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WEAK-SHOCK THEORY FOR
SPHERICAL SHOCK WAVES**

by

W.D. Curtis*
C.E. Rosenkilder†
K.S. Yee*

*Department of Mathematics
Kansas State University
Manhattan, KS 66506

†Lawrence Livermore National Laboratory
Livermore, CA 94550

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Derivation of the Approximate Equations

We begin with the equations for spherically symmetric, isentropic flow,

$$\begin{aligned} \rho_t + u\rho_r + \rho(u_r + \frac{2u}{r}) &= 0 & (a) \\ u_t + uu_r + \frac{1}{\rho} p_r &= 0 & (b) \end{aligned} \quad (1)$$

We introduce new variables, $\Sigma = \sigma r$, $U = ur$, where $\sigma = \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho$.

Then (1) becomes

$$\begin{aligned} U_t + c\Sigma_r + uU_r - \frac{c\Sigma}{r} - \frac{U^2}{r^2} &= 0 & (a) \\ \Sigma_t + cU_r + u\Sigma_r + \frac{cU}{r} - \frac{U\Sigma}{r^2} &= 0 & (b) \end{aligned} \quad (2)$$

Let $F = \Sigma + U$, $G = \Sigma - U$. Then add and subtract the equations (2) to get

$$\begin{aligned} F_t + (c + u)F_r - \frac{cG}{r} - \frac{UF}{r^2} &= 0 & (a) \\ G_t - (c - u)G_r + \frac{cF}{r} - \frac{UG}{r^2} &= 0 & (b) \end{aligned} \quad (3)$$

Acoustic Approximations

If we make the usual acoustic approximations, we get

$$\sigma = \frac{f'(t - \frac{r}{c_0})}{r}$$

and

$$u = \sigma + c_0 \frac{f(t - \frac{r}{c_0})}{r^2}.$$

$$\text{Then } \Sigma = f'(t - \frac{r}{c_0}),$$

$$U = \Sigma + c_0 \frac{f(t - \frac{r}{c_0})}{r},$$

and so

$$F = 2f'(t - \frac{r}{c_0}) + c_0 \frac{f(t - \frac{r}{c_0})}{r}$$

$$G = - \frac{c_0 f(t - \frac{r}{c_0})}{r}.$$

Thus, in the acoustic limit we see that, for large r , G is small compared with F .

Some Order Estimates

We assume the wave deviates from the ambient state only in a narrow "pulse" behind the shock. The width of this pulse is not assumed to be constant but we do assume it is "slowly varying".

(FIG. 1)

If we integrate (3b) across the pulse along a C_0 -characteristic, we obtain the estimate

$$\Delta G \approx O\left(\frac{c_0 F T}{R} + \frac{T F G}{R^2} + \frac{T G^2}{R^2}\right) \quad (4)$$

Explanation of (4): L, T are, as shown in FIG. 1, the spatial and temporal pulse widths and since the rear of the pulse travels with velocity c_0 (The back edge of the pulse is a characteristic adjacent to the ambient state) we see $L = c_0 T$. Thus $\frac{cF}{r} = O\left(\frac{c_0 F T}{R}\right)$. $\frac{UG}{r^2} = \frac{FG - G^2}{2r^2}$ so $\frac{UF}{r^2} = O\left(\frac{FGT}{R^2} + \frac{G^2 T}{R^2}\right)$ so (4) holds.

If we assume, as in the acoustic case, that $G \ll F$ then (4) gives

$$\Delta G = O\left(\frac{LF}{R}\right).$$

If we also make use of the acoustic result that G is very small at the shock we obtain the estimate

$$G = O\left(\frac{LF}{R}\right). \quad (5)$$

Consider the other terms in (3):

$$\frac{cG}{r} = O\left(\frac{c_0}{R} \frac{LF}{R}\right) = \left(\frac{L}{R}\right)^2 \frac{c_0 F}{L},$$

$$F_t = O\left(\frac{F}{T}\right) = O\left(\frac{c_0 F}{L}\right),$$

$$(c + u)F_r = O\left(\frac{c_0 F}{L}\right) \quad (\text{same as } F_t),$$

$$\frac{UF}{r^2} = O\left(\frac{UF}{R}\right) \ll O\left(\frac{c_0 F}{R}\right) = O\left(\frac{L}{R} \frac{c_0 F}{L}\right).$$

Approximate Equations

On the basis of the above estimates we now set $G = 0$ and neglect the terms $\frac{cG}{r}$ and $\frac{UF}{r^2}$ in (3a); and (3) is then replaced by the single equation

$$F_t + (c + u)F_r = 0. \quad (6)$$

Since $G = 0$ we have $F = 2U$ so

$$u = \frac{F}{2r}. \quad (7)$$

We also use the acoustic relation

$$p = \rho_0 c_0 u \quad (p \text{ is the overpressure})^* \quad (8)$$

We are assuming isentropic flow so we have

$$c = c(\rho),$$

$$\sigma = \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho = \sigma(\rho),$$

$$\text{and so } c = c(\sigma).$$

We use the first order approximation to this last equation. That is, we write

$$c = c_0 + c'(0)\sigma.$$

Then the propagation velocity is,

$$\begin{aligned} c + u &= c_0 + c'(0)\sigma + u = c_0 + c'(0) \frac{\Sigma}{r} + \frac{U}{r} = \\ &= c_0 + \frac{c'(0)F}{2r} + \frac{F}{2r} = c_0 + \frac{(c'(0) + 1) F}{2r} \end{aligned}$$

We introduce the notation

*In this report p, p_1, p_i , etc. always represent the difference between the pressure and the ambient pressure ahead of the wave.

$$K = \frac{c'(0) + 1}{2c_0} \quad (9)$$

and we obtain

$$c + u = c_0 \left(1 + \frac{KF}{r}\right). \quad (10)$$

We shall discuss the question of computing K in Appendix A.

