ISLAND FORMATION AND DESTRUCTION OF
FLUX SURFACES IN THREE-DIMENSIONAL MHD EQUILIBRIA

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ABSTRACT

We describe the physics involved in the appearance of islands due to resonant pressure driven currents in three-dimensional MHD equilibria. Estimates of island widths are obtained by an expansion in β . The theory is applied to Princeton's heliac reference design.

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I. INTRODUCTION

In a three-dimensional MHD equilibrium, the diamagnetic and Pfirsch-Schlüter currents driven by the pressure on any given flux surface may resonate with the rotational transform of a flux surface elsewhere in the plasma. This results in the appearance of islands and the destruction of flux surfaces in the equilibrium. Such resonant equilibrium currents are unique to three-dimensional equilibria, being precluded by the symmetry in one or two dimensions (except when q is singular, as described in Sec. II). In one-dimensional (slab) or two-dimensional equilibria, islands may be generated, of course, by the appearance of a (symmetry breaking) tearing instability. In contrast, the islands driven by resonant diamagnetic and Pfirsch-Schlüter currents are intrinsic to the equilibrium. When these islands are sufficiently large that they overlap, the flux surfaces are destroyed, and there is no equilibrium. This confirms the conjecture of Grad that three-dimensional equilibrium solutions do not always exist, even if the vacuum field has good flux surfaces.¹

The dependence of the island widths on β gives an equilibrium β limit, above which no equilibrium solution exists. Although this β limit has not previously been calculated, the convention for stellarators has been to estimate the equilibrium β limit to be that value of β at which the magnetic axis shifts half way to the wall. We will see that this is sometimes a poor estimate.

In this paper we describe the physics involved in the appearance of islands in three-dimensional equilibria, and obtain some estimates for island widths. We specialize to net current free equilibria, as is appropriate for application to stellarators. As a particular application, in Sec. IV we consider Princeton's heliac reference design. Other possible applications

such as the effect of field ripple in an otherwise axisymmetric device, or the nonlinear saturation of MHD instabilities, would require a slight generalization of the theory presented here to equilibria having net current.

The MHD equilibrium equation,

$$\nabla p = \underline{j} \times \underline{B} ,$$

also describes steady flow in an incompressible, inviscid, neutral fluid if we identify $\underline{B} \rightarrow \underline{v}$ and $p + B^2/2 \rightarrow -p^*$, where p^* is the pressure of the neutral fluid.² This equivalence is clear if we rewrite the MHD equilibrium equation in the form

$$\nabla(p + B^2/2) = \underline{B} \cdot \nabla \underline{B} .$$

The MHD equilibrium β limit corresponds to a condition for the onset of stochastic, steady flow.

We will see that resonant pressure driven currents in an MHD equilibrium are associated with the variation of $\oint d\ell/B$ on the corresponding rational surface, where the integral is taken around a closed field line. We make a distinction between direct resonances, due to the variation of $\oint d\ell/B$ in the vacuum field, and nonlinear resonances, due to a variation of $\oint d\ell/B$ that arises in the presence of finite β . We will argue that the amplitude of the direct resonances can be minimized by proper design of the stellarator. The nonlinear resonances, on the other hand, are intrinsic to the three-dimensional nature of the equilibrium, and give a fundamental β limit for each type of stellarator. We show that even if $\oint d\ell/B$ is constant on every rational surface in the vacuum field, it is generally not constant on any rational

surface in the presence of finite β .

In Sec. II we treat the direct resonances, and obtain an estimate for the associated island widths. To calculate the equilibrium, we use an expansion in β^{2-4} . The expansion is valid as long as the nonresonant corrections are small. That is, the finite β shifts and distortions of the flux surfaces must be small.

The nonlinear resonances are dealt with in Sec. III. Some of the general properties of the self-consistent equilibrium solution are examined. We find that the island widths, and therefore the equilibrium β limit, are determined by the increase in the spectral width of the Jacobian with increasing β . Then we make some approximations, and obtain an estimate for the island widths at low order rational surfaces.

The theory is applied to the heliac reference design in Sec. IV.

In obtaining estimates of island widths in this paper, we make some simplifying assumptions which are not strictly necessary. Some geometric factors are ignored to simplify the analysis and clarify the essential physics.

II. LOWEST ORDER FIELD DUE TO PLASMA CURRENTS

In this section we give the lowest order finite β correction to the magnetic field and determine the widths of the magnetic islands due to the resonant components of this field. We obtain the lowest order pressure driven current from the results of Ref. 4. To calculate the field generated by this current, B_1 , we follow the approach of Ref. 4, generalizing that calculation to deal with a helical axis and resonant current.

Adding a pressure $p(\psi)$ to a given vacuum field B , where ψ is constant on the vacuum flux surfaces, the diamagnetic current at low β is approximately

given by

$$\tilde{j}_1 = \frac{1}{B^2} \tilde{B} \times \nabla p . \quad (1)$$

The corresponding Pfirsch-Schlüter current is determined by $\nabla \cdot \tilde{j} = 0$, or

$$\tilde{B} \cdot \nabla \frac{\tilde{j}_1}{B} = - \nabla \cdot \tilde{j}_1 . \quad (2)$$

The total field is approximately given by the vacuum field, B , plus the field driven by these plasma currents, which we call B_1 . If β is sufficiently small, the finite β shifts of the flux surfaces are determined by B_1 . We can iterate the above procedure, calculating the diamagnetic and Pfirsch-Schlüter currents from $B + B_1$. At low β the corrections to the currents are small. The resonant corrections will be discussed in Sec. III.

