MASTER

LECTURES IN ACCELERATOR THEORY*†

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Accelerator theory is a broad area of study involving the behaviour of particle beams. Well known examples are the theory of particle acceleration and the theory of beam stability in linear focusing systems. Out of the many categories, I have chosen to describe three phenomena taken from (1) the theory of single particle motion in nonlinear fields; (2) the theory of high intensity coherent motion and; (3) the theory of current accumulation.

The choice of topics was dictated by the requirement that the physical principles involved be simple so that the ideas and phenomena related to each topic could be described in a single lecture. This implies the use of idealized beam models, where subtle influences are lost in the simplifying approximations. These effects, as well as providing interesting physics, are important in a practical sense as limitations to the performance of high energy physics experiments. However, to fully appreciate such phenomena and their implications requires extended, dedicated effort.

Lecture I deals with the behaviour of particles in the nonlinear field arising from the electromagnetic interaction of colliding beams. The case treated, that of counter-rotating proton beams crossing each other at a non-zero angle, has the

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simple feature that the force between the beam is one dimensional. In lecture II, an analysis of the development of traveling waves on particle beams is presented. The situation studied is that of a uniform beam current in a circular accelerator and the excitation for the coherent motion is induced by the resistivity of the vacuum chamber wall. Finally, in lecture III, a description of the current accumulation process used at the proton storage rings at CERN (The ISR) is given. Particle pulses of rather low average current are injected and stored along the length and width of the vacuum chamber. The efficiency is very high and large currents (over 40 amperes) have been achieved.

We include here a short bibliography which might be helpful in supplementing or expanding on the material presented.

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THE BEAM-BEAM INTERACTION FOR CROSSING BEAMS

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I. THE BEAM-BEAM FORCE

Beam-Beam Geometry

The counter-rotating beams in ISABELLE, as at the ISR, will cross at some small angle, α . The geometry of the collisions is shown in Figure 1.

Fig. 1. Beam-Beam Crossing Geometry

Maxwell's Equations For Static Fields And The Lorentz Force

Problem:

- Write down Maxwell's Equations for static fields and with a charge density ρ and current $j = v \rho$, with v the beam velocity.
- Introduce vector and scalar potentials. Find the electric and magnetic fields in terms of the scalar potential alone.
- Write down the Lorentz Force for a particle of velocity $\vec{v}_p$, with

$$\vec{v}_p \cdot \vec{v} = -v^2 \cos \alpha$$

Show that the force can be written in terms of the Scalar Potential V by

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$$\frac{1}{e} \vec{F} = [1 + \beta^2 \cos \alpha] (-\nabla V) - \beta^2 \sin \alpha \frac{\partial V}{\partial x} \hat{k} \quad (1)$$

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y}$$

where β is the velocity in units of c , x and y are transverse coordinates and $\hat{k}$ is a unit vector in the direction of the beam.

Effect Of A Short Interaction Length

Problem:

- (a) Write down the appropriate relativistic force equation. Assume a linear orbit over the short interaction length (IL). Let $s = vt$ measure the length along the orbit. Then show that the total deflection in one crossing (obtained by integrating over IL) is given in terms of V by

$$\delta x' = - \frac{e}{mv^2 \gamma} [1 + \beta^2 \cos \alpha] \int_{IL} \frac{\partial V}{\partial x} ds$$

$$\delta y' = - \frac{e}{mv^2 \gamma} [1 + \beta^2 \cos \alpha] \int_{IL} \frac{\partial V}{\partial y} ds \quad (2)$$

$$\delta z' = - \frac{e}{mv^2 \gamma} \beta^2 \sin \alpha \int_{IL} \frac{\partial V}{\partial x} ds$$

$$x' \equiv \frac{dx}{ds}$$

and γ is the usual relativistic factor.

- (b) Take a Gaussian transverse charge distribution in x and y , with rms widths σ_H and σ_V respectively. Solve Poisson's Equation for the scalar potential by Fourier Integrals.

- (c) Show by symmetry arguments that

$$\delta x' = \delta z' = 0.$$

Show that $\delta y'$ is given by

$$\delta y' = \frac{2\pi\lambda r_p}{e\gamma\beta^2 \sin \alpha} [1 + \beta^2 \cos \alpha] \phi(y/\sqrt{2}\sigma_V) \quad (3)$$

where $\mathcal{E}$ is the error function,

$$\mathcal{E}(u) = \frac{2}{\sqrt{\pi}} \int_0^u dt e^{-t^2} \quad (4)$$

λ is the line charge density (the current $I = \lambda c$), and r_p is the classical proton radius.

In the relativistic limit ($\beta \rightarrow 1$),

$$\delta y' = B(y) = \frac{2\pi I r_p}{ec \gamma \tan \alpha/2} \mathcal{E}\left(\frac{y}{\sqrt{2}\sigma_v}\right). \quad (5)$$

We have obtained the deflection of a particle at y crossing a beam using a coordinate system with the beam running along the z -axis. The beam is assumed to be fixed (the strong beam) and the particle is one unit of the other beam (the weak beam).

The characteristics of the system are such that a significant simplification has been achieved. The problem has been reduced to a 1-dimensional one. The force is directed vertically. The coordinates defining the interaction plane, x and z , do not appear. Thus only y is affected by the beam-beam interaction. But since the y coordinate is the same in the beam or particle frame of reference, the resulting description of the particle motion in the total accelerator is greatly simplified.

II. MODEL OF THE BEAM-BEAM INTERACTION

Perturbed Equation For ISABELLE

ISABELLE can be represented, when unperturbed, by linear oscillations about a central orbit defined by the magnetic elements which guide the particles around the ring. The linear oscillations are described by a gradient function, $K(s) = \text{Gradient/Momentum}$, which in a strong focusing system such as ISABELLE has a rapid (+ -) cycle. Combining the unperturbed oscillations with the nonlinear deflection, $B(y)$, we obtain the perturbed oscillator equation.

$$y'' + K(s) y = B(y) \delta(s). \quad (6)$$

This equation, with a single point-like perturbation, corresponds to colliding beam systems such as at the ISR or ISABELLE in the case that all the collision regions are assumed to be identical. It describes therefore 1/6th of the ISABELLE ring or 1/8th of the ISR ring.

The Tune Shift

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Problem: Define the Courant-Snyder variables, η , $\dot{\eta}$ by

$$\begin{bmatrix} \eta \\ \dot{\eta}/\nu \end{bmatrix} = \begin{bmatrix} \beta^{-1/2} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \quad (7)$$

where $\theta = \frac{1}{\nu} \int_0^s \frac{d\rho}{\beta(\rho)}$

$$\nu = \frac{1}{2\pi} \int_0^C \frac{d\rho}{\beta(\rho)} \quad (\text{betatron tune}) \quad (8)$$

and $\beta(s)$ is the beta function which has the period of $K(s)$.

a) Show that the perturbed oscillator equation can be expressed in the form:

$$\ddot{\eta} + \nu^2 \eta = 4\pi\nu\xi\delta(\theta) E(\eta) \quad (9)$$

$$\text{where } E(\eta) = \int_0^\eta du e^{-u^2(\beta^*/2\sigma_V^{*2})} \quad (10)$$

$$\xi = \frac{I_r \beta^*}{\sqrt{2\pi} e c \gamma_V \sigma_V^* \tan \frac{\alpha}{2}} \quad (11)$$

and

β^* and σ_V^* are the β -function and the rms beam height at the collision point, respectively.

b) Show that with this definition of the beam-beam strength parameter, ξ , the linear tune shift due to this perturbation is given by

$$(\Delta\nu)_{\text{linear}} = -\frac{1}{4\pi} \int \beta(s) \left\{ \frac{dB(y)}{dy} \Big|_{y=0} \delta(s) \right\} ds = -\xi. \quad (12)$$

Under what conditions is this expression for the tune shift valid?

Luminosity

The luminosity is a measure of how effective the collision geometry is in giving the most events for a given interaction. Let the two particle interaction be represented by the invariant cross section, σ . The event rate for this interaction can there-

fore be written as the product of σ and the luminosity, L , which is thereby a relativistic invariant. With this definition and the crossing geometry described in Figure 2, the luminosity is:

$$L = \left[|\vec{v}_1 - \vec{v}_2|^2 - \frac{1}{c^2} (\vec{v}_1 \times \vec{v}_2)^2 \right]^{\frac{1}{2}} \int \rho(1) \rho(2) dV \quad (13)$$

where $\rho(1)$, $\rho(2)$ are volume particle densities and dV is the 3-D space differential, $dV = dx dy dz$.

Fig. 2. Overlap of Crossing Beams. I_1 , I_2 are currents (uniform) and α is the crossing angle.

Problem:

For Gaussian beams, we have that $\rho(i)$ is Gaussian along y , is Gaussian along the normal to $\vec{v}_i$ and is uniform along $\vec{v}_i$ [for $i = 1, 2$]. Using these properties, evaluate the luminosity in (13) and show that in the relativistic limit ($\beta \rightarrow 1$),

$$L = \frac{I_1 I_2}{2\sqrt{\pi} e^2 c \sigma_v^* \tan \frac{\alpha}{2}} \quad (14)$$

Relationship of Luminosity and Beam-Beam Strength

From the definitions of L and ξ , using (14) and (11), and taking $I_1 = I_2 = I$, we have:

$$\frac{L}{\xi} = \frac{IY}{\sqrt{2} e r_p \beta^*} \quad (15)$$

From this expression we can develop some design rules based on the existence of a limiting value of ξ . In this case, keep ξ at its limiting value and maximize the luminosity. We can increase the luminosity by having a higher I (but we must be careful to keep ξ fixed), or a lower β^* or we can operate at higher γ (higher energy). These rules undoubtedly have a general validity. However, there probably is not a well defined threshold for ξ , although at some level experimental performance will doubtless begin to deteriorate.

