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Ohmic Heating of the Reversed-Field Pinch

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Ohmic Heating of the Reversed-Field Pinch

Richard Gerwin

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OHMIC HEATING OF THE REVERSED-FIELD PINCH

by

Richard Gerwin

ABSTRACT

The stabilized toroidal Z-pinch (Reverse Field Pinch) is presently the lead LASL concept in magnetic confinement fusion. An important potential advantage over Tokamaks is the possibility of ohmic heating a device of large aspect-ratio to ignition. In this report, some basic features of this heating process are discussed.

Simple analytic expressions are found for the global heating rate and the time needed to achieve global power balance with radiation and other losses, in useful agreement with large RFP transport codes. A simple condition is noted, which insures that the heating can be accomplished before appreciable resistive evolution occurs in the pinch profile.

The product of poloidal beta, β_θ , and toroidal current, I , that characterizes a condition of global power balance is derived subject to the above-mentioned condition without making key assumptions used by earlier investigators. First, a perfectly steady state (with local power balance) is not assumed, nor is it appropriate to do so. Secondly, the cross-field resistivity is not required to be classical. Since the value of $(\beta_\theta I)$ plays a fundamental role in determining the kind of device one requires, the foundations of this value are important.

I. Introduction

A stabilized Z-pinch (roughly, toroidal magnetic field inside the pinch and poloidal field outside) has the property that it can be wall-stabilized against the global kink instability. For sub-unity values of beta (ratio of local plasma pressure to local field pressure), it will also be stable to global sausage ($m = 0$) and elliptical ($m = 2$) distortions even without a surrounding wall. Moreover, if the toroidal magnetic field undergoes a small

reversal near the edge of the pinch, sufficient magnetic shear can be introduced as to stabilize the localized pressure-driven modes (Suydam modes), and the localized current-driven modes which are thus deprived of a pitch minimum in which to reside. Such a configuration is then referred to as an RFP (Reverse Field Pinch).

Because of the wall stabilization of the kink mode, rather than the q-stabilization relied upon by Tokamaks, it may be possible to resistively heat this large aspect ratio configuration to ignition by means of its own toroidal current. This constitutes an important potential advantage relative to the Tokamaks, which are presently counting upon unwieldy auxiliary heating schemes, and are presently limited to small aspect ratio geometries that are inconvenient to maintain and repair. Small aspect ratio and large toroidal magnetic field strength conditions are required if the Tokamak is to obtain some benefit from ohmic heating without leaving the stable regime.

Here, simple approximate analytic expressions are derived for the global heating rate and the time needed to reach global radiative power balance in the RFP, in useful agreement with large transport codes. A simple condition is noted that allows the heating to be accomplished without appreciable resistive evolution of the profiles, even if the cross-field resistivity, $\eta_{\perp}$, is anomalously large. Then the heating comes from parallel resistivity. Furthermore, the product of toroidal current and poloidal beta appropriate to a condition of global power balance is derived without relying upon the earlier used assumptions of classical $\eta_{\perp}$ and a perfectly steady state (with local power balance). The latter condition is inappropriate and misleading. Since the value of $(\bar{B}_0 I)$ largely determines the character of the machine, the underlying assumptions behind this value need to be realistic.

In a subsequent report, the present treatment is extended to include both non-radiative losses and heating by alpha particles. In that report, a simple criterion will be displayed that distinguishes "wetwood burners" from RFP's that are capable of ignition.

II. Summary of Results

The results are as follows, all for $\beta \lesssim 30\%$. The ohmic heating rate per unit length is given by

$$\frac{dQ}{dt} \approx D_{\parallel} B_0^2, \quad (1)$$

where $D_{\parallel}$ is the instantaneous diffusion coefficient associated with (uniform) parallel resistivity ($D_{\parallel} = \eta_{\parallel} c^2/4\pi$ in cgs units) and B_0 is the instantaneous field strength on the magnetic axis.

The time needed to heat up to a final condition of global power balance with bremsstrahlung is given by

$$t_{PB} \approx \frac{3}{32} \beta_f \frac{a^2}{D_{\parallel f}}, \quad (2)$$

where "f" means "final" value, and the heating is assumed to be done at constant pinch radius, a , by making continuous minor adjustments of toroidal current. Here, beta refers to the ratio of central plasma pressure to central field pressure, and is distinct from poloidal beta. (Strictly speaking, global power balance is an asymptotic condition, and Eq. (2) is actually the time needed to reach 95% of the asymptotic temperature.) Thus, for a given minor radius and desired final values of beta and temperature, the ohmic heating time can be immediately estimated.

Finally, with $\ln \Lambda = 16$, the product of the poloidal beta and toroidal current at global power balance is found to be

$$\bar{\beta}_\theta I \approx 2 \times 10^6 \text{ amps} \quad (3)$$

A more general result that includes non-radiative energy loss is also given in Chapter X. A similar condition has been derived by Leaf Turner and Derek Robinson (private communications) using the assumptions that both $\eta_{\perp}$ and $\eta_{\parallel}$ are

classical, and that there exists a perfectly steady state ($E_\theta = 0$, E_z is uniform) in which the ohmic heating rate balances the radiation rate locally. Here, we derived Eq. (3) allowing $\eta_\perp$ to be anomalous, and without requiring either a steady state or this pointwise power balance. (Above, $\bar{\beta}_\theta$ is the ratio of the average plasma pressure to the external poloidal field pressure.) We note that a reversal in the toroidal field cannot be maintained in a symmetric steady ohmic state. E. P. Butt and R. S. Pease also treated this problem for a stabilized diffuse pinch (Culham Report CLM-R30, 1963), but they took E_z to be uniform which is inconsistent with the non-steady state.

The significance of this last point should not be overlooked. Because of the short time scale for ohmic heating compared to the time for gradual resistive evolution of the profiles, the present writer believes that the condition of the perfectly steady state, used in the earlier derivations of Eq. (3), will never be achieved in practice. Rather, the pinch will "remember" its initial (post-implosion) profiles during its heating phase.

III. Numerical Examples.

From Eq. (1), we see that an RFP at a temperature near 1 keV with magnetic fields near 20 kgauss, with classical parallel resistivity, will dissipate power at a rate of about 1 Megawatt per meter. If 2 keV were attained, the power dissipation would be only ~0.3 Mwatts/meter. As we shall see later, this power does not come instantaneously from the external circuit, but rather from the magnetic energy already stored in the pinch.

From Eq. (3), we see that $\bar{\beta}_\theta \lesssim 0.5$ (required for ideal MHD stability) implies toroidal currents of at least 4 Mamps for global power balance. For $\beta = 20\%$, "global" pressure balance (derived later) yields $\bar{\beta}_\theta \approx 23\%$, and Eq. (3) then gives $I \approx 8.7$ Mamps.

For Eq. (2), suppose we seek a power balance condition with $\beta = 20\%$ and $T = 2$ keV. Then $D_\parallel \approx 70 \text{ cm}^2/\text{sec}$ and $t_{PB} \approx 2.5 \times 10^{-4} a^2 \text{ sec}$, with the minor radius "a" in centimeters. At this point, an overall view can be displayed by invoking "global" pressure balance. This can be used to express Eq. (3) in the form

$$\bar{\beta}_0 N [\text{cm}^{-1}] T [\text{keV}] = 6.3 \times 10^{18} , \quad (3-a)$$

where N is the line density. From this, one immediately obtains the plasma thermal energy as $\epsilon = (0.3/\bar{\beta}_0) \text{MJ/m}$. Eq. (3-a) will be derived later. Since β (hence $\bar{\beta}_0$) and T are assumed to be given here, the line density is determined by Eq. (3-a). Therefore, values of peak density, with an assumed parabolic profile, determine the radius, hence also the heating time. From Amperes' law, the toroidal current and the radius provide $B_\theta(a)$, and global pressure balance then yields the axial magnetic field from $B_0 = B_\theta(a) \sqrt{2\bar{\beta}_0/\beta}$, derived later. Then, the power dissipation can be obtained from Eq. (1).

In this way, we can construct a table for global power balanced RFP configurations of various densities all with $\beta = 20\%$ and $T = 2 \text{ keV}$ (hence $\bar{\beta}_0 = 23\%$, $I = 8.7 \text{ Mamps}$, $\epsilon = 1.3 \text{ MJ/m}$).