The pressure driven currents are conveniently determined in a set of vacuum flux coordinates (ψ, θ, ϕ) such that

$$\tilde{B} = cg \nabla \phi , \quad (3)$$

where $cg/2$ is the total poloidal current in the coils.^{4,5} The Jacobian is then

$$J = g/B^2 . \quad (4)$$

The currents are obtained in terms of the Fourier decomposition of the Jacobian,

$$\rho = \rho_0 \left[1 + \sum'_{n,m} \delta_{nm} \cos(n\phi - m\theta) \right], \quad (5)$$

where the prime indicates that the term $n = 0, m = 0$ is omitted from the sum. In neglecting the $\sin(n\phi - m\theta)$ terms in Eq. (5) we have assumed a symmetry with respect to double reflection in an appropriately chosen poloidal and equatorial plane. Most stellarator designs have this symmetry. This assumption is made for convenience only, the extension of all results to include the sin terms being completely straightforward.

Because all of our results will be expressed in terms of the δ_{nm} , it is important to note that for any given vacuum field the δ_{nm} can be determined numerically in a straightforward manner by an integration along the field lines.⁴ The physical interpretation of these quantities will be discussed at the end of this section.

The flux coordinate system we have defined only exists, strictly speaking, if the vacuum field has good flux surfaces everywhere. Stellarator vacuum fields generally do not have large islands, but they nevertheless do have islands and stochastic regions if viewed on a sufficiently small spatial scale. In defining vacuum flux coordinates, we construct a set of good flux surfaces which are "nearby" in the sense that they define a field $\tilde{B}_s$ with $|\tilde{B}_s - \tilde{B}| \ll \tilde{B}$.

In solving Eqs. (1) and (2) for the lowest order currents, we take the equilibrium to have zero net current within each flux surface, as is appropriate for stellarators.⁶ For $p(\psi)$ given, the solution of these equations is then³⁻⁵

$$\mathbf{j} = \rho_0 \frac{dp}{d\psi} \left[\nabla\phi \times \nabla\psi + \sum'_{n,m} \frac{\delta_{nm}}{n-m} \cos(n\phi - m\theta) \nabla\psi \times (m\nabla\theta - n\nabla\phi) \right]. \quad (6)$$

This expression can, in principle, have singularities at those rational surfaces $n = \chi(\psi)m$ where $\delta_{nm}(\psi) \neq 0$, corresponding to those surfaces on which $\phi d\ell/B$ is not constant. However, the Pfirsch-Schlüter diffusion coefficient is proportional to $(n - \chi m)^{-2}$ near such a surface, so that the pressure gradient must be proportional to $(n - \chi m)^2$ there.⁵ Also, the resonant currents give rise to a resonant part of $\tilde{B} \cdot \nabla\psi$, which opens up an island at such a rational surface. This is true even if we take $dp/d\psi$ proportional to $(n - \chi m)^2$, so that the resonant current vanishes as we approach the rational surface itself. In fact, we will see that the singular part of the current near the rational surface has little effect on $\tilde{B} \cdot \nabla\psi$, so for our purposes the exact form of $p(\psi)$ near the rational surface is irrelevant. The island width increases as $\sqrt{\beta}$. The width of these islands will be determined later in this section. We expect that the importance of such islands can be minimized by properly designing the vacuum field to minimize δ_{nm} for those n, m corresponding to a rational surface, $\chi = n/m$. Much more difficult to avoid are the resonant currents to be discussed in Sec. III, which are due to a resonant nonlinear coupling of the nonresonant δ_{nm} 's. These points will be clarified further by a discussion at the end of this section, and by the application discussed in Sec. IV.

For a two-dimensional, helically symmetric field, δ_{nm} is nonzero only for $m/n = h$, where h is fixed by the helicity. This follows from the requirement that B^2 not vary in the symmetry direction. Pressure driven currents can resonate only with a rational surface where $\chi = 1/h$, corresponding to a helical transform (transform in the helical reference frame) which is zero, or a helical q which is infinite. In the axisymmetric limit, δ_{nm} is nonzero only for $n = 0$. Pressure driven currents can resonate only with $\chi \approx 0$ ($q = \infty$).

To solve

$$\nabla \times \underline{B}_1 = \underline{j} \quad , \quad (7)$$

we follow the approach of Ref. 4, writing

$$\underline{B}_1 = \underline{H} + \nabla F \quad , \quad (8)$$

where $\underline{H}$ is the particular solution to Eq. (7),

$$\underline{H} = \left(\int_{\psi}^{\psi_a} \ell_0 \frac{dp}{d\psi} d\psi \right) \nabla \psi + \left[\ell_0 \frac{dp}{d\psi} \right]' \frac{\delta_{nm}}{n-m} \sin(n\phi - m\theta) \nabla \psi \quad , \quad (9)$$

with ψ_a the value of ψ at the plasma edge. To determine F we must then solve

$$\nabla^2 F = -\nabla \cdot \underline{H} \quad , \quad (10)$$

subject to the condition that $F \rightarrow 0$ at infinity.

We obtain an approximate analytical solution of Eq. (10) by expanding about the magnetic axis. The minor radius, a , is assumed small compared to the radius of curvature of the magnetic axis. We simplify further by taking the ellipticity of the flux surfaces near the magnetic axis to be small.

We use Mercier's helical coordinate system (ρ, ω, s) ,⁷ where ρ is the distance from the magnetic axis, and s measures the distance along the magnetic axis. The angle ω is defined by

$$\omega = \theta_n + \int_0^s \kappa ds \quad , \quad (11)$$

where θ_n is the angle from the unit normal vector to the magnetic axis,

measured in a plane perpendicular to the axis, and κ is the torsion of the curve formed by the magnetic axis. A general vector is expressed as

$$\tilde{r} = \tilde{x}(s) + \hat{n}(s)\rho \cos \theta_n + \hat{b}(s)\rho \sin \theta_n, \quad (12)$$

where $\hat{n}$ and $\hat{b}$ are, respectively, the unit normal and binormal to the curve $\tilde{x}(s)$ followed by the magnetic axis. The Mercier coordinates are orthogonal.

