27
6-17-80
JWB
AS/ATIS

UCID-18633

MASTER

Suppression of Transverse Beam Breakup Modes
in an Induction Accelerator by Gas Focusing

Richard J. Briggs

April 7, 1980

Lawrence
Livermore
Laboratory

This is an informal report intended primarily for internal or limited external distribution. The opinions and conclusions stated are those of the author and may or may not be those of the Laboratory.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

Suppression of Transverse Beam Breakup Modes
in an Induction Accelerator by Gas Focusing*

R.J. Briggs
April 7, 1980

Abstract

The effect of beam-generated positive ions on the beam-breakup instability in linear electron accelerators is calculated using simple models. The strongly nonlinear dependence of the ion focusing force on radius makes a complete suppression of the mode growth possible, in contrast to solenoidal focusing. The predicted gas pressures for significant stabilization of the mode growth in a machine configuration like ATA is relatively low (10^{-5} to 10^{-4} torr), even for high-Q modes.

*Lawrence Livermore National Laboratory is operated by the University of California for the Department of Energy Under Contract No. W-7405-Eng-48.

This work is performed by LLNL for the Department of Defense under DARPA (DOD) ARPA Order 3718, Amendment #12, monitored by NSWC under Contract No. N60921-80-WR-W0188.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

I. Introduction

The stable self-focused transport of an intense relativistic electron beam in the so-called "low pressure window" was amply demonstrated in the 20 meter Astron beam experiments.¹ In this propagation mode, the Astron beam was injected into a low pressure gas with pressures typically in the 10-50 micron range. Ionization of the background gas, and the ejection of the secondary electrons by the strong radial space charge electric field of the beam, lead to the buildup of an ion density that partially neutralized the electron beam space charge. An ion density greater than $1/\gamma^2$ times the electron beam density results in a net focusing force on the beam electrons, as is well known, and long-distance transport (with radii determined by the beam quality and ion density) is then possible. For the parameters of the Astron beam in the 20 meter tank (5 MeV, 200-500 amps, 0.3 μ sec pulse length, centimeter beam radii) this low pressure gas-focused transport mode was remarkably stable up to pressure-time products where electrical neutralization was complete and a sufficient density of secondary electrons had built up to result in strong two-stream instability.²

In the present report, we discuss the application of partial electrical neutralization by beam-generated ions (at pressures much lower than in the Astron experiments) to the control of transverse instabilities in an induction accelerator such as ATA. The idea of using gas-focussed transport inside an induction accelerator is not new; in fact the NBS program in beam recirculation through an induction machine followed earlier thinking along these lines by Leiss and coworkers. In the present report, we are interested in a level of gas focussing that is only a supplement to an existing

solenoidal magnetic field focussing system, and the main emphasis is on the effect of this extra focussing on transverse (dipole) type instability modes.

The predicted magnitude of the ion neutralization fraction (f_e) that significantly effects dipole oscillations turns out to be quite low. One key reason is that beam-induced dipole fields from the accelerator cavities are proportional to I/b^2 , with b the beam tube radius and I the beam current, whereas the dipole fields from the ion cloud are proportional to $f_e I/a^2$, with " a " the beam radius. The axial extent of the accelerating gaps (w) is also less than the gap spacing (l_g), so a comparison between the transverse forces of the ions and those of the beam-driven cavity modes

defines a critical neutralization fraction $f_{ec} = \frac{a^2}{b^2} \frac{w}{l_g}$ for these forces to be comparable (which is of order 10^{-3} for ATA). A second important point is the fact that the radial focusing forces from the ions can be highly nonlinear in their dependence on radius, resulting in a significant spread in the beam electron's radial oscillation (betatron) frequency. This spread in the particle's betatron frequency can result in a total suppression of the instability growth through the usual phase mixing process. This is in marked contrast to the "stabilization" of the growth by a solenoidal field which can only limit the amplification to a tolerable level within a given machine length and beam pulse width.