We see that low densities are difficult to work with because of the exceedingly large minor bore and excessively long heating time. High densities are difficult because of the uncomfortably large magnetic fields required. Even at intermediate densities, it is clear that we must have a fairly large bore, fairly high current machine. In the subsequent report, we shall see that these conditions are softened by the presence of non-radiative losses.

Thus, Eqs. (1), (2), (3), and (3-a) provide quick and easy estimates of ohmically heated RFP properties associated with conditions of global power balance. It should be obvious that this condition must be reached, maintained,

TABLE I
POWER BALANCED RFP CONDITIONS FOR $\beta = 20\%$, $T = 2 \text{ keV}$

<u>Peak Density</u>	<u>Plasma Radius</u>	<u>Heating Time</u>	<u>Peak Field</u>	<u>Power</u>
$n_0 = 1 \times 10^{13} \text{ cm}^{-3}$	$a = 9.3 \text{ m}$	$t_{PB} = 227 \text{ sec}$	$B_0 = 2.8 \text{ kgauss}$	$\dot{Q} = 5.6 \text{ kwatts/m}$
3×10^{13}	5.4	76	4.9	17
1×10^{14}	3.0	23	9.0	58
3×10^{14}	1.7	7.6	15.5	170
1×10^{15}	0.93	2.3	28.4	570
3×10^{15}	0.54	0.76	49.0	1,700
1×10^{16}	0.30	0.23	88.2	5,550

and then finally overcome (by alpha-particle heating) in a sequence of experiments leading to a reactor. Thus, reaching and maintaining this power balance condition must be the primary goal of a sub-ignition RFP experiment.

IV. Comparison of Time Scales

With classical resistivities ($\eta_{\perp} = 2 \eta_{\parallel}$), the ohmic heating time due to $\eta_{\parallel}$ can be estimated by means of the energy balance equation. (It will be seen that $\eta_{\perp}$ can make only a minor contribution to the heating for conditions of interest.) If $T_e = T_i$, this estimate yields

$$t_{o.H.} \sim 1.5 \beta a^2/D_{\perp} = 0.75 \beta a^2/D_{\parallel} \quad . \quad (4)$$

The final expression is valid at low beta even if $\eta_{\perp}$ is anomalously large.

The use of Ohm's law and pressure balance yields a characteristic time for diffusion of plasma across the field,

$$t_D \sim 2 a^2/\beta D_{\perp} = 2(\beta \eta_{\perp}/\eta_{\parallel})^{-1} (a^2/D_{\parallel}) \quad . \quad (5)$$

Then

$$t_D/t_{o.H.} \sim \beta^{-2} \quad . \quad (6)$$

Thus, low beta insures that the plasma can heat up without appreciable particle diffusion across the magnetic field.

From the ion heat-diffusion equation with classical cross-field thermal conduction, the characteristic time for ion-temperature smoothing is

$$t_{i1} \sim \frac{4a^2}{\beta D_{\perp}} \left(\frac{m_e}{m_i} \right)^{1/2}$$

(We assume that the pinch is not in thermal contact with the wall.) Then

$$\frac{t_{11}}{t_{o.H.}} \sim \frac{3}{\beta^2} \left(\frac{m_e}{m_i}\right)^{1/2} . \quad (7)$$

Therefore, for only moderately small beta values ($\beta \gtrsim 20\%$), the ion temperature will tend to remain flat during the heating. However, at much lower (earlier) beta values, one has to assume that T_i has a flat profile due to set-up conditions, in order to use this flat temperature profile idealization. Otherwise, this idealization has to be regarded as an analytical convenience.

For classical equipartition time, one can obtain

$$t_{eq} \sim \frac{4}{3} \pi \omega_{pi}^{-2} n_i^{-1} ,$$

where ω_{pi} is the ion plasma frequency. Then

$$\frac{t_{eq}}{t_{o.H.}} \sim \frac{2}{9} \beta^{-1} \frac{c^2}{\omega_{pi}^2 a^2} . \quad (8)$$

Thus, a sufficiently small ion skin depth (c/ω_{pi}) guarantees that $T_e \approx T_i$ during the heating phase. This condition will usually be satisfied for conditions of interest, even for very low beta values.

Thus, for low beta, small ion skin depth, and uniform initial or assumed temperature profiles, one can suppose that the plasma heats up without particle diffusion, and that $T_e \approx T_i$ with flat temperature profiles.

Now, we shall turn our attention to anomalous cross-field resistivity, and the associated profile diffusion and heating. For the heating, we shall see later that the current densities are related in order of magnitude by $J_\perp \sim \beta J_\parallel$, so that $n_i J_\perp^2 \sim [(n_i/n_\parallel)\beta^2] [n_\parallel J_\parallel^2]$. Thus, the ohmic heating at low beta will be due primarily to $n_\parallel$, even if n_i is anomalously large. Then, since

Eq. (5) is still valid, $t_D/t_{O.H.} \sim 3 \beta^{-2} (\eta_{\parallel}/\eta_{\perp})$. Thus, the condition $\beta^2 \eta_{\perp} \ll \eta_{\parallel}$ guarantees both that the heating will be mainly due to the parallel current, and also that the characteristic heating time will be much shorter than the plasma cross-field diffusion time.

Finally, we shall consider resistive evolution of the field profiles themselves, a physical process that is quite distinct from the cross-field diffusion mentioned above. (For instance, this process will occur even when $\eta_{\perp} = 0$). If we write Ohm's law in the poloidal direction (with $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}/c$) as

$$E'_{\theta} = \eta_{\parallel} J_{\parallel\theta} [1 + (\eta_{\perp}/\eta_{\parallel}) (J_{\perp\theta}/J_{\parallel\theta})] ,$$

and integrate around a circumference of radius r moving with the plasma fluid, we find from Faraday's law, to order of magnitude,

$$\frac{1}{c} \frac{d}{dt} (\pi r^2 B_z) \sim \eta_{\parallel} (2\pi r J_{\parallel\theta}) [1 + O(\beta \eta_{\perp}/\eta_{\parallel})] .$$

But, from Eq. (20) which will be derived later, $2\pi r J_{\parallel} \sim c B_z$. Then, we can estimate a time scale for resistive evolution of the B_z profile, namely $t_z \sim a^2/D_{\parallel} (1 + O(\beta \eta_{\perp}/\eta_{\parallel}))$. This can be compared to the ohmic heating time. The results are

$$t_z/t_{O.H.} \sim \beta^{-1} \text{ for } \beta \eta_{\perp}/\eta_{\parallel} \lesssim 1 ,$$

$$t_z/t_{O.H.} \sim (\beta^2 \eta_{\perp}/\eta_{\parallel})^{-1} \text{ for } \beta \eta_{\perp}/\eta_{\parallel} \gg 1 .$$

In the second case, then, we would require $\beta^2 \eta_{\perp}/\eta_{\parallel} \lesssim 1$ in order that the heating could be accomplished during a resistive alteration of the B_z profile. In either case, we see again that $\beta \ll 1$ is an indispensable condition for the ohmic heating to be fulfilled before resistive effects destroy the integrity of the RFP profile. To quantify this qualitative result, it is necessary to employ a complete RFP transport code.

As regards the time scale for resistive changes in the B_θ profile, we can again apply Faraday's law. Here, $E'_z(a) - E_z(0) \sim \frac{1}{c} \frac{d}{dt} (a B_\theta)$. But $E_z(0) = \eta_{||} J_{||}(0)$, and $E'_z(a) \sim \eta_{\perp} J_{\perp}(a)$. Furthermore, it can be shown that $J_{||}(0) \sim I/\pi a^2$ and $J_{\perp}(a) \sim \beta I/\pi a^2$. Moreover, $I \sim acB_\theta(a)$. Then, the above Faraday's law can be written in order of magnitude as

$$\eta_{||} (1 + O(\beta \eta_{\perp}/\eta_{||})) \frac{cB_\theta}{\pi a} \sim \frac{1}{c} \frac{d}{dt} (a B_\theta) .$$

From this, one estimates a time-scale for resistive changes of the B_θ profile, as follows,

$$t_\theta \sim a^2/D_{||} (1 + O(\beta \eta_{\perp}/\eta_{||})) .$$

This is the same (qualitative) result that we just obtained for t_z , and the same conclusions apply here as there. Moreover, this result is qualitatively consistent with pressure balance at low beta, from which one finds $B_\theta(a) \sim B_z(0)$. That is, pressure balance at low beta also indicates that the two fields must change on the same sort of time scale.