III. RESONANCE THEORY OF THE BEAM-BEAM INTERACTION

The Hamiltonian

Using the new variable

$$q = \frac{\sqrt{\beta}}{\sqrt{2\sigma}} \eta = \frac{y}{\sqrt{2\sigma}},$$

equation (9) becomes

$$\ddot{q} + \nu^2 q = 4\pi\nu\xi F(q) \delta(\theta), \quad (16)$$

with

$$F(q) = \int_0^q dt e^{-t^2}.$$

The Hamiltonian for this system is

$$H = \frac{\nu}{2} (q^2 + \frac{\dot{q}^2}{\nu^2}) - 4\pi\xi\delta(\theta) \int^q F(q) dq, \quad (17)$$

in the sense that

$$p = \frac{\partial H}{\partial \dot{q}} = \frac{\dot{q}}{\nu}, \quad \dot{p} = - \frac{\partial H}{\partial q}.$$

Action-Angle Transformation

Introduce the new variables J and ϕ :

$$q = \sqrt{2J} \cos \phi, \quad \dot{q}/\nu = \sqrt{2J} \sin \phi.$$

Then the quantity H becomes:

$$H = \nu J - 4\pi\xi\delta(\theta) K (\sqrt{2J} \cos \phi),$$

with,

$$K = \int F(q) dq \Big|_{q = \sqrt{2J} \cos \phi}. \quad (18)$$

Problem:

- (a) Show directly that the equations of motion in the new variables are:

$$\dot{J} = -\partial H / \partial \phi, \quad \dot{\phi} = \partial H / \partial J. \quad (19)$$

Then, use the generating function

$$G = \frac{1}{2} q^2 \tan \phi$$

and show that the transformation is canonical and that the equations of motion are therefore given by (19).

Slow Phase Approximation For A Resonance

θ and ϕ are angle variables, periodic with period 2π . $\delta(\theta)$ and $K(\sqrt{2J} \cos \phi)$ can therefore be written,

$$\delta(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta}, \quad K = \sum_{m=-\infty}^{\infty} K_m(J) e^{-im\phi}$$

where

$$K_m(J) = \frac{1}{2\pi} \int_0^{2\pi} K(\sqrt{2J} \cos \phi) e^{im\phi} d\phi = K_{-m}(J). \quad (20)$$

The Hamiltonian then becomes,

$$H = \nu J - 2\epsilon \sum_{n,m} K_m(J) e^{i(n\theta - m\phi)}.$$

Problem:

Show that $K_m(J)$ can be evaluated and is given by,

$$K_m(J) = \frac{(-1)^{\frac{m}{2} + 1}}{m^2 - 1} e^{-J} \left[\frac{1}{2} I_{\frac{m}{2}} + J \left(I_{\frac{m}{2}} + I_{\frac{m}{2}}' \right) \right] \quad (21)$$

$$m = 2, 4, 6, \dots$$

$$K_m(J) = 0$$

$$m = 1, 3, 5, \dots$$

$$\text{where } I_{m/2}(J) = \frac{1}{\pi} \int_0^\pi e^{J \cos \theta} \cos \frac{m}{2} \theta d\theta$$

is the "I" Bessel function of order $m/2$.

Define a phase variable

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$$a = n\theta - m\phi.$$

If a changes rapidly as a function of time (θ), a systematic change of J in the phase plane will not result from such a term. However, if a changes slowly, then a systematic amplitude (J) change can result from a succession of small changes. The condition for a slow change in a is:

$$\dot{a} \approx 0, \text{ or } \dot{\phi} \approx n/m.$$

Since $\dot{\phi} \approx \nu$, we keep only a single "slow phase" term defined by (m , n , ν). Thus, in this approximation the Hamiltonian is:

$$H = \nu J - 2\xi K_0(J) - 2\xi K_m(J) \cos a \quad (22)$$

and the equations of motion are:

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial J} = \nu - 2\xi K_0'(J) - 2\xi K_m'(J) \cos a, \\ J &= - \frac{\partial H}{\partial \phi} = - 2m\xi K_m(J) \sin a. \end{aligned} \quad (23)$$

We have also retained the $m = n = 0$ term which causes the perturbed betatron frequency to be a function of amplitude, J :

$$\nu_p = \nu - 2\xi K_0'(J). \quad (24)$$

Note that

$$\dot{\phi} \approx \nu_p(J).$$

Invariant in the Slow Phase Approximation

Problem:

Using Equations (22) and (23), show that the quantity

$$\bar{H} = H - \frac{n}{m} J \quad (25)$$

is an invariant of the motion, i.e., $\dot{\bar{H}} = 0$.

We can obtain an approximate invariant which shows the nature of the phase space motion. Since the resonant term in $\dot{\phi}$ in Eq. (23) is small compared to $\nu(J)$, we have at resonance:

$$\dot{a} = n - m\dot{\phi} \approx n - m\nu(J) = 0.$$

Solving for $J = J_p$ we obtain the resonant amplitude:

$$n - m\nu(J_p) = 0. \quad (26)$$

This is an important characteristic of beam-beam resonances in that it ties the resonance frequency to the amplitude J . If J changes significantly from J_p , the resonance condition is lost.

Problem:

Assuming only small deviations of J from J_p , we can obtain the trajectories for particles near $J = J_p$.

(a) Show that the curve

$$\frac{1}{2}(J - J_p)^2 - h_m(J_p) \cos a = C \quad (27)$$

with

$$h_m(J_p) = \frac{K_m(J_p)}{K_o''(J_p)}$$

is an invariant curve in the rotating (J, a) phase plane.

(b) Define a fixed point and show that

$J = J_p$, $a = 0, \pi$ are fixed points.

(c) What are the conditions that define the fixed points to be stable or unstable?

(d) A separatrix is a curve that passes through unstable fixed points. Some trajectories in the (J, a) phase plane are shown in Fig. 3.

Fig. 3. The (J, a) Phase Plane Near the Separatrix.

Show that if $h_m(J_p) < 0$, the separatrix is given by,

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$$\frac{1}{2}(J - J_p)^2 - h_m(J_p) [\cos a - 1] = 0. \quad (28)$$

Show that the maximum range of J of the separatrix is given by,

$$\delta J_m = 4[h_m(J_p)]^{\frac{1}{2}}. \quad (29)$$

This is the resonance width in amplitude or the region of J where the resonance effect is felt.

Phase Space Islands

The resonance can be looked upon as an island in phase space at the amplitude, J_p , and with width δJ_m . For a given order of resonance, m , J_p is a function of the linear tune, ν_{lin} , and the beam-beam strength, ξ . Thus, if ν_{lin} or ξ is time varying, J_p will change. In other words, the islands can move in and out in the phase space. Now, if the time variation is an adiabatic modulation, particles can become trapped in the moving islands and experience a slow drift in the phase plane. This is a potential mechanism for particle phase space transport.

Effects of Moving Islands

The fixed points at J_p are determined by the linear tune, ν_{lin} , by

$$\nu_{lin} = \frac{n}{m} + 2\xi K'_0(J_p). \quad (30)$$

A variation of ν_{lin} corresponds to an island movement in amplitude (J_p) through Eq. (30). If ν_{lin} changes at the rate $\delta \nu_{per}$ (the shift of linear tune per period of the machine), then the rate of island movement, measured by δJ_p , the change in island amplitude per period, is related to $\delta \nu_{per}$ through

$$\delta \nu_{per} = 2\xi K''_0(J_p) \delta J_p,$$

or

$$\delta \nu_{per} = 4\pi \xi K''_0(J_p) J_p. \quad (31)$$

Consider a particle moving within an island near the stable fixed point. The particle's amplitude due to the resonance is changing at the rate,

$$\dot{J} = -2m\xi K_m(J) \sin a.$$

Near the stable fixed point, $|\sin a| \approx 1$, and so

$$|\dot{J}| = |2m\dot{\xi}K_m(J)|. \quad (32)$$

The nature of the motion under these conditions depends on whether the island motion is faster or slower than the amplitude motion due to the resonance. There are two extreme cases:

- (A) If $|\dot{J}| > |\dot{J}_p|$, then we have adiabatic motion of the islands and particles near the island center may be trapped and transported.
- (B) If $|\dot{J}| < |\dot{J}_p|$, we have a condition for resonance splitting. Since the imposed time modulation is fast compared to the resonance motion, we should include this as a modification of the even faster time dependence of the force. If this is done, it will be seen that some "fast oscillating" terms remain. If these are separated out, it will be seen that the resonances in phase space are split by the fast periodic modulation.

Problem:

- (a) If v_{lin} is changing at the rate δv_{per} , show that the trapping condition (A) is given by

$$\delta v_{per} \leq 8\pi m \dot{\xi}^2 |K_0''(J_p) K_m(J_p)|. \quad (33)$$

Show also that if ξ is changing at the rate $\delta \xi_{per}$ (per period), then the trapping condition can be written:

$$\delta \xi_{per} \leq \left| \frac{\xi}{v} \right| 8\pi m \dot{\xi}^2 |K_0''(J_p) K_m(J_p)|, \quad (34)$$

where $\bar{v} = v - \frac{n}{m}$.

- (b) If ξ is modulated at the frequency v_s per revolution, show that each resonance splits into three resonances if (B) is satisfied. Find the widths of these resonances and the resonance spacing. Can they overlap? What is the condition for these "sideband" resonances to overlap?

