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Pulsations of Delta Scuti Stars

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Abstract

A general review of the pulsating δ Scuti variables is given including the observed light curves and positions of the stars in the Hertzsprung-Russell diagram. Theoretical interpretations from evolution and pulsation calculations give their masses, radii, luminosities, and even their approximate internal compositions. Three models of these stars are discussed and used to study the nonlinear hydrodynamic behavior of these stars. The hydrodynamic equations and the Stellingwerf method for obtaining strictly periodic solutions are outlined. Problems of allowing for time-dependent convection and its great sensitivity to temperature and density are presented. Tentative results to date do not show any tendency for amplitudes to grow to large unobserved amplitudes, in disagreement with an earlier suggestion by Stellingwerf. It is found that the very small growth rates of the pulsations may even be too small to be useful in seeking a periodic solution.

I. Introduction

The δ Scuti variables are the most common type of variable star in our galaxy except for the white dwarfs. This is because stars in the mass range from just over one $M_{\odot}$ up to at least several $M_{\odot}$ pass through the yellow giant instability strip in the Hertzsprung-Russell diagram as they evolve off the main sequence to the red. Actually stars up to the maximum main sequence mass also evolve through this region at higher luminosities, but there are so few of them, and they evolve so rapidly to the red, that they are almost unknown. At the higher luminosity, they probably would be called first instability strip crossing Cepheids anyway. Such Cepheids are difficult to separate from those that are on the second blueward instability strip crossing that is much slower. Really the δ Scuti variables are just low luminosity Cepheids.

Stars of both populations start their homogeneous composition evolution on their main sequences, and at ages usually less than 10^9 years they cross the δ Scuti instability

strip. They will be observed as radially and nonradially pulsating stars if their helium content in the surface pulsating layers is large enough for their natural pulsation driving to overcome their deep radiative damping.

Mean radii for these stars when they are pulsationally unstable range from about 1.3 to over $5 R_{\odot}$. During the pulsations the radius changes over a cycle by only a few percent, as contrasted with the much larger pulsation amplitudes of the Cepheids. The radius and light amplitudes are small for these stars, as observed, because the pulsation driving is smaller and more easily saturated relative to the damping, than for the higher luminosity Cepheids.

Even though the masses of these variable stars are not very much larger than that for our Sun, the mean luminosities range from 5 to over $50 L_{\odot}$. This makes these stars easily observable in nearby galactic clusters. Indeed extensive studies of these stars have been done by many including Breger (1979) who has reviewed the pulsations of these stars.

The instability strip in the Hertzsprung-Russell diagram is between about 6900 K and 8100 K. This range is due to the fact that at the hot side of the strip the driving of pulsations from the cyclical variation of the opacity (κ) effect is becoming too shallow to involve enough of the mass, and at the cool side, the time varying convection is disrupting this κ effect driving.

Models of these stars reveal that there is indeed very strong convection carrying, typically, all but a few percent of the luminosity at the maximum. As discussed later, this convection is able to keep up rather well with the changing configuration during the pulsations, and therefore may be able to interfere with the periodic radiation flow blocking caused by the κ effect.

Due to the small size of these stars, their pulsation periods range from just 0.05 to 0.25 day. Only the convection-forced solar oscillations and the white dwarf pulsations have shorter periods for their intrinsic pulsations. Many of these δ Scuti variables display two periods, and often the ratio is between 0.76 and 0.79. This ratio is that expected from the radial fundamental and first overtone pulsation modes. For a few stars the ratio is slightly higher, and it has been suggested that the modes occurring then are the first and second overtone. However, helium depletion in the surface layers may change the stellar structure enough to interpret still the modes as the fundamental and first overtone (Cox, McNamara, and Ryan, 1984). Stars like 4 CVn, and a few

others (Breger et al., 1990), which show 5 pulsation modes, must have some or even all these modes nonradial with low degree so that they are observable in the integrated light of the entire stellar disc.