V. Derivation of the Ohmic Heating Rate

A. The Field Profiles and Pressure Balance

In addition to the assumptions of uniform and equal temperatures, we postulate that the heating occurs in a non-diffusing density profile of the "reasonable" form

$$n = n_0 (1 - r^2/a^2) , \quad (9)$$

which is taken to vanish for $r \geq a$. Thus $r = a$ is the nominal pinch radius. According to the preceeding section, the heating can be considered to take place in a stationary profile provided that beta is sufficiently small.

In the simplest representation of a stabilised Z-pinch, the stabilizing axial field is contained mostly within the pinch. Moreover, in RFP experiments the reversal point is generally located near the "edge" of the pinch, and the amount of field reversal is usually small. We idealize these features by writing the axial magnetic field profile as

$$B_z = B_0 (1 - r^2/a^2) \quad , \quad (10)$$

and which is taken to vanish for $r > a$.

For the purposes of calculating global results, these should be reasonable analytical approximations to the density and toroidal magnetic field profiles of a confined RFP.

The poloidal magnetic field, B_θ , can be obtained from local pressure balance in the form

$$\frac{B_\theta^2}{8\pi} = \frac{1}{X} \int_0^X \left(P + \frac{B_z^2}{8\pi} \right) dX - \left(P + \frac{B_z^2}{8\pi} \right) \quad , \quad (11)$$

where $X \equiv r^2/a^2$. Since $P = 2 n T$, one obtains

$$\frac{B_\theta^2}{8\pi} = n_0 T X + \frac{B_0^2}{8\pi} \left(X - \frac{2}{3} X^2 \right) \quad (12)$$

for $X \leq 1$.

For $X > 1$, one has instead the vacuum field, $B_\theta(r) = a B_\theta(a)/r$.

At $r = a$ ($X = 1$), one then has a nominal, global pressure-balance relation analogous to what would be obtained for a sharp-boundary pinch, namely

$$\frac{B_\theta^2(a)}{8\pi} = n_0 T + \frac{1}{3} \frac{B_0^2}{8\pi} \quad . \quad (13)$$

For low beta, we see that $B_\theta(a) \approx B_0/\sqrt{3}$.

By means of Amperes' law, $B_\theta(a) = 2I/ac$, and in terms of the line density, $N = n_0 \pi a^2/2$, and the embedded axial flux, $\phi_z = B_0 \pi a^2/2$, Eq. (13) can be re-written as follows.

$$\frac{I^2}{c^2} = 4 NT + \frac{1}{3} \frac{\phi_z^2}{\pi^2 a^2} \quad (13-a)$$

Now, the line density, N , is conserved, and by our earlier time-scale analysis, we can claim that the trapped flux, ϕ_z , remains constant during the heating times of interest. [The total toroidal flux, $\phi_{z\text{tot}}$, within the discharge tube, is rigorously conserved when E_θ is shorted out at the wall (crowbarred).]

From Eq. (13-a) there are at least two possible basic scenarios that the pinch can fulfill as it heats up towards a state of power-balance:

1) If the toroidal or axial current, I , is held constant, then the radius, a , must increase, as the temperature, T , increases, so that Eq. (13-a) remains satisfied, or

2) The axial current, I , can be programmed to increase with temperature, T , in just such a way that Eq. (13-a) remains satisfied with a fixed radius, a . This current adjustment is quite minor for low beta values. We shall always assume that the second scenario is carried out, namely that the current is slightly adjusted so as to preserve the low-beta pinch radius during the heating. However, for simplicity, we shall ignore the slight increase in B_θ in our ohmic heating calculation.

Note that the behavior of the current during the heating of a pure Z-pinch must be completely different than scenario #2 for the stabilized Z-pinch. For the pure Z-pinch, $I^2 \approx 4c^2 NT$, so that $I \propto T^{1/2}$ is required by pressure balance.

One can define a poloidal beta as the ratio of peak plasma pressure to external poloidal field pressure,

$$\beta_\theta = 16\pi n_0 T/B_\theta^2(a) = 8 NT/(I^2/c^2) \quad , \quad (14)$$

where N is the line density in cm^{-1} and I is the total toroidal current in stat-amperes.

More usually, the poloidal beta is defined by means of the average plasma pressure, here $\bar{\beta}_\theta$. For the presently-assumed profiles, one has

$$\bar{\beta}_\theta = \frac{1}{2} \beta_\theta \quad . \quad (15)$$

Now, Eq. (13-a) can be written

$$I \sqrt{1 - \bar{\beta}_\theta} = (c/\sqrt{3}) (\phi_z/\pi a) \quad , \quad (13-b)$$

so that an increase of $\bar{\beta}_\theta$ to 30%, say, requires an increase of toroidal current of about 15% during the heating to maintain a constant radius.

For I in amperes, N in meters $^{-1}$ and T still expressed in ergs, Eqs. (14) and (15) imply

$$\bar{\beta}_\theta I^2 = 4 NT \quad . \quad (16)$$

Equation (16) suggests still a third scenario for pinch heating, namely, heating at constant poloidal beta with $I \propto T^{1/2}$, just like the pure Z-pinch. If we put this scenario into Eq. (13-b), we then see that $a \propto T^{-1/2}$, that is, the radius would shrink during heating. This would probably be detrimental because the all-important wall stabilization would tend to be lost during this process.

If one defines a local beta on axis by

$$\beta_0 = 16\pi n_0 T/B_0^2 \quad , \quad (17)$$

then Eq. (13) yields a relation between the on-axis beta and the poloidal beta, namely,

$$\frac{1}{\beta_{\theta}} = \frac{1}{2} + \frac{1}{3\beta_0} \quad , \quad (18)$$

or, also,

$$\frac{1}{\bar{\beta}_{\theta}} = 1 + \frac{2}{3\beta_0} \quad . \quad (18-a)$$

Thus, at low beta, the standard poloidal beta is about 50% larger than the local beta on axis. Note that, for a pure Z-pinch, $\beta_0 \rightarrow \infty$ and $\bar{\beta}_{\theta} \rightarrow 1$.

B. The Current Profiles

For the global ohmic heating calculation, the current profiles are needed. Since the field and pressure profiles are known, the current profiles may be obtained from Amperes' law, and from local pressure balance when convenient.

By means of pressure balance, we find the perpendicular current density to be

$$J_{\perp}^2 = \beta_0 \frac{c^2 n_0 T}{\pi a^2} \frac{X}{1 - (1 - \beta_0/2)X + X^2/3} \quad , \quad (19)$$

where $X \equiv r^2/a^2$.

By means of Amperes' law, we find the total current density to be

$$J^2 = \frac{c^2 B_0^2}{4\pi} \frac{1}{\pi a^2} \left\{ 1 + \frac{\beta_0}{2} - \frac{1}{3}X + \frac{1}{9} \frac{X^2}{1 + \beta_0/2 - 2X/3} \right\} \quad . \quad (20)$$

Ohm's law could now be used to show that the electric field components (θ and z) are comparable and r -dependent unlike the steady state conditions with

$E_\theta = 0$ and E_z uniform. Also, note that $J_\perp \sim \beta_0 J \sim \beta_0 J_\parallel$, by comparison of Eq.'s (19) and (20).

C. The Ohmic Heating Expression

The total ohmic heating rate per cm length is

$$\frac{dQ}{dt} = \int_0^a (\eta_\perp J_\perp^2 + \eta_\parallel J_\parallel^2) 2\pi r dr \quad , \quad (21)$$

which is equivalent to

$$\frac{dQ}{dt} = \pi a^2 \eta_\parallel \left(\frac{\eta_\perp - \eta_\parallel}{\eta_\parallel} \int_0^1 J_\perp^2 dX + \int_0^1 J^2 dX \right) \quad . \quad (21-a)$$

Our assumption of uniform temperature justifies the removal of the classical resistivities from the integral. If $\eta_\perp$ is anomalous and r -dependent, then we are working with its profile-average value, in some sense.