We will rewrite Eq. (10) in Mercier coordinates. To lowest order we can neglect B_ω and B_ρ in Eq. (3),

$$g\nabla\phi \approx B_0 \nabla s, \quad (13)$$

or

$$\phi \approx B_0 s/g. \quad (14)$$

In Eq. (13) we have neglected the small ripple in field strength along the field lines, taking B_0 to be the field on axis. From the equation for $\underline{B}$ in flux coordinates,

$$\underline{B} = \nabla\psi \times \nabla(\theta - \kappa\phi), \quad (15)$$

we conclude that

$$\psi \approx \frac{1}{2} B_0 \rho^2, \quad (16)$$

and

$$\theta = \chi\phi + \omega \approx \omega + \chi B_0 s/g \quad . \quad (17)$$

Equation (14) gives us an expression for the poloidal current,

$$g = B_0 L/2\pi \quad , \quad (18)$$

where L is the length of the magnetic axis. We will be particularly interested in a helical axis stellarator for which the axis is wrapped on a circular torus of minor radius r_0 and major radius R . The length of the axis is then approximately

$$L \approx (1 + k^2 r_0^2)^{1/2} 2\pi R \quad , \quad (19)$$

where $2\pi/k$ is the periodicity length. The total number of periods is $N \approx (1 + k^2 r_0^2)^{1/2} kR$. Equations (18) and (19) give

$$g \approx (1 + k^2 r_0^2)^{1/2} RB_0 \quad . \quad (20)$$

In Mercier coordinates

$$\nabla \cdot \tilde{H} = \frac{1}{\rho h_s} \left[\frac{\partial}{\partial \rho} (\rho h_s H_\rho) + \frac{\partial}{\partial \omega} (h_s H_\omega) + \rho \frac{\partial H_s}{\partial s} \right] \quad , \quad (21)$$

where

$$h_s \equiv 1 - \frac{\rho}{R_C} \cos \theta_n \quad (22)$$

is the scale factor for the s coordinate, and R_c is the local radius of curvature of the magnetic axis. Taking $a/R_c \ll 1$, where a is the circularized plasma radius, we have $h_s \approx 1$. It then follows from Eqs. (9), (13), and (16) that

$$\nabla \cdot \tilde{H} \approx \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\rho) \quad (23)$$

The Laplacian in Mercier coordinates is

$$\begin{aligned} \nabla^2 f &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (h_s \rho \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial}{\partial \omega} (h_s \frac{\partial f}{\partial \omega}) + \frac{\partial}{\partial s} (\frac{1}{h_s} \frac{\partial f}{\partial s}) \\ &\approx \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \omega^2} + \frac{\partial^2 f}{\partial s^2} \end{aligned} \quad (24)$$

Since Eq. (10) is linear, we can restrict our consideration to a single Fourier mode, proportional to $\sin(n\phi - m\theta)$. To simplify the mathematics, and clarify the physics, we will make the further approximation that

$$\frac{|n|}{(1+k^2 r_o^2)^{1/2} R} \ll \frac{|m|}{a} \quad (25)$$

This allows us to neglect the derivatives with respect to s in Eq. (23). Although not essential, this approximation will allow us to express our solutions in terms of algebraic functions rather than Bessel functions.

Substituting our expressions for $\nabla \cdot \tilde{H}$ and $\nabla^2 F$ into Eq. (9), and solving for F , we find

$$F = \frac{1}{2} \rho^m \int_\rho^a \rho^{-m} H_\rho d\rho - \frac{1}{2} \rho^{-m} \int_0^\rho \rho^m H_\rho d\rho \quad (26a)$$

inside the plasma, and

$$F = -\frac{1}{2} \rho^{-m} \int_0^a \rho^m H \, d\rho \quad (26b)$$

outside the plasma. We have taken $m > 0$ without loss of generality. The magnetic field produced by the lowest order pressure driven currents is now determined by Eqs. (8), (9), and (26).

The resonant terms in Eq. (6) give rise to resonant components of $B_1 \cdot \nabla \psi$, which produce magnetic islands. In calculating the island width, we take the net toroidal current to be zero also inside the flux surfaces defined by the islands. During the initial formation of the islands, currents are induced in the islands which retard their growth. These localized currents are rapidly dumped. Since we are interested in Ohmic stellarator equilibria, for which the net toroidal current inside each flux surface is zero, we clearly must take the island currents to be zero.

For a stellarator with nearly circular flux surfaces, the island half-width, w , at $x = n/m$ is

$$w \approx \left| \frac{2L B_1 \rho_{nm}}{\pi B_0 m \, dx/d\rho} \right|^{1/2} \quad (27)$$

All quantities here are evaluated at the rational surface. Our approximate solution has neglected some small coupling of the Fourier terms due to noncircularity of the flux surfaces, and due to the relative shift of these flux surfaces for the vacuum field. This coupling broadens somewhat the spectrum of the resonant field component. The noncircularity and axis shift of the vacuum flux surfaces are generally small, however, compared to that in the finite β equilibria of interest. It follows that the coupling terms we

have neglected here are small compared to those which arise in the next order in β , which we treat in Sec. III.

To evaluate the integrals in Eqs. (26), it is necessary to know the radial dependence of $(dp/d\psi)\delta_{nm}/(n-\chi m)$. Analyticity of $1/B^2$ implies that

$$\delta_{nm} \propto \rho^m \quad (28)$$

near the magnetic axis. Near resonance,

$$n - \chi m = -m(\rho - \rho_0)dx/d\rho, \quad (29)$$

where ρ_0 corresponds to the rational surface. This is the appropriate approximation for calculating the magnetic island widths. Far from resonance, we have $n - \chi m$ approximately constant if the shear is small. It is instructive to evaluate the integrals in both limits, specializing to

$$\rho = \rho_0 (1 - \rho^2/a^2) \quad (30)$$

To handle the singularity in the integrand we take $dp/d\rho = 0$ for $\rho_0 - \epsilon < \rho < \rho_0 + \epsilon$ and let $\epsilon \rightarrow 0$. Because the singular part of the current has opposite signs on the two sides of the rational surface, its effect cancels out.

