MASTER

Final Report on Research Project:

RELATIONSHIPS BETWEEN FRACTURE
TOUGHNESS AND OTHER MATERIAL
PROPERTIES

by

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ABSTRACT

The key experimental and analytical studies which have led to our present understanding of the mechanisms of ductile fracture are reviewed. It is concluded that insufficient progress has been made in the quantitative description of ductile separation mechanisms on a microscale to allow the realistic prediction of fracture toughness from material properties and microstructure. An experimental study of ductile fracture is underway which has the aim of determining the growth rate of voids in known plastic deformation fields as a function of triaxiality of stress and material work-hardening. Novel specimens of particularly well characterized microstructure are utilized.

CHAPTER I

INTRODUCTION

Fracture of engineering structures at nominal stresses below the yield strength may occur by cleavage, ductile fracture, or combinations of both mechanisms. Because many of the high strength alloys used in critical structural applications fail primarily by ductile fracture mechanisms, study of this topic is of considerable practical as well as fundamental importance.

A great deal of emphasis has been placed on determining the microstructural features which affect the cleavage fracture of metals. By contrast, less attention has been focused on the localized mechanisms involved in ductile fracture. The reason for this is that dislocation theory and the other metallurgical concepts necessary to study cleavage have been available for some time whereas a study of ductile fracture depended on the development of continuum studies of void growth on a microscale and clarification of fracture topography provided by the electron microscope. Consequently, little definitive experimental work has been done and several important features of ductile fracture have yet to be adequately described.

The general features of ductile fracture are well established. In nearly all commercial metals ductile fracture proceeds by the initiation, growth and coalescence of voids. An overwhelming body of evidence suggests that the voids responsible for fracture are formed primarily at hard second-phase particles and inclusions. The initiation of voids occurs either by fracture of particles or by decohesion of the particle-matrix interface. The growth of

voids proceeds by plastic deformation. The fracture process is completed by the coalescence of voids. This sequence results in the formation of the dimples that characterize ductile fracture surfaces.

Very little progress has been made in the quantitative description of ductile separation mechanisms on a microscale. Even a thorough characterization of separation mechanisms, however, would be insufficient for fracture prediction. The problem of integrating models of separation with macroscopic crack stress analysis would remain. Ideally, a fracture criterion formulated in terms of local stress-strain history over a small region with characteristic microstructural dimensions is sought.

In the small scale yielding regime where crack extension behavior can be correlated with analytical solutions which characterize the near-tip stress field, some progress has been made in the representation of stresses and strains close to the leading separation elements of the crack. Rice and Johnson [1] have suggested that crack tip blunting plays a crucial role in establishing the intense strain concentrations ahead of cracks which are required for crack advance by ductile mechanisms. For small scale yielding, the mechanics of crack blunting and the mechanisms of ductile separation are central features which must be explored in the prediction of fracture toughness from material properties and microstructure.

In the large scale yielding and fully plastic regimes, most current evidence indicates that no single parameter from elastic-plastic solutions can characterize the near-tip fields. Here, an understanding of local mechanisms and mechanics is required

for the formulation of even approximate theories for crack extension with a sound physical basis.

The primary goal of this investigation is to determine experimentally the growth rate of voids in known plastic deformation fields as a function of triaxiality of stress and workhardening rate. The results will be compared with theoretical solutions based on continuum mechanics. In addition, direct observation of void growth and coalescence ahead of blunted grooves in plane strain fully plastic fracture will be made. Fracture criteria consistent with these observations will be formulated using the knowledge of local mechanisms and mechanics of the separation process.

CHAPTER II

LITERATURE REVIEW: THE MICROSTRUCTURAL BASIS OF DUCTILE FRACTURE

A. Void Initiation

The most significant step toward understanding the microstructural basis of ductile fracture was the discovery that voids nucleate at inclusions and second phase particles. This was not recognized until more than a century after the early work of Joseph Henry [2] who in 1855 suggested that ductile metals fracture by a process of internal necking when stretched. Ludwik [3] in 1927 sectioned a necked but unbroken tensile specimen of aluminum and found that fracture had initiated as an internal crack which had apparently started in the center of the neck. It remained for Tipper [4] to suggest in 1948 that the tensile fracture of ductile metals proceeded by the growth of voids which had initiated by matrix decohesion from nonmetallic inclusions. Unequivocal supporting evidence for this hypothesis was presented in 1959 in the classical work of Puttick [5], which was later corroborated by the work of Rogers [6]. Observation of sectioned tensile specimens of copper and iron under the optical microscope by Puttick revealed that void initiation occurred by decohesion or by fracture of non-metallic inclusions.

Meanwhile, Crussard et al. [7] described numerous cup-like indentations revealed in electron micrographs of carbon replicas of fracture surfaces. He speculated that voids produced the indentations found on the two opposite faces of the fracture. Moreover, he found additional minute indentations in the centers

of many of the dimples which apparently had been produced by the inclusions which had initiated the voids.

Subsequent experiments employed optical micrography of sectioned specimens, transmission electron microscopy of thin sections from the vicinity of fracture surfaces, scanning fractography and fracture surface replication. Results from each of these methods point to the crucial role of inclusions in initiating ductile fracture. The universality of the role of inclusions is apparent when one considers the variety of materials in which it has been demonstrated (internally oxidized copper alloys [8], various maraging and quenched and tempered high strength steels [9], low strength steels above the ductile/brittle transition temperature [10], and many aluminum alloys [11]) and the variety of deformation states (unconstrained or fully constrained) in which it occurs.

There have been some reports of void initiation apparently in the absence of inclusions in very high purity materials and single crystals [12]. However, these observations are probably the results of insufficient magnification of the structure. The indentations observed by fractographic techniques in scanning electron microscopy and the voids observed by transmission electron microscopy of thin foils of deformed metals are frequently several orders of magnitude smaller than those revealed by optical microscopy [11]. Any commercially produced structural alloy contains numerous inclusions or second phase particles that are candidates for nucleating voids.

Experimental observations indicate that void initiation may

occur at any strain between yield [26] and fracture [11]. Voids have been observed to initiate by debonding [8,9] and by cracking [9,10] of brittle inclusions. It has yet to be determined whether interface debonding occurs at critical values of normal stress, shear stress, shear strain, or a combination of these or other variables. Likewise, it is not known whether a critical elastic strain energy criterion or a critical tensile stress criterion can be used to predict particle cracking.

The stresses and strains at which debonding or particle cleavage occur are difficult to establish experimentally. The size, shape, composition and distribution of the second phase and its interfacial energy with the matrix are all variables which influence the failure mechanism, but these are all difficult to investigate experimentally.

The most significant work on void initiation is that of Palmer and Smith [8] who examined interface decohesion in internally oxidized copper containing nearly spherical silica particles. Specimens were subjected to tensile deformation at 77°K and 293°K and viewed by transmission electron microscopy. Fig. 1 is a reproduction of a micrograph from their work. The deformed specimen displayed a dense tangling of dislocations around the particles followed by void formation that occurred first at larger particles. Progressively smaller particles nucleated voids as the strain was increased. Voids were first formed at the particle poles in line with the direction of the tensile axis. Few dislocations remained close to the polar regions after void nucleation. However, the dislocation density remained high at the

particle equators after void nucleation. The dislocations in the polar regions had apparently run into the free surfaces which had been generated. Histograms were presented showing the fraction of particles of a given size on which voids had formed plotted as a function of particle radius. These plots clearly demonstrated that debonding of the larger particles was favored over debonding of the smaller particles. Further, the plots demonstrated that debonding of the smaller particles was more frequent in specimens strained at 77°K than in specimens strained at 293°K. Both of these observations were explained by Palmer and Smith on the basis of cross-slip of dislocations. Before failure, highly localized bands of deformation were developed within which the rate of void growth was increased. Unfortunately, there was some evidence of preferential electropolishing of voids which intersected the surface of the foil. These studies of Palmer and Smith were largely descriptive since details of stress and strain history in the vicinity of particles were unknown.

Recently, Argon and Im [13] reported studies of equiaxed inclusions of Fe_3C in spheroidized 1045 steel, of Cu-Cr inclusions in Cu .6% Cr and of TiC inclusions in unaged maraging steel in plastic deformation fields of known plastic strain and triaxial stress distribution. The elastic-plastic finite element study of Argon, Im and Needleman [14] for tensile bars with various initial neck profiles provided the stresses and strains near the center of the necks of the tensile specimens used in the study.

Optical metallographic observations were made on these specimens after straining and sectioning. The local interfacial stress

for debonding was determined by plotting the density of separated inclusions as a function of position on the central axis of the specimens. The stresses at the point along the axis where the density of separated inclusions dropped to zero were used to evaluate, by means of a bounding analysis, the local interfacial stresses. These authors concluded that in annealed materials without preprocessing the plastic strain required to debond inclusions frequently exceeds one half the total strain to fracture. In addition, they claimed that a critical interfacial stress criterion was appropriate for the separation of inclusions from a deforming matrix. With respect to the observed particle size effects in their experiments, the hypothesis of Argon and Im was that interfacial stress becomes particle size dependent when there exist significant local variations of particle concentration in the matrix. The significance of this work is that it is the first to deduce from experimental observations the actual local conditions for void initiation.

B. Void Growth

Once voids appear in a deforming material they grow and coalesce by plastic deformation of the matrix. The effects of a superimposed triaxial stress are crucial in determining ductile fracture behavior.

Considering tensile behavior, experimental observations show that the voids which initiate fracture grow most rapidly at the center of the necked region and not at the surface. Bridgman's [15] derivation of the stress distribution at the necked region is that of a uniaxially applied tension superimposed on hydrostatic

tension which varies from maximum intensity at the axis to zero at the periphery. Cracks initiate at the center of the necked regions because the triaxial tension accelerates void growth.

After sectioning tensile specimens of copper which had been strained to 30% reduction in area Rogers [6] observed a high density of voids in the plane of the neck area. However, no voids were found in copper that had been cold drawn to 99% reduction in area. The classic experiments are those of Bridgman [16]. By conducting numerous tensile tests on many metals with various levels of superimposed environmental pressure, Bridgman observed that ductility (both residual and retained) and strength increased as the hydrostatic pressure increased.