Upon substitution of (19) and (20), the integrals can be evaluated and one obtains

$$\begin{aligned} \frac{dQ}{dt} = \eta_\parallel \left[\frac{\eta_\perp - \eta_\parallel}{\eta_\parallel} \frac{3}{4} \frac{B_0^2 c^2}{4\pi} \beta_0^2 f_\perp(\beta_0) \right. \\ \left. + \frac{3}{4} \frac{B_0^2 c^2}{4\pi} f_\parallel(\beta_0) \right] \frac{\text{ergs}}{\text{cm sec}} \quad , \quad (22) \end{aligned}$$

where, with $\alpha_\pm \equiv 1 \pm \beta_0/2$, we have

$$f_\perp(\beta_0) = \ln \sqrt{\frac{4}{3} - \alpha_-}$$

$$+ \frac{(3/2)\alpha_-}{\sqrt{3 - (9/4)\alpha_-^2}} \left[\arctan \frac{(3/2)\alpha_-}{\sqrt{3 - (9/4)\alpha_-^2}} - \arctan \frac{(3/2)\alpha_- - 1}{\sqrt{3 - (9/4)\alpha_-^2}} \right], \quad (23)$$

$$f_{\parallel}(\beta_0) = \alpha_+ - \frac{1}{3} + \frac{1}{2}\alpha_+^2 \ln \frac{3\alpha_+}{3\alpha_+ - 2}. \quad (24)$$

The following table gives values of $f_{\perp}(\beta_0)$ and $f_{\parallel}(\beta_0)$.

$\beta_0 = 0.0$	.05	.10	.15	.20	.25	.30
$f_{\perp} = .358$	.343	.329	.317	.306	.295	.286
$f_{\parallel} = 1.22$	1.24	1.27	1.30	1.33	1.36	1.39

We see that for beta values up to 30%, the two functions are rather insensitive to beta, with $f_{\perp} \approx 1/3$, and $f_{\parallel} \approx 4/3$. From Eq. (22), we then observe that the ohmic heating associated with the perpendicular resistivity is a factor $(\beta_0^2/3)$ smaller than the ohmic heating associated with $\eta_{\parallel}$, for classical resistivities. In fact, for the two contributions to be comparable, $(\eta_{\perp}/\eta_{\parallel})$ must be an anomalously large resistivity ratio practically over the whole profile, of order $(3\beta_0^{-2})$. For $\beta_0 = 10\%$, $\eta_{\perp}/\eta_{\parallel} \geq 300$ for $\eta_{\perp}$ heating to be significant.

The conclusion that ohmic heating due to $\eta_{\perp}$ is small for low beta-values is also true locally, as can be seen by comparing $J_{\perp}^2$ and J^2 in Eqs. (19) and (20).

If we therefore ignore the $\eta_{\perp}$ contribution and take $f_{\parallel} \approx 4/3$, the ohmic heating rate becomes, from Eq. (22),

$$\frac{dQ}{dt} \approx \frac{c^2}{4\pi} \eta_{\parallel} B_0^2, \quad (25)$$

valid for $\beta_0 \leq 30\%$.

D. Ohmic Heating of Pure Z-Pinches

For pure Z-pinches, $B_z \rightarrow 0$, $\beta_0 \rightarrow \infty$, and it becomes more convenient to return to the earlier expression preceeding Eq. (22). From either Eq. (19) or Eq. (20),

$$J^2 = J_1^2 = 2 c^2 n_0 T / \pi a^2 \quad , \quad (26)$$

for $r \leq a$.

Then, from Eq. (21),

$$\frac{dQ}{dt} = \eta_1 J_1^2 \pi a^2 = \eta_1 c^2 2 n_0 T = \frac{c^2}{4\pi} \eta_1 \bar{\beta}_\theta B_\theta^2(a) \quad .$$

But, $\bar{\beta}_\theta = 1$ for a pure Z-pinch. Hence,

$$\frac{dQ}{dt} = \frac{c^2}{4\pi} \eta_1 B_\theta^2(a) \quad . \quad (27)$$

This expression bears a satisfying resemblance to Eq. (25).

VI. Poloidal Beta Times Toroidal Current: A Global Power Balance Condition

A. RFP Geometry

If we define $g_\parallel(\beta_0) = \frac{3}{4} f_\parallel(\beta_0) \approx 1$ for $\beta_0 \lesssim 30\%$, and neglect the ohmic heating contribution from $\eta_\perp$, then global power balance can be expressed by setting Eq. (22) equal to the bremsstrahlung radiation rate per cm. The result is

$$D_\parallel B_0^2 g_\parallel = b_r T_{\text{(ergs)}}^{1/2} \int_0^a n^2 2\pi r dr \quad .$$

Here, b_r is the proportionality constant for the radiation power density, $b_r n^2 T^{1/2}$, with T still measured in ergs.

Use of the assumed density profile and also, the classical relation $\eta_{||}^{-1} = \sigma_{||} = 2\sigma_{\perp} = 2 S_{\perp} T_{\text{ergs}}^{3/2}$, enables one to transform the above equation to the following statement of global power balance.

$$\frac{c^2}{4\pi} B_0^2 g_{||} = \frac{2}{3} (b_r S_{\perp}) (n_0^2 T^2) (\pi a^2) \quad . \quad (28)$$

Here, we have related the resistivity $\eta_{||}$ to a classical $\eta_{\perp} = 2\eta_{||}$ for notational reasons only, and have not assumed that the actual $\eta_{\perp}$ is classical. We have only assumed that $\eta_{||}$ itself is classical.

Upon multiplication of Eq. (28) by $4\pi/c^2 B_0^2$, we have

$$g_{||} = \frac{8\pi}{3} \left(\frac{b_r S_{\perp}}{c^2} \right) \left(\frac{n_0^2 T^2}{B_0^4} \right) (\pi a^2 B_0^2) \quad ,$$

and taking the square root yields,

$$\sqrt{g_{||}} = \sqrt{\frac{8}{3} \frac{b_r S_{\perp}}{c^2}} \left(\frac{B_0}{16\pi} \right) (\pi a B_0) \quad . \quad (29)$$

But, from global pressure balance as expressed by Eqs. (13) and (14), one has $B_0 = B_{\theta}(a) \sqrt{\beta_{\theta}/\beta_0}$, so that Eq. (29) becomes

$$\sqrt{g_{||}} = \frac{1}{4} \sqrt{\frac{2}{3} \frac{b_r S_{\perp}}{c^2}} \sqrt{\beta_0 \beta_{\theta}} \frac{I}{c} \quad , \quad (30)$$

in which Amperes' law has been used to relate $B_{\theta}(a)$ to the axial current, I .

Finally, use of global pressure balance in the form of Eq. (18) that relates β_0 to β_θ , and the introduction of $\bar{\beta}_\theta = \beta_\theta/2$ allows Eq. (30) to be rewritten as

$$\bar{\beta}_\theta \frac{I}{c} = \frac{\sqrt{g_\parallel(\beta_0)}}{\sqrt{1 + \frac{3}{2}\beta_0}} \frac{6c}{\sqrt{2b_r S_\perp}} \quad (31)$$

To express the current in amperes, one writes this as

$$\bar{\beta}_\theta I_{\text{amps}} = \frac{\sqrt{g_\parallel(\beta_0)}}{\sqrt{1 + \frac{3}{2}\beta_0}} \frac{60c}{\sqrt{2b_r S_\perp}} \quad (31-a)$$

For $\ln(\Lambda) = 16$, one has $S_\perp = 0.278 \times 10^{31}$ (cgs). Also, $b_r = 1.34 \times 10^{-19}$ (cgs). Then, we have the numerical result

$$\bar{\beta}_\theta I_{\text{amps}} = \left[\frac{\sqrt{g_\parallel(\beta_0)}}{\sqrt{1 + (3/2)\beta_0}} \right] \times 2.09 \times 10^6 \text{ [amps]} \quad (32)$$

The function of β_0 in front, [. .], is very close to unity and only weakly dependent upon β_0 . For example, when $\beta_0 = 0$, it is 0.957, and when $\beta_0 = 30\%$, it is 0.848.

Equation (32) furnishes the toroidal current required to maintain an RFP in global power balance (ohmic heating rate = bremsstrahlung rate) at a given value of the poloidal beta. In the final chapter, this formula is generalized to include losses in addition to bremsstrahlung, and, in a subsequent report, to include alpha particle heating. At that point, numerical investigations of wet-wood burner (unignited) RFPs and ignition conditions will be presented.

Assuming local power balance everywhere, an exact steady state ($E_\theta = 0$), and classical expressions for both $\eta_\perp$ and $\eta_\parallel$, Leaf Turner (private communication) has obtained 1.65×10^6 amps, and D. C. Robinson, with the same

assumptions, has obtained 1.12×10^6 amps (private communication). Here, on the other hand, we have not assumed local power balance, in fact, we have a locally non-steady state ($E_\theta \neq 0$), and $\eta_\perp$ may even be anomalous in our model.