Resonant Bandwidth in Frequency, Resonance Spacing, and a Simple Application of the Chirikov Criterion

Corresponding to a resonance amplitude, δJ_m (Eq. 29), there is a frequency width, δv_m . This arises from the connection between frequency and amplitude, an important property of the beam-beam force. This relationship is given by Eq. (24). If we let δJ_m be a variation in J , then the corresponding variation in frequency, δv_m , is given by

$$\delta \nu_m = -2\xi K_0''(J) \delta J_m.$$

Since the width of the resonance in amplitude is given by Eq. (29), we find for the frequency bandwidth,

$$\delta \nu_m = 8\xi |K_0'' K_m|^{1/2}. \quad (35)$$

Consider two adjacent beam-beam resonances, ν_m and ν_{m+2} . Recall that due to the even nature of the force, odd-ordered resonances disappear. If m is the resonance order, I is the integer value of the betatron frequency and ℓ is an integer which gives the appropriate harmonic of the force ($\ell \leq m$), then the resonance frequency can be written,

$$\nu_m = \frac{\ell + Im}{m}.$$

Problem:

Show that the spacing, $\delta \nu_{sp}$, of two adjacent resonances, ν_m and ν_{m+2} , can be written,

$$\delta \nu_{sp} = |\nu_m - \nu_{m+2}| = \frac{2(m-\ell)}{m(m+2)}.$$

For the set $3/4, 5/6, 7/8, 9/10, \dots$, we have $(m-\ell) = 1$. In this case, then,

$$\delta \nu_{sp} = 2/m(m+2). \quad (36)$$

The condition of resonance overlap corresponds roughly to the onset of "stochastic" or "chaotic" behavior. Resonance overlap occurs when the resonance width is greater than the resonance spacing. Therefore, if

$$\delta \nu_m > \delta \nu_{sp}, \text{ then chaos will result.}$$

From (35) and (36), the beam-beam threshold for "chaos" is given by

$$\xi > \frac{1}{4m(m+2)} |K_0''(J_p) K_m(J_p)|^{-1/2}. \quad (37)$$

A comparison of Eqs. (33) and (37) shows that the trapping condition and the Chirikov condition are somewhat similar in form. Let ξ_{TR} be the trapping limit (i.e., trapping occurs for $\xi > \xi_{TR}$). Let ξ_{CH} be the Chirikov limit (i.e., Chaos occurs if $\xi > \xi_{CH}$). From Eqs. (33) and (37), we find that the ratio ξ_{TR}/ξ_{CH} is given by the simple expression,

$$\frac{\xi_{TR}}{\xi_{CH}} = (m+2) \left| \frac{2m}{\pi} \delta \nu_{per} \right|^{1/2}. \quad (38)$$

Problem: Consider possible operating modes of ISABELLE. Under what conditions would you apply Eq. (38)? Using typical ISABELLE parameters, start to increase ξ from some sufficiently low value. Which effect, trapping or chaos, occurs first? At what value of ξ ?

IV. GENERAL PROBLEM SET

1. Extend the resonance theory to more than one interaction per period. Show how symmetrically located interaction points reduces the number of resonances but increases their strength.
2. With many identical interaction points the resonances discussed here can be termed systematic resonances. This is equivalent to saying that many resonant frequencies have zero strength. Show that by adding small perturbations, these resonances can be "activated". In particular, perform the analysis for resonances induced by random errors in the relative positions of the vertical orbits of the two colliding beams.
3. Perform a resonance analysis for the ISR in the case of 8 crossings, random orbit misalignments and a 5th order resonance. Assume a uniform phase space distribution of particles. Assume δv_{per} is on the order of 10^{-8} to 10^{-7} . What rms error σ_{rms} in orbit alignment leads to particle trapping as the resonance is crossed? To avoid trapping, find a condition on σ_{rms} ? Is this an attainable tolerance -- without information from the beam? With information from beam observation? Taking intrabeam scattering (multiple coulomb scattering within each beam) as the source of momentum diffusion and using ISR parameters, check if the above value of δv_{per} , assumed as an average tune change rate, is reasonable or not. Compare your result on the tolerance on the rms closed orbit error with the ISR correction of the 5th order resonance by orbit control in a weak-beam aperture scan experiment. Can this model of a beam-beam resonance coupled with intrabeam scattering give a loss rate? Are the ISR parameters and observations consistent with your estimated loss rates?
4. Use a similar analysis as in problem 3 for ISABELLE to arrive at a tolerance on orbit misalignment at the crossing points. Show that the rms orbit control should be in the range 0.1 to 0.01 mm. Can this be achieved?
5. If the beams are separated during stacking and acceleration, how best can they be brought into collision?
6. If the frequency $v(J)$ becomes a very weak function of J : $v'(J) \rightarrow 0$, what assumptions in the above theory must be modified? Does the concept of resonance amplitude width still have any

significance? How can the bandwidth be defined in this case?
How does the phase space topology differ from the case where
 $V'(J)$ is large? Can you find a quantitative criterion for
 $V'(J)$ which distinguishes the two limiting cases?

THE RESISTIVE WALL INSTABILITY

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Perfectly Conducting Walls
Finite Resistivity
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Geometry and Source of Fields

Consider the source depicted in Fig. 1: an infinitesimally thin (horizontally) charge located at position x' ($y=0$) with height $2h$ in a circular chamber of radius b .

Fig. 1. Geometry and source of fields

The beam motion is perpendicular to the plane of Fig. 1. Let s be the coordinate measuring distance along the beam axis.

We are seeking a travelling wave solution for the resulting fields, that is with a phase $ks - \omega t$, with ω the coherent frequency and k the wave number. For simplicity, consider motion only in the vertical direction. Therefore the source for such fields is a coherent vertical displacement of the beam, Δy , given by

$$\Delta y = \xi(x') e^{i(ks - \omega t)}, \quad (1)$$

where the oscillation amplitude ξ is a function of horizontal position in the chamber of the source point x' .

The normalized unperturbed charge distribution is:

$$\rho = \frac{I}{\beta c (2h)} \delta(x - x') [\theta(y+h) - \theta(y-h)],$$

where I is the beam current, β is the particle velocity in units of c , and δ and θ are delta and step function respectively. The

current density is simply

$$j_s = \beta c \bar{\rho}.$$

If we take the total charge distribution, including the perturbation (1), then subtract the unperturbed charge, we obtain, to first order in ξ , the perturbed charge density

$$\bar{\rho} = \frac{1}{\beta c (2h)} \xi(x') e^{i(ks - \omega t)} \delta(x - x') [\delta(y - h) - \delta(y + h)]. \quad (2)$$

The perturbed current density is:

$$\bar{j}_s = \beta c \bar{\rho}. \quad (3)$$

Thus, our source model is a simple plane wave travelling along the top and bottom of the thin beam.

We can simplify the problem of finding the fields by recognizing that the frequency ω will be low for transverse beam oscillations. Thus, in first approximation, we can neglect displacement currents. Of course, when wall resistivity is included, this is not a valid approach. Thus, we take the following course: neglect displacement currents to solve the case of a perfectly conducting boundary; then reintroduce the time derivatives to include the effect of finite wall resistivity.

Solution for Perfectly Conducting Walls

For the case of perfectly conducting walls, we have a static problem. Maxwell's equations are

$$\begin{aligned} \nabla \times E &= 0, & \nabla \times B &= \mu \bar{j}_s \hat{t}_s, \\ \nabla \cdot E &= \bar{\rho}/\epsilon, & \nabla \cdot B &= 0, \end{aligned} \quad (4)$$

with $\hat{t}_s$ a unit vector along the beam axis, and $\mu\epsilon = 1/c^2$. The problem separates into independent electrostatic and magnetostatic ones. We introduce the scalar and vector potentials:

$$\begin{aligned} E &= -\nabla V, & B &= \nabla \times A, \\ \nabla \cdot A &= 0. \end{aligned} \quad (5)$$

We, therefore, obtain the equations,

$$\begin{aligned} \nabla^2 V &= -\bar{\rho}/\epsilon, \\ \nabla^2 A_s &= -\mu \bar{j}_s = -(\beta/c) \frac{\bar{\rho}}{\epsilon}. \end{aligned} \quad (6)$$

For a perfectly conducting surface, we have the boundary conditions that the tangential component of E and the normal component of B are zero: that is,

$$\begin{aligned} E_{\theta} &\sim \frac{\partial V}{\partial \theta} = 0, \\ B_r &\sim \frac{\partial A_s}{\partial \theta} = 0 \text{ on the boundary.} \end{aligned} \quad (7)$$

The electric and magnetic problems are identical, with the s component of the vector potential related to the scalar potential by

$$A_s = (\beta/c)V. \quad (8)$$

Recall that the sources for the fields are simply two points. Therefore, we can use the method of images. Consider a complex plane

$$z = x + iy.$$

For a source at $z = z_1$, the image is located at

$$z(\text{image}) = \frac{b^2}{|z_1|^2} z_1,$$

as indicated in Fig. 2.

Fig. 2. z -Plane for solution by images.

If we let,

$$z_1 = x' + ih,$$

then the potential is the difference of two terms, a source at z_1 and one at z_1^* :

$$V = - \frac{IZ_0 \xi e^{i(ks-\omega t)}}{4\pi\sigma h} \left[\ln \left| \frac{z-z_1}{z-b^2/z_1^*} \right| - \ln \left| \frac{z-z_1^*}{z-b^2/z_1} \right| \right], \quad (9)$$

which satisfies the boundary condition, $\partial V / \partial \theta = 0$ at $|z| = b$. The vector potential A_s is given by (8). From V and A_s , the electric and magnetic fields E and B can be found from (5). Z_0 is the impedance of free space.