Shock Conditions

In the usual weak-shock theory the shock velocity, v , is the mean of the acoustic velocities on the front and back sides of the shock. We wish to observe that the present approximate theory can be made to yield the same result as we now show.

Since we neglect terms of order u^2 , i.e., of order $\frac{F^2}{r^2}$ we have

$$\frac{\partial}{\partial r} \left(c_0 F + c_0 K \frac{F^2}{2r} \right) = c_0 F_r + \frac{c_0 K F F_r}{r},$$

where the term $-\frac{c_0 K F^2}{2r^2}$ has been neglected. Thus (6) takes the conservation form

$$F_t + \frac{\partial}{\partial r} \left(c_0 F + c_0 K \frac{F^2}{2r} \right) = 0 \quad (11)$$

The shock velocity corresponding to the conservation law (11) is (compare [4] p. 31).

$$v = \frac{[c_0 F + c_0 K \frac{F^2}{2r}]}{[F]} \quad (12)$$

Recalling that $F = 0$ ahead of the shock we see that (12) is

$$v = c_0 + c_0 K \frac{F}{2r} \quad (13)$$

which is, indeed, the mean of the propagation velocity behind the shock, $c_0(1 + \frac{KF}{r})$, and in front of the shock, c_0 .

Solution In The Boundary Value Case

Suppose $s_0 > 0$ and that, at $r = s_0$, we have a time history for F , that is,

$$F(s_0, \tau) = \begin{cases} F(\tau) & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

We assume $F(0) \neq 0$ so that we may consider that the shock wave has just arrived at $r = s_0$ at $t = 0$.

(FIG. 2)

We have the differential equation

$$F_t + c_0 \left(1 + \frac{KF(r,t)}{r} \right) F_r = 0, \quad (14)$$

which is just (6) using (10). We also have the expression for shock velocity from (13),

$$v = c_0 \left(1 + \frac{KF(s,t(s))}{2s} \right) \quad (15)$$

where, as we shall find convenient, we have written the shock curve as $t = t(s)$.

The characteristics of (14) are solutions of

$$\frac{dr}{dt} = c_0 \left(1 + \frac{KF(r,t)}{r} \right) \quad (16)$$

and along each such curve F is constant.

Consider the characteristic passing through (s_0, τ) , with $\tau > 0$.

Along this curve we have

$$F(r,t) = F(s_0, \tau) = F(\tau), \quad (a)$$

$$\frac{dr}{dt} = c_0 \left(1 + \frac{KF(\tau)}{r} \right), \quad (b)$$

$$\text{so } \frac{dt}{dr} = \frac{1}{c_0 \left(1 + \frac{KF(\tau)}{r} \right)} = \frac{1}{c_0} \left(1 - \frac{KF(\tau)}{r} \right),$$

and hence,

$$\frac{dt}{dr} = \frac{1}{c_0} \left(1 - \frac{KF(\tau)}{r} \right). \quad (18)$$

We integrate (18) to get

$$t = \frac{r}{c_0} - \frac{KF(\tau)}{c_0} \ln r + I(\tau)$$

where $T(\tau)$ is the integration constant, which may vary with τ . The condition that the characteristic pass through (s_0, τ) gives

$$\tau = \frac{s_0}{c_0} - \frac{KF(\tau)}{c_0} \ln s_0 + T(\tau),$$

so

$$T(\tau) = \tau - \frac{s_0}{c_0} + \frac{KF(\tau)}{c_0} \ln s_0. \quad (19)$$

The equation of the characteristic is, then,

$$t = \tau + \frac{r - s_0}{c_0} - \frac{KF(\tau)}{c_0} \ln \left(\frac{r}{s_0} \right). \quad (20)$$

The Equal Area Rule

Given $s > s_0$ there is a unique $\tau > 0$ such that the characteristic through (s_0, τ) also passes through $(s, t(s))$. We shall denote this τ by

$$\tau = \tau_1(s).$$

This is illustrated in FIG. 2.

From (20) we get the equation,

$$t(s) = \tau_1(s) + \frac{s - s_0}{c_0} - \frac{KF(\tau_1(s))}{c_0} \ln \left(\frac{s}{s_0} \right) \quad (21)$$

where, as before, $t = t(s)$ gives the shock front. Thus ($\dot{\tau} = \frac{d\tau}{ds}$)

$$\frac{1}{v} = \dot{t} = \dot{\tau}_1 + \frac{1}{c_0} - \frac{KF(\tau_1)}{c_0 s} - \frac{KF'(\tau_1)}{c_0} \tau_1 \ln \left(\frac{s}{s_0} \right). \quad (22)$$

But from (15)

$$\frac{1}{v} = \frac{1}{c_0 \left(1 + \frac{KF(\tau_1)}{2s} \right)} = \frac{1}{c_0} \left(1 - \frac{KF(\tau_1)}{2s} \right). \quad (23)$$

Comparing (22) and (23) we obtain

$$\begin{aligned} \dot{\tau}_1 - \frac{KF(\tau_1)}{c_0 s} - \frac{KF'(\tau_1)}{c_0} \dot{\tau}_1 \ln\left(\frac{s}{s_0}\right) &= -\frac{KF(\tau_1)}{2c_0 s} \quad \text{or} \\ \dot{\tau}_1 &= \frac{KF(\tau_1)}{2c_0 s} + \frac{KF'(\tau_1)}{c_0} \dot{\tau}_1 \ln\left(\frac{s}{s_0}\right). \end{aligned} \quad (24)$$

Multiply (24) by $2F(\tau_1)$ to get, $2F(\tau_1)\dot{\tau}_1 = \frac{KF^2(\tau_1)}{c_0 s} + \frac{2KF(\tau_1)F'(\tau_1)}{c_0} \dot{\tau}_1 \ln\left(\frac{s}{s_0}\right)$

which is just,

$$2F(\tau_1)\dot{\tau}_1 = \frac{d}{ds} \left(\frac{KF^2(\tau_1)}{c_0} \ln\left(\frac{s}{s_0}\right) \right). \quad (25)$$