It is important to note, from Section IV, that our non-steady state is changing on a time scale of order β^{-1} longer than the heating time, unless $\eta_\perp$ is highly anomalous over most of the profile

The picture that emerges is that the pinch heats up to a condition of global power balance while retaining its initial post-implosion profiles. At this time, the toroidal current must satisfy condition (32), even though an absolutely steady state has not yet been achieved. Then, gradually, during a time of order β^{-1} longer than the heating time, the pinch and the fields (presumably) relax down to more diffuse profiles characteristic of the true steady state.

We conclude this section by deriving some further useful relations for an RFP in global power balance. If we express Eq. (32) as $\bar{\beta}_\theta I = 2 \times 10^6$ amps, and invoke the definition of $\bar{\beta}_\theta$ in the form of Eq. (16), we find

$$I [\text{amps}] = 2N [\text{meters}^{-1}] T [\text{ergs}] \times 10^{-6} \quad , \quad (32-a)$$

or, also,

$$I [\text{amps}] = 3.2 N [\text{meters}^{-1}] T [\text{keV}] \times 10^{-15} \quad , \quad (32-b)$$

or, also upon multiplication by $\bar{\beta}_\theta$,

$$\bar{\beta}_\theta N [\text{meters}^{-1}] T [\text{keV}] = 6.3 \times 10^{20} \quad . \quad (32-c)$$

B. Pure Z-Pinch Geometry

For completeness, we shall now derive the condition on the toroidal current for an unstabilized Z-pinch. Equating the ohmic heating rate as given by Eq. (27) to the bremsstrahlung loss rate, we have

$$\frac{c^2}{4\pi} \eta_{\perp} B_{\theta}^2(a) = b_r T^{1/2} \int_0^a n^2 2\pi r dr = \frac{1}{3} b_r T^{1/2} n_0^2 \pi a^2 .$$

With $\eta_{\perp}^{-1} = \sigma_{\perp} = S_{\perp} T^{3/2}$ for classical perpendicular resistivity, this equation immediately becomes

$$1 = \frac{4}{3} \frac{b_r S_{\perp}}{c^2} \left(\frac{n_0 T}{B_{\theta}^2(a)} \right)^2 \pi^2 a^2 B_{\theta}^2 ,$$

and taking the square root yields,

$$1 = \sqrt{\frac{b_r S_{\perp}}{3c^2}} \left(\frac{\beta_{\theta}}{16\pi} \right) (2\pi a B_{\theta}(a)) .$$

Use of Amperes' law and $\bar{\beta}_{\theta} = \beta_{\theta}/2$ then provides

$$\bar{\beta}_{\theta} \frac{I}{c} = \sqrt{\frac{2}{3}} \frac{6c}{\sqrt{2b_r S_{\perp}}} \quad (33)$$

for the pure Z-pinch.

Now we note that $\bar{\beta}_{\theta} = 1$ for a pure Z-pinch, so that Eq. (33) determines the pinch current for global power balance. By comparison with Eq. (31), we see that this current is $(\sqrt{\frac{2}{3}} \approx 0.8)$ times 2.1 Megamps, that is, 1.7 Megamps. (Pease obtained this value, but with $\ln \Lambda = 10$ and for an exact stationary Z-pinch profile.)

(R. S. Pease, Proc. Phys. Soc. London, 70 B (1957) 11)

VII. The Climb of Temperature to the Bremsstrahlung Limit

Suppose we equate the rate of change of thermal energy to the excess of the ohmic heating rate over the bremsstrahlung loss rate. The rate of change of thermal energy of both electrons and ions can be represented by $3NdT/dt$, where N is the line density. For $\beta_0 \lesssim 30\%$, so $\bar{\beta}_0 \lesssim 31\%$, and assuming that the ohmic heating at low beta is predominantly due to classical $\eta_{\parallel}$, the ohmic heating rate is taken to be $g_{\parallel}(\beta_0) (c^2/4\pi) \eta_{\parallel} B_0^2$ where $g_{\parallel}(\beta_0)$ was defined in Section VI-A. The bremsstrahlung rate can be written as

$$\int_0^a b_r n^2 T^{1/2} 2\pi r dr = \frac{4}{3} \frac{1}{\pi a^2} b_r N^2 T^{1/2} ,$$

where b_r is an absolute constant chosen so that T is in ergs, as in Section VI-A.

Then, expressing time τ in units of the initial diffusion time, $t_0 \equiv a^2/(c^2 \eta_{\parallel}(0)/4\pi)$, where $\eta_{\parallel}(0)$ is the initial (post-implosion) parallel resistivity, and measuring the temperature, T , in terms of its initial value, the above-mentioned equation governing the rate of change of the temperature becomes

$$\frac{dT}{d\tau} = [g_{\parallel}(\beta_0(\tau)) \frac{32}{3} \frac{1}{\beta_0(0)}] T^{-3/2} - [\frac{1}{9} \frac{b_r S_{\perp}}{c^2} a^2 B_0^2 \beta_0(0)] T^{1/2} , \quad (34)$$

where $S_{\perp}$ was defined in Section VI-A.

But, with the aid of pressure balance at $r = a$, Amperes' law, and Eq. (31), it can be shown that

$$\frac{1}{9} a^2 B_0^2 \frac{b_r S_{\perp}}{c^2} = \frac{32}{3} \frac{g_{\parallel}(\beta_{of})}{\beta_{of}^2} ,$$

where β_{of} is the final (global power balance) value of β_o . Then Eq. (34) can be rewritten as

$$\frac{3}{32} \frac{dT}{d\tau} = \frac{g_1(\beta_o(\tau))}{\beta_o(o)} \tau^{-3/2} - \frac{g_1(\beta_{of}) \beta_o(o)}{\beta_{of}^2} \tau^{1/2} \quad (35)$$

Since $\tau(\tau) = \beta_o(\tau)/\beta_o(o)$, we thus verify a self-consistency condition, namely that the right side of Eq. (35) must vanish at large times as a state of global power balance is approached.

Now, the function of $g_1(\beta)$ is a gentle function of β that is always very close to unity in the range of beta-values that we are considering here ($\beta \lesssim 30\%$). Therefore, we can henceforth replace g_1 by unity in Eq. (35) with no significant error incurred. In particular, the just-mentioned consistency condition remains true.

If a new time is now defined by $\tau_1 = (32/3)(\tau/\beta_o(o))$, then Eq. (35) reads

$$\frac{dT}{d\tau_1} = \tau^{-3/2} - \left[\frac{\beta_o(o)}{\beta_{of}} \right]^2 \tau^{1/2} \quad (36)$$

Now, if we define a new temperature $\tilde{T}$ by

$$\tilde{T} = \frac{\beta_o(o)}{\beta_{of}} \tau \equiv \frac{T}{T_f} \quad , \quad (37)$$

which is the temperature normalized to its asymptotic value, and yet a new time by

$$\tau_2 = \left[\beta_o(o)/\beta_{of} \right]^{5/2} \tau_1 \quad , \quad (38)$$

then Eq. (36) becomes

$$\frac{dT}{d\tau_2} = T^{-3/2} - \tilde{T}^{1/2} \quad (39)$$

The solution of this equation can be expressed by the time required to reach a given temperature, namely,

$$\begin{aligned} \tau_2 = & \ln \sqrt{\frac{1 + \sqrt{T}}{1 - \sqrt{T}}} + \arctan \sqrt{T} - 2\sqrt{T} - \ln \sqrt{\frac{1 + \sqrt{\tilde{T}_0}}{1 - \sqrt{\tilde{T}_0}}} \\ & - \arctan \sqrt{\tilde{T}_0} + 2\sqrt{\tilde{T}_0} \quad , \end{aligned} \quad (40)$$

where $\tilde{T}_0 \ll 1$ is the ratio of the initial (post-implosion) temperature to the asymptotic temperature. This solution is plotted in Fig. 1 for different initial temperatures. We see that the curves are insensitive to the initial temperature, and that the temperature has reached 95% of its asymptotic value when $\tau_2 = 1$. Thus it is important to discuss the meaning of the time τ_2 .