Boundary Condition in The Case of Nonperfectly Conducting Walls

We seek a solution for the electromagnetic fields within a conducting medium which has the form of a travelling wave in the direction of the surface. Neglecting the electric displacement current and assuming a conduction current

$$j = \sigma E,$$

with σ the conductivity of the medium, then Maxwell's equations within the conducting medium are given by

$$\begin{aligned} \nabla \times E &= -i\omega B, & \nabla \cdot E &= 0, \\ \nabla \times B &= \mu \sigma E, & \nabla \cdot B &= 0. \end{aligned} \quad (10)$$

All the fields drop immediately to zero in a perfect conductor. Thus, to find an approximate solution in the case of a nonperfect conductor, we make use of the fact that the spatial variation of the fields normal to the surface is much more rapid than the variations parallel to the surface. If $\hat{n}$ is a unit vector normal to the surface, pointing inward, then we can use

$$\nabla \equiv -\hat{n} \frac{\partial}{\partial \xi},$$

where ξ is the coordinate normal to the surface.

From the divergence equations, we have no spatial variation of the normal components of E and B . Therefore, within the conductor we can take approximately

$$n \cdot E = n \cdot B = 0.$$

The curl equations are

$$\frac{\partial}{\partial \xi} (n \times E) = -i\omega B,$$

$$\frac{\partial}{\partial \xi} (n \times B) = -\mu \sigma E.$$

Let $\hat{i}$ be a unit vector along the direction of E . We write,

$$E = E_{\parallel}(\xi) \hat{i}.$$

Then, since $n \times n \times E = -E$, we obtain

$$\frac{d^2 E_{\parallel}}{d\xi^2} = -i \omega \mu \sigma E_{\parallel}.$$

The skin depth δ is given by

$$\delta = \left(\frac{2}{\mu \sigma \omega} \right)^{1/2}.$$

Therefore, the equation for $E_{\parallel}$ in a conducting medium is

$$E''_{\parallel} + \frac{2i}{\delta^2} E_{\parallel} = 0. \quad (11)$$

The B field is also parallel to the surface, but normal to the E field. Writing $B \equiv B_{\parallel}$, we have

$$B_{\parallel} = \frac{i}{\omega} E'_{\parallel}, \quad (12)$$

where the direction of B is such that the Poynting vector is inward, meaning that energy is absorbed into the medium.

Fig. 3. Wall surface. δ is the skin depth for the conducting medium (assumed to have infinite thickness).

The situation at the chamber wall surface is shown in Fig. 3. At the interface between the chamber wall and the inner space, the fields are E_0 and B_0 . What we are seeking is the ratio of electric and magnetic fields just inside the surface - that is, the ratio E_0/B_0 . The continuity at the interface tells us that the fields just outside the metallic surface satisfy the same condition.

The solution of (11) in the medium is

$$E = E_0 e^{-\kappa q}, \quad (13)$$

where

$$\kappa = \frac{\sqrt{2}}{\delta} (-i)^{\frac{1}{2}} = \frac{1-i}{\delta}.$$

This solution satisfies the condition, $E(q) \rightarrow 0$ as $q \rightarrow \infty$. The ratio of fields at the surface is then obtained from (13) and (12), evaluated at $q = 0$, and is given by

$$\frac{E_0}{iB_0} = \frac{\omega\delta}{2} (1 + i). \quad (14)$$

Fields Resulting from Wall Resistivity

The solution for perfectly conducting boundaries results in an electric field at the boundary which is purely radial and a magnetic field which is purely tangential. Longitudinal fields do not enter in the analysis.

Thus, the effect of including resistance in the walls is to introduce a longitudinal E field. The tangential B field must only be continuous at the boundary. If we work only to first order in the resistivity, we can take the tangential magnetic field to be unchanged by the wall resistance. Replacing $B_0 \rightarrow B_\theta$ (the tangential component of B with no wall resistance) and writing E_s for the resulting longitudinal E field, we can obtain E_s on the boundary from (14):

$$E_s(\text{boundary}) = - \frac{(1-i)}{2} \omega \delta B_\theta, \quad (15)$$

where δ is the skin depth.

If we recall that $B = \nabla \times A$, then we have

$$B_\theta(\text{boundary}) = - \frac{\partial A_s}{\partial r} = - \frac{\beta}{c} \frac{\partial V}{\partial r} \Big|_{r=b}, \quad (16)$$

where V is given by (9). Thus,

$$E_s(\text{boundary}) = - \frac{(1-i) \omega \delta I Z_0 \xi e^{i(ks-\omega t)}}{8\pi c h} \frac{\partial K}{\partial r} \Big|_{r=b}, \quad (17)$$

where

$$K(z) = \ln \left| \frac{z-z_1}{z-b^2/z_1^*} \right| - \ln \left| \frac{z-z_1^*}{z-b^2/z_1} \right|. \quad (18)$$

We note that the contributions to K from image and self terms are

equal except for a constant. For the case here, with two poles, the constant term vanishes. From Eq. (18), we find

$$G(z) = \frac{\partial K}{\partial r}(z) = \frac{2}{b} \operatorname{Re} \left\{ \frac{z}{(z-b^2/z_1)} - \frac{z}{z-b^2/z_1^*} \right\}. \quad (19)$$

Now, we are seeking a function $E_s(z)$ which satisfies the Laplace equation in the region within $r = b$ and has the value $E_s(\text{boundary})$ at $r = b$. Since we have manifestly written (19) as a function of z , the function on the right-hand side of (17) satisfies the Laplace equation. Thus, we have, by analytic continuation, the longitudinal electric field $E_s(z)$ everywhere within the chamber:

$$E_s(z) = - \frac{(1-i) \omega \delta I Z_0 \xi e^{i(ks-\omega t)}}{8\pi ch} G(z). \quad (20)$$

The presence of a longitudinal E field at the beam introduces a transverse B field and thus a transverse force. From the curl equation $\nabla \times E = - \partial B / \partial t$,

$$B_x = \frac{1}{i\omega} \frac{\partial E_s}{\partial y}. \quad (21)$$

The Transverse Coupling Impedance

We are considering the effect of a coherent vertical displacement of the beam as a source. The fields produced by this source cause a force which can influence the beam. The component of this force that we require is the vertical one:

$$F_y = E_y + \beta c B_x. \quad (22)$$

The transverse coupling impedance is introduced to provide a simplified force. It is obtained by averaging the coefficient of the rf wave over the circumference. Thus, the transverse impedance $Z_\perp$ is

$$Z_\perp = \frac{-i(2\pi R) F_y}{\beta I \xi}, \quad (23)$$

with R the average machine radius.

Perfectly Conducting Walls

In this case, $E_y = - \partial V / \partial y$ and $B_x = \partial A_s / \partial y$, and therefore from (8) and (22), we have

$$F_y = - (1-\beta^2) \frac{\partial V}{\partial y} = - \frac{1}{\gamma^2} \frac{\partial V}{\partial y}. \quad (24)$$

Using (9), we obtain

$$Z_{\perp}(x, x', y) = - \frac{iRZ_o}{2\beta^2 \gamma^2 h} \frac{\partial K}{\partial y}. \quad (25)$$

Since all field quantities contain the factor $e^{i(ks - \omega t)}$, we suppress it and simply recall it when it plays a role.

Evaluating $Z_{\perp}$ at $y = 0$, and setting $h = 0$ in the image term, we find

$$Z_{\perp}(x, x') = \frac{RZ_o i}{\beta^2 \gamma^2} \left[\frac{1}{(x-x')^2 + h^2} - \frac{h^2}{(b^2 - xx')^2} \right]. \quad (26)$$

To relate this to the "standard" transverse impedance, we must average this over the beam. Consider a small beam of width $\pm \Delta$, centered at $x = x_o$. The "standard" impedance is thus

$$\tilde{Z}_{\perp} = \frac{1}{2\Delta} \int_{x_o - \Delta}^{x_o + \Delta} Z_{\perp}(x_o, x') dx'.$$

Setting $x' = x_o$ in the second term, we have

$$\tilde{Z}_{\perp} = \frac{RZ_o i}{\beta^2 \gamma^2} \left[\frac{1}{2\Delta} \int_{-\Delta}^{\Delta} \frac{du}{u^2 + h^2} - \frac{b^2}{(b^2 - x_o^2)^2} \right],$$

or

$$\tilde{Z}_{\perp} = \frac{RZ_o i}{\beta^2 \gamma^2} \left[\frac{1}{\Delta h} \tan^{-1} \frac{\Delta}{h} - \frac{b^2}{(b^2 - x_o^2)^2} \right]. \quad (27)$$

For a centered "square" beam,

$$\tilde{Z}_{\perp} = \frac{-iRZ_o}{\beta^2 \gamma^2} \left[-\frac{\pi}{4\Delta^2} + \frac{1}{b^2} \right],$$

which is essentially the result for a circular beam centered in a circular chamber.

Coupling Impedance in the Case of a Wall With Finite Resistivity

The added field due to the wall resistance causes the force

[from Eq. (21)],

$$F_y = \epsilon c B_x = \frac{\beta c}{i\omega} \frac{\partial E_s}{\partial y}.$$

The coupling impedance corresponding to this force is given by,

$$Z_{\perp}(x, x', y) = \frac{-iR Z_o \delta (1+i)}{4h} \frac{\partial G}{\partial y}. \quad (28)$$

In the limit $y \rightarrow 0$ and $h \rightarrow 0$, the added impedance resulting from the wall resistivity becomes

$$Z_{\perp}(x, x') = -iR Z_o (1+i) \frac{\delta}{b^3} \frac{b^4 (b^2 + xx')}{(b^2 - xx')^3}. \quad (29)$$

We can interpret this in terms of field point and source point. We use the relation between impedance and force,

$$F_y = \frac{i \beta I_e \xi}{2 \pi R} Z_{\perp}.$$

Therefore, the force at a point x due to a source at x' is

$$F_y(x, x') = \frac{i \beta I_e}{2 \pi R} Z_{\perp}(x, x') \xi(x'). \quad (30)$$

Remember that the phase factor $e^{i(ks - \omega t)}$ is implied.