Integrate (25) with respect to s from s_0 to s to get,

$$\frac{KF^2(\tau_1(s))}{c_0} \ln\left(\frac{s}{s_0}\right) = 2 \int_0^{\tau_1(s)} F(\xi) d\xi. \quad (26)$$

Equation (26) is called the Equal Area Rule. If we solve (26) for s in terms of τ we obtain $s = s(\tau)$, which is the inverse function of our $\tau = \tau_1(s)$. This is as follows:

$$s = s_0 \exp \left[\frac{2c_0}{KF^2(\tau)} \int_0^{\tau} F(\xi) d\xi \right]. \quad (27)$$

Determination of the Shock Path

From (27) we have s explicitly in terms of τ . Then we rewrite (21) as

$$t = \tau + \frac{s(\tau) - s_0}{c_0} - \frac{KF(\tau)}{c_0} \ln\left(\frac{s(\tau)}{s_0}\right)$$

which gives t in terms of τ along the shock. Invert this to get τ in terms of t along the shock and combine it with $s = s(\tau)$ to get

$$s = s(t)$$

along the shock.

Asymptotic Forms

Suppose that the initial pulse, $F(\tau)$, is as shown in FIG. 3.

(FIG. 3)

The initial pulse is extended over the interval $[0, \tau^*]$ of length τ^* so τ^* may be regarded as an initial pulse duration. We shall consider later an alternate, perhaps more useful definition, which does not require that the pulse drop off to zero.

Note that $\int_0^{\tau^*} F(\xi) d\xi$ has dimension L^2 so we may regard

$l_0 = (\int_0^{\tau^*} F(\xi) d\xi)^{1/2}$ as some sort of characteristic length for the problem.

For a pulse as in FIG. 3 the characteristics look as in FIG. 4.

(FIG. 4)

The point is that the characteristic through (s_0, τ^*) never catches the shock because that characteristic is, from (20), with $F(\tau^*) = 0$,

$$t = \tau^* + \frac{r - s_0}{c_0}.$$

Hence $\frac{dr}{dt} = c_0 < \dot{s}$ along the whole curve. As $\tau \rightarrow \tau^*$ we see $s \rightarrow +\infty$. Thus, if we let $s \rightarrow +\infty$ in (26) we get

$$\lim_{s \rightarrow +\infty} \frac{KF^2(\tau_1(s)) \ln\left(\frac{s}{s_0}\right)}{c_0} = 2 \int_0^{\tau^*} F(\xi) d\xi,$$

$$\text{so} \quad \lim_{s \rightarrow +\infty} F^2(\tau_1(s) \ln(\frac{s}{s_0})) = \frac{2c_0}{K} I_0^2 \quad (28)$$

This gives the asymptotic relation

$$F(\tau_1(s)) \sim \frac{I_0 \sqrt{\frac{2c_0}{K}}}{(\ln(\frac{s}{s_0}))^{1/2}} \quad (29)$$

Since $u(s, t(s)) = \frac{F(\tau_1(s))}{2s}$, $p = \rho_0 c_0 u$, we have the following asymptotic formulae for the particle velocity and pressure at the shock front:

$$u_1(s) \sim \frac{I_0 \sqrt{\frac{c_0}{2K}}}{s (\ln(\frac{s}{s_0}))^{1/2}} \quad (30)$$

$$p_1(s) \sim \frac{\rho_0 c_0 I_0 \sqrt{\frac{c_0}{2K}}}{s (\ln(\frac{s}{s_0}))^{1/2}} \quad (31)$$

Note that (30) and (31) can be normalized to give:

$$\frac{u_1(s)}{c_0} \sim \frac{A}{s (\ln(\frac{s}{s_0}))^{1/2}} \quad (32)$$

$$\frac{p_1(s)}{p_0} \sim \frac{A}{s (\ln(\frac{s}{s_0}))^{1/2}} \quad (33)$$

where

$$A = I_0 (2Kc_0)^{-1/2}$$

$$p_0 = \rho_0 c_0^2 \quad (\text{characteristic pressure})$$

It may appear that the characteristic length,

$$I_0 = \left(\int_0^{\tau^*} F(\xi) d\xi \right)^{1/2},$$

depends on the initial point s_0 . Suppose, however, that we consider another point $\bar{s}_0$. Of course the shock will not reach $\bar{s}_0$ at $t = 0$ as was assumed for s_0 but no essential use was ever made of that assumption except that, had the shock arrived at $t = \tau_0$, the integral for I_0 would have had τ_0 as its lower limit of integration. Thus let the shock reach $\bar{s}_0$ at $t = \bar{\tau}_0$. Let $\bar{F}(\bar{s}_0, \tau) = \bar{F}(\tau)$. Then define $\bar{I}_0^2 = \int_{\bar{\tau}_0}^{\bar{\tau}^*} \bar{F}(\xi) d\xi$. Now just as before we get the asymptotic relation

$$p_1(s) \sim \frac{\rho_0 c_0 \bar{I}_0 \sqrt{\frac{c_0}{2K}}}{s \left(\ln \left(\frac{s}{\bar{s}_0} \right) \right)^{1/2}}. \quad (34)$$

Comparing this with (31) we see we must have

$$I_0 = \bar{I}_0.$$

Now $p(\bar{s}_0, \tau) = \frac{\rho_0 c_0}{2\bar{s}_0} \bar{F}(\bar{s}_0, \tau)$, so

$$\int_{\bar{\tau}_0}^{\bar{\tau}^*} p(\bar{s}_0, \tau) d\tau = \frac{\rho_0 c_0}{2\bar{s}_0} \int_{\bar{\tau}_0}^{\bar{\tau}^*} \bar{F}(\tau) d\tau = \frac{\rho_0 c_0}{2\bar{s}_0} I_0^2.$$

The integral, $\int_{\bar{\tau}_0}^{\bar{\tau}^*} p(\bar{s}_0, \tau) d\tau = I(\bar{s}_0)$, is the impulse per unit area at $r = r_0$ delivered by the wave. Thus we have established the result

$$I(s) = \frac{\rho_0 c_0 l_0^2}{2s} . \quad (35)$$

The impulse per unit area, at range s, drops off as $\frac{1}{s}$. Our characteristic length is then $l_0 = \left(\frac{2sI(s)}{\rho_0 c_0} \right)^{1/2}$ which we have shown is independent of s.

