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PRACTICAL APPLICATIONS OF ADAPTIVE

MESH REFINEMENT (REZONING)

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ABSTRACT

A collection of finite element problems that have been examined with a posteriori convergence measures is presented. Results show these local convergence measures can be used to indicate the quality of a finite element solution and to suggest boundaries for mesh rezones.

1. INTRODUCTION

Historically, finite element analyses have required a relatively extensive dedication of time, personnel and money. Usually, once a solution is available few of these resources are left for pursuing the important question "How good is the answer?" Within the last few years, more advanced modeling techniques have eased the time and personnel problems and have somewhat reduced the cost factor; however, with no apparent definitive answer currently available to the "goodness" question, it appears that provisional convergence measures such as those employed by Humphrey and Carley [1], Guerra [2], and Melosh and Marcal [3] must be relied on. These element level measures are qualitative however, and their interpretations may differ with problem/element type and the proximity of boundary conditions.

It is the intent of this paper to present a collection of problems that have been examined with convergence measures, and, where possible, to present conclusions on the meanings of the measures. In the first discussion (Sections 2.0 through 2.5), two one dimensional problems are considered -- one linear and one non-linear. Sections 3.0 through 3.3 touch on regularly elliptic boundary value problems with specific reference to Poisson's and Laplace's equations, and to an elasticity problem. Finally, Sections 4.0 through 4.2 address convergence measures applied to thin plate bending problems.

2. MESH EVALUATION AND REFINEMENT FOR ONE DIMENSIONAL PROBLEMS

In the course of approximating the solution of a boundary value problem with finite element methods, one should naturally ask how good is the approximation. This question has received extensive attention in mathematical literature in the form of a priori error estimates. While these estimates provide valuable insight into the approximation method, their validity usually does not extend to the basically nonlinear problems of interest in this section.

In this work a simple algorithm is presented which provides information about the quality of an approximate solution and how the mesh might be modified if a more precise solution is desired. After its mathematical basis is mentioned, the method is applied to an example linear problem and results for a nonlinear problem are cited.

2.1 The Residual Function

A minimum of notation will be required for the following discussion. For a function $f(x)$, $x \in R$, the "root-mean-square" or " L_2 " norm of $f(x)$ on the interval (a,b) is denoted

$\|f(x)\|_{(a,b)}$ and is defined by

$$\|f(x)\|_{(a,b)} = \left\{ \int_a^b [f(x)]^2 dx \right\}^{1/2} \quad (1)$$

Some norms will be taken only on the interior of elements; $\|f\|_i$ will denote the L_2 -norm of $f(x)$ on the open interval defining element i . The brackets $[\cdot]_x$ will denote the absolute jump obtained in evaluating the enclosed quantity when approaching the point x from either direction. Clearly, if f is continuous at x the jump is zero, while if f has values dependent on the approach, the jump is nonzero.

Let $v(x)$ be an approximate solution to the general operator equation

$$Au = f \quad (2)$$

where $u(x)$ is the desired unknown function. The residual function $R(x)$ is defined to be what $v(x)$ lacks in satisfying the equation, i.e.,

$$R(x) = Av - f. \quad (3)$$

It seems natural to assume that $R(x)$ gets smaller as $v(x)$ better approximates $u(x)$. It also seems reasonable to assume that $R(x)$ increases as the approximation deteriorates. The mathematical conditions and theories required to prove these propositions are quite involved and unnecessary for the algorithmic nature of this presentation. For detailed analysis, the reader is referred to Humphrey and Carey [1] for

the particular operator treated in this section or Guerra [2] for a general elliptic operator and elastostatics problem.