Evaluating the integrals in Eq. (26a), using Eqs. (28) through (30), we find that the resonant radial component of $B_{\sim 1}$ at the rational surface with $\chi = n/m$ is

$$\frac{B_{1\rho}}{B_0} = \beta_0 \frac{L}{4\pi a^2} \delta_{nm} \left[\ln\left(\frac{a-\rho_0}{\rho_0}\right) + \sum_{j=1}^{2m+1} \frac{1}{j} + \frac{a}{\rho_0} \right] \sin(n\phi - m\theta) \quad (31)$$

where

$$\beta_0 \equiv 2p_0/a_0^2$$

Note that although the $\ln(a - \rho_0)$ term blows up if we evaluate Eq. (31) for rational surfaces closer and closer to the plasma edge, the singularity is cancelled by the $\ln(m)$ dependence of the following term. The field itself is well-behaved. Equations (27) and (31) together determine the island widths due to the direct resonances.

We will need the nonresonant contribution to $B_{\sim 1}$ for the calculations of Sec. III. Taking $n - m$ constant, we get

$$\frac{F}{B_0} \approx - \frac{B_0 L \delta_{nm}}{8\pi(n-m)} \left(1 - \frac{m+2}{m+1} \frac{\rho^2}{a^2} \right) \sin(n\phi - m\theta) \quad (32)$$

All of the results obtained in this section have been expressed in terms of the Fourier amplitudes of the Jacobian, the δ_{nm} . To understand these results, it is necessary to understand the physical significance of the δ_{nm} . The toroidal curvature of the stellarator gives the Jacobian a $\cos \theta$ dependence, and thus contributes to the nonresonant δ_{01} term in Eq. (5). The resulting plasma field gives a toroidal shift of the flux surfaces. This is the well-known toroidal Shafranov shift, which exists even in an axisymmetric device such as the tokamak. In a helical axis stellarator, the helical curvature gives a $\cos(\theta - N\phi)$ dependence, contributing to (the nonresonant) δ_{N1} . The resulting field gives a helical flux surface shift. The shape of the flux surfaces is determined by the $m \geq 2$ contributions to δ_{nm} . In stellarator vacuum field designs, the resonant harmonic content of the flux surface shapes is kept small by the requirement that no large islands be

present in the vacuum field. As we will see in Sec. IV, this condition is not sufficient to preclude the presence of sizable resonant δ_{nm} 's. However, these resonant terms are not intrinsic to the stellarator design, so we expect that they can be suppressed. The amplitudes of the δ_{nm} for the vacuum field decay exponentially with increasing m and n , so that at most a few such resonant terms need to be suppressed. Unfortunately, there are resonant corrections to the field which arise from the nonlinear coupling of components that are intrinsic. This is the subject of the next section.

III. RESONANT CORRECTIONS TO THE PLASMA FIELD

In the previous section we calculated the lowest order magnetic field due to plasma currents. Although this field can have Fourier components which resonate with rational surfaces, we argued that such components can be minimized by proper design of the stellarator. In higher order, however, the nonresonant vacuum amplitudes couple nonlinearly to give resonances. In particular, the nonresonant components which are intrinsic to the design of the stellarator couple and give resonant fields.

We first consider what the fully self-consistent equilibrium solution must look like, and then we make some approximations. Equations (1) and (2) determine the exact plasma current if $\tilde{B}$ is the exact equilibrium field, with p constant on the equilibrium flux surfaces. If the equilibrium islands are not too large, we can write

$$\tilde{B} = \tilde{B} + \tilde{B} ,$$

where $\tilde{B}$ is a nearby field with good flux surfaces, and $\tilde{B}$ is the (small) resonant part of the field.⁹ We have argued in the previous section that ∇p

should vanish on those rational surfaces where $\phi dl/B$ is not constant. It follows that we can replace $\underline{B}$ by $\tilde{\underline{B}}$ in Eqs. (1) and (2). The resulting error in $\underline{j}$ can be made arbitrarily small if $\tilde{\underline{B}}$ is chosen to vanish everywhere except in the immediate vicinity of the resonant surfaces. Note that it is not strictly necessary that ∇p vanish at resonant surfaces because, as we saw in the previous section, the currents in the immediate vicinity of the rational surface do not have much effect. Working in a flux coordinate system (ψ, θ, ϕ) for $\tilde{\underline{B}}$ with Jacobian $1/\tilde{B}^2$, the solution for the plasma current is given by Eq. (6), with χ , $\underline{j}_0$, and δ_{nm} determined from $\tilde{\underline{B}}$.

The self-consistent set of equations is completed by determining $\underline{B}$ in terms of $\underline{j}$, $\nabla \times \underline{B} = \underline{j}$. The self-consistent equilibrium field is $\underline{B} = \underline{H} + \nabla F$, with $\underline{H}$ and F determined by Eqs. (9) and (10).

The island structure may be determined by solving for the perturbed flux function, $\psi + \tilde{\psi}$, from $\tilde{\underline{B}} \cdot \nabla \tilde{\psi} = -\tilde{\underline{B}} \cdot \nabla \psi$. This gives

$$\frac{\partial \tilde{\psi}}{\partial \phi} + \chi \frac{\partial \tilde{\psi}}{\partial \theta} = -\tilde{\underline{B}} \cdot \nabla \psi \quad , \quad (33)$$

where χ is the rotational transform for the field $\tilde{\underline{B}}$, and where we have used $\tilde{\underline{B}} \cdot \nabla \psi = \underline{j}$. Given $\tilde{\underline{B}}$, Eq. (33) determines the island widths, including those discussed in the previous section and those arising from higher order corrections.