For a "small" beam, we can set $x = x' = x_o$ and combine (27) with (29) to find the total impedance,

$$\begin{aligned} \tilde{Z}_{\perp} = -i R Z_o \left[\frac{1}{\beta^2 \gamma^2} \left\{ -\frac{1}{\Delta h} \tan^{-1} \frac{\Delta}{h} + \frac{b^2}{(b^2 - x_o^2)^2} \right\} \right. \\ \left. + (1+i) \frac{\delta}{b^3} \frac{b^4 (b^2 + x_o^2)}{(b^2 - x_o^2)^3} \right]. \quad (31) \end{aligned}$$

For a centered beam this becomes

$$\tilde{Z}_{\perp} = -i R Z_o \left[\frac{1}{\beta^2 \gamma^2} \left\{ -\frac{1}{\Delta h} \tan^{-1} \frac{\Delta}{h} + \frac{1}{b^2} \right\} + (1+i) \frac{\delta}{b^3} \right].$$

II. DISPERSION RELATION

Lorentz Force and Integral Equation for the Perturbed Amplitude

The motion of a particle of momentum p in a force field F is

$$\frac{dp}{dt} = F \text{ (total).}$$

If y is the vertical displacement of a particle of rest mass m , then the transverse momentum is

$$P_y = m \gamma \dot{y}.$$

Separating out from F (total) the focusing field due to the magnetic structure around the ring, we find that the transverse betatron motion of a particle is described by the perturbed harmonic oscillator equation

$$\ddot{y} + (\nu \omega_0)^2 y = F_y / m\gamma, \quad (32)$$

where ν is the particle tune and F_y is the force due to the beam induced fields.

We are looking for a solution where all the particles travel along the ring tube in a wave of the form $e^{i(ks - \omega t)}$. To obtain the acceleration of a particle, we work in a coordinate frame travelling with the particle. Thus, the time dependence is obtained by setting $s \rightarrow R \omega_0 t$. Introducing the mode number n ,

$$n = k R,$$

the phase factor takes the form $e^{i(n \omega_0 - \omega)t}$.

Therefore, the time derivative in (32), for the collective solutions being sought, can be replaced by $i(n \omega_0 - \omega)$: that is,

$$\frac{d}{dt} \rightarrow i(n \omega_0 - \omega). \quad (33)$$

We have then

$$[-(n \omega_0 - \omega)^2 + (\nu \omega_0)^2] y = F_y / m\gamma. \quad (34)$$

We can now omit the phase factor since it appears on both sides of (34) and cancels.

Consider an ensemble of N particles y_i in a strip of height $2h$ and small (infinitesimal) width at the horizontal position x . The mean displacement of these particles is simply the amplitude ξ defined in (1):

$$\frac{1}{N} \sum_{i=1}^N y_i = \xi. \quad (35)$$

We assume that F_y is slowly varying in y and simply evaluate it at $y = 0$. Further, we take ν to be y independent. Therefore, averaging (34), we obtain

$$\left[- (n \omega_0 - \omega)^2 + (v \omega_0)^2 \right] \xi(x) = F_y / m\gamma \Big|_{y=0}. \quad (36)$$

Now, the force on a given strip at the point x has contributions from the complete beam. The particular contribution from a source at x' has been written in the form [Eq. (31)]

$$F_y(x, x') = \frac{i \beta I e}{2 \pi R} Z_{\perp}(x, x') \xi(x').$$

This is, in other words, the force at a point x due to a source at x' , with $\xi(x')$ the vertical coherent displacement in the beam at the source point, x' . The total force is obtained by summing over the source "strips". Introducing a density function $\rho(x)$ normalized to unity,

$$\int_{\text{Beam Width}} \rho(x) dx = 1, \quad (37)$$

(36) can be written

$$\left\{ - (n \omega_0 - \omega)^2 + [v(x) \omega_0]^2 \right\} \xi(x) = \frac{F_y(x)}{m\gamma}, \quad (38)$$

where

$$F_y(x) = \int F_y(x, x') \rho(x') dx' \Big|_{y=0}, \quad (39)$$

and we have explicitly noted that the betatron tune, v , is considered a function of the beam position, x .

We can simplify the equation (38) by noting that the two solutions for ω correspond to fast and slow waves, $\omega \approx (n+v) \omega_0$ and $\omega \approx (n-v) \omega_0$, respectively. Although the fast travelling wave is a possible solution, it is one that does not lead to growing coherent oscillations, but rather to damped oscillations. Since over a long time such waves do not persist, we can simply ignore this mode. Thus, we can simplify (38) and write it in the form,

$$\left[v(x) - \left(n - \frac{\omega}{\omega_0} \right) \right] \xi(x) = \frac{F_y(x)}{2m \gamma \omega_0^2 v_0},$$

where v_0 is the betatron tune at the beam center. Inserting (30) and using the expression for the classical proton radius,

$$r_p = \frac{e^2}{4\pi \epsilon_0 mc^2} = \frac{e^2 Z_0}{4\pi mc^2},$$

we obtain

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$$\left[\nu(x) - \left(n - \frac{j}{\omega_0} \right) \right] \xi(x) = i \left(\frac{I}{e} \right) \frac{r_p}{\gamma \omega_0 \nu_0 Z_0} \int Z_{\perp}(x, x') \rho(x') \xi(x') dx', \quad (40)$$

which is a homogeneous integral equation for the perturbed coherent amplitude, $\xi(x)$, with the transverse impedance function $Z(x, x')$ acting as a sort of Green's function, providing the internal coupling for the system of oscillators.

Model with a Linear Correspondence of Tune and Horizontal Position

The equation for the perturbed amplitude can be simplified by invoking a direct correspondence between tune and chamber location. This does, in fact, correspond to reality in that the operating machine chromaticity does indeed lead to such a relationship. The connection is normally linear and we will make this assumption.

Let the beam have its center at $x = x_0$ and a finite range given by

$$\text{Range } [x - x_0] = \pm \Delta.$$

In tune space, we take for the range

$$\text{Range } [\nu - \nu_0] = \pm d,$$

with ν_0 the beam tune center corresponding to the beam position, x_0 . Taking a linear relation between ν and x , we have

$$\nu(x) = \nu_0 + \frac{d}{\Delta} (x - x_0). \quad (41)$$

Inserting x in place of ν in (40), we can write

$$(x - x_c) \xi(x) = \int G(x, x') \rho(x') \xi(x') dx', \quad (42)$$

where the coherent frequency in x is

$$x_c = x_0 - \frac{\Delta}{d} \left(\nu_0 - n + \frac{\omega}{\omega_0} \right), \quad (43)$$

and where the oscillator coupling function or effective Green's function is given by

$$G(x, x') = i \left(\frac{I}{e} \right) \frac{r_p \Delta}{\gamma \omega_0 \nu_0 Z_0 d} Z_{\perp}(x, x'). \quad (44)$$

We can perform one more simplification by using a normalized space variable, u , defined by

$$u = (x - x_0) / \Delta. \quad (45)$$

The variable u thus defines the extent of the beam in space and has the value 0 at the beam center and the range,

$$\text{Range } [u] = \pm 1.$$

If we redefine the density to be normalized to unity with respect to u , that is,

$$\int_{-1}^1 \rho(u) du = 1, \quad (46)$$

then (42) can be expressed by

$$(u - u_c) \xi(u) = \int G(u, u') \rho(u') \xi(u') du', \quad (47)$$

where the coherent frequency u_c is

$$u_c = \frac{1}{d} (n - v_0 - \frac{\omega}{\omega_0}), \quad (48)$$

and the coupling function is redefined to be

$$G(u, u') = i \left(\frac{I}{ed} \right) \frac{r_p}{\gamma \omega_0 v_0 Z_0} Z_{\perp}(u, u'), \quad (49)$$

with $Z_{\perp}(u, u')$ the same function as $Z_{\perp}(x, x')$ with x and x' replaced by u and u' via (45).

The Case of a Small Beam

We start with the integral equation for ξ , Eq. (47). If the beam is small, we can approximate $G(u, u')$ by its value at the beam center, except that the beam term is averaged as previously described. Thus, from (49), and using the "small" beam impedance function, Eq. (31), we obtain

$$G(u, u') \equiv \tilde{G} = \left(\frac{I}{ed} \right) \frac{r_p R}{\gamma \omega_0 v_0} \left\{ \frac{1}{\beta^2 \gamma^2} \left[-\frac{1}{\Delta h} \tan^{-1} \frac{\Delta}{h} + \frac{1}{b^2} \frac{1}{(1-t_0^2)} \right] + (1+i) \frac{\delta}{b^3} \frac{(1+t_0^2)}{(1-t_0^2)^3} \right\}, \quad (50)$$

where we have introduced r , the ratio of space position to chamber radius, in place of x . In particular, the beam center is designated by

$$r_0 = x_0/b.$$

Thus, the integral equation becomes,

$$(u-u_c) \xi(u) = \tilde{G} \int \rho(u') \xi(u') du'. \quad (51)$$

We recall that the phase factor $e^{-i\omega t}$ describes the time evolution of the beam amplitude as well as the induced fields. Since we are discussing a physical situation which is assumed to have existed for a long time and which will continue in the future, solutions that damp are not admissible. In other words, we are looking for solutions with $\text{Im}(\omega) \geq 0$. The quantity $\text{Im}(\omega)$, when it exists, is just the e-folding growth rate,

$$\text{Im}(\omega) = 1/\tau. \quad (52)$$

By a threshold solution is meant the limit of zero growth rate, or $\text{Im}(\omega) \rightarrow 0^+$. From (48), we can see that u_c and ω differ in sign. In fact,

$$\text{Im}(\omega) = -\omega_0 d \text{Im}(u_c). \quad (53)$$

Thus, a threshold solution is found by taking the limit,

$$\text{Im}(u_c) \rightarrow 0. \quad (54)$$

Under unstable conditions the mean observed signal is simply

$$S = \int \rho(u') \xi(u') du'. \quad (55)$$

Thus, Eq. (51) becomes

$$\xi(u) = \frac{\tilde{G} S}{u-u_c}. \quad (56)$$

Multiplying by $\rho(u)$ and integrating, we obtain

$$S \left[1 - \tilde{G} \int \frac{\rho(u)}{u-u_c} du \right] = 0.$$

Therefore, for a nonzero signal to survive, we must satisfy the condition on the coherent frequency, u_c :

$$1 = \tilde{G} \int \frac{\rho(u)}{u-u_c} du. \quad (57)$$

For a sustained but nongrowing signal, that is, for a threshold, we must define passage through the singular point $u = u_c$ by taking the limit, $\text{Im}(u_c) \rightarrow 0$.