In this presentation the operator equation is taken to be the two-point boundary value problem

$$-\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x) u(x) = f(x) \quad (4)$$

with $u(0)=u(1)=0$. An approximate solution is obtained by applying Galerkin's method with Lagrange polynomials as basis functions. Then on each element the residual function is continuous and its square may be integrated. However, $\frac{dv}{dx}$ is not continuous across boundaries and the residual will have delta functions at the element interfaces. This precludes calculation of $\|R\|_{(0,1)}$ since the integral is undefined, but the element residual $\|R\|_i$ may be calculated on element i as

$$\|R\|_i = \|R\| + \frac{1}{2} \left[P \frac{dv}{dx} \right]_m + \frac{1}{2} \left[P \frac{dv}{dx} \right]_n \quad (5)$$

where m and n are, respectively, the left and right end points of element i , and the global residual $\|R\|$ then defined by

$$\|R\| = \left(\sum_{i=1}^N \|R\|_i^2 \right)^{1/2}. \quad (6)$$

Under the proper restrictions on the operator the a posteriori error bound

$$\|e\|_{(0,1)} \leq C_1 \|R\| + C_2 \quad (7)$$

may be derived. Although the constants C_1 and C_2 are unknown, this expression is significant since it is valid for non-uniform meshes.

2.2 Convergence of the Residual

The problem used in this work is (4) with

$$\begin{aligned} p(x) &= \frac{1}{\alpha} + \alpha(x-\bar{x})^2 \\ q(x) &= 0 \\ f(x) &= 2 \left\{ 1 + \alpha(x-\bar{x}) [\tan^{-1} \alpha(x-\bar{x}) + \tan^{-1} \alpha\bar{x}] \right\}. \end{aligned} \quad (8)$$

The analytic solution is given by

$$u(x) = (1-x) [\tan^{-1} \alpha(x-\bar{x}) + \tan^{-1} \alpha\bar{x}]$$

where the parameters x and $\bar{x}$ are prescribed to control the character of the solution. Examples are shown in Figure 1.

An important property of the residual is illustrated by a uniform mesh refinement sequence for the smooth problem ($\alpha=0.5$ and $\bar{x}=0.2$ in equation 8). Figure 2 shows that, similar to the error norm, the residual depends on the mesh size h . These relationships have the form

$$\begin{aligned} \|e\|_{(0,1)} &= C_3 h^{k+1} \\ \|R\| &= C_4 h^{k-1} \end{aligned}$$

Figure 1. Solution Profiles for the Linear Problem of Section 2.4

Figure 2. Global Residuals for Various Lagrange Interpolating Polynomials, Equation (4), Uniform Mesh

where k is the order of interpolating polynomial and $k-1$ is called the asymptotic rate of residual convergence. For problems with more rapidly varying solutions, these rates of convergence are not attained until the mesh has sufficient resolution in all areas.

2.3 An Adaptive Mesh Refinement Algorithm

While (7) relates the residual and error norms on the entire domain, a related question is whether the element residuals $|||R|||$ indicate the solution quality of one element relative to another. Mathematical support does exist for this proposition, [2], as do numerical results. On these bases, a refinement or rezoning algorithm may be formulated which inserts new elements where element residuals are largest. In one dimensional problems, element connectivities are simple so that they may be split arbitrarily without upsetting inter-element compatibility. For this reason mesh refinement will be studied here while mesh rezoning for two dimensional problems is treated in following sections.

In the method used here, each element residual is normalized by the element length, then the mean residual, $\bar{R}$, and standard deviations, s , are calculated. Each normalized residual must fall in one of the intervals $(0, \bar{R}+s), (\bar{R}+s, \bar{R}+2s), \dots, (\bar{R}+(t-1)s, \bar{R}+ts)$, and elements are refined accordingly (eg. elements with residuals in interval

$t-1$ to t standard deviations from the mean are refined into t elements). This algorithm has been implemented in a one-dimensional finite element code so that refinement is automatic until the global residual is less than some prescribed value.

2.4 Example Linear Problem Results

The interior layer problem ($\alpha=100.0$ and $\bar{x}=0.36388$ in equation 8) is used to demonstrate the characteristics of the refinement algorithm. The initial mesh consists of four equal length quartic elements. The solution to this problem yields a global residual of 59.99 with normalized element residuals of 33.08, 237.2, 14.8 and 1.0. The algorithm splits the second element since the adverse solution behavior there requires a denser mesh. As refinement proceeds, the second original element is refined exclusively until the first element is also refined on the eighth refinement. Table 1 displays several element distributions. Some elements become very short (0.001 in length) while others remain large (0.25 in length). The algorithm distributes elements so that element sizes vary smoothly from dense to sparse regions.