It is unlikely that the use of gas focusing will be required for ATA, based on the recent success Dan Birx has had in lowering the cavity mode Q 's to insignificant levels. It provides a useful "backup" however in case any unanticipated modes of interaction appear.

II. Ion focusing

In the low pressure (few micron or less) regime, background hydrogen gas is ionized by the high energy beam electrons at a rate of order $0.25 p_u$ ions per microsecond per beam electron with p_u the pressure in microns. The numerical constant of 0.25 is for hydrogen at 5 MeV; larger molecules have larger constants (e.g. factor of 6 larger for Argon), and the rate also increases slowly with particle energy.

The secondary electrons generated by impact ionization tend to be expelled from the beam by the strong radial electric field,

$$(E_r)_{\max} = 60 I/a$$

$$\approx 600 \text{ kV/cm at } 10 \text{ kA and } a = 1 \text{ cm.} \quad (2-1)$$

The B_θ field of the beam plus the external solenoidal fields (B_z) can keep the electrons from moving to the wall (since $E_r < |B|c$), but the excursions away from the parent ion will still be quite large. Using an estimate from elementary orbit theory

$$\delta r \sim 2mE_r/eB^2 \quad (2-2)$$

we find $\delta r \sim 1 \text{ cm}$ for $B_\theta \sim B_z \sim 2-3 \text{ KG}$. For simplicity, in the following we will neglect any reduction in the neutralization of the electron beam space charge by ions because of secondary electrons retained in the beam region (i.e. we will assume that they have escaped). The fractional electrical neutralization is therefore of order

$$f_e \approx 0.25 p_u 10^6 t; \text{ for } H_2 \text{ and}$$

$$\approx P_u 10^6 \tau; \text{ for } N_2, \text{ Argon.} \quad (2-3)$$

The ions will oscillate in the strong radial electric field at a frequency

$$\omega_{osc}^2 = \frac{q_i}{M_i} \frac{J_B}{2 \epsilon_0 c} \quad (2-4)$$

with $J_B = I/\pi a^2$. With a 10 kA beam current the oscillation period is

$$\tau_{osc} = \frac{2 \pi}{\omega_{osc}} = 2 \times 10^{-10} \left(\frac{M_i}{m_e} \right)^{1/2} a(\text{cm}) \quad (2-5)$$

For protons, the period is about 8 nsecs for $a \approx 1$ cm. With nitrogen (N_2^+) it is about 30 nsecs. This oscillation of the ions does not negate their neutralization of the beam space charge, of course. The potential harm is the likelihood of a "new" instability being introduced at a frequency characteristic of this natural ion oscillation motion if the beam pulse length is many oscillation periods in duration. In cases of interest for ATA, this does not appear to be a problem, particularly with gases like N_2 , unless the beam radius is very small.

In the presence of an external solenoidal field and fractional neutralization by ions, the equation for the equilibrium beam radius is

$$\frac{\omega_c^2}{4 c^2} a + \frac{(I/I_A) f_e}{a} = \frac{q}{a^3} \quad (2-6)$$

with $\omega_c = e B_z / \gamma m_0$, $I_A = 17 \gamma$ kiloamps, and q the beam quality. We take a value anticipated for ATA,

$$q = q_0/\gamma$$

(2-7)

with $q_0 = 0.3$ rad-cm and $I \approx 10$ kA. The equilibrium beam radius is plotted as a function of γf_e in Figure 1 for several values of B_z . With $B_z \approx 3$ kG, we can see from the figure that $\gamma f_e > 1$ is required before the equilibrium radius is significantly decreased (i.e. - $f_e \approx 1\%$ at 50 MeV).

With adiabatic entrance into the gas region from "hard vacuum" conditions, the time-dependent beam radius through the pulse at a given pressure can be derived from Figure 1 using the rate of neutralization in Eq. (2-3).