Table 1 gives observational data and the theoretically interpreted masses, radii, and luminosities for 9 δ Scuti variable stars that have 2 modes, apparently the fundamental and first overtone. There has been some question as to whether there should be an enhancement of the opacity of the stellar material between temperatures like 150,000 K and 800,000 K that would make theoretical period ratios larger (Andreasen, 1988). In the earlier work presented in Table 1, these effects have not been included, but the derived masses, radii, and luminosities would not be changed much from those given, because the data depend mostly on theoretical evolutionary track data in the Hertzsprung-Russell diagram.

These data could be derived even more simply for a given variable star by noting the luminosity intersections of the observed effective temperature line and the evolutionary tracks and the luminosity intersections of the constant temperature line with lines of the observed constant period for a series of masses. The stellar luminosity and mass would then be at that intersection luminosity where the mass given by the evolution track is the same as that given by the constant (observed) period line. Actually the exhaustion of the central hydrogen causes a jog in the evolution tracks just after the main sequence stage, and there are sometimes three possibilities for the luminosity at a given effective temperature. The blueward track, however is very rapid, and it is not likely that we see δ Scuti stars there. Fortunately the theoretical period ratio for these observed stars at the listed mass, luminosity, and radius is within 0.01 that observed. One should note how accurately the masses, radii, and luminosities are constrained.

Breger, in his review, plots the light curves for two extreme examples of δ Scuti variables. These are shown in Figure 1. Unlike the larger amplitude Cepheids, these stars do not show bumps in their light curves other than those that are the result of multimode superpositions. With such small amplitudes, it is very accurate to compare observed periods with theoretical predictions, because the linear theory used is not as far from reality as for many other classes of variable stars. The several tenths of a magnitude light variation corresponds to ± 10 km/s approximately, and that corresponds to only $\pm$ several percent radius variation, typically.

Figure 2 gives Breger's observations of δ Scuti variable candidates, and shows the instability strip in the Hertzsprung-Russell diagram. Note that a large fraction of the stars do not pulsate at the minimum observable level of a few thousandths of a magnitude. It could be that the amplitudes are even smaller, or, as I prefer to think, these stars have lost helium by slow diffusive settling from the surface pulsation driving region near 40,000 K.

Figure 3 discusses the helium abundance situation as published by Cox, King, and Hodson (1979). With advancing age, the helium content (plotted on the left ordinate) decreases by settling downward, unless the star rotates quite rapidly to keep it mixed. The decreased helium, to about 0.1 in Y, eventually causes the helium convection zone to disappear. Then the single well-mixed hydrogen convection zone is considerably more shallow, and radiation levitation processes can work for some metals at much lower densities. Then the surface hydrogen convection zone quickly can become contaminated with metallicity, producing δ Del stars or Am stars. The theoretical prediction is then that the simultaneous occurrence of both pulsations and metallicity can be only for the coolest δ Scuti variables. This prediction has been realized recently for HD 1097, where Kurtz (private communication) has found pulsations in an Am star, which Breger reports is just at the red edge of the pulsation instability strip.

Another prediction is that near the red edge, stars do not need much helium to pulsate, and therefore stars there should all be variable. This does not accord with the Breger observations of Figure 2 unless the stars are all extremely helium poor and display conspicuous metallicity. Observation of double-mode δ Scuti variables near the red edge should show an occasional anomalous period ratio as predicted by Cox, McNamara, and Ryan (1984) for low helium abundance in the surface layers.

On the right ordinate of Figure 3 is an indication of the luminosity of the models for a fixed period of 0.11 day. At higher effective temperature the luminosity is higher for fixed period, that is, almost for a fixed radius.

The motivation for our hydrodynamic studies to be explained below is that Stellingwerf (1979) suggested that there is a theoretical "catastrophe" that predicts a very large limiting pulsation amplitude of these stars. The possibility of this giving significant mass loss was discussed by Willson, Bowen, and Struck-Marcell (1987). This then led to calculations (Guzik, thesis 1988) of evolution tracks for stellar models for masses just above $1 M_{\odot}$. They indicated that standard theoretical predictions of evolution tracks

may need major modifications. Can we, with the latest techniques, predict the observed low pulsation amplitudes, or, do we still have the theoretical problem?