We see that again the resonant terms in the Jacobian give rise to magnetic islands. We will find that a finite β broadens the spectrum of the Jacobian for the equilibrium solution, thus increasing the number of islands and the island widths. The spectrum of resonant perturbations of $\tilde{\psi}$ is broadened somewhat further by the presence of $\underline{j}$ on the right-hand side in Eq. (33) and by the transformation from $\underline{B}_\psi$ to $\underline{B}^\psi$. The additional broadening gives

corrections to the resonant Fourier amplitudes of order $(\beta_s/\beta)(\rho/R)$, where β_s is the value of β at which the magnetic axis shifts half way to the outer flux surface. These corrections are small when β/β_s is large compared to the inverse aspect ratio, which is generally the case for β values of interest. The island widths, and the equilibrium β limit, are essentially determined by the spectral width of the Jacobian.

Although we have thus far specialized to the particular coordinate system satisfying Eq. (4), the amplitude of the resonant part of the Jacobian, δ_{nm} , can be determined in any flux coordinate system. The general transformation from one flux coordinate system to another leaves this quantity invariant.⁵ To evaluate the Jacobian, we consider the transformation from our vacuum flux coordinate system $(\psi_v, \theta_v, \phi_v)$, to a conveniently chosen set of flux coordinates for the equilibrium field. Denote the flux coordinates in the presence of plasma by $(\psi_p, \theta_p, \phi_p)$. Let B_v be the vacuum field, and let b be the nonresonant part of the field due to the plasma currents. The flux coordinates for $\bar{B}$ must satisfy

$$(\bar{B}_v + b) \cdot \nabla \psi_p = 0 \quad , \quad (34)$$

and

$$(\bar{B}_v + b) \cdot \nabla \theta_p = \chi_p (\bar{B}_v + b) \cdot \nabla \phi_p \quad , \quad (35)$$

where χ_p is the rotational transform in the presence of plasma. We choose $\phi_p = \phi_v$. The remaining coordinates are then determined by Eqs. (34) and (35). The Jacobian, as usual, is given by $1/J_p = \nabla \psi_p \cdot \nabla \theta_p \times \nabla \phi_p$.

In transforming from the vacuum flux coordinate system to the plasma flux

coordinate system, the Fourier spectrum of the Jacobian is modified in two different ways. First, the local value of the Jacobian is different in the two coordinate systems. Second, the poloidal angle variables with respect to which we are Fourier transforming are different. Even if the local value of the two Jacobians were the same at each point, the Fourier amplitudes would be different due to the difference in the θ 's. The increase in the width of the spectrum is, in fact, due predominantly to the second effect.

Writing $B_{\sim V}$ in contravariant form,

$$B_{\sim V} = \nabla\psi_V \times \nabla\theta_V + r_V \nabla\phi_V \times \nabla\psi_V ,$$

it follows that $B_{\sim V} \cdot \nabla\phi_V = 1/\ell_V$. Similarly, $(B_{\sim V} + b_{\sim}) \cdot \nabla\phi_V = 1/\ell_P$, so that

$$1/\ell_P - 1/\ell_V = b_{\sim} \cdot \nabla\phi_V . \quad (36)$$

In going from the current, Eq. (6), to b^ϕ , there is some coupling of the Fourier components due to the transformation from b_ϕ to b^ϕ . This coupling is small compared to that due to the difference in the θ 's.

We now approximate $b_{\sim}$ by the nonresonant part of B_1 (calculated in Sec. II), and solve explicitly for the δ_{nm} . The neglect of the Fourier coupling due to corrections to the magnetic field which are higher order in β is reasonable for low order rational surfaces (low m and n), but may give an underestimate of the amplitudes in the tail of the Fourier spectrum. The calculation is therefore of particular relevance to helical axis stellarators, where the relatively large transform per period allows the presence of low order rational surfaces. A particular design for such a device will be discussed in Sec. IV. More generally, the approximate calculation of the δ_{nm}

in this section may be regarded as giving a necessary condition for the existence of good flux surfaces.

Define

$$\theta_p = \theta_v + \tilde{\theta} \quad , \quad \psi_p = \psi_v + \tilde{\psi} \quad ,$$

where we will assume $\tilde{\theta} \ll \theta_v$ and $\tilde{\psi} \ll \psi_v$ (low β). Equation (35) gives

$$\underline{B}_v \cdot \nabla \tilde{\theta} = -\underline{B}_1 \cdot \nabla \theta_v + \chi_{v1} \underline{B}_1 \cdot \nabla \phi_v + (\chi_p - \chi_v) \underline{B}_v \cdot \nabla \phi_v \quad . \quad (37)$$

The change in the rotational transform is due to the change in ψ , and due to the change in the functional form of the transform itself as a function of ψ ,

$$\chi_p - \chi_v = \Delta \chi(\psi_v) + \frac{d\chi_v}{d\psi_v} \tilde{\psi} \quad . \quad (38)$$

Equation (34) gives

$$\underline{B}_v \cdot \nabla \tilde{\psi} = -\underline{B}_1 \cdot \nabla \psi_v \quad .$$

Fourier expanding in the vacuum flux coordinates, we find

$$\tilde{\psi}_{nm} = B_{1\rho nm} \frac{\rho \tilde{L}_z}{2\pi(n - \chi m)} \quad , \quad (39)$$

where

$$B_{1\rho} = \sum_{n,m} B_{1\rho nm} \sin(n\phi_v - m\theta_v) \quad .$$

and

$$\tilde{\psi} = \sum_{n,m} \tilde{\psi}_{nm} \cos(n\phi_v - m\theta_v) .$$

The radial component of $\underline{B}_1$ is determined by Eqs. (8), (9), and (32),

$$B_{1\rho nm} = - \frac{g_v \beta_o m \delta_{nm}^v}{4\rho(n-\chi m)} \left(1 - \frac{m}{m+1} \frac{\rho^2}{a^2}\right) , \quad (40)$$

where δ_{nm}^v is the Fourier amplitude of the vacuum Jacobian, $\underline{J}_v$.

