equations there are terms of different nature: "smooth" and "oscillating"; small (of the order $1/\omega$), "normal" (of the order 1) and large (of the order ω). For the "oscillating" terms of the order ω , from equation (25) we obtain:

$$\frac{\partial \xi_i}{\partial t} = \lambda_i [1 - \alpha(T - T_0)] \omega^2 \mu \sin \omega t. \quad (27)$$

For the "oscillating" terms of the order 1 from equation (26), we have:

$$\frac{\partial \zeta}{\partial t} = -\xi_i \frac{\partial \tau}{\partial x_i}. \quad (28)$$

Solving equations (27) and (28) under the condition $\partial \tau / \partial x_3 = \text{const}$, we obtain:

$$\xi_i(t) = \begin{cases} 0 & \text{for } i=1,2 \\ -[1 - \alpha(T - T_0)] \omega \mu \cos \omega t, & i=3, \end{cases}$$

$$\zeta(t) = -\frac{\partial \tau}{\partial x_3} [1 - \alpha(T - T_0)] \mu \sin \omega t.$$

Substituting these functions into (25), (26) and averaging over the fast "oscillating" motions, we obtain:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + [1 - \alpha(T - T_0)] X_i + \nu \nabla^2 v_i \quad (29)$$

$$\frac{\partial \tau}{\partial t} + v_j \frac{\partial \tau}{\partial x_j} = \kappa \nabla^2 \tau, \quad (30)$$

$$X_i = \begin{cases} 0 & \text{for } i=1,2, \\ -(g - \frac{\alpha \mu^2 \omega^2}{2} \frac{\partial \tau}{\partial x_3}), & i=3. \end{cases} \quad (31)$$

System (29)-(31) is of the same form as system (20)-(22) without oscillation. Therefore, if

$$\mu^2 \omega^2 > \frac{2g}{\alpha} \left(\frac{\partial \tau}{\partial x_3} \right)^{-1}, \quad \frac{\partial \tau}{\partial x_3} > 0, \quad \omega \gg 1, \quad (32)$$

the direction of gravitation in averaged equations (29)-(31) is inverted and, consequently, all the properties of the Benard problem are inverted: a vibrating layer heated from below behaves as a layer without vibrations heated from above. In other words, if condition (32) is satisfied the Benard instability does not occur.

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