The effect of triaxial tension was again demonstrated by Marshall and Shaw [17], who stopped their tensile tests after the neck had formed and machined the specimen so that it was returned to cylindrical shape before the tensile test was resumed. By relieving the hydrostatic tension in this way they were able to achieve increased values of reduction in area. The large triaxial tension in front of a Mode I crack in plane strain tension inevitably has a pronounced influence on the local strains to fracture.

McClintock [18] and Rice and Tracey [19], among other, have presented continuum analyses (to be discussed in Chapter III) which indicate that void growth depends strongly on stress state. Rosenfield [20] pointed out in his review of the metallurgical aspects of ductile fracture that the few direct studies of void growth have not yet yielded the necessary quantitative data to

provide adequate evaluation of the several theories which have been proposed for void growth. Only minor progress has been made since his review was written.

Ductile fracture in 200 ksi yield strength AISI 4340 and 18 Ni maraging steel was recently studied by Cox and Low [9]. These authors employed smooth and circumferentially notched tensile specimens in their studies of void initiation, growth and coalescence. Void initiation in AISI 4340 was found to occur first by decohesion of large MnS inclusions from the matrix. These voids grew until void initiation at the much smaller carbide dispersion occurred in sheets which linked the large voids along planes oriented at approximately 45° to the tensile axis. In the 18 Ni maraging steel, initiation occurred by the cracking of the titanium carbonitride inclusions. These authors apparently observed that the larger inclusions fractured at lower strains. With increasing strain, both the number and size of voids observed in the necked regions of sectioned tensile specimens increased. The increase in the number of voids as the strain increased resulted from cracking of progressively smaller inclusions.

In the smooth tensile specimens at small strains voids grew primarily in the direction of the applied tensile stress. In the notched specimens and in the initially smooth specimens after substantial necking had taken place, voids in the center of the neck exhibited lateral growth under the prevailing triaxial stress state.

If void initiation occurs at lower strains for larger inclusions, as Cox and Low have apparently observed, then the

size of a void present at a particular location not only is a function of the stress and strain history at that location, but also is a function of the size of the inclusion which initiated the void. Considering the probability of the existence of inclusions of one particular size, shape, and orientation at specific locations, the problem of correlating void sizes quantitatively with the stress and strain histories even in relatively simple metallurgical systems is formidable. Even if the inclusions were roughly spherical so that no orientation effects would arise, errors would occur in the study of void growth because a metallurgical section would only infrequently cut a void precisely at its midplane.

In order to circumvent some of these difficulties Cox and Low selected the cross sectional area of the largest void present in the midplane as a measure of the extent of void growth. The reasoning behind this choice was that the largest void present would be the one which had initiated first (at the largest inclusion) and thus represented the best available indicator of the extent of void growth. A plot of the radius of the largest void size vs. true strain for 18 Ni maraging steel from the work of Cox and Low is shown in Fig. 2. These results clearly demonstrate that the increased level of triaxial stress in the notched specimens greatly increases the rate of void growth. These data are compared with the analytical predictions of McClintock [18] in the figure. The agreement is poor. Cox and Low suggested that the discrepancy results from the fact that McClintock's analysis assumes no interaction between voids.

It is also possible that the selection of the largest void to measure the extent of void growth biases the results toward high rates of growth - the largest void could in fact be the product of two or more adjacent voids which have coalesced at an early stage. A third possible source of error is that the Bridgman [15] solution for the distribution of stress and plastic strain across the plane of the neck used by Cox and Low is not adequate for initially machined grooves in round bars with appreciable strain hardening. Clausen [21] has demonstrated that in such instances the axial plastic strain increments may not be uniform along the plane of the neck as is necessary for the Bridgman solution. In any event, the data of Cox and Low do not provide adequate evaluation of McClintock's theory of void growth.

Mackenzie and his coworkers [22] have studied the influence of stress state on the effective plastic strain to cause void coalescence on a large scale in the necks of circumferentially notched tensile specimens. The influence of stress state on the strain to failure initiation was investigated for several high strength steels. Because void coalescence on a large scale is accompanied by a sharp loss in load-bearing cross-section, this event was used as a convenient measure of failure initiation.

Specimens of varying ratio a/R , where a is the radius of minimum cross-section and R is the radius of curvature of the circumferential notch, were used to control the severity of triaxial stress in the plane of the neck. Bridgman's analysis [15] was used to estimate the stress state parameter $\sigma_m/\bar{\sigma}$

in the central region of the specimen where failure initiated.

The mean stress, σ_m , is defined as

$$\sigma_m = \sigma_{kk}/3$$

and the effective stress, $\bar{\sigma}$, is defined as

$$\bar{\sigma} = \sqrt{\frac{3}{2} S_{ij} S_{ij}}$$

where

$$S_{ij} = \sigma_{ij} - \delta_{ij} \sigma_m.$$

The equivalent strain to failure $\bar{\epsilon}^f$, was also estimated from Bridgman's analysis.

The influence of notch severity on failure strains is seen in Fig. 3 for Q1 (HY80) steel and in Fig. 4 for Swedish iron (a commercial low carbon iron with higher inclusion content). Comparisons of these data with the predictions of the void growth models of McClintock [18] and Rice and Tracey [19] were made.

The empirical constants α and β were introduced to the theories and were adjusted so that the curves generated by the expressions most closely coincided with the experimental data. Mackenzie's modification of McClintock's equation is

$$\bar{\epsilon}^f = \frac{(1-n)\alpha}{\sin h[\sqrt{3}(1-n) \sigma_m/\bar{\sigma}]}$$

where n is the index of workhardening in the equation $\bar{\sigma} = \sigma_0 (\bar{\epsilon}^p)^n$.

The modified equation of Rice and Tracey was given as

$$\bar{\epsilon}^f = \beta \exp\left[-\frac{3}{2} \frac{\sigma_m}{\bar{\sigma}}\right].$$

Although these equations predict the stress state dependence of failure strain for Q1 steel rather well, they predict the

stress state dependence of failure strain poorly for Swedish iron. Mackenzie suggested that in Swedish iron voids may be initiated at large strains and that this strain may be represented by $\bar{\epsilon}_i$ in the equation

$$\bar{\epsilon}^f = \bar{\epsilon}_i + \beta \exp\left[-\frac{3}{2} \frac{\sigma_m}{\sigma}\right].$$

This equation predicts the stress state dependence of Swedish iron well but the physical significance of the empirical constants is not distinct enough for the approach to have importance. We know that there is no unique strain at which voids initiate.

C. Void Coalescence

The apparent mechanisms by which adjacent voids coalesce fall into two categories. The original concept was that voids joined by continuing plastic flow, reducing the cross-section of the ligament between the voids to zero. A second possibility is the activation of a mechanical instability between voids. As void size becomes comparable to void spacing, the possibility arises of a localization of plastic flow in a band between the voids. The intervening matrix in this band would undergo much larger local strains than the remote matrix. Under fully plastic conditions, stress fields between voids can interact even when the voids are several diameters apart and elevate the mean stress above that in the remote matrix. Localization of shear and elevation of stress would lead to accelerated initiation, growth and coalescence of voids in the intervening matrix.

Evidence in the literature indicates that coalescence of voids frequently consists of localization of plastic flow in

bands of intense shearing deformation [8,23]. In Baker and Smith's [24] study of the ductile fracture of polycrystalline copper containing copper oxide inclusions, both modes of coalescence were noted. Coalescence occurred either by the reduction of the ligament to zero cross section by continuing plastic flow or by the localization of plastic flow in the matrix in directions approximately 45° to the tensile axis. In the work of Cox and Low [9], 18 Ni maraging steel exhibited coalescence by impingement of growing voids. In AISI 4340 however, the growth of voids which had been initiated at relatively large MnS inclusions was interrupted before impingement by initiation at the much smaller carbides, resulting in fine cracks connecting the larger voids. Very similar observations of microcracking between voids when the voids are several diameters apart have been reported in Q1 (HY80) steel by Mackenzie [22] who arrived at the conclusion that large inclusions markedly reduce the ductility of the material. These authors speculated that if the large voids were not present to amplify the local stresses and strains, the matrix would fail at higher overall strains.

Fracture surface topography has been shown to reflect the deformation history of the material close to the path of fracture. A model of dimple formation was proposed by Rogers [6] and further developed by Crussard [7] and Beachem [25]. Recently Beachem [27] presented evidence obtained by matching of fracture surfaces which indicates that mating dimples seldom have identical shapes. Several of the proposed modes of dimple formation are given in Fig. 5A. Other modes of dimple formation

were predicted but have not been confirmed experimentally. The modes of loading designated by I, II and III in Fig. 5 refer to local directions of loading and not necessarily to macroscopic loading. Column 3 of Fig. 5A shows the mating shapes of the dimples before the fracture surfaces are opened up, with the dimples being shown as solid lines on the bottom surface and as dashed lines on the top surface. The fracture surface appearance is given in column 4. Fig. 5B shows a progression from pure mode II loading on the left to pure mode I loading on the right. The vertical arrows depict normal motion and the slanted arrows depict shear motion.

Broek [11] has shown that misinterpretation of dimple types is easily possible if the normal to the area of the fracture surface under examination forms a large angle with the optical axis of the electron microscope. Under these conditions equiaxed dimples, for example, could appear as parabolic dimples. Tilting is required before the dimple types can be clearly distinguished.

CHAPTER III

Literature Review: The Analysis of Ductile FractureA. Void Initiation

1. Assumptions

The mechanics of void initiation of inclusions has received little attention although the problem of determining the stresses and strains in and near a non-yielding inclusion in a ductile matrix is certainly within the capability of existing analysis methods. The use of the continuum plasticity approach in the study of void initiation is justified for cases in which the particles of interest are greater than or equal to about 1μ in size. For particles much smaller than 1μ the dislocation structure about a rigid particle in a deforming matrix has been observed to be highly organized, at least for small strains. Fig. 6 shows the rows of prismatic loops formed near 0.05μ alumina particles in Cu 30% Zn deformed 6% which were observed in the study of Hirsch and Humphreys [28]. If initiation occurs in the presence of such an organized dislocation structure, the analysis for initiation must necessarily be based on dislocation rather than continuum principles. However, in many commercial alloys the particles responsible for initiation are large, and dense dislocation tangling about the particles is observed prior to initiation. Void initiation is commonly the result of large plastic flow on a microscale, supporting the use of continuum solutions.