In Eq. (37), $\underline{B}_1 \cdot \nabla \phi_v$ can be neglected relative to $\underline{B} \cdot \nabla \theta_v$. To solve the resulting equation we use Eq. (38), and we again Fourier transform in the vacuum flux coordinates. The $\Delta\chi(\phi_v)$ contributes only to the $n = 0, m = 0$ term, which is uninteresting. Using Eqs. (8), (9), (32), and (39), we find for the rest of the Fourier terms (except $m = 1$)

$$\tilde{\theta}_{nm} = - \frac{mL^2 \beta_o \delta_{nm}^v}{16\pi^2 \rho^2 (n-\chi m)^2} \left[1 - \frac{m+2}{m+1} \frac{\rho^2}{a^2} + \frac{\rho}{n-\chi m} \frac{d_1}{d\rho} \left(1 - \frac{m}{m+1} \frac{\rho^2}{a^2}\right)\right] , \quad (41a)$$

where

$$\tilde{\theta} = \sum_{n,m} \tilde{\theta}_{nm} \sin(n\phi_v - m\theta_v) . \quad (41b)$$

In Eq. (41), the term proportional to $dx/d\rho$ is roughly $\rho(dx/d\rho)/(n-\chi m)$ times the remaining terms. It can be neglected in equilibria with low shear.

We have noted that Eq. (41) is not valid for $m = 1$. That is because the $m = 1$ terms give an axis shift, which complicates the analysis. To get these components correctly we must subtract off from $\underline{B}_1$ the uniform part of the field, which just gives a rigid shift of the flux surfaces and does not affect

the Jacobian (except to change slightly the value of L). This renormalization (we incorporate the uniform part of B_1 in B_v) allows us to treat $\tilde{\theta}_{n1}$ perturbatively near the magnetic axis. The result is

$$\tilde{\theta}_{n1} = \frac{L^2 \beta_0 \delta_{n1}^v}{32\pi^2 a^2 (n-\chi)^2} \left(3 + \frac{\rho}{n-\chi} \frac{d\chi}{d\rho} \right) . \quad (42)$$

In deriving this expression we have neglected the corrections to δ_{n1}^v , which go like ρ^3 and are therefore small.

To solve for the Fourier amplitudes, δ_{nm} , of the finite β Jacobian in terms of the δ_{nm}^v of the vacuum Jacobian, we want to substitute $\theta_v(\theta_p)$ into the expression for J_v . Equations (41) and (42) give us θ_p in terms of θ_v , or

$$\theta_v = \theta_p - \tilde{\theta}(\theta_v, \phi) , \quad (43)$$

where $\tilde{\theta}$ is small and $\phi = \phi_v = \phi_p$. To obtain θ_v in terms of θ_p we can iterate this equation, at each step substituting Eq. (43) for θ_v wherever it appears on the right-hand side. At each step, the error we make in simply substituting θ_p for θ_v on the right-hand side gets smaller and smaller. This procedure broadens somewhat the spectral width of $\tilde{\theta}$. To get an analytic estimate of the spectral width of the Jacobian, we retain just the lowest order correction to θ_v ,

$$\theta_v \approx \theta_p - \tilde{\theta}(\theta_p, \phi) ,$$

or

$$\tilde{\theta} = \sum_{n,m} \tilde{\theta}_{nm} \sin(n\phi - m\theta_p) . \quad (44)$$

This approximation will underestimate somewhat the spectral width of the Jacobian, as do all the approximations we make. The approximation is reasonable for low order rational surfaces, but may not be accurate if m or n is large.

Substitute Eq. (44) into Eq. (5) and expand

$$\begin{aligned} \cos[n\phi - m\theta_p + m \sum_{j,k} \tilde{\theta}_{jk} \sin(j\phi - k\theta_p)] &= \text{Re} \{ \exp[i(n\phi - m\theta_p)] \} \pi_{j,k} \\ &\times \left[\sum_{\ell=-\infty}^{\infty} \exp[i\ell(j\phi - k\theta_p)] J_{\ell}(m\tilde{\theta}_{jk}) \right] \end{aligned} \quad (45)$$

where Re denotes the real part and J_{ℓ} is the Bessel function of order ℓ . At small β , $\tilde{\theta}_{jk}$ is small and we can approximate

$$J_{\ell}(m\tilde{\theta}_{jk}) = \frac{1}{\ell!} \left(\frac{m\tilde{\theta}_{jk}}{2} \right)^{\ell} \quad (46)$$

For a given set of δ_{nm}^V , the finite β δ_{nm} are determined by Eqs. (5), (41), (42), and (45). Analyticity of $1/B_V^2$ implies that the δ_{nm}^V get exponentially small with increasing m and n , so that in practice the $\tilde{\theta}_{jk}$ with small j and k dominate the coupling in Eq. (45).

It is instructive to specialize to a helical axis stellarator with very nearly circular vacuum flux surfaces and small field ripple at the magnetic axis, so that only δ_{01}^V and δ_{N1}^V are nonnegligible (where N is the number of periods). The finite β Jacobian is then

$$\begin{aligned} J &\approx J_0 \left\{ 1 + \delta_{01}^V \sum_{\ell,j=-\infty}^{\infty} J_j(\tilde{\theta}_{N1}) J_{\ell}(\tilde{\theta}_{01}) \cos[jN\phi - (j + \ell + 1)\theta_p] \right. \\ &\quad \left. + \delta_{N1}^V \sum_{\ell,j=-\infty}^{\infty} J_j(\tilde{\theta}_{N1}) J_{\ell}(\tilde{\theta}_{01}) \cos[N(j + 1)\phi - (j + \ell + 1)\theta_p] \right\} \quad (47) \end{aligned}$$