Now consider the spatial width of the pulse when the shock is at $r = s$. We have the width, $W(s)$, given as,

$$W(s) = s - s_0 - c_0(t(s) - \tau^*) . \quad (36)$$

From (21) we have

$$W(s) = s - s_0 - c_0(\tau_1(s) + \frac{s - s_0}{c_0} - \frac{KF(\tau_1(s))}{c_0} \ln(\frac{s}{s_0}) - \tau^*)$$

or,

$$W(s) = -c_0\tau_1(s) + KF(\tau_1(s))\ln(\frac{s}{s_0}) + c_0\tau^* . \quad (37)$$

From the Equal Area Rule, (26), we can write (37) as

$$W(s) = c_0(\tau^* - \tau_1(s)) + \left(\ln \frac{s}{s_0}\right)^{1/2} 2c_0K \left(\int_0^{\tau_1(s)} F(\xi) d\xi \right)^{1/2} . \quad (38)$$

We now let $s \rightarrow \infty$ and use the fact that $\tau_1(s) \rightarrow \tau^*$ to get from (38)

$$W(s) \sim l_0 \sqrt{2c_0K} \left(\ln \frac{s}{s_0}\right)^{1/2} . \quad (39)$$

Finally consider the temporal duration of the pulse as a function of s . We denote this by $\theta(s)$. Then

$$\theta(s) = \tau^* + \frac{s - s_0}{c_0} - t(s) . \quad (40)$$

But, referring to (3E), we note that

$$W(s) = c_0 \theta(s) , \quad (41)$$

so that we have

$$\theta(s) \sim l_0 \sqrt{\frac{2K}{c_0}} \left(\ln\left(\frac{s}{s_0}\right) \right)^{1/2} . \quad (42)$$

Exact Solution For Exponential Pulse.

Now let us consider the case of an exponentially decaying profile for p at some $r = s_0$. That is,

$$p(s_0, t) = p_i e^{-t/\theta_i} \quad (43)$$

so

$$u(s_0, t) = u_i e^{-t/\theta_i} \quad (44)$$

where

$$p_i = \rho_0 c_0 u_i .$$

Consider the value of F at the front when the shock is at $r = s$, denoted $F_1(s)$. We have

$$F_1(s) = F(\tau_1(s)),$$

$$U_1(s) = \frac{F_1(s)}{2s},$$

$$p_1(s) = \frac{\rho_0 c_0 F_1(s)}{2s}.$$

From the Equal Area Rule, (26), we have

$$F_1(s) = \left(\frac{2c_0}{K} \int_0^{\tau_1(s)} F(\xi) d\xi \right)^{1/2} / \left(\ln\left(\frac{s}{s_0}\right) \right)^{1/2}. \quad (45)$$

This can be explicitly evaluated in the present case. Since $p(r,t) = \rho_0 c_0 \frac{F(r,t)}{2r}$ we see

$$F(\xi) = \frac{2s_0 p_i}{\rho_0 c_0} e^{-\xi/\theta_i}. \quad (46)$$

We let $B = \frac{2s_0 p_i}{\rho_0 c_0}$. Then

$$\int_0^{\tau_1(s)} F(\xi) d\xi = -\theta_i B e^{-\xi/\theta_i} \Big|_0^{\tau_1(s)} = \theta_i B \left[1 - \exp\left(-\frac{\tau_1(s)}{\theta_i}\right) \right],$$

So

$$F_1^2(s) = \frac{2c_0 \theta_i B}{K} \left[1 - \exp\left(-\frac{\tau_1(s)}{\theta_i}\right) \right] / \ln\left(\frac{s}{s_0}\right). \quad (47)$$

From (46) $F_1(s) = B \exp\left(-\frac{\tau_1(s)}{\theta_i}\right)$, so (47) becomes

$$F_1(s)^2 = \left[\frac{2c_0 \theta_i B}{K} - \frac{2c_0 \theta_i}{K} F_1(s) \right] / \ln\left(\frac{s}{s_0}\right). \quad (48)$$

We regard (48) as a quadratic equation to be solved for $F_1(s)$ and the solution is

$$F_1(s) = \frac{\frac{-2c_0 \theta_i}{K} + \left(\frac{4c_0^2 \theta_i^2}{K^2} + \frac{8c_0 \theta_i B}{K} \ln\left(\frac{s}{s_0}\right) \right)^{1/2}}{2 \ln\left(\frac{s}{s_0}\right)} \quad (49)$$

Using $p_1 = \rho_0 c_0 F_1 / 2s$, $p_0 = \rho_0 c_0^2$, $B = \frac{2s_0 p_i}{\rho_0 c_0}$ and some algebra we rewrite (49) as

$$p_1(s) = p_i \frac{\left[1 - \left(1 + \frac{4Ks_0 p_i}{\rho_0 \theta_i} \ln\left(\frac{s}{s_0}\right) \right)^{1/2} \right]}{-\frac{2sKp_i}{\rho_0 \theta_i} \ln\left(\frac{s}{s_0}\right)}. \quad (50)$$

Define the dimensionless quantity

$$D = \frac{2Ks_0 p_i}{\rho_0 \theta_i} \quad (51)$$

and (50) becomes,

$$p_1(s) = p_i \frac{\left[1 - \left(1 + 2D \ln\left(\frac{s}{s_0}\right) \right)^{1/2} \right]}{-D \frac{s}{s_0} \ln\left(\frac{s}{s_0}\right)}. \quad (52)$$

It should be noted that the present case corresponds to taking $\tau^* = \infty$ in the discussion of the preceding section. If so, the asymptotic form (31) should coincide with the asymptotic form

$$p_1(s) \sim \frac{2p_i s_0}{\sqrt{2D}} \frac{1}{s \left(\ln \frac{s}{s_0} \right)^{1/2}} \quad (53)$$

which is readily derived from (52). It is easily checked that this is indeed the case.