Criteria for particle fracture and decohesion are usually formulated in terms of distance, strain, stress, and energy parameters. However the physical significance of these is frequently

unclear. Consequently it is difficult to test the theories conclusively. It was pointed out in Chapter II that it is not known whether interface debonding occurs at a critical value of normal stress, shear stress or strain, or a combination of these or other variables. Likewise, it is not known whether a critical elastic strain energy or a critical tensile stress criterion can be used to predict particle cracking.

The most straightforward criterion for void initiation is the occurrence of particle cracking or decohesion when the maximum radial stress at the interface reaches a critical value. This simplicity arises from the assumption that void initiation does not depend on the stress and strain history of the deforming matrix very close to the particle. Inclusions generally possess larger elastic moduli than the surrounding matrices and are stronger in all but special instances. Alumina and silica particles, for example, are extremely strong and possess shear moduli ten times that of aluminum or copper. When rigid inclusions are dispersed in deforming media the differences between the moduli and strengths lead to amplification of stress and strain at the inclusions.

Complete plasticity solutions which incorporate arbitrary deformation of a strain hardening matrix about particles are not available. Rhee and McClintock, [29] however, have provided experimental evidence to support the hypothesis that if plastic deformation is not confined to small bands, the strain concentrations in hardening media are bounded by the limiting idealizations of non-hardening rigid plasticity and linear hardening plasticity. Argon et al. [30] have made similar assumptions concerning stress

concentrations in hardening media.

When considering linear hardening plasticity, $\bar{\sigma} = H/d\bar{\epsilon}^P$, elastic solutions are of interest because deformation plasticity theories which specify linear hardening are actually linear elasticity theories. This equivalence can be established by noting that if the ratios and directions of the components of plastic strain increments remain constant, and if no unloading is permitted, then the incremental relationships

$$\begin{aligned} S_{ij} &= \sigma_{ij} - \delta_{ij} \sigma_m \\ d\epsilon_{ij} &= \frac{3}{2} \frac{d\bar{\epsilon}}{\bar{\sigma}} S_{ij} & \begin{aligned} S_{ij} &= \sigma_{ij} - \delta_{ij} \sigma_m \\ d\bar{\epsilon} &= \sqrt{\frac{2}{3}} \frac{d\epsilon_{ij}^P}{d\epsilon_{ij}^P} \\ \bar{\sigma} &= \sqrt{\frac{3}{2}} S_{ij} S_{ij} \end{aligned} \end{aligned} \quad 3-1$$

can be integrated. The result,

$$\epsilon_{ij} = \frac{3}{2} \frac{\bar{\epsilon}}{\bar{\sigma}} S_{ij} = \frac{3}{2H} S_{ij} \quad 3-2$$

is identical with the stress-strain relations for an isotropic incompressible linear elastic material if $H = 3G$.

Complete elasticity solutions are available for circular cylindrical [31], elliptical cylindrical [32], and spheroidal inclusions (spherical [31], oblate and prolate [33]) in an infinite matrix. Many of these solutions have been compiled by Savin [34]. The model used in these treatments is restricted to the case of a single particle embedded in an infinite matrix with both matrix and particle assumed to be isotropic and elastic. The solutions to inclusion problems have been obtained in a general form which allows the elastic properties of the inclusion to be taken into account. If the inclusion has elastic properties similar to those

of the matrix, there is little disturbance to the stress field. The solution for a rigid inclusion gives an indication of the greatest disturbance that is likely to occur.

2. Rigid Circular Inclusions in a Linear Hardening Matrix

Consider the simplest case of a rigid circular cylindrical inclusion in the applied stress field, shown in Fig. 7, for a linear incompressible ($\nu = 1/2$) elastic matrix or a linear hardening plastic matrix. Assume that adhesion of the matrix to the inclusion is sufficiently strong so that there are no matrix displacements at the interface. Because the rigidity of the inclusion requires the axial strain to vanish this is a plane strain solution.

The expressions for the stresses due to the uniaxial stress at infinity are

$$\begin{aligned}\sigma_r &= \frac{\sigma_{y\infty}}{2} \left\{ 1 - \left[1 + 4\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4 \right] \cos 2\theta \right\} \\ \sigma_\theta &= \frac{\sigma_{y\infty}}{2} \left\{ 1 + \left[1 - 3\left(\frac{a}{r}\right)^4 \right] \cos 2\theta \right\} \\ \tau_{r\theta} &= \frac{1}{2} (\sigma_r - \sigma_\theta) = \frac{\sigma_{y\infty}}{2} \left[1 - 2\left(\frac{a}{r}\right)^2 \cos 2\theta \right] \\ \tau_{r\theta} &= \frac{\sigma_{y\infty}}{2} \left[-1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4 \right] \sin 2\theta .\end{aligned}\tag{3-4}$$

We need to obtain the interface stresses at inclusions in more general applied stress fields. We can find the stresses due to the remote applied stress $\sigma_{x\infty}$ by replacing θ in Eq. 3-4 with $\theta + 90^\circ$. Defining $\sigma_{x\infty}/\sigma_{y\infty} = \kappa$ and superposing the two solutions, we find that

$$\sigma_r = \kappa \sigma_{y\infty} + (1 - \kappa) \frac{\sigma_{y\infty}}{2} \left\{ 1 - \left[1 + 4\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4 \right] \cos 2\theta \right\} .\tag{3-5}$$

At the interface

$$\sigma_r(r=a) = \kappa \sigma_{y\infty} + (1 - \kappa) \frac{\sigma_{y\infty}}{2} \left\{ 1 - 2 \cos 2\theta \right\} \quad 3-6$$

The interfacial tensile stresses around the inclusion are plotted in Fig. 8 as a function of θ and κ . Interfacial tensile stress concentrations are maximized at $\theta = 90^\circ$ for $-1 \leq \kappa < 1$.

In triaxial tension ($\kappa = 1$), $\sigma_r^{\max}/\sigma_{y\infty} = 1$. In uniaxial tension ($\kappa = 0$), $\sigma_r^{\max}/\sigma_{y\infty} = \frac{3}{2}$. In pure shear ($\kappa = -1$), $\sigma_r^{\max}/\sigma_{y\infty} = 2$.

Because our aim is to investigate stress concentrations near inclusions in plastically flowing matrices, it is most instructive to restrict our attention to combinations of applied stress which give the same values of effective stress, $\bar{\sigma}$. When $\kappa = -1$, let $\sigma_{y\infty} = -\sigma_{x\infty} = \tau$, the current flow stress in shear. The interfacial stress for any remote state of stress can be generated by superposing a mean stress on the interfacial stress due to shear, so that

$$\sigma_r^{\max} = 2\tau + \sigma_m. \quad 3-7$$

3. Rigid circular Inclusions in a Rigid Nonhardening Matrix.

The idealization of nonhardening rigid plasticity places a second bound on stress and strain concentrations in hardening media. Unfortunately, the only solution for a rigid inclusion in a rigid plastic matrix which is available is that of McClintock [35], for the case of longitudinal shear (shear strain applied parallel to the inclusion axis), shown in Fig. 9. The most important feature of this plastic limit analysis is that although the radial stress on the interface is equal to the yield strength of the matrix, strain

concentrations in the matrix near the inclusion can become infinite. McClintock argued that similar strain concentrations could occur near rigid inclusions in plane strain deformation and proposed that in materials with very low strain hardening rates, void initiation by matrix decohesion may follow a critical local strain criterion. Further use of this line of thinking has not been made.

Limited progress has been made in determining the stress and strain states near non-yielding inclusions in rigid nonhardening matrix using finite difference and finite element methods. A numerical study considering the plane deformation of a rigid plastic nonhardening matrix about a rigid circular cylindrical inclusion has been performed by Argon, et al. [30]. Attention was restricted to pure shear deformation in the plane normal to the inclusion axis. Contrary to the expectation of McClintock that strain concentrations could become infinite, these authors found maximum interfacial shear strain concentrations of about 1.1. The maximum normal stress at the interface was 1.5 times the yield strength in shear and occurred in the principal tensile direction.

A solution for the more general problem of the plane deformation of an elastic plastic workhardening matrix about a cylindrical inclusion has been presented by Orr and Brown [36]. Even in the limit of a nonhardening matrix, no steady state values of interfacial stress concentration were calculated for plastic strains in the remote matrix as great as 16 times the yield strain.

Accepting the result of Argon that the maximum normal stress at the interface of a rigid cylinder in a nonhardening material is

$\frac{3}{2} \tau_0$ for plane strain, and recalling Eq. 3-7 for linearly hardening materials, we can say that for materials of intermediate work hardening characteristics

$$\sigma_m + \frac{3}{2} \tau_0 \leq \sigma_r^{\max} \leq 2\tau + \sigma_m. \quad 3-8$$

This is a statement of Argon's hypothesis that stress concentrations in strain hardening materials can be bounded by those in rigid perfectly plastic materials and by those in linearly hardening materials.

4. Particle Size Effects.

In continuum analyses, stress concentrations at a particle of a particular shape are independent of the size of the particle. Consequently, continuum models predict that small particles initiate voids at the same values of imposed stress and strain as large particles. Most, but not all, experimental observations indicate that the large particles fail first.

One possible explanation for this discrepancy is that the particle or the particle-matrix interface contains pre-existing flaws. Weibull's probabilistic treatment of brittle strength [37] predicts that for bodies of differing sizes with a given stress field, the larger the body the higher the probability of finding a critical flaw. Although the Weibull approach agrees qualitatively with experimental observations, there are insufficient data to test the hypothesis.