Upon using the preceeding definitions of t_0 , τ_1 , and τ_2 , we conclude that τ_2 is the physical time expressed in terms of the time unit, t_f , where

$$t_f = \frac{3}{32} \beta_{of} \frac{a^2}{D_{lf}} \quad , \quad (41)$$

where $D_{lf} = (c^2/4\pi)\eta_{lf}$, and the subscript f refers to the long-time steady (global power balance) value of the temperature. Thus, it is the final temperature which is most significant in the representation of the time to achieve global power balance by means of ohmic heating.

Thus, we find the result that the temperature climbs to practically its final power balance value (95%) in a time given by Eq. (41). This characteristic time scales as $NT_f^{5/2}/\beta_0^2$, where N is line density. However, if use is made of Eqs. (32-a) and (32), and also if (β_0/β_0) is treated as an

Fig. 1.

Temperature (abscissa) vs time (ordinate), both normalized, for an ohmically-heated reversed-field pinch with bremsstrahlung losses.

approximate constant, the heating time given by Eq. (41) is found to scale roughly as $(T_f^{1/2}/n)$. Thus, the higher-density plasmas will reach global power balance sooner than the lower-density plasmas (where n means volume density)! (The catch is that, for a given current, the final temperature will be lower for the higher density; see Eq. (32-b).) It is worth remarking that the earlier-mentioned time τ_1 represents physical time normalized to a time unit identical to Eq. (41) except that initial values appear instead of final

values. Such a time unit is far too small to constitute a useful measure of the time to reach global power balance.

Finally, we have to qualify the heating time result obtained here due to the neglect of magnetic field depletion (wherein field energy becomes plasma thermal energy during ohmic heating). The large transport codes [which have self-consistent field depletion] obtain heating times about a factor of two or three shorter [without impurities] than the present expression, although the scaling with the various parameters does seem to be predicted correctly (Richard S. Christian, private communication). The presence of impurity line radiation complicates matters still further, and the present theory probably applies only at temperatures above those required to strip the impurities, perhaps a few hundred eV for light ions (E. Caramana, private communication).

VIII. Volt-Seconds Required for Heating

In this section we shall estimate the number of volt-seconds required in order to heat the pinch to a condition of global power balance. Let us now work in MKS units and use Faraday's law to express the toroidal electric field at the wall, $r = r_w$, in terms of the internal fields.

$$E_z(r_w) = E_z'(a) + a \ln\left(\frac{r_w}{a}\right) \frac{\partial B_\theta(a)}{\partial t} ,$$

where "a" is the (nominal) pinch radius, assumed to be held constant on the heating time scale by adjustments of the toroidal current, or, equivalently, by adjustment of $B_\theta(a)$. Here, we have, for simplicity, ignored a term arising from the slow resistive motion of the radius "a(t)", which term ultimately can be shown to be of the same order of magnitude as the logarithmic term itself, for small beta values. At any rate, both the logarithmic term retained and the term neglected prove to correspond to only a very modest number of volt-seconds. Then, upon multiplication by the pinch length, l , and integration over time, $t = 0$ to t_f , we find the required number of volt-seconds at the wall (considered to be a distributed poloidal gap through which the B_θ flux lines penetrate), namely,

$$V.S.(r_w) \equiv \ell \int_0^{t_f} E_z(r_w) dt = \ell \int_0^{t_f} E'_z(a) dt + \ell a \ln\left(\frac{r_w}{a}\right) [B_\theta(a, t_f) - B_\theta(a, 0)] ,$$

or

$$V.S.(r_w) = V.S.(a) + \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{r_w}{a}\right) [I(t_f) - I(0)] . \quad (42)$$

Now, $E'_z(a) = \eta_\perp J_\perp(a)$ where E' means the electric field relative to the moving plasma, and $J_\perp(a)$ can be found from Eq. (19). After calling upon pressure balance and Amperes' law at $r = a$, it can be written in MKS units as

$$J_\perp(a) = \frac{I}{\pi a^2} \bar{\beta}_\theta . \quad (43)$$

Therefore, Eq. (42) can be written as

$$V.S.(r_w) = \ell \int_0^{t_f} \eta_\perp \frac{I}{\pi a^2} \bar{\beta}_\theta dt + \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{r_w}{a}\right) [I(t_f) - I(0)] . \quad (44)$$

The first term represents the poloidal flux actually absorbed by the plasma at $r = a$, and the second term represents the flux required to replenish the vacuum region around the plasma.

Now, by means of pressure balance at $r = a$, it has been shown that the toroidal current must be programmed during the heating so as to make $(I\sqrt{1 - \bar{\beta}_\theta})$ independent of time. This programming is required in order to keep the pinch radius constant. Then Eq. (44) becomes

$$\begin{aligned}
V.S.(r_w) = & \left\{ \frac{\ell}{\pi a^2} \int_0^{t_f} \frac{\eta_{\perp}}{\eta_{\perp}(t_f)} \frac{\bar{\beta}_{\theta}}{\sqrt{1 - \bar{\beta}_{\theta}}} dt \right. \\
& \left. + \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{r_w}{a}\right) \left[\frac{1}{\sqrt{1 - \bar{\beta}_{\theta}(t_f)}} - \frac{1}{\sqrt{1 - \bar{\beta}_{\theta}(0)}} \right] \right\} \times I(t_f) \sqrt{1 - \bar{\beta}_{\theta}(t_f)} \quad (45)
\end{aligned}$$

If we now neglect the small, finite beta corrections, and use $\bar{\beta}_{\theta}(t_f) I(t_f) = 2 \times 10^6$ amperes, Eq. (45) becomes approximately,

$$\begin{aligned}
V.S.(r_w) \approx & \left\{ \frac{\ell}{\pi a^2} \int_0^{t_f} \frac{\eta_{\perp}}{\eta_{\perp}(t_f)} \frac{\bar{\beta}_{\theta}}{\bar{\beta}_{\theta}(t_f)} dt \right. \\
& \left. + \frac{\mu_0 \ell}{4\pi} \ln\left(\frac{r_w}{a}\right) \right\} [\text{ohm-sec}] \times 2 \times 10^6 [\text{amps}] \quad (46)
\end{aligned}$$

As an example, for about a ten-meter major circumference, the logarithmic second term represents a modest number of volt-seconds, about 2 volt-seconds times $\ln(\frac{r_w}{a})$. In any case, it is directly calculable for a given set of dimensions, ℓ , r_w , a . Therefore, we now concentrate on the first term of Eq. (46), namely, the flux absorbed by the plasma itself.

The first term of Eq. (46), $V.S.(a)$, involves the cross-field resistivity, $\eta_{\perp}$, which could well be anomalous, particularly near the edge of the pinch where the drift velocities are large. If so, this term cannot be truly estimated until we have some knowledge of the value of this resistivity, and its time behavior during the heating phase. Therefore, we shall perform a detailed estimate of this first term only under the assumption that the resistivity is strictly classical.

Then, with the aid of Eq. (39), the integral in Eq. (46) can be done easily and takes the form $t_f \ln(1 - \tilde{T}^2(t_f))^{-1/2}$, where the small initial temperature has been neglected, and $\tilde{T}$ is (T/T_f) . Actually, within the present model, it takes an infinite time to reach global power balance and strictly

speaking, the required number of volt-seconds is then also infinite. But, practically speaking, we shall take $t = t_f$ to be the time at which the temperature has reached 95% of its asymptotic value, so $\tilde{T}(t_f) = 0.95$ at $t = t_f$ as given by Eq. (41). Then, the logarithm is very close to unity, and the integral in Eq. (46) essentially becomes just t_f . Then, the first term of Eq. (46) reads

$$V.S.(a) \approx \ell \frac{\eta_{\perp}(t_f)}{\pi a^2} t_f \times 2 \times 10^6 \text{ volt-seconds} \quad (47)$$

for classical cross-field resistivity with $T = 0.95 T_f$.

A naive application of Ohm's law, $\ell E_z = I \ell \eta / \pi a^2$, integrated over time from $t = 0$ to $t = t_f$, would have given a number of volt-seconds larger than Eq. (47) by the factor $(\beta_0)^{-1}$. The reason this larger result is incorrect is that this simpler form of Ohm's law does not apply to a non-steady state. We shall discuss this more fully below, when we carefully examine Poynting's theorem.