III. Model of Ion Focusing Effects on Beam-Breakup Modes

We describe the transverse beam-cavity interaction using the same general formalism and notation of Neil, Hall, and Cooper³. The coherent transverse beam displacement in the x (y) direction is denoted by ξ_x (ξ_y). At the nth cavity, this beam displacement drives a transverse cavity mode ($A_z \propto \cos \phi$) which in turn results in a change in transverse momentum of the beam electrons passing through the cavity (Δp_x) described by (c.f. Eq. 3.10, 3.11, of Ref. 3)

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_0}{Q} \frac{\partial}{\partial t} + \omega_0^2 \right) \frac{\Delta p_x}{p_0} = \omega_0^2 k_0 \frac{I}{I_a} \xi_x \quad (3-1)$$

In (3-1), $p_0 = \gamma m_0 c$, ω_0 is the resonant frequency of a particular cavity mode, Q is the quality factor, and (in MKS units)

$$k_0 = \frac{\omega_0}{9 \times 10^{11}} \frac{Z_1}{Q} \text{ cm}^{-1} \quad (3-2)$$

with Z_1/Q the transverse interaction impedance of the mode in question (in ohms).

To describe the beam dynamics, we use an "equivalent" continuous system. This model is adequate as long as the cavity spacing, l_g , is much less than the focusing wavelength ($\sim 2\pi c/\omega_c$). The change in transverse momentum due to the cavity fields can then be approximated by

$$\frac{\Delta p_x}{\Delta t} = \frac{c(\Delta p_x)_{\text{one gap}}}{l_g} \quad (3-3)$$

The fields created by the displaced beam charge and current in the smooth pipe regions have negligible transverse forces (i.e. the dipole fields of a beam in a smooth pipe cancel within $1/\gamma^2$ just as the monopole fields do). The ion cloud being created by the beam, on the other hand, has a monopole radial electric field associated with it which we can write in the form

$$(E_r)_{\text{ions}} = f_e \frac{J_B}{2\epsilon_0 c} r \quad (3-4)$$

for a uniform density inside $r = a$. (We will consider the spread in focusing forces introduced by nonuniform densities later). This electric field causes a transverse restoring force on the displaced beam equal to

$$-k_{\beta i}^2 \gamma m_0 \xi_{x,y} \text{ with}$$

$$k_{\beta i}^2 = f_e \frac{2}{a} \frac{I}{I_a} \equiv f_e k_{\beta}^2 \quad (3-5)$$

We should also consider, in general, the transverse force on the beam due to a displacement of the ion cloud center of mass. If the frequency of the beam oscillation in the lab frame is much higher than the ion oscillation frequency (discussed in Section 2) or the ion generation rate, this force should be ignorable compared to the above force.

Including the transverse force due to an axial magnetic field, we can now write the overall force equation for $\xi_{x,y}(\tau, z)$ with

$$\tau = t - z/c \quad (3-6)$$

as

$$\frac{\partial^2 \xi_x}{\partial z^2} = \left(\frac{\Delta p_x}{p_0} \right) \frac{1}{l_g} + \frac{1}{\rho} \frac{\partial \xi_y}{\partial z} - k_{\beta i}^2 \xi_x \quad (3-7)$$

$$\frac{\partial^2 \xi_y}{\partial z^2} = \left(\frac{\Delta p_y}{p_0} \right) \frac{1}{l_g} - \frac{1}{\rho} \frac{\partial \xi_x}{\partial z} - k_{\beta i}^2 \xi_y \quad (3-8)$$

with $\rho = c/\omega_c$.