The work reported here is an ongoing collaboration with Joyce Guzik and Dale Ostlie. The goals are to produce a time-dependent convection hydrodynamic program and to apply it to δ Scuti and RR Lyrae variable stars.

II. Equilibrium Models

A homogeneous composition model of $1.8 M_{\odot}$ with composition $X=0.70$ and $Z=0.02$ was evolved using the Iben (1965) evolution program from the main sequence to the middle of the δ Scuti instability strip at an age of 0.94×10^9 years. This gives a luminosity of $23.2 L_{\odot}$. The evolution was advanced enough to exhaust the hydrogen completely from the inner 8 % of the mass, and hydrogen is burned slightly all the way out to a temperature of about 5×10^8 K, with 80 % of the mass interior. In spite of this enhancement of helium at the bottom of our pulsation models by about 0.02 in Y , we assume that the pulsation envelopes are homogeneous with the surface Y of 0.28.

Table 2 gives the properties of three models that we have considered for linear theory radial and nonradial pulsations as well as nonlinear 60 mass shell hydrodynamic calculations. All these studies are nonadiabatic by including the energy conservation equation with radiation allowed to flow in the diffusion approximation. For the convection model, the standard mixing length theory was used. To keep the convection as weak as possible for our hydrodynamics, the surface effective temperature for that model was set at 7800 K, rather than the 7263 K for the deep and shallow radiation models.

III. Linear Radial and Nonradial Pulsations

The 553 mass shell evolution model was reproduced by our model code with 1600 zones to calculate selected radial and nonradial linear periods and growth rates. This model includes the entire mass of the model, and it includes the detailed composition change due to nuclear burning in the central regions. We were able to confirm the general behavior of the growth rates that was previously given by Lee (1985) and shown in Figure 4a. Figure 4b shows the Guzik calculations of the variation of the kinetic energy growth rate (labeled here work) as a function of the angular frequency in units

of radians per second (labeled ω). Only the radial and dipole ($l=1$) p-modes are given.

Lee shows that the variations with frequency are even larger as the l value is increased to 2 and 3. Presumably these variations are caused by the relative sampling of the driving and damping regions by the modes. We will see these regions later with the much coarser hydrodynamic zones.

The evolution model was also recast into three different 60 mass shell models for the hydrodynamic calculations. Table 3 gives the periods, period ratios for adjacent modes, and the growth rates in units of kinetic energy change per period for the first 7 radial modes with this coarse zoning. Comparison with Figure 4b shows that the deep radiative model reproduces the more accurate growth rates well, but the shallow model does not properly sample the fundamental mode damping that is deeper than where it has resolution. It also senses much more damping for the higher order modes, because it has better resolution in the surface layers than the deep model.

The shallow model mode periods given in Table 3 are a few percent smaller than for the deep model. This is because the immovable bottom boundary condition is too shallow to represent the true non-zero normal mode motions. Note that the convection model data in Table 3 with the smaller radius has shorter periods for all its modes, as expected.

The temperature variation of the opacity is given for the deep radiative model in Figure 5. This quantity is the key one for determining the pulsational stability of stars, because it indicates whether the opacity during the compression stage of a pulsation cycle increases or decreases. If the opacity is higher during compression, radiation is temporarily blocked. Then during the re-expansion phase this radiation can flow easier to increase the internal energy and pressure of the overlying mass layers as they move outward. This phasing gives pulsation driving. However, if the opacity is decreased during compression, radiation can leak away and not be available when the expansion phase occurs. Then this gives radiation damping, which is common in the deep layers of all stars.