This determines the finite β Fourier amplitudes

$$\begin{aligned} \delta_{nm} = & \delta_{01}^v J_{n/N}(\tilde{\theta}_{N1}) [J_{m-n/N-1}(\tilde{\theta}_{01}) + (-1)^{n/N} J_{m-n/N-1}(\tilde{\theta}_{01})] \\ & + \delta_{N1}^v J_{m-n/N}(\tilde{\theta}_{01}) [J_{n/N-1}(\tilde{\theta}_{N1}) + (-1)^{m-n/N} J_{n/N-1}(\tilde{\theta}_{N1})] , \end{aligned} \quad (48)$$

where n must be a multiple of N . At a resonant surface we have $x = n/m$ with $x/N < 1$, so that $n/N < m$. If $n \neq 0$, it follows that

$$\delta_{nm} \approx \delta_{01}^v J_{n/N}(\tilde{\theta}_{N1}) J_{m-n/N-1}(\tilde{\theta}_{01}) + \delta_{N1}^v J_{m-n/N}(\tilde{\theta}_{01}) J_{n/N-1}(\tilde{\theta}_{N1}) . \quad (49)$$

At low β , δ_{nm} is proportional to β^{m-1} .

Even though the vacuum Jacobian has no resonant terms in this case, the finite β Jacobian resonates with every rational surface. That is, even though $\phi d\lambda/B$ is constant on every rational surface in the vacuum field, $\phi d\lambda/B$ is not constant on any rational surface in the presence of finite β . The islands do not overlap at small β because of the exponential decay of the δ_{nm} ; but they increase in width with increasing β to give an equilibrium β limit. This behavior is intrinsic to the three-dimensional nature of the vacuum field. The Jacobian of a fully three-dimensional vacuum field must have at least two Fourier amplitudes with incommensurate n/m .

For a given resonant δ_{nm} , the corresponding resonant field component is approximately given by Eq. (31), where $B_{1\rho}$ now includes second order corrections to the plasma field. The corresponding island width is approximately given by Eq. (27). The width scales as $\beta^{m/2}$. The equilibrium β limit is reached when there is significant island overlap, so that the field lines are stochastic over much of the plasma volume, and the pressure gradient

can no longer be maintained.

The finite β corrections to δ_{nm} that we have calculated determine, in turn, corrections to the plasma current and field. The correction to the plasma field can be used to improve the approximation to $\tilde{b}$ in Eqs. (34) and (35), allowing us to systematically carry our approximation to higher order in β by iterating the whole procedure. In doing so, it is necessary also to iterate further the solution to Eq. (41b). The exponential decay of the δ_{nm}^V implies that the finite β corrections to δ_{nm} due to coupling to larger n and m are small. It follows that there is a set of dominant δ_{nm}^V 's, with low n and m , which have negligible finite β corrections. The low order (in n, m) resonant δ_{nm} are determined by the first order (in β) coupling to the dominant δ_{nm}^V through Eqs. (5) and (45), with small higher order finite β corrections. For large n or m it may be necessary to calculate the higher order corrections to determine δ_{nm} accurately. For a given equilibrium, we generally expect the largest islands to form at the lowest order rational surfaces, because they couple most readily to the nonresonant δ_{nm}^V . These general considerations will be clarified by our treatment of a particular stellarator field in the following section.

IV. APPLICATION AND DISCUSSION

We apply our analysis to the Princeton heliac reference design, for which an engineering feasibility study has been done. The transport in this device has also been studied.¹¹

In applying the analysis of the previous sections to the heliac reference design, with its quite noncircular flux surfaces, we will be stretching some of our approximations, which assume nearly circular flux surfaces. We will see that the results are sufficiently decisive that small geometric factors

would not make much difference; and we will be particularly interested in the neighborhood of the magnetic axis, where the approximations are most nearly valid. Our confidence in the results is bolstered by the remarkably good agreement between the helical axis shift calculated from the lowest order plasma field, and numerically calculated axis shifts for straight heliac.¹² Our main point, in any case, is to show that the physical effects we have described can play an important role in stellarator equilibria of interest.

The heliac reference design is a three-period, helical axis stellarator. It has a relatively large rotational transform which implies a relative small finite β magnetic axis shift. The value of x increases from slightly above 1.5 at the magnetic axis to about 1.7 at the edge.¹³ The axis shifts halfway to the outer flux surface at a β of about 15%.

We have argued that the δ_{nm}^v for a general vacuum field should decrease rapidly in amplitude as n or m increases. This is borne out for the heliac reference design by a numerical Fourier decomposition of the vacuum field.^{11,14} The largest components correspond to $(n,m) = (3,0)$, $(0,1)$, and $(3,1)$. Of course, n must be a multiple of 3 because of the periodicity. The δ_{30} component corresponds to the field ripple on the magnetic axis. It is very nearly equal to $2r_0/R = 0.5$, where r_0 is the radius of the helix formed by the magnetic axis and R is the major radius. The δ_{01} term, which is due to the toroidal curvature, is about $2.5 \rho/R$, where ρ is taken to be the circularized flux surface radius. The helical curvature gives the δ_{31} term, which has a value of about $\delta_{31} = 1.3\rho/r_{ch}$, where r_{ch} is the helical radius of curvature of the magnetic axis,

$$r_{ch} = (1 + k^2 r_0^2) / (k^2 r_0) \quad .$$

As in Sec. II, $2\pi/k$ is the periodicity length.

The decrease of δ_{nm}^V with increasing (n,m) implies that the most dangerous direct resonances are those with low n and m . Higher order resonances are due to coupling of the δ_{nm}^V . The coupling is strongest for resonances with low n and m . So here again the low order resonances are the most dangerous. For the heliac reference design, the most serious problem is posed by the $n = 3, m = 2$ resonance, which lies at the magnetic axis. (The transform at the axis is actually very slightly above 1.5. It is convenient for the following discussion to take χ there as 1.5 exactly, which has only a small effect on the results.) The radial $m = 2$ component of the plasma field goes to zero at the axis; but $dx/d\rho$ also vanishes there, so that the island width can nonetheless be finite. It is necessary to modify Eqs. (27) and (31) to take into account the vanishing $dx/d\rho$.