It is also possible to predict a particle size effect employing energy criteria for initiation. It has been proposed that the elastic strain energy stored in the particle during straining and

released when the particle fractures must be great enough to provide the energy of the resulting crack surface. Gurland and Plateau [38] have formulated the criterion:

$$1/2 \frac{(q\sigma)^2}{E} V > \gamma A \quad 3-9$$

where σ is the applied normal stress, q is the maximum elastic stress concentration factor at the inclusion, V is the inclusion volume, E is Young's modulus and A is the area of the newly created crack surface. For particle fracture,

$$\gamma = 2 \gamma_p$$

where γ_p is the surface energy for the particle. For interface separation

$$\gamma = \gamma_m + \gamma_p - \gamma_{mp}$$

where γ_m is the surface energy of the matrix and γ_{mp} is the relative interfacial energy. For spherical particles A/V would be proportional to $1/d$, so that the stress necessary for void initiation would be inversely proportional to the square root of the particle diameter. According to this approach, large inclusions would fail at a lower applied stress than small inclusions.

Several objections to the criterion of Gurland and Plateau can be raised. It is only a necessary condition for initiation and it does not take into account the matrix contribution to the release of strain energy. In addition, the possibility of plastic accommodation in the matrix was not considered.

5. Dislocation Mechanisms

There have been several attempts to formulate void initiation

criteria based on dislocation models. Most notably, Ashby [39] has proposed a dislocation mechanism and formulated a criterion for decohesion of the matrix from a non-deformable spherical inclusion. The matrix surrounding the particle is subjected to a shearing deformation as shown in Fig. 10A. If the particle had possessed elastic and plastic properties identical to those of the matrix, compatibility with the matrix would have required a cross section of the particle to assume an ellipsoidal shape. But since the particle is rigid the distortions are suppressed, giving rise to tensile stresses at the poles and compressive stresses at the equator. The stresses exerted on the matrix by the particle eventually become sufficient to cause plastic flow in the matrix. Ashby proposed that these stresses would be relieved by interstitial prismatic loops being punched into the matrix at the particle equator and by vacancy loops being formed at the particle poles. Ashby's result for the stress exerted on the interface from a pile-up of n prismatic loops is

$$\sigma = \alpha \frac{\epsilon d}{bL} \quad 3-12$$

where ϵ is the matrix strain; d is the particle diameter, b is the magnitude of the Burgers vector, L is the pile-up size and α is a constant. For the case in which decohesion occurs at a critical value of interfacial stress, σ_c ,

$$\epsilon_c = 1/\alpha \frac{bL}{d} \sigma_c \quad 3-13$$

The strain required for decohesion is predicted to vary inversely with the particle size. An important consequence of Ashby's analysis is the prediction that decohesion will occur regardless of the sign of the applied stress at infinity. For example, uniaxial compression produces the same strain pattern as uniaxial tension, rotated by 90° . The more likely case is that the elastically induced interfacial stress augments the slip-induced interfacial stress.

Ashby's model is particularly interesting because it predicts decohesion in terms of the strain history of the material. An objection to Ashby's mechanism is that it requires slip by prismatic loops, which can only be caused by high stresses. Broek [11] has suggested a minor modification of Ashby's model based on the accumulation of prismatic loops near particles by the cross slip of primary dislocations. Broek has also suggested that the development of high stresses at and within inclusions by direct impingement of dislocation pile-ups leads to decohesion or particle fracture. The development of high stresses by direct impingement of pile-ups on a particle is unlikely in metals which possess many equivalent slip systems. In deformed dispersion hardened alloys after strains of a few percent, dislocation tangles are observed near particles. It is arguable whether or not these dense tangles are effectively the same as the idealized arrays.

Initiation criteria employing dislocation models have the weakness of requiring over-simplification to evaluate the parameters in the problem. These criteria do have the advantage of predicting a size effect, which is more difficult to explain on the basis of

continuum models.

B. Void Growth

1. Assumptions

Once a void has been initiated its free surface acts as a dislocation sink, causing the void to expand. Continuum plasticity solutions are justified by the relatively large plastic strains and regular nature of the flow observed experimentally around growing voids. The regular flow around small silica particles in copper is apparent in Fig. 1, from Palmer and Smith [8]. In order to formulate ideal ductile fracture criteria it would be necessary to describe the large changes in void size, shape and orientation under arbitrary imposed deformation for arbitrary material behavior. The ideal analysis of ductile fracture would also address the crucial matters of interactions between neighboring voids and the unstable coalescence of neighboring voids or void arrays. The general solution is exceedingly difficult.

McClintock and co-workers [18, 40] have focused attention on a few extremes of material behavior, including nonhardening plasticity and linear hardening plasticity, and have interpolated between these extremes to give insight into intermediate behavior. McClintock's basic postulate is that void growth behavior of elliptical holes in moderately strain hardening materials lies between the limiting void growth solutions in nonhardening and linear hardening plasticity. The solution for linear hardening plasticity can be obtained but the solution for nonhardening plasticity must be estimated.

2. Circular Cavities in a Linear Hardening Matrix

Elastic solutions for the stress and strain distributions around

cylindrical holes under axisymmetric stress, Fig. 11, are well known. Incompressible elastic solutions can be used directly to determine void growth in linear hardening materials if deformation theory is assumed to apply for the plastic material. For the case in which the strain in the axial direction is a constant, the axial strain does not affect the radial or tangential components of stress and the axial stress is a constant. The components of stress are [41]

$$\begin{aligned}\sigma_r &= \sigma_{r\infty} \left(1 - \frac{a^2}{r^2}\right) \\ \sigma_\theta &= \sigma_{r\infty} \left(1 + \frac{a^2}{r^2}\right) \\ \sigma_z &= H\epsilon_z + \sigma_{r\infty}\end{aligned}\tag{3-14}$$

where, as before, $H = \frac{d\bar{\sigma}}{d\bar{\epsilon}}$.

The stress-strain relations are

$$\begin{aligned}\epsilon_r &= \frac{1}{H} \left[\sigma_r - \frac{1}{2} (\sigma_z + \sigma_\theta) \right] \\ \epsilon_\theta &= \frac{1}{H} \left[\sigma_\theta - \frac{1}{2} (\sigma_r + \sigma_z) \right].\end{aligned}\tag{3-15}$$

At the free surface

$$\begin{aligned}\sigma_r(r = a) &= 0 \\ \sigma_\theta(r = a) &= 2 \sigma_{r\infty} \\ \sigma_z &= H\epsilon_z + \sigma_{r\infty}\end{aligned}$$

so that

$$\epsilon_{\theta a} = \frac{3}{2H} \sigma_{r\infty} - \frac{\epsilon_z}{2}.\tag{3-16}$$

Far from the hole both the radial stress and tangential stress reach the value of the applied biaxial stress at infinity,

$$\sigma_{\theta\infty} = \sigma_{r\infty}.$$

The radial and tangential strains at infinity become equal so by incompressibility

$$\epsilon_{\theta\infty} = \epsilon_{r\infty} = -\frac{\epsilon_z}{2}.$$

From the stress-strain relations, then,

$$\epsilon_{r\infty} = \frac{1}{2H} \left[\sigma_{r\infty} - \sigma_{z\infty} \right].$$

Substituting into Eq. 2-16 we find that

$$\epsilon_{\theta a} = \ln \frac{a}{a_0} = \epsilon_{r\infty} \left[\frac{3\sigma_{r\infty}}{\sigma_{r\infty} - \sigma_{z\infty}} + 1 \right] \quad 3-17$$

which is Eq. 21 in McClintock's paper. Because we are restricted to radial loading in stress space, $\sigma_{r\infty}/\sigma_{z\infty} = \text{constant} = \alpha$, we can write

$$\frac{1}{a} \frac{da}{d\epsilon_{r\infty}} = \left[\frac{3\alpha}{\alpha-1} + 1 \right]. \quad 3-18$$

The relative hole growth is amplified over the remote strain, by a factor which depends on the ratio of the remote stresses. This factor is plotted in Fig. 12 as a function of α . Of course, infinite hole growth does not occur at $\alpha = 1$ because

$$\lim_{\frac{\sigma_{r\infty}}{\sigma_{z\infty}} \rightarrow 1} \ln \frac{a}{a_0} = \lim_{\frac{\sigma_{r\infty}}{\sigma_{z\infty}} \rightarrow 1} \left(\frac{1}{2H} \right) (\sigma_{r\infty} - \sigma_{z\infty}) \left[\frac{3\sigma_{r\infty}}{\sigma_{r\infty} - \sigma_{z\infty}} + 1 \right] = \frac{3\sigma_{r\infty}}{2H}$$

3. Circular Cavities in a Rigid Nonhardening Matrix

Most metals are more closely approximated by a rigid plastic

idealization than by a linear hardening idealization. Although solutions for circular holes in non-hardening materials are not known in general, an exact solution is available for circular holes under axisymmetric deformation [18].