The definition of temporal duration of the pulse as given in (40) cannot be used in the present setting. Therefore we define the duration, θ , as a function of shock position, s , by

$$\theta(s) = \frac{-p(s, t(s))}{\frac{\partial p}{\partial t}(s, t(s))} . \quad (54)$$

We note that, in view of (43), we have $\theta(0) = \theta_i$. We shall derive an explicit formula for $\theta(s)$. Recall from (17) we have

$$p(r, t) = \rho_0 c_0 \frac{F(\tau)}{2r} , \quad (55)$$

$$t = \tau + \frac{r - s_0}{c_0} - \frac{KF(\tau)}{c_0} \ln\left(\frac{r}{s_0}\right), \quad (56)$$

where in the present case

$$F(\tau) = \frac{2s_0 p_i}{\rho_0 c_0} e^{-\tau/\theta_i}, \quad (57)$$

$$\frac{\partial p}{\partial t} = \rho_0 c_0 \frac{F'(\tau)}{2r} \frac{\partial \tau}{\partial t} = -\frac{1}{\theta_i} \rho_0 c_0 \frac{F(\tau)}{2r} \frac{\partial \tau}{\partial t} .$$

Thus

$$\theta(s) = \frac{\theta_i}{\partial \tau / \partial t} . \quad (58)$$

From (56) we get

$$1 = \frac{\partial \tau}{\partial t} - \frac{KF'(\tau)}{c_0} \frac{\partial \tau}{\partial t} \ln\left(\frac{r}{s_0}\right) \quad \text{so that}$$

$$\frac{\partial \tau}{\partial t} = \frac{1}{1 - \frac{K}{c_0} F'(\tau) \ln\left(\frac{r}{s_0}\right)} \quad \text{or, using (57),}$$

$$\frac{\partial \tau}{\partial t} = \frac{1}{1 + \frac{K}{\theta_i c_0} F(\tau) \ln\left(\frac{r}{s_0}\right)} \quad (59)$$

Evaluating at the shock and using (58), we get

$$\begin{aligned} \theta(s) &= \theta_i \left(1 + \frac{K}{\theta_i c_0} F(\tau_1(s)) \ln\left(\frac{s}{s_0}\right) \right) \\ &= \theta_i \left(1 + \frac{K}{\theta_i c_0} \frac{2s}{\rho_0 c_0} p_1(s) \ln\left(\frac{s}{s_0}\right) \right) \\ &= \theta_i \left(1 + \frac{K}{\theta_i c_0} \frac{2s}{\rho_0 c_0} \ln\left(\frac{s}{s_0}\right) p_i \frac{[(1 + 2D \ln \frac{s}{s_0})^{1/2} - 1]}{D \frac{s}{s_0} \ln\left(\frac{s}{s_0}\right)} \right) \\ \theta(s) &= \theta_i (1 + 2D \ln(\frac{s}{s_0}))^{1/2} . \end{aligned} \quad (60)$$

Equation (60) is our exact expression of θ in terms of s .

Solution in the Initial Value Case

Suppose we have an initial profile given by

$$F(\xi, 0) = \begin{cases} F(\xi) & 0 < \xi \leq s_0 \\ 0 & \xi > s_0 \end{cases}.$$

So if we assume $F(s_0) \neq 0$ we may interpret this as saying that at $t = 0$ the shock has just arrived at $r = s_0$.

(FIG. 5)

Consider the characteristic passing through $(\xi, 0)$, $0 < \xi \leq s_0$. Along this curve we have :

$$F(r,t) = F(\xi,0) = F(\xi), \quad (a)$$

(61)

$$\frac{dr}{dt} = c_0 \left(1 + \frac{KF(\xi)}{r}\right), \quad (b)$$

$$\text{so } \frac{dt}{dr} = \frac{1}{c_0 \left(1 + \frac{KF(\xi)}{r}\right)} = \frac{1}{c_0} \left(1 - \frac{KF(\xi)}{r}\right),$$

$$\frac{dt}{dr} = \frac{1}{c_0} \left(1 - \frac{KF(\xi)}{r}\right), \quad (52)$$

and so

$$t = \frac{r}{c_0} - \frac{K}{c_0} F(\xi) \ln r + T(\xi).$$

Since $t = 0$ when $r = \xi$, we see

$$T(\xi) = -\frac{\xi}{c_0} + \frac{K}{c_0} F(\xi) \ln \xi, \quad (63)$$

so that the equation of the characteristic is

$$t = \frac{r - \xi}{c_0} - \frac{K}{c_0} F(\xi) \ln \frac{r}{\xi}. \quad (64)$$

The Equal Area Rule

Given $s > s_0$ there is a unique $\xi < s_0$ such that the characteristic through $(\xi, 0)$ also passes through $(s, t(s))$. We shall denote this ξ by

$$\xi = \xi_1(s).$$

This is illustrated in FIG. 5.