On the ninth mesh the global residual is 0.77. Since the analytic solution is known, $\|e\|_{(0,1)}$ can be calculated

Table 1. Element Distribution for Several Meshes

Interval	Mesh Number				
	5	6	7	8	9
0.0 – 0.25	1	1	1	1	2
0.25 – 0.3125	1	1	1	2	4
0.3125 – 0.375	5	5	8	11	12
0.375 – 0.4375	2	4	4	4	4
0.4375 – 0.5	1	1	1	1	1
0.5 – 1.0	2	2	2	2	2
Total Elements	12	14	17	21	25
Global Residual	2.02	1.35	1.25	1.17	0.77

(2.99×10^{-5}) . This small value suggests that the error is strongly bounded by the global residual.

2.5 Nonlinear Problems

Although not supported by mathematical proof, the above residual analysis and the statistical refinement method have been applied to a nonlinear problem with a severe boundary layer. The global residual on the first mesh of four quartic elements is 2419. Refinement proceeds until the global residual is reduced to 0.715 on the eleventh mesh. The final mesh contains 35 elements, some of lengths as short as 0.0001.

Due to the iterative nature of solving nonlinear problems, the work required to obtain a converged solution on a mesh may be substantially reduced when an improved starting solution is available. During refinement, such an improved starting solution is available on a refined mesh by interpolation from the parent mesh. Experiments confirm that for the problem considered, fewer operations are required to obtain a given solution accuracy if local refinement is used than if the problem involving refined global mesh/original boundary conditions is solved.

3. MESH EVALUATION AND REZONING FOR REGULARLY ELLIPTIC BOUNDARY VALUE PROBLEMS

The class of problems to be considered in this section are finite element approximations to second-order regularly elliptic boundary value problems of the form,

$$\begin{aligned} Au &= f \quad \text{in } \Omega \\ u &= g_0 \quad \text{on } \partial\Omega \end{aligned} \tag{9}$$

where A is a second-order elliptic differential operator, Ω is a smooth open bounded domain in $\mathbb{R}^2$, the boundary $\partial\Omega$ is smooth, and the data f and g_0 are given.

3.1 The Rezone Problem

The rezone problem consists of solving a "coarse" grid finite element problem, computing boundary conditions from this grid for a region of interest, and solving the reduced problem of the refined region with computed boundary conditions. In [2], it is shown that if P_1 is the rezone region, ∂P_1 the rezone boundary, U_h^0 the coarse grid solution, and $\epsilon_h^0 = U_h^0 - u$ the error in the finite element approximation, then

$$\begin{aligned} \|\epsilon_h^1\|_{H^1(P_1)} &\leq C_1 (h_1^{\gamma_1} \|f\|_{H^0(P_1)} + h_1^{\gamma_2} \|g_e\|_{H^{1/2}(\partial P_1)}) \\ &\quad + C_1 h_1^{\gamma_2} \|\epsilon_h^0\|_{H^{1/2}(\partial P_1)} + C_2 \|\epsilon_h^0\|_{H^{1/2}(\partial P_1)} \end{aligned} \tag{10}$$

where h_1 is the maximum element diagonal in P_1 , γ_1 and γ_2 are constants depending on the space of data and the degree of the polynomial interpolation, g_e is the exact boundary data on ∂P_1 , and C_1 and C_2 are positive constants. The error estimate (10) for rezone problems shows that if the rezone region is refined sufficiently, i.e. $h_1 \rightarrow 0$, then the error ϵ_h^i does not go to zero unless the rezone boundary error ϵ_h^0 (from coarse grid) is also small. On the other hand, since the problem is elliptic, the effects of the boundary error ϵ_h^0 decay into the interior of the rezoned region. These observations are clearly demonstrated in the subsequent numerical results.

In view of (10), to adequately apply a rezoning scheme, one needs to estimate the error, ϵ_h^0 . In [2] it is shown that if we define the global residual as

$$r_h = AU_h - f = AU_h - Au = A\epsilon_h \in H^{-1}(\Omega), U_h \in H^1 \quad (11)$$

then

$$\|\epsilon_h\|_{H^1(\Omega)} \leq \bar{C}_1 \|r_h\|_{\Omega} + \bar{C}_2 \|\epsilon_h\|_{H^{1/2}(\partial\Omega)} \quad (12)$$

where

$$\|r_h\|_{\Omega}^2 = \sum_{e=1}^N \|AU_h - f\|_{H^0(\Omega_e)}^2 + Ch_e^{-1/2} \|[\gamma_1 U_h]\|_{H^0(\partial\Omega_e/\partial\Omega)}^2. \quad (13)$$

$[\gamma_1 U_h]$ denotes the jump in normal derivatives, h_e is the

maximum diagonal in element e , and $\partial\Omega_e/\partial\Omega$ are the element edges not on $\partial\Omega$.