Consider again the circular cylindrical hole in Fig. 11. Assume that the material is rigid, perfectly plastic, and is of the Mises type. The uniform axial strain increment is $d\epsilon_z$. If u is the radial displacement,

$$d\epsilon_\theta = \frac{du}{r}$$

$$d\epsilon_r = \frac{d(du)}{dr}$$

Combining these with the condition of incompressibility and solving,

$$d\epsilon_r + d\epsilon_\theta + d\epsilon_z = 0$$

$$\frac{d(du)}{dr} + \frac{du}{r} + d\epsilon_z = 0$$

$$du = \frac{A}{r} - \frac{rd\epsilon_z}{2}$$

Far from the hole, $d\epsilon_{r\infty} = d\epsilon_{\theta\infty} = -\frac{d\epsilon_z}{2}$ because $d\epsilon_z$ is not a function of r . At the inner radius the tangential strain increment is $d\epsilon_{\theta a}$, so that

$$du(r=a) = a d\epsilon_{\theta a} \approx \frac{A}{a} + a d\epsilon_{r\infty}$$

or

$$A = a^2(d\epsilon_{\theta a} - d\epsilon_{r\infty})$$

We may then write the radial and tangential strain increments in the form

$$d\epsilon_{\theta} = \frac{a^2}{r^2} (d\epsilon_{\theta a} - d\epsilon_{r\infty}) + d\epsilon_{r\infty}$$

3-19

$$d\epsilon_r = -\frac{a^2}{r^2} (d\epsilon_{\theta a} - d\epsilon_{r\infty}) + d\epsilon_{r\infty}$$

The equivalent strain increment

$$d\bar{\epsilon} = \left\{ \frac{2}{9} \left[(d\epsilon_r - d\epsilon_{\theta})^2 + (d\epsilon_r - d\epsilon_z)^2 + (d\epsilon_{\theta} - d\epsilon_z)^2 \right] \right\}^{1/2}$$

then becomes

$$d\bar{\epsilon} = \left[\frac{4}{3} \left(\frac{A}{r^2} \right)^2 + 4 d\epsilon_{r\infty}^2 \right]^{1/2}$$

The stress/incremental strain relations are

$$d\epsilon_r = \frac{d\bar{\epsilon}}{\sqrt{3} \tau_0} \left[\sigma_r - 1/2 (\sigma_{\theta} + \sigma_z) \right]$$

3-20

$$d\epsilon_{\theta} = \frac{d\bar{\epsilon}}{\sqrt{3} \tau_0} \left[\sigma_{\theta} - 1/2 (\sigma_r + \sigma_z) \right]$$

Taking the difference $d\epsilon_r - d\epsilon_{\theta}$ leads to

$$(\sigma_r - \sigma_{\theta}) = \frac{2\sqrt{3} \tau_0}{3 d\bar{\epsilon}} (d\epsilon_r - d\epsilon_{\theta})$$

3-21

Combining with the equilibrium equation,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_{\theta}}{r} = 0$$

3-22

we find that

$$\frac{d\sigma_r}{dr} = \frac{2\sqrt{3} \tau_0}{3 d\bar{\epsilon}} \frac{1}{r} (d\epsilon_{\theta} - d\epsilon_r)$$

3-23

By substituting from the expressions for the strain increments, Eq. 3-19, we obtain

$$d\sigma_r = 2 \tau_0 \frac{dr}{r^3} \left[\frac{1}{r^2} + 3 \left(\frac{d\epsilon_{r\infty}}{A} \right)^2 \right]^{-1/2}.$$

Integrating this relation from the free boundary where $\sigma_r = 0$ to infinity where $\sigma_r = \sigma_{r\infty}$, we find

$$\sigma_{r\infty} = \tau_0 \sinh^{-1} \left[\frac{1}{\sqrt{3}} \left(1 - \frac{d\epsilon_{\theta a}}{d\epsilon_{r\infty}} \right) \right].$$

Inverting and rearranging,

$$\frac{d\epsilon_{\theta a}}{d\epsilon_{r\infty}} = \left[\pm \sqrt{3} \sinh \frac{\sigma_{r\infty}}{\tau_0} + 1 \right] \quad 3-24$$

where the + sign applies for the case $d\epsilon_{r\infty} > 0$ (i.e., $\sigma_{r\infty} - \sigma_{z\infty} = \sqrt{3} \tau_0$) and the - sign applies for the case $d\epsilon_{r\infty} < 0$ (i.e., $\sigma_{z\infty} - \sigma_{r\infty} = \sqrt{3} \tau_0$).

Fig. 13 is a plot of

$$\frac{d\epsilon_{\theta a}}{d\epsilon_{r\infty}} = \frac{1}{a} \frac{da}{d\epsilon_{r\infty}} \quad \text{versus} \quad \frac{\sigma_{r\infty}}{\tau_0}$$

for $\sigma_{r\infty} - \sigma_{z\infty} = \sqrt{3} \tau_0$ and for $\sigma_{z\infty} - \sigma_{r\infty} = \sqrt{3} \tau_0$.

4. Elliptical Cavities in a Linear Hardening Matrix

In the investigation of the mechanics of ductile fracture, the study of elliptical holes and their growth under arbitrary biaxial loading is more pertinent than the study of circular cylindrical holes under axisymmetric loading. In connection with the growth of elliptical cavities in linear hardening materials McClintock has examined Berg's solution [42] for the growth of elliptical cavities

in an incompressible linear viscous material. The stress strain rate equations for an incompressible linear viscous material are

$$\dot{\epsilon}_{ij} = \frac{1}{2\eta} S_{ij} \quad 3-25$$

where η is the coefficient of viscosity and the dot denotes differentiation with respect to time. In an incompressible linear elastic material

$$\epsilon_{ij} = \frac{1}{2G} S_{ij} \quad 3-26$$

According to Rayleigh's analogy, if the same boundary tractions are specified on linear viscous and linear elastic bodies of the same shape, the stress fields will be identical. Accordingly, strain rates in the viscous body can be obtained from strains in the elastic body by substituting η for G in Eq. 3-26.

To understand McClintock's work we need to establish a connection between linear hardening plasticity and linear viscosity. McClintock transformed Berg's viscous solution to a plastic solution by making the substitution,

$$\frac{t}{\eta} = \frac{3\bar{\epsilon}}{\bar{\sigma}}$$

Such a substitution means that McClintock chose to let total viscous strains (assuming constant stress) be analogous to total plastic strains (assuming deformation theory). To see this, note that for S_{ij} not a function of time, we may integrate Eq. 3-25 to obtain

$$\epsilon_{ij}^v = \frac{t}{2\eta} S_{ij} \quad 3-28$$

Deformation plasticity theory implies that we can integrate the stress incremental strain relations to obtain

$$\epsilon_{ij}^p = \frac{3}{2} \frac{\bar{\epsilon}}{\bar{\sigma}} S_{ij}.$$

If ϵ_{ij}^p is analogous to ϵ_{ij}^v , it would at first seem as though we could write $\frac{t}{\eta} = \frac{3\bar{\epsilon}}{\bar{\sigma}}$. If we write

$$\frac{t}{2\eta} S_{ij} = \frac{3}{2} \frac{\bar{\epsilon}}{\bar{\sigma}} S_{ij}, \quad 3-29$$

we see on closer examination that on the left side of Eq. 3-29, S_{ij} is required to be constant during the interval of time in which viscous deformation occurs. On the right side of Eq. 3-29, however, if S_{ij} is constant, no plastic deformation can occur. In addition, for a linear hardening material obeying deformation theory, $\bar{\sigma} = H\bar{\epsilon}$ so that McClintock's substitution becomes

$$\frac{t}{\eta} = \frac{3}{H}$$

which means that time does not proceed. We must conclude that an analogy between viscous and plastic deformation cannot be formulated in this way.

Assuming linear hardening deformation plasticity theory to apply, it is more reasonable to omit all reference to linear viscosity and allow total plastic strains to be analogous to elastic strains with $H = 3G$, as before.

Consider an elliptical hole in an isotropic elastic material in an infinite plane subject to remote biaxial inplane tensions $\sigma_{1\infty}$ and $\sigma_{2\infty}$, Fig. 14. The remote principal stresses are tilted at

a fixed angle β from the principal axes of the ellipse.

Muskhelishvili [43] has solved the problem by mapping conformally the exterior of an ellipse in the z plane to the exterior of a unit circle in the ζ plane. The biharmonic equation was solved in terms of two analytic functions of a complex variable. Using standard elasticity relations, the conditions of a free surface at the boundary and of biaxial tension at infinity were imposed. The stress function $\phi(\zeta)$ which satisfied this condition is

$$\phi(\zeta) = \frac{R}{4} \left[(\sigma_{1\infty} + \sigma_{2\infty})\zeta + \frac{2(\sigma_{1\infty} - \sigma_{2\infty})e^{2i\beta} - (\sigma_{1\infty} + \sigma_{2\infty})m}{\zeta} \right] \quad 3-30$$

where $R = \frac{a+b}{2}$ and $m = \frac{a-b}{a+b}$. For plane strain, the elastic free surface displacements can be shown to be

$$U + iV = \frac{1}{G} \phi(e^{i\theta}),$$

where U and V are displacements in the x and y directions, respectively. A straightforward check verifies that the deformed shape of the initially elliptical free boundary is also an ellipse. The boundary displacements in the corresponding plastic problem are

$$U + iV = \frac{3\bar{\epsilon}}{\sigma} \cdot \frac{R}{4} \left[(\sigma_{1\infty} + \sigma_{2\infty})e^{i\theta} + \frac{2(\sigma_{1\infty} - \sigma_{2\infty})e^{2i\beta} - (\sigma_{1\infty} + \sigma_{2\infty})m}{e^{i\theta}} \right] \quad 3-32$$

The incremental change of the mean size, dR , of the ellipse with imposed equivalent strain increment, $d\bar{\epsilon}$, is

$$\begin{aligned}
\frac{dR}{d\bar{\epsilon}} &= \lim_{\bar{\epsilon} \rightarrow 0} \frac{R(\bar{\epsilon}) - R(0)}{\bar{\epsilon}} \\
&= \lim_{\bar{\epsilon} \rightarrow 0} \frac{(a-a_0) + (b-b_0)}{2\bar{\epsilon}} \\
&= \lim_{\bar{\epsilon} \rightarrow 0} \frac{U(1) + V(i)}{2\bar{\epsilon}} = \frac{3}{4} R \frac{(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}} .
\end{aligned}$$

Therefore,

$$\frac{dR}{d\bar{\epsilon}} = \frac{3}{4} R \frac{(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}}$$

and

$$\frac{dR}{R} = \frac{3}{4} d\bar{\epsilon} \frac{(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}} \quad 3-33$$

Integrating, and remembering that the loading is radial, we obtain

$$\ln \frac{R}{R_0} = \frac{3}{4} \frac{\bar{\epsilon}(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}} \cdot (\text{plane strain}) \quad 3-34$$