We assume that the cavities are circularly-symmetric, so modes in the x or y direction are identical. We can then let

$$\xi = \xi_x + i \xi_y \quad (3-9)$$

$$\Delta = \frac{1}{l_g} \left(\frac{\Delta p_x + i \Delta p_y}{p_0} \right) \quad (3-10)$$

$$G = \frac{k_0}{l_g} \frac{I}{I_a} \quad (3-11)$$

The system of equations for our continuous model is then

$$\frac{\partial^2 \xi}{\partial z^2} = \frac{-i}{\rho} \frac{\partial \xi}{\partial z} - k_{\beta i}^2 \xi + \Delta \quad (3-12)$$

$$\left(\frac{\partial^2}{\partial \tau^2} + \frac{\omega_o}{Q} \frac{\partial}{\partial \tau} + \omega_o^2 \right) \Delta = \omega_o^2 G \xi \quad (3-13)$$

The model as derived thus far neglects the important effect of a spread in $k_{\beta i}^2$ due to a nonuniform ion cloud density. We will model this in exact parallel with the "well-established" techniques in hose instability theory⁴. Divide the beam into "beamlets" with different unperturbed (incoherent) radial oscillation amplitudes, having an equivalent frequency from the ion restoring force

$$k_\eta^2 = \eta \left(k_{\beta i}^2 \right)_{\max} \quad (3-14)$$

with $0 \leq \eta \leq 1$, and a coherent oscillation amplitude of ξ_η . Each beamlet then satisfies a force equation

$$\frac{\partial^2 \xi_\eta}{\partial z^2} + \frac{i}{\rho} \frac{\partial \xi_\eta}{\partial z} + k_\eta^2 \xi_\eta = \Delta \quad (3-15)$$

while

$$\left(\frac{\partial^2}{\partial \tau^2} + \frac{\omega_o}{Q} \frac{\partial}{\partial \tau} + \omega_o^2 \right) \Delta = \omega_o^2 G \langle \xi \rangle \quad (3-16)$$

with the average displacement defined by⁴

$$\langle \xi \rangle = \int g(\eta) \xi_\eta \, d\eta \quad (3-17)$$

Here $g(\eta)$ is a "suitable" distribution function, normalized so that

$$\int_0^1 g(\eta) \, d\eta = 1 \quad (3-18)$$

Examples of $g(\eta)$ are given later.

IV. Beam-breakup growth in the absence of ion focusing

We first analyze our model with $k_{\beta i} = 0$ to show that it can recover the well-known results³. With the usual form

$$\exp(i \omega \tau - i k z) \quad (4-1)$$

we can derive the following dispersion relation from Eq. (3-12) and Eq. (3-13), appropriate for ω close to ω_0 and $Q \gg 1$ (which is already implicit in taking only one term in the mode expansion for the cavity fields)

$$k \left(k - \frac{1}{\rho} \right) = \frac{\omega_0 G}{2 (\omega - \omega_0 - i \omega_0 / 2Q)} \quad (4-2)$$

For a long beam pulse duration, the growth of the perturbations down the machine would approach the steady-state growth rate in z obtained from a solution of Eq. (4-2) for $\text{Im } k$ with ω real. The maximum spatial amplification rate is at $\omega = \omega_0$ in the limit of "strong enough" focusing;

$$k = \frac{1}{\rho} + \frac{\omega_0 G \rho}{2 (\omega - \omega_0 - i \omega_0 / 2Q)} \quad (4-3)$$

so

$$(\text{Im } k)_{\max} = G \rho Q \quad (4-4)$$

After n accelerator units, $z = n l_g$, the growth approaches

$$\xi \sim \exp(G \rho Q z) = \exp \Gamma_{\max} \quad (4-5)$$

with

$$\Gamma_{\max} = n k_o \rho Q I / I_a \quad (4-6)$$

in agreement with Ref. 3.

With a finite pulse duration, we need to consider an initial value problem with $\xi(\tau)$ given at $z = 0$, say:

$$\xi(\tau, z) = \int \hat{\xi}(\omega) e^{i\omega\tau - i k(\omega)z} \frac{d\omega}{2\pi} \quad (4-7)$$