To estimate the growth rate, we take a delta function density distribution,

$$\rho(u) = \delta(u),$$

which, from (57), leads to

$$u_c = -\tilde{G}. \quad (58)$$

Recalling that the growth rate is

$$\frac{1}{\tau} = \text{Im}(\omega) = -\omega_0 d \text{Im}(u_c),$$

we have that the growth rate far from threshold is roughly given by

$$\frac{1}{\tau} = \omega_0 d \text{Im}(\tilde{G}). \quad (59)$$

To see how the dispersion relation is to be dealt with in finding the current threshold, take a uniform density distribution,

$$\rho(u) = \frac{1}{2}, \quad -1 \leq u \leq 1. \quad (60)$$

Equation (57) becomes

$$\frac{2}{\tilde{G}} = \int_{-1}^1 \frac{du}{u - u_c}.$$

To find the threshold, take u_c to have a small, negative imaginary part,

$$u_c = u_R - i\epsilon.$$

Taking the limit $\epsilon \rightarrow 0$, we obtain for the threshold equation,

$$\frac{2}{\tilde{G}} = P \int_{-1}^1 \frac{du}{u - u_R} - i\pi, \quad \text{for } -1 < u_R < 1,$$

where P stands for Principal Value Integral. Thus, we have for the threshold condition,

$$\frac{2}{\tilde{G}} = \ln \left| \frac{1 - u_R}{1 + u_R} \right| - i\pi, \quad -1 < u_R < +1. \quad (61)$$

Setting the real part of (61) to zero gives the real oscillation frequency, u_R , while taking the imaginary part of both sides leads to the current threshold. With

$$\tilde{G} = \tilde{G}_R + i \tilde{G}_I,$$

the threshold current is found immediately from

$$\frac{\tilde{G}_I}{\tilde{G}_R^2 + \tilde{G}_I^2} = \pi/2, \quad (62)$$

while the real frequency shift, u_R , is obtained from

$$\frac{\tilde{G}_R}{\tilde{G}_R^2 + \tilde{G}_I^2} = \frac{1}{2} \ln \left| \frac{1-u_R}{1+u_R} \right|, \quad (63)$$

where the current used in (63) is the threshold value found from (62).

In the case of a more general density distribution function, $\rho(u)$, the threshold current and frequency shift are found from the complex equation,

$$\frac{2}{\tilde{G}} = P \int_{-1}^1 \frac{\rho(u)}{u-u_R} du - i \pi \rho(u_R), \quad -1 < u_R < 1. \quad (64)$$

III. EXAMPLE OF COHERENT INSTABILITY FOR A POINT BEAM

ISABELLE Parameters

The results we have obtained can be applied to ISABELLE when beam is injected. The chamber is circular and beam is injected in small current pulses off axis. Thus, we may inquire about the stability of small beams at various horizontal positions in the vacuum chamber. ISABELLE parameters which can be used to evaluate thresholds and growth rates are given in Tables I, II and III.

Table I: General ISABELLE Parameters

Parameter	Value	Expression or comment
Average radius, R	599.5 m	$C = 4 \frac{2}{3} C_{AGS}$
Central tune, ν_0	22.62	
Revolution frequency, f_0	79.64 kHz	$f_0 = \beta c / 2\pi R$
Beam energy, γ	31.3	$\gamma = E/mc^2$ ($E = 29.4$ GeV)
Chamber radius, b	4.4 cm	
Resistivity, ss, ρ_1	$1.0 \times 10^{-6} \Omega m$	
Resistivity, Cu, ρ_2	$1.7 \times 10^{-8} \Omega m$	
Azimuthal mode, n	23	n = 24, 25...also possible
Resonant frequency for mode n, f_r	30.26 kHz	$f_r = (n - \nu_0) f_0$ (n = 23)
Skin depth, ss, δ_1	2.89 mm	{ $\delta = (\rho c / \pi Z_0 f_r)^{1/2}$ (n = 23)
Skin depth, Cu, δ_2	0.38 mm	

Problems:

1. Take the uniform tune distribution defined in Eq. (60). Assume a small centered beam with properties defined in Tables I, II and III. Find the threshold value of Current-tune density, $i = (I/2d)$. Use a stainless steel chamber. Compare it with the value given in Table I.
2. Do problem 1 for a beam off center, at position $t_0 = x_0/b$. Let i_0 be the threshold at the chamber center and $i(t_0)$ be the off-center threshold. Plot $\eta(t_0) = i(t_0)/i_0$ for $0 < t_0 \leq 1$. At what value of t_0 is the threshold down by two orders of magnitude? Also, plot the real frequency shift, $\omega_R(t_0)$, as a function of t_0 .
3. Assume the beam is above threshold. Plot the e-folding growth time as a function of t_0 . At what value of t_0 is this time period for growth reduced by two orders of magnitude?
4. ISABELLE will have a double layered chamber, with 1 mm of stainless steel and a copper envelope. Show that in this case the skin depth becomes a complex quantity. Assuming an infinite copper thickness, find an expression for this skin depth. Do the threshold and growth rate analyses in this case:
 - (a) Find the threshold value of i for a centered beam (i_0).
 - (b) Plot $\eta(t_0) = i(t_0)/i_0$ for $0 < t_0 \leq 1$.
 - (c) Plot $\omega_R(t_0)$ for $0 < t_0 \leq 1$.
 - (d) Find the e-folding growth time as a function t_0 .

Table II: ISABELLE Beam Parameters.

Parameter	Value	Expression or comment
Minimum β , $\beta_{\min}$	10 m	{ Cells (excluding dispersionless insertions)
Maximum β , $\beta_{\max}$	65 m	
Average vertical β , β_V	28.3 m	$\beta_V = (2/3) \beta_{\min} + (1/3) \beta_{\max}$
Max. dispersion, $X_{p,\max}$	2.7 m	{ Cells
Min. dispersion, $X_{p,\min}$	1.4 m	
Average dispersion, X_p	2.27m	$X_p = (2/3) X_{p,\max} + (1/3) X_{p,\min}$
Average horizontal β , β_H	46.7 m	$\beta_H = (2/3) \beta_{\max} + (1/3) \beta_{\min}$
Beam Emittance, $E_V = E_H$	$15\pi \times 10^{-6}$ rad-m	
Vertical $\frac{1}{2}$ -size, h	3.63 mm	$h = (E_V \beta_V / \pi \gamma)^{\frac{1}{2}}$
Horizontal, betatron $\frac{1}{2}$ -size, Δ_B	4.73 mm	$\Delta_B = (E_H \beta_H / \pi \gamma)^{\frac{1}{2}}$
Nominal full beam current, I	8 A	
Nominal full tune spread, $\pm d$	$\pm 9.2 \times 10^{-3}$	
Normalized current - tune density, i_0	433.5 A/unit tune	$i = I/2d$

5. If the beam is wide, the small beam analysis fails. Write down the integral equation in this case. Show that it can be solved by interpolation. Show that the threshold can be determined by solving an eigenvalue problem. What is the impact on ISABELLE performance due to the required aperture utilization for the accumulation of sufficient current?

Table III: Beam Parameters with Point Beam Model

Parameter	Value	Expression or comment
Beam Current, I	1.6 A	20% of Nominal full current
Tune spread, $\pm d$	$\pm 1.84 \times 10^{-3}$	20% of Nominal spread
Momentum spread, $\pm \delta_p$	$\pm 0.1\%$	20% of Nominal stack spread
Momentum $\frac{1}{2}$ size, Δ_p	2.27 mm	
Horizontal $\frac{1}{2}$ size, Δ	7.0 mm	$\Delta = \Delta_\beta + \Delta_p$

CURRENT ACCUMULATION IN MOMENTUM BY PHASE DISPLACEMENT

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I. CURRENT ACCUMULATION TECHNIQUES

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Accelerators and Storage rings for the different types of stable particles have utilized a variety of techniques to intensify the circulating beams. A list is given in Table I.

Table I: Existing Current Accumulation Techniques

Method	Particles Accumulated
Radiation Damping of Electron Beams (Electron-Positron Colliding Beams)	Electrons
Multiturn Injection (AGS)	Protons
Negative Hydrogen Ion (H^-) Injection (ANL And Fermilab Booster)	
Phase Displacement of Momentum Technique (ISR, ISABELLE)	
Short Bunch Injection - Boxcar Accumulation (Fermilab Main Ring)	
Beam Cooling to Accumulate $\bar{p}$'s (CERN $\bar{p}$ -Project Under Construction) (Fermilab $\bar{p}$ -Project Proposed)	Antiprotons

Problem:

The AGS is an injector for ISABELLE. This source produces a sequence of 12 equally separated short bunches of 30 GeV particles every $1\frac{1}{2}$ seconds. Compare the use of multiturn stacking in betatron phase space, the use of negative hydrogen ions, and momentum stacking by radiofrequency phase displacement. Explain why the first two are not suitable with the AGS as a source. Show also that the performance of the ISABELLE colliding beams is optimal with momentum stacking.