Three temperature regions are seen in Figure 5 where the opacity either increases with an increase of temperature, that is, during compression, or at least does not decrease very much. Even a negative value of the temperature partial derivative is not necessarily damping, because upon compression the density increases with always an

opacity increase from this density effect. The zones near the surface on the right are in the hydrogen and first helium ionization region, with typically the opacity increasing as the 12th power of the temperature. Near zone 47, the second helium ionization is occurring in the model. Finally, the second helium ionization coupled with the increasing photon energy at about 150,000 K gives the Stellingwerf (1978) "bump" near zone 36. All these opacity "bumps" give the well-known κ effect. The low gamma of the material in the hydrogen and helium ionization regions promotes very large compressions and not very large temperature increase there so that the opacity does indeed increase in the compression stage. The hiding of the luminosity from the γ effect is another process that adds to the driving. Deeper, the material is "harder" without the ionization occurring, and radiative damping always occurs in stars at temperatures greater than those for the ionization of an abundant element like hydrogen or helium.

Figure 6 shows the eigensolution for the work per zone each pulsation cycle as a function of the zone numbers again. One can see that the hydrogen ionization region is driving, but there is not much mass involved to move the entire stellar model. Most of the driving for δ Scuti variables is due to the ionization of helium. There is a non-negligible amount due to the Stellingwerf bump near zone 31 also. The integral, that one can make by eye in Figure 6, is positive, giving pulsation instability for this deep radiative model.

IV. Hydrodynamic Methods and Results

The 60 zone models are run in the hydrodynamics program with the equilibrium hydrostatic structure plus the linear $\delta r/r$ eigenvector superimposed as the velocity structure. All problems have been started with an outward surface velocity of 10 km/s. This velocity amplitude is equivalent to the few tenths of a magnitude light variation that is seen in many δ Scuti variables.

The completely implicit hydrodynamic method uses either three or four equations, depending upon whether convection is allowed or is completely ignored in both the equilibrium model and the hydrodynamics calculation. The equations are the conservation of energy with units of erg/g , the conservation of momentum with units of cm/s^2 , the velocity integral equation with units of cm/s , and the convective velocity equation with again units of cm/s . The form used has all four G_j values equal to zero when the time step, dt , between the previous time p and the final time f is completely converged.

$$\begin{aligned}
G_1 &= Ef(i) - Ep(i) + \frac{Pf(i) + Pp(i)}{2} (Vf(i) - Vp(i)) - dt \frac{(sourcef(i) + sourcep(i))}{2} \\
&\quad + \frac{dt}{mass1(i)} (\theta(Lf(i) - Lf(i-1)) - (1-\theta)(Lp(i) - Lp(i-1))), \\
G_2 &= \frac{area(i)}{mass2(i)} \left\{ \frac{Pf(i+1) + Pp(i+1)}{2} - \frac{Pf(i) + Pp(i)}{2} \right\} - g(i) + \frac{(\dot{r}p(i) - \dot{r}f(i))}{dt}, \\
G_3 &= \frac{2}{dt} (rf(i) - rp(i)) - (\dot{r}f(i) + \dot{r}p(i)), \\
G_4 &= v_c f(i) - vel f(i).
\end{aligned}$$

In these equations for *mass1* shell-centered and *mass2* interface-centered, *E* is the internal energy including turbulent energy, *P* is pressure including turbulence, turbulent viscosity, and artificial viscosity, *V* is the specific volume of the shell mass, *source* is the nuclear energy source, θ (usually set to 2/3) is the off-centering factor on luminosity for stabilization of the implicit method, *L* is the luminosity including both radiation and convection, *area* is a specially time-centered interface area (Fraley, 1968), *g* is the local acceleration of gravity, $\dot{r}$ is the interface velocity, with *r* the interface position, and *vel0*, and *vel* the previous iteration convection velocity and the current convection velocity that is a function of all local variables.