We again assume $\omega \sim \omega_o$, $k \sim 1/\rho$, and let

$$\Omega = \omega - \omega_o \quad (4-8)$$

$$K = k - \frac{1}{\rho} \quad (4-9)$$

Then

$$\xi(z, \tau) = e^{i\omega_o\tau - i z/\rho} \int \hat{\xi} e^{i\Omega\tau - i Kz} d\Omega \quad (4-10)$$

with

$$\rho = i\Omega\tau - i Kz, \quad (4-11)$$

and

$$K = \frac{\omega_o G \rho}{2(\Omega - i\alpha)} \quad (4-12)$$

with

$$\alpha = \omega_o/2Q \quad (4-13)$$

We look for the major contributions to the integral in Eq. (4-10) from saddle points of $\phi(\Omega)$, or $\phi(K)$, when z and τ are large; i.e. from (4-11) and (4-12) we have

$$\phi(K) = -\alpha \tau + \frac{i\omega_0 G \rho \tau}{2 K} - i K z ; \quad (4-14)$$

and

$$\frac{\partial \phi}{\partial K} = 0 \quad (4-15)$$

gives

$$K_s^2 = - \frac{\omega_0 G \rho}{2} \frac{\tau}{z} \quad (4-16)$$

and

$$\phi_{\text{saddle}} = -\alpha \tau + \left(2 \omega_0 G \rho \tau z \right)^{1/2} \quad (4-17)$$

Eq. (4-17) agrees with the asymptotic calculation in Ref. 3 (their Eq. (5-13)). If we maximize ϕ_{saddle} with respect to τ , we again obtain the maximum "steady-state" exponent (Eq. 4-6) as we must. With a limited pulse duration, the growth at the end of the machine can be less than this steady-state value.

V. Ion Focusing Effects on Beam-Breakup Growth

First consider the effect of the ions described within the framework of Equations (3-12) and (3-13); namely, a constant ion density in radius. For simplicity, we also consider f_e to be constant in the first part of this section; the case $f_e \sim \tau$ is treated at the end.

Considering $k_{\beta i}^2$ to be constant, the mode growth in the simple case (no spread) is essentially equivalent to an increase in the solenoidal focusing field. For example, the steady-state growth rate becomes (compare Eq. 4-4)

$$\begin{aligned} (\text{Im } k)_{\max} &= \frac{GQ}{\sqrt{1/\rho^2 + 4 k_{\beta i}^2}} \\ &= \frac{GQ\rho}{\sqrt{1 + 4 k_{\beta i}^2 \rho^2}} \end{aligned} \quad (5-1)$$

The decrease in the growth rate with $f_e \neq 0$ is significant only when the neutralization fraction is large enough to decrease the equilibrium radius (Eq. (2-6) and Figure I). Similar conclusions apply to the growth with finite pulse duration.

We now include the frequency spread using the formalism described in Section III, first considering $(k_{\beta i}^2)_{\max}$ to be independent of τ . With the dependence for $\xi(z, \tau)$ in Eq. (4-1), we have the dispersion relation

$$\omega - \omega_0 - i\alpha = \frac{\omega_0 G}{2} \left\langle \frac{1}{\left(k - \frac{1}{\rho}\right) k - k_{\eta}^2} \right\rangle \quad (5-2)$$

We consider in the body of this paper the distribution function appropriate to self-forces of Bennett charge distributions⁴

$$g(\eta) = 6\eta(1-\eta) \quad (5-3)$$

(A different example is worked out in the Appendix.) We then have

$$\omega - \omega_0 - i\alpha = - \frac{\omega_0 G}{2} \int_0^1 \frac{6\eta (1-\eta) d\eta}{\eta k_{\beta i}^2 - k(k - \frac{1}{\rho})} \quad (5-4)$$

with $k_{\beta i}^2$ the maximum frequency, as given by Eq. (3-5). Let

$$\psi = \frac{k(k - \frac{1}{\rho})}{k_{\beta i}^2} \quad ; \quad (5-5)$$

then

$$\omega - \omega_0 - i\alpha = - \frac{3 \omega_0 G}{2 k_{\beta i}} \left\{ \frac{1}{2} - \psi + \psi(1-\psi) \left(\ln \frac{\psi-1}{\psi} \right) \right\} \quad (5-6)$$

Considering k as a Laplace transform variable in z , along the real k axis we have $k = k_r - i0^+$ and

$$\ln \frac{\psi-1}{\psi} = \ln \frac{1-\psi}{\psi} + i\pi \quad (5-7)$$

along $0 \leq \psi \leq 1$.