Similarly, because

$$\begin{aligned}
\frac{d(Rm)}{d\bar{\epsilon}} &= \lim_{\bar{\epsilon} \rightarrow 0} \frac{(a-a_0) - (b-b_0)}{2\bar{\epsilon}} \\
&= \lim_{\bar{\epsilon} \rightarrow 0} \frac{U(1) - V(i)}{2\bar{\epsilon}} ,
\end{aligned}$$

we see that

$$\frac{d(Rm)}{d\bar{\epsilon}} = \frac{3}{4} R \left[2(\sigma_{1\infty} - \sigma_{2\infty}) e^{2i\beta} - (\sigma_{1\infty} + \sigma_{2\infty}) m \right] \quad 3-35$$

Integrating, with Eq. 3-33, we obtain

$$m = \left(\frac{\sigma_{1\infty} - \sigma_{2\infty}}{\sigma_{1\infty} + \sigma_{2\infty}} \right) e^{2i\beta} \left[m(0) - \left(\frac{\sigma_{1\infty} - \sigma_{2\infty}}{\sigma_{1\infty} + \sigma_{2\infty}} \right) e^{2i\beta} \right] \exp \left[- \frac{3}{2} (\sigma_{1\infty} + \sigma_{2\infty}) \frac{\bar{\epsilon}}{\bar{\sigma}} \right] \quad 3-36$$

A solution for generalized plane strain can be obtained by superposing a uniform stress σ_3 on the body. The resulting fractional change in hole size is $-\epsilon_3/2$. From incompressibility

$$-\frac{\epsilon_3}{2} = \frac{\epsilon_1 + \epsilon_2}{2}$$

so that

$$\ln \frac{R}{R_0} = \frac{3}{4} \frac{\bar{\epsilon}(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}} + \frac{(\epsilon_1 + \epsilon_2)}{2} \quad \text{(generalized plane strain)} \quad 3-37$$

These equations for R and m are identical to McClintock's equations 19 and 20, respectively. Because infinitesimal strain tensors were used in the Muskhelishvili formulation, it is apparent that the use of these expressions to describe the large ductile expansion of holes in linear hardening plastic materials is not justified. Further, the use of deformation theory requires the loading at each material point to be proportional in stress space, which will most certainly not be the case near the ellipse.

Considering plane strain solutions and remembering that loading is radial, we let $\alpha = \sigma_{1\infty}/\sigma_{2\infty}$

so that

$$\sigma_{3\infty} = \frac{1}{2} (\sigma_{1\infty} + \sigma_{2\infty}) = \frac{\sigma_{1\infty}}{2} (1 + \alpha)$$

and

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \sigma_{1\infty} |1 - \alpha|$$

Combining these with the expression

$$\frac{1}{R} \frac{dR}{d\epsilon} = \frac{3}{4} \frac{(\sigma_{1\infty} + \sigma_{2\infty})}{\bar{\sigma}}$$

we obtain

$$\frac{1}{R} \frac{dR}{d\bar{\epsilon}} = \frac{3}{2} \frac{(1 + \alpha)}{|1 - \alpha|} . \quad 3-38$$

A plot of $\frac{1}{R} \frac{dR}{d\bar{\epsilon}}$ as a function of $\alpha = \sigma_{1\infty}/\sigma_{2\infty}$ is given in Fig. 15.

5. Elliptical Cavities in a Rigid nonhardening Matrix

Solutions for the growth of elliptical cavities in nonhardening rigid plastic materials under remote biaxial tension have not been obtained. We see from Table 3-1 that the solution for circular cavities in non-hardening rigid materials, Eq. 3-24, can be obtained from the solution for circular cavities in linear hardening materials, Eq. 3-17, by replacing $\frac{\sqrt{3} \sigma_{r\infty}}{\sigma_{r\infty} - \sigma_{z\infty}}$ with $\sinh \frac{\sqrt{3} \sigma_{r\infty}}{\sigma_{r\infty} - \sigma_{z\infty}}$. McClintock argued that an analogous transformation for the solution for elliptical cavities in linear hardening materials, Eq. 3-37, would lead to a reasonable estimate for elliptical cavities in non-hardening rigid materials, Eq. 3-39.

Intermediate behavior was determined by using n , the work hardening index in the empirical relation $\bar{\sigma} = \sigma_0 \bar{\epsilon}^n$, to interpolate between the extremes of nonhardening ($n = 0$) and linear hardening ($n = 1$) plasticity. We see that Eq. 3-40 and Eq. 3-41 reduce to the correct nonhardening and linear hardening solutions in the respective limits of $n \rightarrow 0$ and $n \rightarrow 1$. (Recall that $\lim_{\alpha \rightarrow 0} \frac{\sinh \alpha}{\alpha} = 1$.)

7. Elliptical Cavities Under Shear Deformation and Spherical Cavities Under Uniaxial Deformation.

The growth of holes may be accelerated by the localization of applied deformation into thin bands either as a result of strain softening or mechanical instability due to the growth of holes

themselves. McClintock, Kaplan and Berg [40] have extended the McClintock analysis for non-rotational flow fields to hole growth in shear bands. In shear deformation each element rotates with respect to the principal axes of stress. The mean radius of holes in shear bands was found to increase according to

$$\frac{1}{R} \frac{dR}{d\gamma} = \frac{1}{2(1-n)} \sinh \left[(1-n) \frac{\sigma}{\tau} \right] \quad 3-44$$

where τ is the shear stress in the band, σ is the normal stress across it and n is the strain hardening coefficient in $\tau = \tau_0 \gamma^n$.

Because of the difficulty of obtaining exact plasticity solutions other than those for spheres and cylinders under symmetric loading, Rice and Tracey [19] used an energy formulation based on the Rayleigh-Ritz variational method to obtain a solution for the growth of initially spherical holes under uniaxial deformation. A non-hardening Mises material was considered. For moderately high triaxiality, their results show that the dilatational growth dominates and there is little change in shape of the void. An expression given for high triaxiality which describes the numerical results closely is

$$\frac{1}{R} \frac{dR}{d\bar{\epsilon}} = 0.283 \exp \left(\frac{\sqrt{3}}{2} \frac{\sigma_m}{\tau_0} \right) . \quad 3-45$$

McClintock [44] has made an order of magnitude estimate for ellipsoidal holes from his analysis by taking the averages of the three principal stresses rather than the two transverse ones,

$$\frac{1}{R} \frac{dR}{d\bar{\epsilon}} = \frac{\alpha}{(1-n)} \sinh(1-n) \frac{\sigma_m}{\tau_0} , \quad 3-46$$

where R is the mean of the semiaxes and where α is a constant on the order of unity. In the nonhardening limit, McClintock's equation predicts a stress state dependence which is similar to that predicted by Rice and Tracey's equation, especially at high triaxiality.

8. Limitations of Continuum solutions for Small Deformations

The common feature of continuum solutions for hole growth in nonhardening media is that relative growth rates are amplified over imposed strain rates by a factor rising exponentially with the ratio of the mean normal stress to the yield strength. The second basic result is that the presence of workhardening decreases void growth rates for a given imposed strain rate. The limitations of these analyses are clear. They are based on the growth of isolated holes in infinite media and have assumed macroscopic incompressibility, so their use can be justified only for the case of holes which are very small compared to the hole spacing. The works of McClintock [18] and Rice and Tracey [19] have neglected void interactions altogether. An approximate method of accounting for interaction effects has been presented by Tracey [45]. The analysis of void growth should address the crucial matters of finite shape changes, interactions between neighboring voids and the unstable coalescence of neighboring voids or void arrays. This necessarily involves numerical formulations for large deformations. Some promising work of this type has been reported by Needleman.

9. Continuum Solutions for Large Deformations.

Needleman's work [46] is a finite element solution based on a variational principle for the large ductile expansion of a periodic

array of initially cylindrical voids in an elastic plastic power law hardening material. The array of voids which was considered, shown in Fig. 16, was periodic in the x^1 and x^2 directions. The uniaxial deformation of the array was studied under plane strain conditions. Needleman examined only deformations such that the straight lines of the type $\overline{EB}$ $\overline{BF}$ $\overline{FC}$ and $\overline{CE}$ remained straight after deformation. Due to the symmetry of the array and of the applied deformations, void interaction effects were accounted for rigorously.

One quadrant of a cell was strained in the x^2 direction. The overall tensile behavior of the material was calculated for differing combinations of initial hole spacing and material work hardening properties. Needleman's results for the deformed shape of a quadrant are given in Figs. 17, 18 and 19 for the three conditions:

$$\text{Fig. 17 } \frac{L_0}{R_0} = 2, \quad n = \frac{1}{4}, \quad \epsilon_2 = 0.318$$

$$\text{Fig. 18 } \frac{L_0}{R_0} = 2, \quad n = \frac{1}{8}, \quad \epsilon_2 = 0.231$$

$$\text{Fig. 19 } \frac{L_0}{R_0} = 4, \quad n = \frac{1}{8}, \quad \epsilon_2 = 0.318$$

where n is the work hardening coefficient in $\bar{\sigma} = \sigma_0 \bar{\epsilon}^n$.

The cross hatched regions superimposed on these figures are McClintock's hole growth predictions based on Eq. 3-41 and Eq. 3-43. Assuming Needleman's results to be nearly correct, this comparison indicates that for uniaxial deformation McClintock's equation severely underestimates the transverse rate of void growth.

Needleman also obtained results for the development of plastic zones about the voids under increasing overall strain. Plastic unloading was found to begin in the corner of the cell most distant

from the void and encompassed approximately one third of the cell at $\epsilon_2 = 0.3$. Significant error can arise in McClintock's analysis, which assumed deformation theory, as a result of this unloading.

The extension of Needleman's work to the biaxial deformation of an elastic plastic medium containing a periodic square array of cylindrical holes is a problem of major significance which has not yet been solved.

C. Void Coalescence

1. Assumptions

The final stage in the fracture of ductile metals is the coalescence of adjacent voids or void arrays. Rosenfield in his 1968 review [20] concluded that void coalescence is the least understood aspect of the fracture process. It is usually assumed that coalescence is a result of a mechanical instability of the ligaments between adjacent voids. Others have neglected mechanical instability and have assumed that voids grow uniformly prior to coalescence.