From (64) we have the equation

$$t(s) = \frac{s - \xi_1(s)}{c_0} - \frac{KF(\xi_1(s))}{c_0} \ln\left(\frac{s}{\xi_1(s)}\right). \quad (65)$$

Thus, using '.' to denote differentiation with respect to s , we get

$$\frac{1}{v} = \dot{t} = \frac{1 - \dot{\xi}_1}{c_0} - \frac{K}{c_0} F'(\xi_1) \dot{\xi}_1 \ln\left(\frac{s}{\xi_1}\right) - \frac{K}{c_0} F(\xi_1(s)) \left(\frac{1}{s} - \frac{\dot{\xi}_1}{\xi_1}\right). \quad (66)$$

But, using (15), we also have

$$\frac{1}{v} = \frac{1}{c_0 \left(1 + \frac{KF(\xi_1)}{2s}\right)} = \frac{1}{c_0} \left(1 - \frac{KF(\xi_1)}{2s}\right). \quad (67)$$

Comparing (66) and (67) we get

$$\frac{-\dot{\xi}_1}{c_0} - \frac{K}{c_0} F'(\xi_1) \dot{\xi}_1 \ln\left(\frac{s}{\xi_1}\right) - \frac{K}{c_0} F(\xi_1) \left(\frac{1}{s} - \frac{\dot{\xi}_1}{\xi_1}\right) = \frac{-KF(\xi_1)}{2c_0 s}$$

or,

$$-K \frac{F(\xi_1)}{2s} + KF(\xi_1) \frac{\dot{\xi}_1}{\xi_1} - KF'(\xi_1) \dot{\xi}_1 \ln\left(\frac{s}{\xi_1}\right) = \dot{\xi}_1. \quad (68)$$

Multiply (68) by $2F(\xi_1)$ and rearrange to get

$$\frac{KF^2(\xi_1)}{s} + 2KF(\xi_1)F'(\xi_1) \ln s \dot{\xi}_1 = -2F(\xi_1) \dot{\xi}_1 + 2K \frac{F^2 \dot{\xi}_1}{\xi_1} + 2KFF' \ln \xi_1 \dot{\xi}_1.$$

Bring the terms $K \frac{F^2 \xi_1}{\xi_1} + 2KF' \ln \xi_1 \xi_1$ to the left side and we can rewrite the above equation as

$$\frac{d}{ds} (KF^2(\xi_1) \ln s - KF^2(\xi_1) \ln \xi_1) = -2F(\xi_1) \xi_1 + \frac{KF^2(\xi_1) \xi_1}{\xi_1}. \quad (69)$$

Integrate (69) from s_0 to s

$$KF^2(\xi_1(s)) \ln \left(\frac{s}{\xi_1(s)} \right) = \int_{s_0}^s \left(-2F(\xi_1) \xi_1 + \frac{KF^2(\xi_1) \xi_1}{\xi_1} \right) ds,$$

or

$$KF^2(\xi_1(s)) \ln \left(\frac{s}{\xi_1(s)} \right) = \int_{s_0}^{\xi_1(s)} (-2F(\xi) + K \frac{F^2(\xi)}{\xi}) d\xi. \quad (70)$$

We refer to (70) as the Equal Area Rule in the present problem. It is the analog of (26) and plays a similar role in the determination of asymptotic relations.

Let

$$\Phi(\xi) = \int_{s_0}^{\xi} (-2F(\xi) + K \frac{F^2(\xi)}{\xi}) d\xi.$$

then (70) becomes

$$KF^2(\xi_1(s)) \ln \left(\frac{s}{\xi_1(s)} \right) = \Phi(\xi_1(s)). \quad (71)$$

If we write (71) as

$$s = \xi \exp \left(\frac{\Phi(\xi)}{KF^2(\xi)} \right), \quad (72)$$

we have s as a function of ξ which is the inverse of $\xi = \xi_1(s)$.

Determination of the Shock Path

Substitute (72) into (65) to get t explicitly in terms of ξ . Since (72) gives s in terms of ξ we then have a parametric representation of the shock path or, solving for ξ in terms of t , a relation giving s as a function of t .

Asymptotic Forms

Suppose that the initial pulse, $F(\xi)$, is as shown in FIG. 6.

(FIG. 6)

We may regard $s_0 - \xi^*$ as initial pulse width. We have

$$\lim_{s \rightarrow \infty} \xi_1(s) = \xi^* . \quad (73)$$

Therefore, from (71) we get

$$\lim_{s \rightarrow \infty} K F^2(\xi_1(s)) \ln\left(\frac{s}{\xi^*}\right) = \phi(\xi^*) \quad (74)$$

We have

$$\phi(\xi^*) = \int_{s_0}^{\xi^*} (-2F(\xi) + K \frac{F^2(\xi)}{\xi}) d\xi, \quad (75)$$

which has dimensions $L^3 T^{-1}$. Thus, $K\phi(\xi^*)$ has dimensions L^2 , so we define a characteristic length

$$\bar{\ell}_0 = (K\phi(\xi^*))^{1/2}. \quad (76)$$

Then from (74),

$$F(\xi_1(s)) \sim \left(\frac{\bar{\ell}_0^{-2}/K^2}{\ln s} \right)^{1/2}. \quad (77)$$

Since $u_1(s) = \frac{F(\xi_1(s))}{2s}$ and $p = \rho_0 c_0 u$ we see

$$u_1 \sim \frac{\bar{\ell}_0/K}{s(\ln s)^{1/2}} \quad (78)$$

$$p_1(s) \sim \frac{\rho_0 c_0 \bar{\ell}_0/K}{s(\ln s)^{1/2}}. \quad (79)$$

Comparing (79) with (31)

$$\rho_0 c_0 \bar{\ell}_0 / K = \rho_0 c_0 \ell_0 \sqrt{\frac{\epsilon_0}{2K}},$$

so that

$$\bar{\ell}_0 = \ell_0 \sqrt{\frac{c_0 K}{2}}. \quad (80)$$

Combining (35) and (80) we get for the impulse per unit area at range s

$$I(s) = \frac{\rho_0 \bar{x}_0^{-2}/K}{s} . \quad (81)$$

As before $\bar{x}_0$ may be expressed in terms of impulse as

$$\bar{x}_0 = \left(\frac{KsI(s)}{\rho_0} \right)^{1/2} . \quad (82)$$

Numerical Methods

Initial Value Formulation

Consider the initial value problem with initial profile

$$F(\xi, 0) = \begin{cases} F(\xi) & \xi \leq s_0 \\ 0 & \xi > s_0 \end{cases} .$$