It can be shown, in analogy to the resistive hose calculations⁴, that the peak growth of the perturbation for all z , at a given point τ in the pulse, is given by $\exp(\omega_g \tau)$, where ω_g is the maximum of $-\text{Im } \omega$ over all real k . A critical distance in z is required for growth to this maximum to occur, and an explicit calculation is presented in the appendix. From Eq. (5-6) and (5-7), the maximum $\text{Im } \omega$ occurs at $\psi = 1/2$, and is equal to

$$\omega_g = \frac{3\pi}{4} \frac{\omega_0 G}{k_{\beta i}} - \alpha \quad (5-8)$$

When $k_{\beta i}^2 (f_e) \rightarrow 0$, this maximum amplification is unlimited, in agreement with the $\exp \sqrt{z\tau}$ dependence for this case derived in Section IV. With a spread in the radial focusing frequencies of various particles, the growth of perturbations in a given pulse length is bounded, in close analogy again with the resistive hose mode⁴.

Using the definitions in Eqs. (3-5), (3-11), and (4-13), the maximum growth factor can be written

$$\omega_g \tau = \frac{3\pi}{8} \frac{k_o a^2 \omega_o \tau}{l_g f_e} - \frac{\omega_o \tau}{2Q} \quad (5-9)$$

Consider a numerical example with parameters roughly equivalent to the lowest transverse mode measured in the ETA/ATA cavities⁵:

$$l_g = 28 \text{ cm (11 inches)}$$

$$\frac{\omega_o}{2\pi} = 300 \text{ MHz}$$

$$Z_L/Q = 10 \text{ ohms}$$

From Eq. (3-2);

$$k_o = 2.1 \times 10^{-2} \text{ cm}^{-1}$$

Complete stability is predicted when $\omega_g \tau < 0$, which requires

$$f_e > \frac{3\pi}{4} \frac{k_o a^2}{l_g} Q \quad (5-10)$$

or, for the above parameters,

$$f_e > 1.8 \times 10^{-3} a^2 Q \quad (5-11)$$

with a in centimeters. A Q of 10 and a λ 0.3 cm would therefore have no growth for $f_e > 2 \times 10^{-3}$, which for N_2 or Argon would occur within 20 nsec at pressures around 10^{-4} torr (see Section II).

We now generalize the results to the case $f_e = \frac{\partial f_e}{\partial \tau} \tau$, with $\frac{\partial f_e}{\partial \tau} = \text{const.}$, and we neglect variation of " a " with τ . In Eqs. (3-15) and (3-16), we can still perform a Laplace transform with respect to z ($\exp(-ikz)$ dependence for ξ_η), and obtain

$$\langle \xi \rangle = \Delta \int_0^1 \frac{g(\eta) d\eta}{\eta k_\beta^2 f_e(\tau) - k(k - 1/\rho)} \quad (5-12)$$

We assume that the oscillations are close to ω_0 , as before, and let

$$\Delta = A(\tau) \exp(i\omega_0 \tau) \quad (5-13)$$

with $\partial A / \partial \tau \ll \omega_0 A$. Equation (3-16) can then be approximated by

$$\left(2i\omega_0 \frac{\partial A}{\partial \tau} + i \frac{\omega_0^2}{Q} A \right) e^{i\omega_0 \tau} = \omega_0^2 G \langle \xi \rangle \quad (5-14)$$