In equation (12), $|||r_h|||_{\Omega}$ can be viewed as the error due to modeling, and $||\epsilon_h||_{H^{1/2}(\partial\Omega)}$ as the boundary interpolation error. This last term has been shown to vary as d^{-1} (d being the distance from $\partial\Omega$ to an interior region Ω'). For regions Ω' sufficiently distant from $\partial\Omega$,

$$||\epsilon_h||_{H^1(\Omega)} \leq \bar{C}_1 |||r_h|||_{\Omega}. \quad (14)$$

3.2 Example Problems

The first problem consists of solving Poisson's equation on $\Omega = (0,1) \times (0,1)$ such that,

$$u(x,y) = x^2 y^2 e^{-2(x+y)},$$

and

$$f = -2 \left\{ x^2 (1 + 4y^2 - 4y) + y^2 (1 - 4x) \right\} e^{-2(x+y)}.$$

The second problem is concerned with solving Laplace's equations on an L-shape domain (Figure 7) such that,

$$u = r^{2/3} \sin(2\phi/3)$$

where

$$r^2 = x^2 + y^2, \phi = \tan^{-1} y/x$$

and the L-shape domain is in

$$\Omega = (-1, 1) \times (-1, 1).$$

This problem has been studied extensively in [4] and [5]. It was chosen for this study to investigate the effects of a singularity on the residual and rezoning schemes used.

Problem 3 consists of a cracked plate under uni-axial tension. Figure 10a shows the coarse grid used and the associated boundary conditions.

3.3 Results

Appendix A shows several of the computed errors for the example problems. Numerical studies for problem 1 showed rates of convergence of h^3 , h^2 , and h for E_1 , H_1 , and R , respectively. Moreover, when a 2x2 Gauss quadrature was used to compute H_1 , the rate of convergence was h^3 . Figures 3 and 4 show the correlation of the element residuals R_e , and the element errors, E_3^e , for several coarse grids. Figures 5 and 6 show the segregation of the error and residual near the rezone boundaries and a decay into the interior of the rezone regions. Figures (a), (b), and (c) are for rezones of the lower right 2x2 elements in Figure 3 (or 4) b, whereas (d) is a rezone of the lower right 3x3 elements in Figure 3 (or 4) c.

Observed rates of convergence for problem 2 are $h^{4/3}$ for E_1 , $h^{2/3}$ for H_1 and $h^{-1/3}$ for R (where a 2x2 Gauss quadrature

(a) 2x2 coarse grid

(b) 4x4 coarse grid

(c) 6x6 coarse grid

(d) 8x8 coarse grid

Figure 3. Coarse Grid Contour Plots of E_3^6 , Problem 1, Section 3.2

(a) 2x2 coarse grid

(b) 4x4 coarse grid

(c) 6x6 coarse grid

(d) 8x8 coarse grid

Figure 4. Coarse Grid Contour Plots of R^0 , Problem 1, Section 3.2

(a) Contours for rezone of
4x4 coarse grid

(b) Contours for rezone of
4x4 coarse grid

(c) Contours for rezone of
4x4 coarse grid

(d) Contours for rezone of
6x6 coarse grid

Figure 5. Rezoned Grid Plots of E_3^H , Problem 1, Section 3.2

(a) Contours for rezone of
4x4 coarse grid

(b) Contours for rezone of
4x4 coarse grid

(c) Contours for rezone of
4x4 coarse grid

(d) Contours for rezone of
6x6 coarse grid

Figure 6. Rezoned Grid Contour Plots of R^6 , Problem 1, Section 3.2

Figure 7. Problem 2, Section 3.2, and Two Rezoned Regions

(a) Rezone Grid (1)

(b) Rezone Grid (2)