These equations express the basic concepts of mass, momentum, and energy conservation plus a constraint for the convection luminosity. The mass conservation equation is not one of the four above, because it is not a differential equation. Yet it relates the fixed Lagrangian shell mass with density using the time-changing interface radii. The energy equation is the standard first law of thermodynamics. The *PdV* work term allows for the energy to be converted to or from kinetic and potential energy. This role of the *PdV* term can be seen from consideration of the momentum equation. The momentum equation comes in two parts because the pressure gradient produces an acceleration, a second order time derivative. Thus the second and third equations are both expressions of the momentum conservation. The convection luminosity equation is a constraint that assures that during the iterations at the *f* step, the convective velocity at the previous iteration is the same as that given by its analytic expression at the current iteration. Convergence is attained when these equations have their *G* less than typically 10^{-6} of their largest term.

The bottom boundary condition is an unmovable interface so that the conditions like temperature, density, pressure, and energy interior do not play any role. The luminosity across that bottom interface is the surface luminosity. The surface boundary conditions are that the gas pressure beyond the mesh is zero while the radiation pressure is the same as in the outmost mass shell. There is no convection luminosity across the surface, and the radiation luminosity is given by the standard Eddington approximation

$$Lf(I) = 4\pi r f(I)^2 \sigma T_e^4$$

with

$$Tf(I)^4 = 3/4 T_e^4 (\tau + 2/3)$$

where τ is the Rosseland mean opacity optical thickness of the last mass shell of temperature $Tf(I)$ and specific volume $Vf(I)$. The last interface *mass2(I)* is taken traditionally 7.7 times the last shell mass for the momentum equation G_2 to roughly account for the exterior chromosphere and coronal mass.

The solution method for these equations is to assume than an extrapolated p -step is a good approximation for the equations above. Then the increments to the four independent variables $T, r, \dot{r}$, and v_c are calculated to make the basic hydrodynamic equations become zero. Thus:

$$\begin{aligned} G_j + \sum_i \frac{\partial G_j}{\partial T_i} \delta T_i + \sum_i \frac{\partial G_j}{\partial r_i} \delta r_i \\ + \sum_i \frac{\partial G_j}{\partial \dot{r}_i} \delta \dot{r}_i + \sum_i \frac{\partial G_j}{\partial v_{c,i}} \delta v_{c,i} = 0. \end{aligned}$$

The sums consists of 15 terms for the analytic G_1 derivatives, 19 terms for the analytic G_2 derivatives, 2 terms for the analytic G_3 derivatives, and 11 terms for the analytic G_4 derivatives. Solution of the derivatives matrix for a correction vector for the four independent variables with the residual G_j vector as the right hand side is done repeatedly with new derivatives calculated each iteration. This Newton-Raphson method results in all $G_j = 0$ at convergence.

This method works extremely well for purely radiative models, but there is considerable trouble when the convection is allowed to transport luminosity. In all cases, it is necessary to limit the corrections to reasonable values so that the solution iterations

do not diverge. The convection luminosity is highly nonlinear and very sensitive to the values of the independent variables. Thus the allowance for complicated and extreme nonlinearity by recalculating the matrix elements each iteration is not necessarily enough to assure convergence. Frequently the time step needs to be reduced to perhaps 1/400 or less of the pulsation period to allow convergence.

Analysis of convection zones in pulsating stars reveals that there is a good deal of sensitivity of the convection luminosity to the independent variables T and ρ . The convection luminosity goes as the 10th power of T and as the 10⁴ power of ρ , but only as the 10th power of density.

The convection luminosity is calculated allowing for the time delay that occurs when the convective elements are forced to adjust to the changing pulsation configuration. The convective velocity is interpolated with a three-point Lagrangian method between the velocity at time step $n - 2$, step $n - 1$, and the current time n , where the current time instantaneous value v^o is given directly by the mixing length formulas. For any zone at the interpolated time, t' ,

$$v_c^n = \frac{(t' - t^{n-1})(t' - t^{n-2})}{(t^n - t^{n-1})(t^n - t^{n-2})} v_c^o + \frac{(t' - t^n)(t' - t^{n-2})}{(t^{n-1} - t^n)(t^{n-1} - t^{n-2})} v_c^{n-1} \\ + \frac{(t' - t^n)(t' - t^{n-1})}{(t^{n-2} - t^n)(t^{n-2} - t^{n-1})} v_c^{n-2},$$

where

$$t' = t^{n-1} + \tau(t^n - t^{n-1}),$$

with

$$\tau = \frac{v_c^n(t^n - t^{n-1})}{\ell} lfac,$$

and τ , the fraction of a mixing length the mean convective eddy can move in one time step, is limited to be between 0 and 1. We have used $lfac$ as unity, but it is available for any adjustment of the lagging.