The solution to Eq. (5-14), using (5-12), is of the form

$$A(\tau) = A_0 e^{N(\tau)} \quad (5-15)$$

with the exponent

$$N(\tau) = -\alpha \tau + i \int_0^\tau \frac{\omega_0 G}{2 k_\beta^2 f_e(\tau')} \int_0^1 \frac{g(\eta) d\eta}{\eta - \phi(\tau')} d\tau' \quad (5-16)$$

and

$$\phi(\tau) = \frac{k(k - 1/\rho)}{k_\beta^2 f_e(\tau)} \quad (5-17)$$

Using the Bennett beam distribution for $g(\eta)$, the integration over η gives the following growth factor ($\text{Re } N$) for real k ;

$$\text{Re } N(\tau) = -\alpha \tau + \frac{3 \pi \omega_0 G}{k_\beta^2} \int_a^b \frac{\phi(1-\phi)}{f_e(\tau')} d\tau' \quad (5-18)$$

where the limits a and b are constrained by the requirement $0 \leq \phi \leq 1$. We take the linear variation of f_e with τ , and let

$$\phi = \frac{d}{\tau} \quad (5-19)$$

with

$$d = \frac{k(k - 1/\rho)}{k_\beta^2 \left(\frac{\partial f_e}{\partial \tau} \right)} \quad (5-20)$$

encompassing all the dependence on k . The integration

$$\int \frac{\phi(1-\phi)}{\tau'} d\tau' = \int_d^\tau \frac{d}{\tau'} \left(1 - \frac{d}{\tau'} \right) \frac{d\tau'}{\tau'} \quad (5-21)$$

becomes, with $x = \tau'/d$,

$$= \int_1^{\tau/d} \frac{1}{x^2} \left(1 - \frac{1}{x}\right) dx \quad (5-22)$$

It has a maximum value of $1/2$, for $\tau \gg d$. The maximum number of e-foldings of the instability even for very high Q is therefore bounded by

$$\text{Re}(N) \leq \frac{3 \pi \omega_o G}{2 k_\beta^2 \frac{\partial f_e}{\partial \tau}} \quad (5-23)$$

for all z and all τ (constrained by the limit $\tau < \left(\frac{\partial f_e}{\partial \tau}\right)^{-1}$ of course). Using the definitions of G and k_β^2 this becomes

$$\text{Re} N \leq \frac{3\pi}{4} \frac{\omega_o k_o a^2}{\frac{1}{g} \frac{\partial f_e}{\partial \tau}} \quad (5-24)$$

If we allow a possible exponentiation through N_o e-folds, we require

$$\frac{\partial f_e}{\partial \tau} \geq \frac{3 \pi}{4 N_o} \frac{\omega_o k_o a^2}{\frac{1}{g}} \quad (5-25)$$

Using the same numerical example, we have

$$\frac{\partial f_e}{\partial \tau} \geq 3.3 \times 10^6 \frac{a^2}{N_o} \text{ sec}^{-1} \quad (5-26)$$

or a pressure of nitrogen (argon, air, etc.) of order

$$p_u \geq 3.3 a^2 / N_0 \quad \text{microns} \quad (5-27)$$

With $a = 0.3$ cm, a pressure of 3×10^{-5} torr would keep the worst possible amplification within 10 e-foldings.

Figure 1. Beam radius vs γf_e

Appendix: Growth calculations with an alternate betatron distribution function

In this appendix, we consider a distribution function of the form shown below:

This distribution function can model relatively constant ion densities by choosing a small value for s while with $s \sim 1$ it should be roughly equivalent to the broad distribution (Eq. 5-3) shown⁴ to be appropriate for Bennett charge distribution. It is simpler for analytic work, and we show below the full solutions of the saddle point calculation for growth in z .

The previous dispersion relation, Eq. (5-6), now becomes

$$\omega - \omega_0 - i\alpha = \frac{\omega_0 G}{2 k_{\beta i}^2} \frac{1}{s} \ln \left(1 + \frac{s}{\psi - 1} \right) \quad (\text{A-1})$$

with ψ defined in (5-5). With ψ (or k) real and $1 - s \leq \psi < 1$, we have

$$\ln \left(1 + \frac{s}{\psi - 1} \right) = \ln \left[\frac{\psi + s - 1}{1 - \psi} \right] - i\pi \quad (\text{A-2})$$

and the peak $\text{Im}(-\omega)$ for real k is now

$$\omega_g = \pi \frac{\omega_0 G}{2s k_{\beta i}^2} - \alpha \quad (\text{A-3})$$

Comparing (A-3) with (5-8), we see that a choice $s = 2/3$ will give the same peak growth rate as the Bennett distribution.