2. Mechanical Instability

The concept of void coalescence suggested by Cottrell [47] is, that of internal necking of the ligament between adjacent voids due to tensile plastic instability in the matrix. The problem of ligament instability has not been solved.

Needleman's work accounts for the unstable coalescence of neighboring voids, but the numerical procedure used proved to be inadequate well before coalescence occurred. Needleman used two procedures to obtain an approximation of the strains at coalescence. In the first the load-strain curves were extrapolated to zero load. A second estimate was obtained by estimating the strain at which

the remaining ligament vanished. Such estimates necessarily overestimate the fracture strain. Nevertheless, Needleman's estimates are considerably smaller than those determined from the McClintock analysis with no transverse stress. McClintock's equation does not predict normal fracture at all, but rather delamination.

Needleman's numerical formulation predicts dilatation due to finite void growth. Berg [48] has suggested that macroscopic plastic constitutive laws which properly incorporate dilatational strains could themselves predict ductile fracture. Usually, ductile fracture criteria are supplementary to the analysis of stress and strain. Berg's idea is that constitutive laws which include dilatation due to finite void growth may permit localization of flow in a narrow band. Flow localization could occur if material hardening in an increment of deformation exactly balanced softening due to void growth.

The idea of a criterion for tensile plastic instability of the ligament between voids has been developed by Thomason [49]. Consider an element of unit thickness, width δ and height δ , Fig. 20. The element contains a doubly periodic array of square prismatic holes. Plane strain deformation of this element is assumed and the matrix is a nonhardening Mises material.

If the cavities are separated by a distance many times their diameter, the critical net section stress, σ_n^* , required to cause localized flow in the region between cavities may not be reached before uniform flow occurs, so that internal necking is prevented. For this geometry the actual net section stress, σ_n , is related to the nominal stress, σ_1 , by the relation

$$\sigma_n(\delta - na) = \sigma_1 \delta ,$$

or

$$\sigma_1 = \sigma_n \left(1 - \frac{na}{\delta}\right) . \quad 3-47$$

For square cavities arranged in a square array the cavity volume fraction is

$$\frac{(na)^2}{\delta^2} = V_f . \quad 3-48$$

Recalling that for a Mises material in plane strain

$$\frac{\sqrt{3}}{2} (\sigma_1 - \sigma_2) = Y = \sqrt{3} \tau_0$$

or,

$$\sigma_1 = \sigma_2 + 2 \tau_0 . \quad 3-49$$

Substituting Eq. 3-48 and Eq. 3-49 into Eq. 3-47, we obtain

$$\sigma_2 + 2 \tau_0 = \sigma_n (1 - \sqrt{V_f}) .$$

Ductile fracture by internal necking will occur if

$$\frac{\sigma_2}{2\tau_0} + 1 \geq \frac{\sigma_n^*}{2\tau_0} (1 - \sqrt{V_f}) . \quad 3-50$$

Thomason's equation is slightly more complicated because he considers the effect of a superimposed hydrostatic pressure. If the inequality is not satisfied the cavities must grow until the constraint factor $\frac{\sigma_n^*}{2\tau_0}$ becomes sufficiently small to allow internal necking. Thomason derived an expression for $\frac{\sigma_n^*}{2\tau_0}$ in terms of an

upper bound solution for constrained flow, the value of $\frac{\sigma_n^*}{2\tau_0}$ closely approaching unity for $\frac{a}{b} \geq 1$.

We may apply Thomason's criterion for internal necking to the void array in Needleman's analysis if we take the transverse stress in Thomason's equation to be zero. Select for examination the particular case in which $\frac{L_0}{R_0} = 4$, $n = \frac{1}{8}$ and $\epsilon_1 = .182$, Fig. 21A. According to Needleman's calculations the array had reached a state of macroscopic tensile instability at $\epsilon_1 = .120$. The workhardening rate of the material at these strains is small ($\frac{d\bar{\sigma}}{d\epsilon} < E/200$) so that comparison with Thomason's theory (for nonhardening materials) is not unreasonable. We make the assumption that Thomason's half neck length a and half cavity spacing b are given as shown in Fig. 21A. The constraint factor corresponding to $a/b = 0.64$ is $\frac{\sigma_n^*}{2\tau_0} = 1.1$. The criterion then becomes

$$\frac{\sigma_n^*}{2\tau_0} \left(1 - \frac{c}{d}\right) = 1.1 \left(1 - \frac{17}{64}\right) = 0.803 < 1.$$

Therefore, the criterion for internal necking is satisfied and according to Thomason's approach subsequent deformation should be confined to the region between the voids. But in Fig. 21B, which is the same cavity at $\epsilon_1 = 0.318$ we see that substantial general deformation occurred between $\epsilon_1 = 0.182$ and $\epsilon_1 = 0.318$. The conclusion must be that even minute amounts of strain hardening postpones internal necking far beyond the point anticipated in Thomason's analysis.

3. Uniform Growth Prior to Coalescence

Criteria for coalescence that neglect mechanical instability

have been developed primarily by McClintock and his co-workers [18,40]. This scheme assumes uniform growth of voids prior to merge of their free surfaces. McClintock considered elliptical holes with axes in the z direction having the semi-axes of length a and b as in Fig. 22. Each hole is surrounded by a cell whose dimensions are of the order of the mean spacing between the holes. The spacings are l_a and l_b for this ideal array. As growth proceeds the ratio of hole size to hole spacing increases. A relative growth factor for transverse growth can be defined as

$$F_a = (a/l_a)/(a^0/l_a^0) \quad 3-51$$

where superscript zero denotes initial values.

McClintock determined an upper limit to fracture strain by assuming that each cavity would grow as if it were isolated and that the cells surrounding the cavities would deform as if the cavities were not present. Neither of these assumptions is reasonable as hole size approaches the cell size. A more reasonable approach might be to invoke an internal necking criterion, such as the Thomason criterion after some critical relative hole growth factor has been obtained.

Nevertheless fracture certainly would occur if a semi-axis of the ellipse became equal to the corresponding cell half size

$$a^f = \frac{l_a^f}{2},$$

so that

$$F_a^f \leq \frac{l_a^0}{2a^0}.$$

McClintock presented his results in terms of a cumulative damage parameter

$$d\eta_a = \frac{d \ln F_a}{\ln F_a} \quad 3-52$$

The parameter η accumulates to unity at coalescence. The growth rates are provided by Eqs. 3-41 and 3-43 and the hole spacings are given in terms of the appropriate component of applied strain.

The final result is

$$\frac{d\eta_a}{d\bar{\epsilon}} = \frac{1}{\ln F_a} \left\{ \frac{\sqrt{3}}{2(1-n)} \sinh \left[\frac{\sqrt{3}(1-n)}{2} \frac{(\sigma_{a\infty} + \sigma_{b\infty})}{\bar{\sigma}} \right] + \frac{3}{4} \frac{(\sigma_{b\infty} - \sigma_{a\infty})}{\bar{\sigma}} \right\} \quad 3-53$$

McClintock's objective in presenting Eq. 2-53 was to allow the summing up of damage over increments of strain during which the changes in the ratios of the stress components were small. At the outset in his analysis McClintock assumed deformation theory, which specifies radial loading. Therefore the intended function of Eq. 3-53 is inconsistent with this initial assumption. Given radial loading and assuming that the term $(\sigma_{b\infty} - \sigma_{a\infty})/\bar{\sigma}$ is small compared to the hyperbolic sine term (high triaxiality), Eq. 3-53 can be integrated to give

$$\bar{\epsilon}^f = \sqrt{\frac{2}{3}} \frac{(1-n) \ln F_a^f}{\sinh \left[\frac{3(1-n)(\sigma_{a\infty} + \sigma_{b\infty})}{2\bar{\sigma}} \right]} \quad 3-54$$

Mackenzie [22] further simplified this to

$$\bar{\epsilon}^f = \frac{(1-n)\alpha}{\sinh \left[(1-n)\sqrt{3} \frac{\alpha_m}{\bar{\sigma}} \right]}$$

for ellipsoidal holes, where σ_m is the mean stress.

A criterion for coalescence based on similar principles has been developed for failure by growth of holes in a shear band by McClintock, Kaplan and Berg [40].

CHAPTER IV

WORK IN PROGRESS

Obtaining definitive experimental results on the nature of ductile separation process on a microscale is crucial in the further development of fracture criteria. Most experimental investigations have remained either at the observational level or have been confined to documenting the dependence of fracture toughness or strain to fracture on inclusion concentration.

Fundamental experimental studies of ductile fracture require specimens with controlled states of stress, especially triaxiality, in regions where the strain distribution is known. In cracked structural metals, the extent of the plastic zone is invariably larger than the extent of the interinclusion spacing. Close to the crack tip, the plastic parts of the strain are large compared to the elastic. Fully plastic specimens may therefore be used to determine ductile fracture criteria, especially since the elastic-plastic distributions are not well understood near crack tips. Introducing the triaxial state of stress, which strongly affects fracture, requires grooves which introduce strain concentrations. Since the maximum triaxiality is subsurface, the strains at the point of initiation must often be inferred from overall extensions. In the double notch plane strain tensile specimen, strain varies gradually so that the effects of geometry changes and strain hardening are relatively unimportant and the stress and strain at the site of fracture initiation can be estimated reasonable well from overall measurements of force and deformation.

As part of this research we are conducting a study of the double notch plane strain tensile specimen using a finite element analysis for large deformations. Analysis is being performed for both large and small included notch angles considering widely varying material workhardening characteristics.

In any fundamental experimental study of ductile fracture there are several metallurgical obstacles which must be dealt with. Studies of ductile fracture in complex metallurgical systems necessarily involve complications due to the nucleation of voids throughout the loading history of the specimen. Incomplete decohesion and particle cracking both lead to an initial void size which is considerably less than the particle size. In most commercial materials inclusions are not uniformly distributed. There may even be systematic variations in spacing arising from the segregation of second-phase particles. Variations of particle size, shape and non-random variations of particle orientation make precise descriptions of microstructure additionally complex. Because these parameters are known to exert a profound influence on the ductility and toughness of alloys, their control is an important requirement for the quantitative study of ductile fracture. Solution treatment and aging usually provide suitable means for the control of morphology of intermetallic phases but the control of nonmetallic inclusions is more difficult.