This convective velocity at the current timestep is checked to see that it has not increased or decreased more than physically possible by the buoyancy or drag forces. This increment is:

$$\delta v_c = gdt|\Delta\rho/\rho|afac = gdtQ|\Delta T/T|afac$$

where ΔT is the excess temperature above or below the adiabatic gradient between adjacent mass shells, Q is the thermodynamic partial derivative of density with respect to temperature at constant pressure and composition, and $afac$, the acceleration factor, is another adjustable parameter near unity.

There is also spatial averaging for the mean convective element velocity. The velocity used in the mixing length formula for the convection luminosity is then:

$$\bar{v}_{c,i} = a_{i-1}v_{c,i-1}^{n-1} + a_{i+1}v_{c,i+1}^{n-1} + (1 - a_{i-1} - a_{i+1})v_{c,i}^n$$

with $v_{c,i}^n$ the lagged convective velocity. The weighting factor for zone k is

$$a_k = (1 - \frac{|r_k - r_i|}{\ell})avgfac,$$

subject to the constraint that a_k lie between zero and $avgfac$, and $avgfac$ being less than or equal to 1/3.

The turbulent pressure and energy per gram are:

$$P_t = \rho \bar{v}_c^2 t fac \quad \text{and} \quad E_t = \bar{v}_c^2 t fac.$$

Here $t fac$ is taken as 1/3. Further the turbulent viscosity pressure, applied both on compression and expansion depends on the velocity gradient and is:

$$Q_t = \rho \bar{v}_c H_p \Delta \dot{r} / \Delta r q fac,$$

with H_p being the pressure scale height in the model and $q fac$ being of the order of unity.

In addition, there are two other viscosities. Hydrodynamic calculations that produce strong shocks require artificial viscosity to maintain numerical stability. This artificial viscosity is applied only when a zone is being compressed, and it uses the same expression as above with the convection element velocity replaced with the zone interface velocity gradient. There is also a linear viscosity used for the deepest mass shell in our hydrodynamics program, and it is calculated as above but with the sonic velocity replacing the convective element velocity. For both of these viscosities, we have cancelled the pressure scale height with the interface position difference. We have reduced the coefficient in these viscosities to be sure that the artificial damping is small

compared to the natural driving and damping of the stellar pulsations. We also use a modified Stellingwerf (1974) procedure of not applying the artificial quadratic viscosity until the relative interface closure velocity exceeds a threshold of 0.02 times the sonic velocity.

The hydrodynamics with convection is made difficult because of the temperature gradient becoming suddenly sub- or superadiabatic. This gives a large change in the convective and total luminosities. When the gradient becomes subadiabatic after being superadiabatic, the convective velocity is not put directly to zero, because physically there will be a drag on the convective elements that is not able to stop the turbulence immediately. The convective luminosity in a subadiabatic region then should be put in as a negative (backwards) value, but we have not developed this procedure yet.

The time behavior of the convection luminosity for the convection model is given in Figure 7. At this early stage of our calculations we cannot tell whether this convection luminosity is lagging enough to produce pulsation driving. Apparently it does not lead the maximum compression to nullify the periodic luminosity blocking from the κ effect. The convection does reach about 90% of the total luminosity each cycle, but for only a brief time. Note that the convection behavior with almost all the luminosity as convection luminosity at the start of the run is greatly modified later by the time lagging and spatial averaging.