The relatively simple dispersion relation in Eq. (A-1) allows a full analytic solution of the saddle point calculation for growth in τ and z (with $k_{\beta i}^2$ independent of τ), similar to the one presented in Section IV. Using the same variables Ω and K defined in Equations (4-8) and (4-9), and taking $|\Omega| \ll \omega_0$ and $|K| \ll 1/\rho$, we can write (see Eq. 4-10 and Eq. 4-11)

$$\phi = i\Omega\tau - iKz \quad (\text{A-4})$$

as a function of ψ , where

$$K = \left(k_{\beta i}^2 \rho \psi \right) \quad (\text{A-5})$$

We let

$$A \equiv \frac{\omega_0^2 G}{2k_{\beta i}^2} \quad (\text{A-6})$$

and normalized distance be

$$Z = k_{\beta i}^2 \rho z, \quad (\text{A-7})$$

so

$$\phi = -\alpha\tau + \frac{iA\tau}{s} \ln \left(1 + \frac{s}{\psi-1} \right) - iZ\psi \quad (\text{A-8})$$

The saddle point, $\partial\phi/\partial\psi = 0$, is at

$$\psi_s - 1 = -\frac{s}{2} + i \left[\frac{A\tau}{2} - \frac{s^2}{4} \right]^{1/2}$$

$$\equiv -\frac{s}{2} + i R \quad (A-9)$$

Note that

$$\ln \frac{\psi - 1 + s}{\psi - 1} = \ln \left[\frac{\frac{s}{2} + i R}{-\frac{s}{2} + i R} \right]$$

$$= -i(\pi - 2\theta) \quad (A-10)$$

where

$$\cos \theta = \left[\frac{s^2}{4 A \tau} \right]^{1/2} \quad (A-11)$$

Using these relations for the saddle point ψ_s in Eq. (A-8), we can write the growth factor

$$\text{Re } \rho = -\alpha\tau + \frac{A\tau}{s} F(z) \quad (A-12)$$

with

$$F(z) \equiv \pi - 2\theta + 2 \sqrt{z_0 - z_0^2}, \quad (A-13)$$

$$\theta = \cos^{-1} \sqrt{z_0'} \quad (A-14)$$

and

$$z_0 \equiv \frac{s^2 Z}{4 A \tau} \quad (\text{A-15})$$

At fixed τ , the growth of the perturbation with distance z (or the normalized variable z_0) has the form

The main conclusion from this saddle point analysis is that one must go a distance $z_0 = 1$ (or $Z = 4A\tau/s^2$) to have the perturbation grow to its maximum amplitude, correctly given by $\omega_g \tau$ (in Eq. A-3) obtained from maximizing $-\text{Im } \omega$ over real k .

References

1. Beal, Briggs, Pearlstein, "Beam Research Program -- Recent Results and Status", UCID-16169, December 1972.
2. Briggs, Clark Fessenden, Hester, and Lauer, "Transport of Self-Focused Relativistic Electron Beams", Proc. of 2nd Int. Topical Conference on High Power Electron & Ion Beam Research and Technology, Oct. 3-5, 1977, pg. 319.
3. Neil, Hall, and Cooper, "Further Theoretical Studies of the Beam Breakup Instability", Particle Accelerators, Vol. 9, Pg. 213-222, 1979.
4. Edward P. Lee, "Resistive Hose Instability of a Beam with the Bennett Profile", Physics Fluids 21, August 1978, 1327-1343.
5. Daniel L. Bix, "Microwave Measurements of the ETA Accelerating Cavity", UCID-18582, March 12, 1980.