Upon substitution of Eq. (41) into Eq. (47), with due conversion of (41) into MKS units, and recalling that $\eta_{\perp} = 2 \eta_{\parallel}$ classically, we find that (47) becomes

$$V.S.(a) \approx 0.15 \ell \beta_f \approx R \beta_f \text{ volt-seconds} \quad , \quad (48)$$

where R is the major radius in meters.

Thus, with $\ell = 10$ meters for the major circumference and $\beta_f = 0.3$, only 0.5 volt-seconds would be required to reach global power balance. Thus, Eq. (48) constitutes a modest requirement and applies when the perpendicular resistivity is classical.

However, we emphasize that if $\eta_{\perp}$ is highly anomalous, the required number of volt-seconds could be much larger than given by Eq. (48). Some of these volt-seconds get used up in ohmic dissipation, but also, some of them get used in restructuring the magnetic field profile on the time scale a^2/D_{eff} as discussed earlier in the paper. Here D_{eff} is an effective resistive diffusion

coefficient of the form $D_{\parallel}(1 + O(\beta\eta_{\perp}/\eta_{\parallel}))$. To roughly estimate the magnitude of this effect, we suppose $\eta_{\perp}$ is some anomalous constant value, $\eta_{\perp}^*$, and we write the first term of Eq. (46) as (again using Eq. (39))

$$\begin{aligned} V.S.^*(a) &= \left[\frac{\ell \eta_{\perp}^*}{\pi a^2} \int_0^{t_f} \tilde{T} dt \right]_{\text{ohm-sec}} * 2 \cdot 10^6 \text{ [amps]} \\ &= \left[\frac{\ell \eta_{\perp}^*}{\pi a^2} t_f \int_0^u \frac{\tilde{T}^{5/2} d\tilde{T}}{1 - \tilde{T}^2} \right]_{\text{ohm-sec}} * 2 \cdot 10^6 \text{ [amps]} . \end{aligned}$$

The integral can be done exactly for $u < 1$, and if $u = 0.95$, then the value of the integral is about $3/4$. Then

$$V.S.^*(a) \approx \left(\frac{\ell \eta_{\perp}^* t_f}{\pi a^2} \right)_{\text{ohm-sec}} * 1.5 \times 10^6 \text{ [amps]} . \quad (49)$$

After division of Eq. (49) by (47), we find to within a factor close to unity,

$$\frac{V.S.^*(a)}{V.S.(a)} \approx \frac{\eta_{\perp}^*}{\eta_{\perp}(t_f)} ,$$

so that if $\eta_{\perp}^*$ is anomalously large, say by a factor of 100 compared to the final classical resistivity, one could wind up with 50 volt-seconds for $\ell = 10$ meters, and $\beta_f = 1/3$.

Now we shall examine Poynting's theorem in order to clarify the reason that only a modest number of volt seconds is required to heat to power balance, when $\eta_{\perp}$ is not highly anomalous.

IX. The Contributions to Ohmic Heating

Poynting's theorem applied to a cylindrical screw pinch with a moving boundary can be put into the following form,

$$\begin{aligned} \int (\eta_{\perp} J_{\perp}^2 + \eta_{\parallel} J_{\parallel}^2) dV = & - \oint \frac{c}{4\pi} \vec{E}' \times \vec{B} \cdot d\vec{s} - \frac{d}{dt} \left(\int \frac{B^2}{8\pi} dV \right) \\ & - \int \frac{1}{c} \vec{J} \times \vec{B} \cdot \vec{v} dV - \oint \frac{B^2}{8\pi} \vec{v} \cdot d\vec{s} \quad , \end{aligned} \quad (50)$$

where the right side is composed of the electromagnetic energy flux across the moving boundary, the rate of change of magnetic energy in the pinch volume, the rate of work by the pondero-motive force on the bulk plasma, and the rate of work done on the pinch by the external magnetic pressure. Here, $\vec{v}$ is the radial plasma velocity, whereas V denotes the volume of integration.

For a low beta, perfectly steady state RFP the last three terms would vanish and Eq. (50) would reduce to a statement of equality between the circuit power and the resistive dissipation,

$$\int (\eta_{\perp} J_{\perp}^2 + \eta_{\parallel} J_{\parallel}^2) dV = D_{\parallel} B_0^2 = 2 E_z(a) I \quad , \quad (51)$$

from which we would find $E_z(a) \approx \eta_{\parallel} I / \pi a^2$. This expression will generally differ from $E_z'(a)$ calculated directly from Eq. (43) and Ohm's law, namely, $E_z'(a) = \bar{B}_{\theta} \eta_{\perp} I / \pi a^2$. The reason for this paradox is the neglect of terms, caused by the incorrect assumption just made that our hypothetical post-implosion profiles constitute a perfectly steady state. As emphasized earlier in this paper, the profiles in fact slowly evolve on a resistive time scale, and it is essential to include this feature for a correct interpretation of Poynting's theorem, as we shall now see.

We shall assume low beta ($\beta \lesssim 30\%$), and, according to our time scale estimates in Section IV, we shall take the resistive time scale to be $a^2/D_{\parallel}$ ($1 + O(\beta \eta_{\perp}/\eta_{\parallel})$), with the associated plasma velocity $a^{-1} D_{\parallel} (1 + O(\beta \eta_{\perp}/\eta_{\parallel}))$. Then, recalling that the left side of Eq. (50) is just $(D_{\parallel} B_0^2)$, we can estimate the

order of magnitude of each of the four terms on the right side of Eq. (50) as follows.

$$\text{Term (1)} \sim (D_{\parallel} B_0^2)(\beta \eta_{\perp}/\eta_{\parallel})$$

$$\text{Term (2)} \sim (D_{\parallel} B_0^2)(1 + O(\beta \eta_{\perp}/\eta_{\parallel}))$$

$$\text{Term (3)} \sim \beta(D_{\parallel} B_0^2)(1 + O(\beta \eta_{\perp}/\eta_{\parallel})).$$

$$\text{Term (4)} \sim (D_{\parallel} B_0^2)(1 + O(\beta \eta_{\perp}/\eta_{\parallel}))$$

We see that if $\beta \eta_{\perp}/\eta_{\parallel} \ll 1$ (classical or moderately anomalous $\eta_{\perp}$), the primary contributions to resistive dissipation are Terms (2) and (4), namely, the magnetic energy change and the work done by the external magnetic pressure. Only for strongly anomalous $\eta_{\perp}$, $\beta \eta_{\perp}/\eta_{\parallel} \gtrsim 1$, does Term (1), the power flux across the pinch edge, contribute to the same order as Terms (2) and (4).

We conclude that, even though the ohmic heating time may be short ($\beta \ll 1$) compared to the slow resistive time scale for the profile evolution, it is nevertheless essential to take the latter effect into account in the process of understanding the various contributions to power balance that appear in Poynting's theorem.

We further reiterate that this resistive heating can take place even with negligible electromagnetic power flow across the pinch boundary, and indeed does not even require power flow into the discharge tube (after the set-up phase). As shown by large RFP transport codes, sufficient ohmic heating can take place just from conversion of ambient ohmic magnetic energy even in a totally crow-barred system ($E_{\theta} = 0$, $E_z = 0$, at the wall). This is also the reason why the number of volt-seconds absorbed by the plasma from outside is small. (The main reason for not crowbarring the toroidal voltage would seem to be that this voltage allows some control of the toroidal current, which, in turn, allows some control of the plasma radius. As mentioned earlier, the ideal approach would be to maintain the radius constant during the heating phase.)

In summary, an equation essentially equivalent to Eq. (51) is usually employed in steady state derivations of the value of $\bar{\beta}_0 I$. However, even slow resistive evolution of the profiles renders Eq. (51) invalid, and additional terms appear which are more important. Therefore, a different procedure has to

be used to derive the Pease relation. This amounts to the use of pressure balance, and the resulting Pease relation proves to be practically the same as given by the steady-state approach.

X. Additional Losses

A. Introduction

The condition of global power balance represented by Eq. (31-a) assumed that bremsstrahlung was the only energy loss mechanism. However, it is easy to take extra losses, such as radial heat-conduction or anomalous transport, into account in an approximate manner by means of a phenomenological energy loss time, τ_E .

Here, we shall do this for the nonsteady state with anomalous $\eta_{\perp}$ in Section B. For comparison, we shall reproduce Leaf Turner's elegant treatment (private communication) of the exact steady state with classical $\eta_{\perp}$ in Section C. Both models yield very similar results for the product $\beta_0 I$. However, the reversed-field pinch could never relax to such steady-state profiles during the relatively short ohmic heating time. In particular, these steady state profiles can be shown to admit no toroidal field reversal, which is disadvantageous from the standpoint of ideal MHD stability.