V. Periodic Solution Methods

Stellingwerf (1974) developed a method for quickly finding a full amplitude nonlinear solution for stellar pulsations. We have added it to our regular hydrodynamics code so that we can reach limiting amplitude in only a few trial periods instead of calculating the initial value problem for possibly millions of periods of the mode of interest. Not only does this method yield the desired solutions, but it also can be used to study the stability of the full amplitude solution against switching to another of the normal modes. Thus, if double-mode behavior occurs, the full amplitude solutions of the two modes should be unstable against switching to each other.

The method starts by postulating that there is a periodic condition, such that, $Z_j^0 - Z_j^N = 0$, where Z_j stands for T_j , r_j , $\dot{r}_j$, and $v_{c,j}$, and Z_j is the desired periodic solution with complete closure of all variables at the end of the period. Since we want to correct all the independent variables before each trial period, and we also want to

adjust the period each trial, we need another equation for this additional quantity $\delta\Pi$. This extra equation is a uniqueness condition $r_i^0 - r_i^N = 0$. In other words, zone i will always be started with the same radius value for all the trial periods and in the final periodic solution. We find that the method works best when this fiducial zone is deep enough to have a temperature at the starting time between 100,000 and 500,000 K.

We use a Newton-Raphson method to converge on the periodic solution:

$$Z_j^0 - Z_j^N + \left(\frac{\partial Z_j^0}{\partial Z_j^0} \right) \delta Z_j^0 - \sum_k \left(\frac{\partial Z_j^N}{\partial Z_k^0} \right) \delta Z_k^0 - \left(\frac{\partial Z_j^N}{\partial \Pi} \right) \delta \Pi = 0,$$

where

$$Z_j^0 = Z_j^0 + \delta Z_j^0$$

and

$$\tilde{\Pi} = \Pi + \delta \Pi$$

The method adjusts Z_j^0 and Π to get both the periodicity and the uniqueness. For this latter condition,

$$r_i^0 - r_i^N + \delta r_i^0 - \sum_k \left(\frac{\partial r_i^N}{\partial Z_k^0} \right) \delta Z_k^0 - \frac{\partial r_i^N}{\partial \Pi} \delta \Pi = 0.$$

One can see that the period correction is the fiducial radius closure error divided by the derivative of the fiducial radius with respect to the period:

$$\delta \Pi = \frac{r_i^0 - r_i^N}{\frac{\partial r_i^N}{\partial \Pi}},$$

since the postulated correction to r_i is:

$$\delta r_i^0 = \sum_k \frac{\partial r_i^0}{\partial Z_k^0} \delta Z_k^0 = 0.$$

With this $\delta \Pi$ known, it is now possible to solve for the δZ_j^0 for all j . The above Newton-Raphson equation can be written in the form:

$$\delta Z_j^0 = c_j \delta \Pi + d_j$$

where the c_j vector carries information about how each zone reacts to the period change, and the d_j vector indicates how good the closure was at the end of the trial period for zone j . These vectors are the result of a matrix solution, where the matrix is the sum of the unity matrix $\frac{\partial Z_j^0}{\partial Z_j^0}$ and the Floquet matrix $\frac{\partial Z_j^N}{\partial Z_k^0}$. Thus the matrix here is the Floquet matrix modified by having unity subtracted from each main diagonal element.

The problem now is to see how to relate each independent variable at the end of the trial period to all of them at the beginning and to relate all variables to a change in the period. This is done by induction during each trial period using always a fixed set of hydrodynamic timesteps. The equations of energy, momentum and velocity for zone j in the regular hydrodynamics are:

$$G_j^n(Z_k^n, Z_k^{n-1}, \Pi) = 0,$$

where k denotes all neighboring zones and zone j . Taking partial derivatives of this G_j , we get:

$$\begin{aligned} \delta G_j^n &= \left(\frac{\partial G_j^n}{\partial Z_k^n} \frac{\partial Z_k^n}{\partial Z_j^0} + \frac{\partial G_j^n}{\partial Z_k^{n-1}} \frac{\partial Z_k^{n-1}}{\partial Z_j^0} \right) \delta Z_j^0 \\ &+ \left(\frac{\partial G_j^n}{\partial Z_k^n} \frac{\partial Z_k^n}{\partial \Pi} + \frac{\partial G_j^n}{\partial Z_k^{n-1}} \frac{\partial Z_k^{n-1}}{\partial \Pi} + \frac{\partial G_j^n}{\partial \Pi} \right) \delta \Pi \\ &= 0. \end{aligned}$$