II. LONGITUDINAL PHASE SPACE

Particle motion in an accelerator in the presence of an applied radiofrequency system can be described in terms of particle momentum, p , and position along the length of the accelerator, θ . If we take θ to be travelling with the particles as they circulate, the phase space motion is slow and is typically as shown in Fig. 1. In this case the momentum of the central bucket is also changing (acceleration of particles within the bucket). Thus, the trajectories actually describe slow changes from the dominant particle motion, which is circulation around the ring and energy increase from the applied rf. As can be seen, only a limited region of the phase space accommodates stable acceleration - i.e., those particles

Fig. 1. Longitudinal Phase Space.

whose initial conditions place them in the buckets. Particles outside the buckets are affected by the buckets when their momentum is close to that of the central bucket momentum. This is described in Fig. 1. which shows roughly the region of bucket influence.

There are two main characteristics of bucket motion through the phase space (i.e., particles of a beam when occupied):

1. The phase space acts like an incompressible fluid, with phase space displaced downward by an upward passing bucket. There are no long-range effects. When the bucket passes, the displacement results and then the phase fluid remains passive.
2. This is of course not the whole story. There is a scatter effect of the passing bucket. Particles passing near the unstable fixed points cannot maintain their phase space integrity and a dilution of the phase space density results. The distinction here is between adiabatic and non-adiabatic behavior. Sufficiently close to the unstable fixed points, particles behave non-adiabatically. The resulting phase space filamentation causes overall density dilution.

III. INCOMPRESSIBLE FLUID MODEL (IFM) OF PHASE DISPLACEMENT STACKING

Parameters Describing The Phase Displacement Stacking Process

The process of phase displacement stacking within the IFM is described schematically in Fig. 2. The storage ring aperture is defined in terms of a set of rectangular channels each of the same width as the bucket width. Buckets filled with phase fluid (beam) of a given density are deposited in a predetermined channel in the storage ring aperture. The rules for the process are:

- As a bucket enters a channel, it displaces the area of the channel entered to the channel below.
- For channels not yet entered, there is no effect.
- For channels passed, but once removed, there is no further effect.
- After a bucket passes a channel, the channel mixes its contained fluid so that each channel always has uniform density before the next stacking cycle. The same applies for the channel in which the fluid is deposited.

From this description we see that there are only two independent parameters: J_T , the number of channels defining the aperture; and f , the ratio of the area of the bucket to the area of the channel. For simplicity and with no loss of generality, we take the channel width equal to the bucket width.

INCOMPRESSIBLE FLUID MODEL

Fig. 2. Incompressible Fluid Model.

Thus, by neglecting any scatter effects due to the unstable fixed points, we find that the process is purely geometric, depending only on J_T and f , and we can derive a set of simple equations which describe the stacking.

Stacking Equations

Let N_B be the number of particles carried by a bucket. This then is the maximum density of the phase fluid. Let $N_J(I)$ be the number of particles in channel J after I stacking cycles. Now, after a bucket passes channel J :

- Channel J adds the fluid from channel $(J-1)$; and
- Channel J loses its own fluid; with the fluid changes corresponding to that amount in the bucket area. Thus, the resulting change in the number of particles in channel J during the cycle $(I+1)$ is given by

$$\Delta N_J(I) = f [N_{J-1}(I) - N_J(I)] \quad (1)$$

where we have defined

$$\Delta N_J(I) = N_J(I+1) - N_J(I).$$

These equations are clear except for channel 1. However, they become true for channel 1 as well if we define for the 0th channel,

$$N_0(I) = N_B/f, \text{ a constant.}$$

In the process we are considering, we drop fluid in channel 1 and there is no fluid present prior to the first cycle, $I = 1$. Therefore, we have to impose the conditions on $N_J(I)$ that

$$N_J(I) = 0, \text{ for } I < J$$

and

$$N_J(0) = 0.$$

Stacking Equations and The Channel Efficiency

We have that N_B is the number of particles added per stacking cycle and that N_B/f is the maximum number of particles per channel (since the channel density must be less than the maximum fluid density, N_B/f).

If we define the channel efficiency as the fraction of the maximum number of particles per channel present in channel J after cycle I , then we have for the efficiency,

$$\eta_J(I) = N_J(I) / (N_B/f). \quad (2)$$

The equations for $\eta_J(I)$ are analogous to those for $N_J(I)$ and are given by,

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$$\Delta\eta_J(I) = f[\eta_{J-1}(I) - \eta_J(I)] \quad (3)$$

$$\eta_0(I) = 1$$

$$\eta_J(I) = 0, I < J$$

$$\eta_J(0) = 0.$$

Thus, we have obtained a set of difference equations for the channel efficiency. For a large number of channels, these difference equations can be approximated by differential equations. In this case the equations for $\eta_J(I)$ are

$$\frac{d\eta_J}{dI} = f[\eta_{J-1}(I) - \eta_J(I)] \quad (4)$$

with initial conditions,

$$\eta_0(I) = 1$$

$$\eta_J(0) = 0.$$

[The condition $\eta_J(I) = 0$ for $I < J$ is not relevant in this approximation, which is not very good for $I < J$]. The use of a continuous stacking process permits a further simplification. Introducing $x = fI$, we can write the channel efficiency equation as,

$$\frac{d\eta_J}{dx} = \eta_{J-1}(x) - \eta_J(x) \quad (5)$$

with initial conditions,

$$\eta_0(x) = 1$$

$$\eta_J(0) = 0.$$

To solve this set of equations we note that the solution to the equation

$$\frac{df}{dx} + f = g$$

with,

$$f(0) = 0$$

is

$$f = e^{-x} \int_0^x g(u) e^u du.$$

Thus, we have immediately

$$\eta_J(x) = e^{-x} \int_0^x \eta_{J-1}(u) e^u du.$$

If we continue the substitution, we obtain,

$$\eta_J(x) = e^{-x} \int_0^x du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \dots \int_0^{u_{J-1}} du_J e^{u_J}$$

where we have used,

$$\eta_0(x) = 1.$$

Performing the integrals, there results,

$$\eta_J(x) = 1 - e^{-x} \sum_{\ell=0}^{J-1} \frac{x^\ell}{\ell!}. \quad (6)$$

The channel efficiency is therefore described in terms of the incomplete gamma function, by

$$\eta_J(x) = \frac{1}{(J-1)!} \gamma(J, x). \quad (7)$$

The incomplete γ function has the well known integral representation,

$$\gamma(J, x) = \int_0^x e^{-t} t^{J-1} dt.$$

The Stacking Efficiency For A Fixed Aperture

For J_T channels, the maximum number of particles that can be stacked in the given aperture is

$$N_{\max} = \frac{N_B}{f} J_T [\text{max. density per channel} \times \text{the number of channels in the aperture}] \quad (8)$$

The number of particles actually accumulated in the aperture after the stacking cycle I is

$$N_{AP}(I) = \sum_{J=1}^{J_T} N_J(I). \quad (9)$$

It should be remembered that some particles could be phase displaced downward out of the aperture and are lost.

We define the stacking efficiency, after I cycles, to be the fraction of the maximum number of particles remaining in the aperture, which is given by

$$\eta(J_T, I) = \frac{N_{AP}(I)}{N_{\max}} = \frac{1}{J_T} \sum_{J=1}^{J_T} \eta_J(I) \quad (10)$$

or,

$$\eta(J_T, x) = \frac{1}{J_T} \sum_{J=1}^{J_T} \eta_J(x). \quad (11)$$

Now, since the total number of particles "injected" after I cycles is

$$N_{INJ}(I) = N_R I, \quad (12)$$

then the number lost is

$$N_{LOST}(I) = N_{INJ}(I) - N_{AP}(I), \quad (13)$$

As a fraction of the ideal or maximum number of particles injected into the stack, we have the fractional loss,

$$E(J_T, I) = \frac{N_{LOST}(I)}{N_{\max}} = \frac{N_{INJ}(I)}{N_{\max}} - \frac{N_{AP}(I)}{N_{\max}},$$

or,

$$E(J_T, I) = \frac{fI}{J_T} - \eta(J_T, I) = \frac{x}{J_T} - \eta(J_T, x). \quad (14)$$

Problems:

- (a) Show that the channel efficiency $\eta(J_T, x)$ can be expressed in the form,

$$\eta(J_T, x) = \frac{x}{J_T} - \frac{1}{J_T!} \int_0^x dt \gamma(J_T, t). \quad (15)$$

- (b) Prove the following properties of the γ function:

$$\int_0^x dt \gamma(J_T, t) = x \gamma(J_T, x) - \gamma(J_T + 1, x),$$

$$x \gamma(J_T, x) - \gamma(J_T + 1, x) = (x - J_T) \gamma(J_T, x) + e^{-x} x^{J_T}.$$

Show then that the stacking efficiency and fractional loss can be written as

$$\eta(J_T, x) = \frac{x}{J_T} - \frac{e^{-x} x^{J_T}}{J_T!} - \frac{1}{(J_T - 1)!} \left(\frac{x}{J_T} - 1 \right) \gamma(J_T, x),$$

and,

$$E(J_T, x) = \frac{e^{-x} x^{J_T}}{J_T!} + \frac{1}{(J_T-1)!} \left(\frac{x}{J_T} - 1\right) \gamma(J_T, x).$$