Our experimental approach, which simultaneously reduces metallurgical complexity and controls the local distribution of stress and strain, is to embed non-adhering circular

cylindrical fibers in a periodic array in the double notch plane strain tensile specimen. Deformation takes place in the plane perpendicular to the cylinder axes. With a non-adhering combination, voids exist from the outset and growth of these voids occurs from the beginning of the test. In this case, confusion is eliminated as to whether events observed are consequences of void initiation or of void growth. This, in effect, separates the stages of fracture so that appropriate fracture criteria can be proposed.

Copper and graphite are essentially insoluble in each other and are non-reactive. The degree of interaction is extremely small because copper does not wet graphite even at temperatures as high as 1285°C. Contact angles approaching 180° have been reported for this combination. The nucleation of voids at secondary sites is suppressed by using a highly pure (99.999%) matrix.

These specimens are being constructed by placing an array of 10 μ graphite filaments between two carefully cleaned blocks of copper which are then placed in a mold. A continuous matrix is produced by melting in a vacuum resistance furnace while the filaments are held in place at their ends. The resulting cubical block is strained in its three principal directions and subsequently recrystallized. Further pre-straining determines the workhardening characteristics of the material during the test. Fig. 23 shows the appearance of the specimen after final machining. The experiments themselves consist of straining, sectioning and metallographically observing these specimens.

With these specimens we are testing the proposition that inclusion spacing is the dominant microstructural variable affecting ductile fracture. The quantitative effects of stress state and strain hardening on the rate of void growth are also being studied. Such a study has not been made for metals and it is not known whether the rates of void expansion predicted by the McClintock and Needleman approaches are realistic. Internal necking is being observed directly. Observation of fracture surface topography in a scanning electron microscope will eventually enable a correlation of the depth to breadth ratio of the dimples with stress and strain history and initial inclusion spacing.

In contrast to the complexity of usual metallurgical systems used in the study of ductile fracture, the initial state of this system is particularly simple and well characterized. Subsequent observations are being related directly to this known initial state and the number of postulates needed to formulate fracture criteria is therefore limited.

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NONHARDENING $n = 0$	QUENCHED STATE $0 \leq n \leq 1$	LINEAR HARDENING $n = 1$
$\ln \frac{a}{a_0} = \sqrt{3} \epsilon_r \sin \left(\frac{\sqrt{3} \sigma_{res}}{\sigma_{res} - \sigma_{res}^0} \right) + \epsilon_r$ <p>Eg 3-24 $\sigma_{res} - \sigma_{res}^0 = \sqrt{3} \tau_0$</p>	$\ln \frac{a}{a_0} = \sqrt{3} \epsilon_r \sin \left[\frac{(1-n) \sqrt{3} \sigma_r}{\sigma_{res} - \sigma_{res}^0} \right] + \epsilon_r$ <p>Eg 3-20</p>	$\ln \frac{a}{a_0} = \sqrt{3} \epsilon_r \left(\frac{\sqrt{3} \sigma_{res}}{\sigma_{res} - \sigma_{res}^0} \right) + \epsilon_r$ <p>Eg 3-17</p>
$\ln \frac{R}{R_0} = \frac{\sqrt{3}}{2} \left[\sqrt{3} \left(\frac{\sigma_a + \sigma_b}{2\sigma} \right) + \left(\frac{\epsilon_a + \epsilon_b}{2} \right) \right]$ <p>Eg 3-39</p>	$\ln \frac{R}{R_0} = \frac{\sqrt{3}}{2} \sin \left[\frac{(1-n) \sqrt{3} (\sigma_a + \sigma_b)}{2\sigma} \right] + \frac{\epsilon_a + \epsilon_b}{2}$ <p>Eg 3-41</p>	$\ln \frac{R}{R_0} = \frac{\sqrt{3}}{2} \left[\sqrt{3} \left(\frac{\sigma_a + \sigma_b}{2\sigma} \right) + \left(\frac{\epsilon_a + \epsilon_b}{2} \right) \right]$ <p>Eg 3-37</p>
$m = \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right) e^{2i\theta} + \left[m_0 \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right] \exp \left[- \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right]$ <p>Eg 3-42</p>	$m = \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^{2i\theta} + \left[m_0 \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right] \exp \left[- \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right]$ <p>Eg 3-38</p>	$m = \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right) e^{2i\theta} + \left[m_0 \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right] \exp \left[- \left(\frac{\sigma_{res} - \sigma_{res}^0}{\sigma_{res} - \sigma_{res}^0} \right)^2 \right]$ <p>Eg 3-36</p>

FIG. 1. VOID IN AN LATE STAGE OF DEFORMATION
 OF AN ANISOTROPIC POLYMER SPECIMEN
 STRETCHED AT 215°C. X 102,000. REFRACTED
 LIGHT POLARIZATION [8].

Fig. 2. Void Growth Relationship for Commercial 18 Ni Maraging Steel. After Cox [2]

FIG. 3. COMPARISON OF VOID GROWTH FOUNDATIONS WITH DATA FROM VENTURE

FIG 4. COMPARISON OF VOID GROWTH EQUATIONS WITH SWEDISH IRON DATA
MAGGENBIEFFER

LOCAL FRACTURE MODE		DIMPLE CONFIGURATIONS	
COLUMN 1	COLUMN 2	SEEN THROUGH THE METAL COLUMN 3	SEEN ON THE FRACTURE SURFACE COLUMN 4
UNIFORM STRAIN			
MODE I			
MODE II			
MODE III			
COMBINED MODES I AND II			

A

B.

FIG. 5. PROPOSED MODES FOR DIMPLE FORMATION.
AFTER BENNEM. [27]

FIG. 3. $\text{Cu} + 30\% \text{Zn} + 0.049\% \text{Al}$,
 550 MPa, 10³ K, $\epsilon = 0.06$,
 10³ s, 10³ K. THE PARTICLE A SHOWS
 A CROWAN LOCK (NEARLY NORMAL TO
 THE SURFACE), AND A TANGENTIAL PRismatic
 DISLOCATION FROM HIRSH AND FOLCHERTS[23].

FIG. 7. A RIGID CIRCULAR CYLINDRICAL INCLUSION UNDER UNIAXIAL TENSION

FIG. 8. THE DISTRIBUTION OF INTERFACIAL TENSILE STRESS AS A FUNCTION OF θ FOR A RIGID CIRCULAR CYLINDRICAL INCLUSION IN A LINEAR VISCOELASTIC MATRIX.

FIG. 9. AN INCLUSION UNDER LONGITUDINAL SHEAR

B. DISLOCATION LOOPS

C. CAVITY DUE TO COALESCENCE OF LOOPS

FIG. 10. VOID INITIATION (FROM LINDS TO PARRY [59]).

FIG. 11. A CIRCULAR CYLINDRICAL CAVITY, UNDER
AXISYMMETRIC LOADING.

FIG. 12. THE HOLE GROWTH AMPLIFICATION FACTOR FOR CYLINDRICAL CAVITIES UNDER AXISYMMETRIC LOADING AS A FUNCTION OF STRESS RATIO. LINEAR HARDENING-MATERIAL

FIG. 13. THE HOLE GROWTH AMPLIFICATION FACTOR FOR CYLINDRICAL CAVITIES UNDER AXISYMMETRIC LOADING AS A FUNCTION OF σ_{fm}/σ_0 , NONHARDENING MISES MATERIAL.

FIG. 14. AN ELLIPTICAL CAVITY SUBJECT TO THE REMOTE PRINCIPAL STRESSES σ_{100} AND σ_{200} .

FIG. 15. THE VOID GROWTH AMPLIFICATION FACTOR FOR THE PLANE STRAIN DEFORMATION OF CIRCULAR CAVITIES IN A DUCTILE, HOMOGENEOUS MATERIAL.

FIG. 15. DOUBLY PERIODIC ARRAY OF VOIDS OR INCLUSIONS

FIG. 16. AN ISOLATED CELL

FIG. 16. THE ARRAY OF VOIDS IN NEEDLEMAN'S ANALYSIS.

$$\epsilon_2 = 0.0$$

$$\epsilon_2 = 0.318$$

FIG. 17. NEEDLEMAN'S PREDICTION AND MCCLINTOCK'S
PREDICTION (CROSS-HATCHED) FOR VOID GROWTH WITH
 $\frac{l_0}{R_0} = 2$ AND $n = \frac{1}{4}$

$$E_2 = 0.00$$

$$E_2 = 0.231$$

FIG. 18. NEEDLEMAN'S PREDICTION AND MASCHLINTOCK'S PREDICTION (CROSS HATCHED) FOR VOID GROWTH WITH $L_0 = 1.2$ AND $n = \frac{1}{6}$.

FIG. 19. NEEDLE-TIP FILTRATION (PL. FIGURE 2)
 PROJECTION (OR, INVERSELY, FOR VOID GROWTH WITH
 $\frac{L_0}{R_0} = 4$ AND $n = \frac{1}{8}$.

FIG. 20. SQUARE PNEUMATIC CAVITIES
ARRANGED IN A SQUARE PATTERN, AFTER THOMASON [49].

A.

$c_1 = 0.187$

B.

$c_1 = 0.318$

FIG. 21. NEEDLEMAN'S VOID ARRAY $\frac{b_0}{R_0} \approx 4$, $n = \frac{1}{8}$ WITH THOMPSON'S NOTATION FOR THE GEOMETRY.

FIG.22. THE VOID ARRAY UTILIZED BY MCCLINTOCK IN THE DEVELOPMENT OF A FRACTURE CRITERION BASED ON UNIFORMLY GROWING ELLIPTICAL CAVITIES.

FIG. 23. THE DOUBLE NOTCH PLATE STRAINER AND SPECIFICALLY IN FIG. 23, THE NOTCHES OF THE TAPERED CORNER PLATE. THE NOTCH DEPTH $2a_n$ AND DEPTH OF THE TAPER 2ω .