Specializing to $n = 3, m = 2$, we estimate

$$n - m\chi \approx -2 \frac{dx}{d\psi} \Delta\psi \approx -2 \Delta\chi \rho^2/a^2, \quad (50)$$

where $\Delta\chi$ is the change in χ across the minor radius, a . We replace Eq. (29) with Eq. (50), and reevaluate the integrals in Eq. (26a) to obtain

$$\frac{B_1\rho}{B_0} \approx \beta_0 \frac{L}{4\pi\Delta\chi} \frac{\delta_{32}}{\rho} \left(\ln \frac{a}{\rho} + \frac{1}{4} \right) \sin(3\phi - 2\theta). \quad (51)$$

This is the required modification of Eq. (31). Since the flux surfaces of the reference design are quite noncircular, some comments are necessary concerning the approximations that went into Eq. (51). Our approximations for ∇^2 and ∇ in Mercier coordinates follow from an expansion about the magnetic axis. These should be accurate, since we are interested in describing the

neighborhood of the axis. Our transformation from flux coordinates to Mercier coordinates, and back to flux coordinates, on the other hand, does assume that the surfaces of constant ψ nearly coincide with those of constant ρ . For noncircular flux surfaces, $\nabla\psi$ has a substantial $\hat{e}_\omega$ component. Overall, we might expect the calculated field to be off by a geometric factor of order 2. Finally, we note that our neglect of the toroidal derivative is well-justified in the neighborhood of the magnetic axis.

To treat the island at the magnetic axis, it is necessary to introduce a renormalized zeroth order flux coordinate.³ We take

$$\psi_0 = (\rho/a_1)^3$$

where $a_1 \approx 1.28a$ satisfies $\ln(a/\rho) + 1/4 = \ln(a_1/\rho)$. Perturbing $\psi = \psi_0 + \psi_1$, and solving $\vec{E}_y \cdot \nabla\psi_1 = -B_1 \cdot \nabla\psi_0$, we find

$$\psi_1 = \varepsilon \frac{\rho}{a_1} \ln\left(\frac{\rho}{a_1}\right) \cos(3\phi - 2\theta) \quad , \quad (52a)$$

where

$$\varepsilon = \frac{3L^2}{16\pi} \frac{\delta_{32}}{\rho^2 (\Delta x)^2} \frac{a^2}{a_1^2} \beta_0 \quad . \quad (52b)$$

The separatrix is defined by $\psi = 0$. This gives an island width, w , determined by

$$(w/a_1)^2 + \varepsilon \ln(w/a_1) = 0 \quad . \quad (53)$$

The numerical Fourier decomposition of the vacuum field gives $\delta_{32}^V \approx$

$1.4\rho^2/R^2$.¹⁴ Using Eq. (19) for L, with $kr_0 = 0.75$ for the reference design, Eq. (52b) gives $\epsilon \approx 25 \beta_0$ for the direct resonance. The island width due to the direct resonance is equal to half the minor radius at $\beta_0 \approx 0.006$. For such a large island, the presence of the island itself would cause distortion of the flux surfaces and, consequently, further broadening of the Jacobian, an effect that we have not included. It is clear that the β limit due to the direct resonance is unacceptably low. Serious as this problem is, it is probably easily remedied. It could presumably be greatly alleviated by slight adjustments in the positions of the coils to minimize δ_{32}^V . If necessary, an additional coil can be added to suppress the $n = 3, m = 2$ helical component of the vacuum field. This Fourier component of the vacuum field does not contribute in any way to its favorable properties. Turning now to the nonlinear resonances, we will see that they pose a more difficult problem.

The δ_{31}^V and δ_{01}^V components couple directly to give a nonlinear δ_{32} component. Equations (49) and (42) give $\delta_{32}^{NL} \approx 31 \beta_{0P}^2/R^2$. The corresponding island width is equal to half the minor radius at $\beta_0 \approx 0.017$. Because the δ_{31}^V and δ_{01}^V Fourier components are intrinsic to the heliac vacuum field, the solution of this problem calls for more drastic design changes than does the elimination of the direct resonance. One possibility is a helical equilibrium coil whose current would be adjusted as a function of β to suppress the resonant $n = 3, m = 2$ part of the equilibrium field. This would be analogous to adjusting the current in the vertical field coil of a tokamak to maintain equilibrium as β is increased. A few helical equilibrium coils would suffice to suppress the islands at all the low order rational surfaces. However, because the island widths grow like $\beta^{m/2}$, it would not be practical to control them in this way for large m . It would probably be preferable, in any case, to modify the χ profile so that such coils are not necessary.

In the general design of stellarator vacuum fields, we might have expected the requirement of good vacuum flux surfaces to suppress the resonant field amplitudes. Our calculation for the heliac reference design shows that the amplitudes of the direct resonances may nonetheless be unacceptably large. We conclude that it is necessary to incorporate the constraints on the resonant δ_{nm}^V directly in the design procedure. Our application also shows that coupling of nonresonant components can give large islands, even for values of β at which the axis shift is small relative to the minor radius.

Our approximations in this paper have allowed us to obtain analytic estimates of island widths, for the direct resonances and for low n, m nonlinear resonances. Our simplifying approximations have also allowed us to explore and clarify the basic physics of island formation and surface destruction in three-dimensional equilibria. Clearly, the theory we have presented can be used to do more accurate calculations, both analytically and using numerical methods. An extension of the calculation to higher order in β would allow the determination of island widths for higher m, n , and give a more complete description of the equilibrium β limit.

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