Choose $\Delta\xi > 0$ and set

$$\xi_i = s_0 - (i - 1)\Delta\xi . \quad (83)$$

$$\text{Then set } s_i = \xi_i \exp\left(\frac{\phi(\xi_i)}{KF^2(\xi_i)}\right) . \quad (84)$$

Also define

$$t_i = \frac{s_i - \xi_i}{c_0} - \frac{KF(\xi_i)}{c_0} \ln\left(\frac{s_i}{\xi_i}\right) . \quad (85)$$

The value of shock pressure at $r = s_i$ is

$$p_i = \frac{\rho_0 c_0 F(\xi_i)}{2s_i} . \quad (86)$$

The pulse duration, $\phi(s)$, is defined by evaluating at the shock front the expression

$$\frac{-p}{\partial p / \partial t} = \frac{-F(s, t(s))}{\frac{\partial F}{\partial t}(s, t(s))} . \quad (87)$$

But $F(r, t) = F(\xi_1(r, t))$ where $\xi_1(r, t)$ is determined by the requirement that the characteristic through $(\xi_1(r, t), 0)$ also passes through (r, t) . Then

$$\frac{\partial F}{\partial t} = F'(\xi_1) \frac{\partial \xi_1}{\partial t} . \quad (88)$$

The function ξ_1 satisfies the equation

$$t = \frac{1}{c_0} \left[r - \xi_1 - KF(\xi_1) \ln \left(\frac{r}{\xi_1} \right) \right] . \quad (89)$$

Differentiating with respect to t we get

$$1 = \frac{1}{c_0} \left(- \frac{\partial \xi_1}{\partial t} - KF'(\xi_1) \frac{\partial \xi_1}{\partial t} \ln \left(\frac{r}{\xi_1} \right) + \frac{KF(\xi_1)}{\xi_1} \frac{\partial \xi_1}{\partial t} \right) ,$$

so

$$\frac{\partial \xi_1}{\partial t} = \frac{c_0}{\frac{KF(\xi_1)}{\xi_1} - KF'(\xi_1) \ln \left(\frac{r}{\xi_1} \right) - 1} . \quad (90)$$

We want to evaluate (90) at the shock as a function of s and, if we do, $\xi_1(s, t(s))$ is just the function $\xi_1(s)$ previously used. Thus we have,

$$\theta(s) = \frac{-F(\xi_1(s))}{F'(\xi_1(s))} \frac{1}{c_0} \left[\frac{K_1(\xi_1(s))}{\xi_1(s)} - KF'(\xi_1(s)) \ln\left(\frac{s}{\xi_1(s)}\right) - 1 \right]. \quad (91)$$

For numerical computation we have the following version of Eq. (91)

$$\theta_i = \frac{-F(\xi_i)}{F'(\xi_i)} \frac{1}{c_0} \left[\frac{KF(\xi_i)}{\xi_i} + KF(\xi_i) \frac{(F(\xi_{i+1}) - F(\xi_{i-1}))}{2\Delta\xi} \ln\left(\frac{s_i}{\xi_i}\right) - 1 \right]. \quad (92)$$

For $i = 2, 3, \dots$

Actually, for $i = 1$, since ξ_0 is not defined, we replace the centered difference in (92) by the simple difference quotient $\frac{F(\xi_{i+1}) - F(\xi_1)}{\Delta\xi}$ and similarly at the last ξ value.

Boundary Value Formulation:

Consider a boundary profile

$$F(s_0, \tau) = \begin{cases} F(\tau) & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}.$$

Choose N a positive integer, $\Delta\tau > 0$. For $i = 1, \dots, N$ define

$$\tau_i = (i - 1)\Delta\tau. \quad (93)$$

Then define

$$s_i = s_0 \exp \left[\frac{2c_0}{KF^2(\tau_i)} \int_0^{\tau_i} F(\xi) d\xi \right] . \quad (94)$$

The integral is evaluated numerically using a simple trapezoidal approximation.

Then we can define

$$t_i = \tau_i + \frac{s_i - s_0}{c_0} - \frac{KF(\tau_i)}{c_0} \ln\left(\frac{s_i}{s_0}\right) \quad (95)$$

$$p_i = \frac{\rho_0 c_0 F(\tau_i)}{2s_i} . \quad (96)$$

The formula for pulse duration, $\theta(s)$, is obtained as follows.

$$\theta(s) = \frac{-p}{\frac{\partial p}{\partial t}} = \frac{-F}{\frac{\partial F}{\partial t}} , \quad (97)$$

where everything is evaluated at the shock.

$F(r, t) = F(\tau_1(r, t))$ where $\tau_1(r, t)$ is determined by the requirement that the characteristic through $(s_0, \tau_1(r, t))$ also passes through (r, t) . Then

$$\frac{\partial F}{\partial t} = F'(\tau_1) \frac{\partial \tau_1}{\partial t} . \quad (98)$$

The function τ_1 satisfies the equation

$$t = \tau_1(r, t) + \frac{r - s_0}{c_0} - \frac{KF(\tau_1(r, t))}{c_0} \ln\left(\frac{r}{s_0}\right) , \quad (99)$$

so

$$1 = \frac{\partial \tau_1}{\partial t} - \frac{K}{c_0} \ln\left(\frac{r}{s_0}\right) F'(\tau_1) \frac{\partial \tau_1}{\partial t}$$

and thus

$$\frac{\partial \tau_1}{\partial t} = \frac{1}{1 - \frac{K}{c_0} F'(\tau_1) \ln\left(\frac{r}{s_0}\right)} .$$