(c) Rezone Grid (2) with exact nodal boundary conditions

Figure 8. E_2^0 , Contour Plots for Rezones of Problem 2, Section 3.2

(a) Rezone Grid (1)

(b) Rezone Grid (2)

(c) Rezone Grid (2) with exact nodal boundary conditions

Figure 9. R^e Contour Plots for Rezones of Problem 2, Section 3.2

Figure 10a. Coarse Grid for Problem 3, Section 3.2

Figure 10b. Coarse Grid Residual Contours for Problem 3, Section 3.2

(a) Coarse grid region of interest

(b) First rezone

(c) Second rezone

Figure 11. Residual Contours in Rezoned Regions of Interest, Problem 3, Section 3.2

Figure 12. σ_y and v Perpendicular to Crack Tip, Problem 3, Section 3.2

rule was used for the area integrals). Figure 7 shows a coarse grid with two rezone regions that were considered. Figures 8 and 9 show the rezone contour plots for R^e and E_2^e where approximate boundary conditions were used on (a) and (b) and exact values on (c).

For problem 3, Figure 10a shows the coarse grid considered and Figure 10b the element residual contour plots. Figure 11 shows the element residual contours on the rezone region. Finally, Figure 12 compares the rezone and theoretical values near the crack tip.

4. MESH EVALUATION AND REZONING FOR THIN SHELL/PLATE PROBLEMS

This section considers the finite element approximation to the solution of the fundamental equation of uniform thickness, linear, thin plate bending

$$\nabla^4 u = \frac{1}{D} (q + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y}). \quad (15)$$

To simplify the study, the following regularly elliptic homogeneous boundary value problem is examined:

$$\left. \begin{array}{l} \nabla^4 u = 0 \\ u = g_0 \\ \frac{\partial u}{\partial \eta} = g_1 \end{array} \right\} \begin{array}{l} \text{in } \Omega \\ \text{on } \partial\Omega. \end{array} \quad (16)$$

As in the previous section, Ω is a smooth open bounded domain in $\mathbb{R}^2$, the boundary $\partial\Omega$ is smooth, and g_0 and g_1 are given. For certain conditions on the data g_0 and g_1 , equations (16) have a unique solution and there exists an associated variational boundary value problem with a unique solution. The interested reader is referred to [7] for general mathematical finite element information.

It is demonstrated in this section that by following procedures similar to the ones outlined previously, the quality of the finite element approximation of this variational problem (and therefore equation 16) can be shown. Furthermore, it is demonstrated that this measure of quality can be used to select, in an a posteriori fashion, boundaries within Ω along which the solution can be used for boundary conditions in problem rezoning.

4.1 Residual Calculations

By successive integrations by parts and finite element partitioning of (16), one arrives at a statement of the element residual (cf. equation 13)

$$\|r_h\|_{\Omega_\theta} = \|AU_h - f\|_{H^0(\Omega_\theta)} + C(\Omega_\theta) \|[\gamma, U_h]\|_{H^0(\partial\Omega_\theta/\partial\Omega)}. \quad (17)$$

For this work, the element boundary normal derivative jump terms $[\gamma, U_h]$ are ignored and any element residual is assumed to be caused by the inability of that particular element to com-

pletely capture the energy state. This appears to be in the spirit of [3], where the "specific energy difference" (SED) measures element energy as a function of allowable strain modes.

For the a posteriori application of (17), one must establish a set of shape functions that are consistent with the original analysis element formulation. For this paper, the complete and compatible Clough-Felippa Q-19 flat plate quadrilateral element provided the necessary analysis data at four interior sub-triangle integration points. The quadrilateral element stress field was assumed to be linear along element boundaries and quadratic on the element interior, and the element residuals, $|||r_h|||_{S_0}$, were calculated using 2x2 Gauss quadrature.

4.2 Example Problem and Results

The example problem selected was that of a flat plate simply supported on all edges, with a point load applied perpendicular to the center (Figure 14). The stress solution to this problem exhibits exponential behavior similar to that encountered in the neighborhood of tubular joint intersections.