B. The Non-steady State With Possibly Anomalous Perpendicular Resistivity

For uniform and equal species temperatures, global power balance can be written as

$$g_{\parallel} D_{\parallel} B_0^2 - b_r T^{1/2} \int_0^a n^2 2\pi r dr - \frac{3NT}{\tau_E} = 0 \quad , \quad (52)$$

where N is the line density, and the three terms are ohmic heating with low beta, radiation loss, and other unspecified energy losses with a characteristic time τ_E . We recall that $g_{\parallel}(\beta_0) = \frac{3}{4} f_{\parallel}(\beta_0)$ with $f_{\parallel}(\beta_0)$ defined in Eq. (24). We also recall that $D_{\parallel} = (c^2/4\pi)\eta_{\parallel}$ and write $\eta_{\parallel}^{-1} = \sigma_{\parallel} \equiv 2 S_{\perp} T^{3/2}$. (The last step does not assume $\eta_{\perp}$ is classical!)

After some manipulation, this reads

$$\sqrt{g_{\parallel}} = \sqrt{\frac{2}{3} \frac{b_r S_{\perp}}{c^2} + 3 \frac{S_{\perp}}{\tau_E c^2} \frac{T^{1/2}}{n_o}} (2\pi a B_o) \frac{n_o T}{B_o^2} . \quad (53)$$

Upon use of Amperes' law, $2\pi B_o(a) = 4\pi I/c$, together with $B_o/B_o(a) = \sqrt{\beta_o/\beta_o}$ obtained from pressure balance, one finds

$$\sqrt{g_{\parallel}} = \frac{1}{4} \sqrt{\frac{2}{3} \frac{b_r S_{\perp}}{c^2}} \sqrt{1 + \frac{9}{2} \frac{1}{b_r \tau_E} \frac{T^{1/2}}{n_o}} \frac{I}{c} \sqrt{\beta_o \beta_o} . \quad (54)$$

Finally, noting that $\beta_o = 2 \bar{\beta}_o$ and that $\beta_o/\bar{\beta}_o = (2/3)(1 + 3\beta_o/2)$ from pressure balance, and that $I_{amps} = 10 I/c$, this becomes

$$\bar{\beta}_o I_{amps} = \left\{ \frac{\sqrt{g_{\parallel}(B_o)}}{\sqrt{1 + 3\beta_o/2}} \frac{60c}{\sqrt{2} b_r S_{\perp}} \right\} \frac{1}{\sqrt{1 + (9/2) T^{1/2}/b_r \tau_E n_o}} . \quad (55)$$

For τ_E very large, we recover Eq. (31-a). Thus, for finite τ_E , the product $\bar{\beta}_o I$ becomes smaller. We observe that a knowledge of toroidal current, poloidal beta, density and temperature, at power balance conditions, enables one to estimate the value of τ_E .

C. The Steady-State With Classical Perpendicular Resistivity

The elegant derivation presented here is due to Leaf Turner. It assumes a steady state ($E_{\theta} = 0$, E_z uniform), and entails a nonuniform temperature profile. From Ohm's law in the form $\vec{J} = \sigma_{\perp} \vec{E}_{\perp} + \sigma_{\parallel} \vec{E}_{\parallel}$, we have

$$J_z = E_z (\sigma_{\perp} \sin^2 \alpha + \sigma_{\parallel} \cos^2 \alpha) = E_z S T^{3/2} , \quad (56)$$

where α is the pitch angle of magnetic field lines and classical conductivities are assumed, so that

$$S \equiv S_{\perp} (1 + \cos^2 \alpha) \quad . \quad (57)$$

Here, $\sigma_{\parallel} = 2 \sigma_{\perp}$ and $\sigma_{\perp} = S_{\perp} T^{3/2}$.

Equating the local steady power input to the steady heating rate (similar to the local form of Eq. (51)) and thence to radiation and other losses, we have

$$E_z J_z = E_z^2 S T^{3/2} = b_r n^2 T^{1/2} + 3nT/\tau_E \quad . \quad (58)$$

This is a quadratic equation for $(n/\sqrt{T})$, with the solution

$$\frac{n}{T^{1/2}} = \frac{1}{2} \left\{ -\frac{3}{\tau_E b_r} + \sqrt{\left(\frac{3}{\tau_E b_r}\right)^2 + 4E_z^2 \frac{S}{b_r}} \right\} \quad , \quad (59)$$

so that the plasma pressure is

$$p = 2nT = \{ \dots \} T^{3/2} = \{ \dots \} \frac{J_z}{E_z S} \quad , \quad (60)$$

where the braces refer to Eq. (59) and we have used Eq. (56). Since we seek a global result, we shall ignore the weak profile-dependence of S and use an average value, $1.5 S_{\perp} \approx S$. According to Eq. (59), this implies that the ratio $(n/\sqrt{T})$ has only a weak dependence on radial position. Then, the average pressure is immediately

$$\bar{p} = \{ \dots \} I / \pi a^2 E_z S \quad . \quad (61)$$

But the poloidal beta is $\bar{\beta}_\theta = 8\pi\bar{p}/B_\theta^2(a)$. Amperes' law and $\bar{p}$ from Eq. (61) then immediately produce $\bar{\beta}_\theta I/c = 2c \{ . . . \}/E_z S$, or

$$\bar{\beta}_\theta I_{\text{amps}} = \frac{20c}{E_z S} \{ . . . \} \quad . \quad (62)$$

But

$$\{ . . . \} = 2n/T^{1/2} \approx 2 n_o/T_o^{1/2} \quad ,$$

where the subscript "o" refers to the value on the magnetic axis, and, from Eq. (58),

$$\frac{(E_z S)^2}{S b_r} = \left(\frac{n}{T^{1/2}} \right)^2 + \frac{3}{b_r \tau_E} \left(\frac{n}{T^{1/2}} \right) \quad ,$$

so that

$$\frac{\{ . . . \}}{E_z S} = \frac{2}{\sqrt{S b_r}} \frac{1}{\sqrt{1 + (3/\tau_E b_r) T_o^{1/2}/n_o}} \quad . \quad (63)$$

Substitution of Eq. (63) into (62) yields

$$\bar{\beta}_\theta I_{\text{amps}} = \frac{40c}{\sqrt{1.5 S} b_r} \frac{1}{\sqrt{1 + (3/\tau_E b_r) T_o^{1/2}/n_o}} \quad . \quad (64)$$

The numerical pre-factor is about $(3/4)$ of that shown in Eq. (55). In the τ_E term, the numerical factors also differ by about $(3/4)$, but this is not all that significant if we interpret T as a profile-average temperature in Eq. (55) and note that T_0 is the central temperature in Eq. (64). It is remarkable that two such distinct physical models produce such similar results.

An energy loss time smaller than $(3/b_r)(T^{1/2}/n)$ can therefore have an appreciable effect on the conditions associated with global power balance. For $T = 1.6 \times 10^{-9}$ ergs = 1 keV, $n = 10^{14} \text{ cm}^{-3}$, we find $\tau_E = 10$ seconds as the critical loss time

It is worth noting that alpha particle heating can roughly be expressed in terms of a negative τ_E , in which case the produce $\bar{\beta}_0 I$ becomes larger than 2 Megamps. Alpha particle heating will be treated more carefully in a subsequent report. There, it will be shown that losses much greater than bremsstrahlung cannot be tolerated if ignition of the RFP is to be achieved by ohmic heating alone.

XI. Conclusion

A state of global power balance between ohmic heating and bremsstrahlung must be reached and then surpassed by alpha heating or other supplementary heating in a progression of experiments leading to the RFP reactor. Therefore, the achievement of the sub-ignition power balance condition is itself an important milestone on the way towards reaching reactor conditions in the RFP.

In this report, we have re-examined the assumptions leading to the characterizations of this milestone. The earlier results for $\bar{\beta}_0 I$ were here found to remain valid for more realistic assumptions than were used before. In addition, simple analytical results for heating rates and heating times were uncovered. Finally, the expression $\bar{\beta}_0 I$ was generalized to include non-radiative loss processes.

Thus, the coupled pictures of heating and resistive evolution of the RFP plasma have been clarified.

XII. Acknowledgments

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