(c) Consider the case $x = J_T$ or $I = J_T/f$. This is just the number of cycles when the injected phase area exactly equals the aperture phase area. Show that the stacking efficiency and fractional loss in this case are given by

$$\eta(J_T, J_T) = \eta_{ID} = 1 - \frac{e^{-J_T} J_T^{J_T}}{J_T!}, \quad (16)$$

and,

$$E(J_T, J_T) = E_{ID} = \frac{e^{-J_T} J_T^{J_T}}{J_T!}. \quad (17)$$

From the asymptotic form of the factorial function,

$$J_T! \rightarrow \sqrt{2\pi J_T} J_T^{J_T} e^{-J_T} \text{ for } J_T \rightarrow \infty,$$

show that for large J_T , η_{ID} and E_{ID} become

$$\eta_{ID} \approx 1 - \frac{1}{\sqrt{2\pi J_T}}, \quad (18)$$

and,

$$E_{ID} \approx \frac{1}{\sqrt{2\pi J_T}}. \quad (19)$$

IV. VALIDITY OF THE IFM

Deviations From The IFM

We wish to compare the IF model with computer simulation of stacking. We must therefore relate the parameters J_T and f to a physical situation. If we let

A = bucket area,

ΔE_T = energy width of a bucket,

and f_{rf} = rf (bucket) frequency,

then we have,

$$\text{CHANNEL AREA} = \frac{\Delta E_T}{f_{rf}},$$

and

$$\frac{\text{BUCKET AREA}}{\text{CHANNEL AREA}} = f = \frac{A f_{rf}}{\Delta E_T}$$

Using rf bucket theory, we can evaluate f in terms of Γ , the acceleration parameter ($\Gamma = 0$ corresponds to a bunch stored but not accelerated). We find,

$$f = \frac{2}{\pi} \frac{\alpha(\Gamma)}{F(\Gamma)}, \quad (20)$$

where α and F are the standard moving bucket factors of rf theory. The number of channels is related to f because we have chosen for simplicity in formulating the problem that the channel width be the same as the bucket width. In terms of physical parameters,

$$J_T = \frac{(\Delta p/p)_{ap}}{(\Delta p/p)_{ch}} = \frac{(\Delta p)_{ap}}{A f_{rf}} f, \quad I_T = \frac{(\Delta p)_{ap}}{A f_{rf}} \quad (\text{independent of } \Gamma). \quad (21)$$

In Fig. 3. we compare the IFM with a computer simulation. A direct numerical solution of (3) is used for $\eta_J(I)$. For the computer simulation a standard program developed at CERN to simulate ISR stacking was used. With $\Gamma = 0.5$, the parameter $f = 1/3$. For ISABELLE, we can take $J_T = 15$ as a typical case. Note that the so-called ideal value of I is $I_T = J_T/f = 45$. The fits are reasonably good. However, attempts to fit the computer results for higher Γ do not work so well.

To demonstrate the deviation with increasing Γ , we plot the stacking efficiency in the "ideal" case, η_{ID} , taken from (18), as a function of Γ . We take the case of fixed $I_T = 45$ and changing J_T determined from $J_T = I_T f$. The computation of η_{ID} for some typical values of Γ is shown in Table II.

Table II. Computation of η_{ID}

$\eta_{ID} = 1 - \frac{1}{\sqrt{2\pi I_T f}}, \quad f = \frac{2}{\pi} \frac{\alpha(\Gamma)}{F(\Gamma)}$				
Γ	$\alpha(\Gamma)$	$F(\Gamma)$	$f(\Gamma)$	η_{ID}
0.17	0.70	0.86	0.51	0.92
0.34	0.49	0.72	0.43	0.91
0.5	0.33	0.59	0.36	0.90
0.6	0.25	0.49	0.32	0.89
0.7	0.17	0.39	0.28	0.88
0.77	0.12	0.33	0.23	0.87
0.87	0.06	0.22	0.17	0.85

Fig. 3. Channel Efficiency. Comparison of Computer Simulation With The IFM.

A comparison of the stacking efficiency η_{ID} between computer simulation and the IFM is shown in Fig. 4. In the region up to $\Gamma \approx 0.5$, the incompressible fluid model is good. However, for Γ higher than 0.5, a deviation from the IFM develops and worsens, with the process of stacking becoming progressively less efficient than predicted by the IFM.

The Effect Of Scatter Due To A Passing Bucket

The incompressible fluid model maintains the "large scale" conservation of area and so neglects the nonadiabatic behavior of particles passing close to the bucket unstable fixed point. The actual situation results in phase space filamentation with the ultimate consequence of "large scale" dilution or "large scale" expansion of the fluid.

Using computer simulation, a simple rule for determining the dilution has been developed. It can be shown that the mean square increase in the energy width of a channel of width $(\Delta E)_{ch}$, when a bucket of the same energy width passes through, is mainly a function of Γ . Thus, we can approximately express the final energy width, when a bucket of width $(\Delta E)_{ch}$ passes through the channel, in the simple form,

$$\frac{(\Delta E)_{ch, final}}{(\Delta E)_{ch}} = \left[1 + \left(\frac{2}{\pi} \frac{\Gamma}{F(\Gamma)} \right)^2 \right]^{\frac{1}{2}} \quad (22)$$

Modified IFM

The effect described in Eq. (22) can be interpreted as a dilution of the phase fluid. If we guess that the effect could be described by a modification of the parameter J_T or f , we simply replace f by fd in η_{ID} , where d is the dilution factor taken from Eq. (22):

$$d = \left[1 + \frac{4}{\pi^2} \frac{\Gamma^2}{F^2} \right]^{-\frac{1}{2}} \quad (23)$$

With this modified IFM, we can compute the stacking efficiency η_{ID} again and compare it with the computer simulation. The computations of η_{ID} are given in Table III and the comparison with the computer simulation is shown in Fig. 5.

Fig. 4.. Stacking Efficiency, η_{ID} . A Comparison of Computer Simulation And The IFM.

Fig. 5. Stacking Efficiency η_{ID} For The Modified IFM

Table III. Computations of η_{ID} for Modified IFM

Γ	$f_o(\Gamma)$	$F(\Gamma)$	$d(\Gamma)$	$f(\Gamma)$	$\eta_{ID}(0)$	$\sqrt{d}$	η_{ID}
0.17	0.51	0.86	0.99	0.51	0.92	0.99	0.92
0.34	0.43	0.72	0.96	0.41	0.91	0.98	0.89
0.5	0.36	0.59	0.88	0.32	0.90	0.94	0.84
0.6	0.32	0.49	0.79	0.25	0.89	0.89	0.79
0.7	0.28	0.39	0.66	0.18	0.88	0.81	0.71
0.77	0.23	0.33	0.56	0.13	0.87	0.75	0.65
0.87	0.17	0.22	0.37	0.06	0.85	0.61	0.52

The computer simulation agrees well with the modified IFM even for quite high Γ . Thus, we could conclude that phase displacement stacking can be approximately described by an IF model modified by a dilution factor which takes account of bucket scatter.

V. PROBLEMS

1. In transferring beam from the AGS to ISABELLE, what factors enter in diluting the phase fluid of the incoming bunches? How is this dilution taken into account in the IF model?
2. Suppose there was 1% of momentum aperture available for the stack. Suppose also 6×10^{14} particles were required for ISABELLE operation. Assume further a 20% dilution of the phase fluid in transferring beam from the AGS to ISABELLE. Using the IF model, show how to find what the properties of the phase fluid provided by the AGS must be to accomplish this objective. How efficient is the process in terms of beam loss out of the aperture? What are the results for $\Gamma = 0.3, 0.5$ and 0.7 ? Estimate the stacking time in each case. (Use the ISABELLE parameters given in the January 1978 proposal - BNL 50718).
3. One of the assumptions in the IF model is a uniform channel distribution at the beginning of each new cycle. Is this assumption valid? What is the major parameter that determines how the evolution toward channel uniformity develops? What is the debunching time? How does the use of suppressed buckets influence the assumption of channel uniformity?
4. The microwave instability is a concept which developed from a set of observations at the PS and ISR at CERN. Basically there were three striking observations:
 1. Unstable longitudinal motion had an extremely rapid growth rate - much larger than could be explained by the resistive wall.
 2. A broad band of coherent signals were observed in the microwave region during the unstable period.

3. The instability occurred in bunched beams.

A radical proposal was made to attempt to describe the instability with coasting beam theory. It was successful. The coasting beam theory puts a limit on the allowable longitudinal impedance in the ring. Derive an expression for this limit. Show how to extrapolate this to bunched beams. Is this procedure physically valid? What conditions could make it applicable in a transient sense? Find the impedance limit as a function of Γ for bunches being stacked in ISABELLE just before the bunches enter the stack. Is the impedance limit better for high Γ ? How would you choose an optimum value of Γ taking into account both the requirements on impedance and the requirements on beam loss into the superconducting material in the magnets?

5. Using ISABELLE cell parameters, draw a schematic of the aperture utilization during stacking. Show that ideally the beam during stacking will require ≥ 7 cm. Assume that the vacuum chamber diameter is 8.8 cm and the superconducting coil diameter is 13 cm.

From your knowledge of single particle resonances and the resistive wall theory, show the following:

1. For particles on orbits off the central orbit, there is an enhancement of the resistive wall instability. What happens if you are very close to the vacuum chamber aperture? What conclusions can you make with regard to how much aperture can be used by beam?
2. For particles on orbits off the central orbit, there is an enhancement of the error multipoles due to coil block placement errors. See if you can find a rough estimate of the enhancement for any given multipole. What limits does this imply with regard to magnet construction tolerances? If the magnet construction techniques are fixed, what implications does this enhancement have with regard to how much aperture can be used by the beam?