Evaluating (99) at the shock $\tau_1(s, t(s))$ becomes the function $\tau_1(s)$ previously discussed and then

$$\frac{\partial \tau_1}{\partial t} = \frac{1}{1 - \frac{K}{c_0} F'(\tau_1(s)) \ln\left(\frac{s}{s_0}\right)} . \quad (100)$$

Therefore

$$\theta(s) = \frac{-F(\tau_1(s))}{F'(\tau_1(s))} \left[1 - \frac{K}{c_0} F'(\tau_1(s)) \ln\left(\frac{s}{s_0}\right) \right] . \quad (101)$$

For numerical computation we have

$$\theta_i = \frac{-F(\tau_i) 2\Delta\tau}{(F(\tau_{i+1}) - F(\tau_{i-1})))} \left[1 - \frac{K}{c_0} \frac{(F(\tau_{i+1}) - F(\tau_{i-1})))}{2\Delta\tau} \ln\left(\frac{s_i}{s_0}\right) \right] , \quad (102)$$

where for $i = 1$ and $i = N$ the centered differences are replaced by simple difference quotients.

Appendix A: Computing K

$$K = \frac{1}{2c_0} (c'(0) + 1) \text{ where } c = c(\sigma) \text{ and } \sigma = \int_{p_0}^p \frac{c}{p} dp. \text{ But}$$

$$\frac{\partial p}{\partial p} = c^2, \text{ so } \frac{\partial^2 p}{\partial p^2} = 2c \frac{\partial c}{\partial p} \quad (\text{Entropy fixed}) \text{ and then}$$

$$\frac{dc}{d\rho} = \frac{dc}{dp} \frac{dp}{d\rho} = \frac{1}{2c} \frac{\partial^2 p}{\partial \rho^2} \frac{p}{c} = \frac{1}{2} \frac{\partial^2 p}{\partial \rho^2} \frac{p}{c^2}.$$

$$\text{So } c'(0) = \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2} \right)_s (\rho_0) \frac{p_0}{c_0^2}.$$

The case $p = A \rho^\gamma$

$$\text{Here: } A = \frac{p_0}{\rho_0^\gamma}$$

$$\frac{\partial p}{\partial \rho} = \gamma A \rho^{\gamma-1} = c^2, \quad \frac{\partial^2 p}{\partial \rho^2} = \gamma(\gamma-1) A \rho^{\gamma-2}$$

$$c = \sqrt{\gamma A \rho^{\frac{\gamma-1}{2}}}$$

$$\frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2} \right) \frac{p}{c^2} = \frac{1}{2} \frac{\gamma(\gamma-1) A \rho^{\gamma-2} p}{\gamma A \rho^{\gamma-1}} = \frac{\gamma-1}{2}.$$

$$\text{So } c'(0) = \frac{\gamma-1}{2}.$$

Note the result $\left(\frac{\partial^2 p}{\partial \rho^2} \right)_s = (\gamma-1) \frac{c^2}{\rho}$. Using $c_0 = 1500$ m/s we get for $\gamma = 7$,
 $K = \frac{2}{c_0} = .00133$

Calculation of K using Beyer's data [6].

Expand isentropic EOS as

$$p = p_0 + \rho_0 \left(\frac{\partial p}{\partial \rho} \right)_s (\rho_0) x + \frac{1}{2!} \rho_0^2 \left(\frac{\partial^2 p}{\partial \rho^2} \right)_s (\rho_0) x^2 + \dots$$

In Beyer's notation (We use x in place of his s) we have

$$A = \rho_0 \left(\frac{\partial p}{\partial \rho} \right)_s (\rho_0) = \rho_0 c_0^2$$

$$B = \rho_0^2 \left(\frac{\partial^2 p}{\partial \rho^2} \right)_s (\rho_0) \quad .$$

But $c'(0) = \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2} \right)_s (\rho_0) \frac{\rho_0}{c_0^2}$, so

$$B = \rho_0^2 c'(0) \frac{c_0^2}{\rho_0} = \rho_0 c_0^2 c'(0),$$

$$\frac{1}{2} \frac{B}{A} = c'(0) \quad .$$

Beyer's Table 30 - 1 gives,

$$c'(0) = 2.6 \quad \text{for distilled } H_2O \text{ at } 30^\circ C \quad .$$

$$c'(0) = 2.7 \quad \text{for sea water at } 30^\circ C \quad .$$

The corresponding values of K are

$$K = .0012 \quad \text{distilled } H_2O \text{ at } 30^\circ C.$$

$$K = .00123 \quad \text{sea water at } 30^\circ C.$$

Calculation of K from Approx. Gruneisen:

$$p = \frac{\rho_0 c_0^2 \left[1 + \left(1 - \frac{\gamma_0}{2} \right) \mu - \frac{a}{2} \mu^2 \right]}{\left[1 - (s_1 - 1) \mu - s_2 \frac{\mu^2}{(\nu + 1)} - s_3 \frac{\mu^3}{(\nu + 1)^2} \right]^2} + (\gamma_0 + a\mu)E \quad ,$$

where

$$\mu = \frac{\rho - \rho_0}{\rho_0} \quad .$$

For H_2O

$$c_0 = 0.14823$$

$$s_1 = 2.56$$

$$s_2 = -1.986$$

$$s_3 = 0.2268$$

$$\gamma_0 = 0.11$$

$$a = 0$$

Neglecting higher powers of μ we get

$$p = \rho_0 c_0^2 \left(\mu + \left(s_1 - 1 - \frac{\gamma_0}{2} \right) \mu^2 \right)$$

$$= \rho_0 c_0^2 (\mu + 4.065 \mu^2) .$$

So $A = \rho_0 c_0^2$, $B = 2(4.065) \rho_0 c_0^2$. Thus $c'(0) = \frac{1}{2} B/A = 4.065$

and $K = .00168$

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