Figure 13 demonstrates how the global residual varies with uniform coarse mesh refinement. With the aid of the

Figure 13. Global Mesh Residual ($|||r_h|||_{\Omega}$) and Error ($||\epsilon_h||_{H^0(\Omega)}$) for Simple Support Sample Problem of Section 4.2 (Uniform Mesh)

Figure 14. Global Modelling Error Vs. Total Number of Elements,
Simple Support Sample Problem, Section 4.2

theoretical displacement solution [8] we obtain the second figure 13 curve, where the error is defined

$$\|e_h\|_{H^0(\Omega)} = \left\{ \sum_{e=1}^N \left(\int_{\Omega_e} (U_h - u)^2 d\Omega_e \right) \right\}^{1/2}.$$

Here, U_h is the finite element approximation and u is from [8]. This relationship between the H^0 model error and global residual can be used to give an indication of the model quality as a function of the total number of elements (Figure 14). The systematic uniform mesh refinement represented by the solid line indicates that the global error is proportional to N^{-1} . Figures 15a and b are orthographic projections of three dimensional representations of the element residuals for two coarse meshes (residuals shown here as constant on each element), and Figure 16 compares the Figure 15b symmetry line stress resultants with those from [8].

Three rezones of the Figure 15b mesh resulted in global modeling errors as shown in Figure 14. Because the refinement was conducted where the element residuals were largest, Figure 14 indicates a more rapid rate of convergence to a better solution. Rezone 1, not pictured here, was a refinement of the four elements closest to the point load in Figure 15b. Figures 15c and d show element residuals for rezones 2 and 3, and Figure 16 shows comparisons of stress resultants.

a. 3x3 coarse grid

b. 6x6 coarse grid

c. 6x6 rezone 2

d. 6x6 rezone 3

Figure 15. Scaled Element Residuals, Simple Support Problem of Section 4.2 (X25)

Figure 16. X Moment (Stress Resultant) Along Symmetry Lines, Rezone Meshes, Simple Support Problem of Section 4.2

Finally, Figure 17 presents for comparison the "SED" parameter from [3] using type 50 elements, and Figure 18 a practical example to be discussed in more detail during the conference.

Figure 17. MARC SED Contour for Simple Support Problem of Section 4.2

Figure 18. Element Residuals, 7m Leg/1m Brace Intersection

APPENDIX A

Let $e_h = U_h - u$, Ω_e be a typical element, and $\phi = \sum_{i=1}^N \phi_i$ where $\partial\Gamma_e^i$ is an element edge. Then:

$$E_1^e = \left(\int_{\Omega_e} |e_h|^2 da \right)^{1/2}, \quad E_1 = \left(\sum_{e=1}^N (E_1^e)^2 \right)^{1/2}$$

$$H_1^e = \left(\int_{\Omega_e} (|e_h|^2 + |\nabla e_h|^2) da \right)^{1/2}, \quad H_1 = \left(\sum_{e=1}^N (H_1^e)^2 \right)^{1/2}$$

$$R^e = \left(\int_{\Omega} |AU_h - f|^2 da \right)^{1/2} + \left(h_e^{-1} \phi \int_{\partial\Gamma_e} |\gamma_1 U_h|^2 ds \right)^{1/2}, \quad R = \left(\sum_{e=1}^N (R^e)^2 \right)^{1/2}$$

$$E_2^e = \max_{G_e} |\nabla \hat{U}_h - \nabla u|, \quad E_2 = \left(\sum_{e=1}^N (E_2^e)^2 \right)^{1/2}$$

$$G_e = \left\{ \text{set of 2 Gauss points on the element edges} \right\}$$

$$\nabla \hat{U}_h = \text{averaged gradients}$$

$$|\nabla \hat{U}_h - \nabla u| = \left(\left(\frac{\partial \hat{U}_h - u}{\partial x} \right)^2 + \left(\frac{\partial \hat{U}_h - u}{\partial y} \right)^2 \right)^{1/2}$$

$$E_2^e = \left(\oint_{\partial\Gamma_e} |\nabla \hat{U}_h - \nabla u|^2 ds \right)^{1/2}, \quad E_2 = \left(\sum_{e=1}^N (E_2^e)^2 \right)^{1/2}$$

For the area integrals, in E_1^e , H_1^e , and R^e a 2x2 point Gauss quadrature was used in most cases, and a 2-point Gauss quadrature was used for all line integrals.

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