Here the multiplications and additions are matrix operations. Note that for this to be zero regardless of the corrections to Z_j and Π ,

$$\frac{\partial Z_k^n}{\partial Z_j^0} = - \left(\frac{\partial G_j^n}{\partial Z_k^n} \right)^{-1} \frac{\partial G_j^n}{\partial Z_k^{n-1}} \frac{\partial Z_k^{n-1}}{\partial Z_j^0}$$

and

$$\frac{\partial Z_k^n}{\partial \Pi} = - \left(\frac{\partial G_j^n}{\partial Z_k^n} \right)^{-1} \left(\frac{\partial G_j^n}{\partial \Pi} + \frac{\partial G_j^n}{\partial Z_k^{n-1}} \frac{\partial Z_k^{n-1}}{\partial \Pi} \right)$$

The G_j derivatives are given by analytic formulae in the computing program, and for the derivatives at timestep $n + 1$, they are already known from their use in the implicit hydrodynamics.

The value of $\frac{\partial Z_1^{n-1}}{\partial Z_j}$ for the first timestep is a matrix with zero everywhere except ones along the main diagonal, and the matrix $\frac{\partial Z_1^{n-1}}{\partial \Pi}$ is zero. Each timestep the solution matrix from the previous timestep can be used on the right hand side to give the updated solution matrix. This induction gives $\frac{\partial Z_N}{\partial Z_j}$ after the fixed N timesteps of each trial period.

This periodic technique has been implemented including the time-dependent convection. However, the only results to report here are for the purely radiative models. An example of how the looping of a pulsation driving zone appears in the P-V plane is given as Figure 8. As the iterations converge, the loops become identical. While convergence in this example is not yet attained to a very small error, it does seem that convergence will occur at this amplitude very close to that observed for these variable stars.

VI. Current Conclusions

Linear theory studies of δ Scuti variables have reasonably well mapped the behavior of models in both radial and nonradial modes across the instability strip. The current frontier is to understand the nonlinear behavior of these stars, especially when proper allowance for time-dependent convection is made. An interesting question will be why the amplitudes are limited to such small values. Near-main sequence pulsators like the B stars, the δ Scuti stars, and even the Sun, are low amplitude pulsators. Are they still limited in their amplitudes by nonlinear effects as the classical yellow giants are? When our studies are finished, we should be able to answer this question, at least for the δ Scuti stars.

It may be that Stellingwerf did not include in his nonlinear calculations enough deep damping. We expect that our models are deep enough to consider properly this effect, and the low amplitudes that we get may be more nearly correct. When our models are tightly converged, we will be able to comment more completely about this question.

Convection has been implemented in the periodic Stellingwerf method, and runs are being made. The extreme sensitivity of the convection luminosity has not yet been overcome, however.

From early runs for very slowly growing modes, it appears that we may not be able to use the periodic method if the growth rate in kinetic energy is less than 10^{-4} per period. We know this because the Floquet matrix $\frac{\delta Z^N}{\delta Z^0}$, of rank 3 or 4 times the number of Lagrangian mass shells of the model, can be analyzed for its eigensolutions, which are the normal modes of the stellar model. They represent all the ways that the other modes can either grow or decay in the environment of the current hydrodynamic cycle. Among these eigensolutions is one that represents only a phase change of the current solution. That is, it corresponds to the vector δZ_j^0 at the trial period start, which is exactly the same as the actual timestep increment of all the independent variables times a real normalization constant. The eigenvalue of this eigensolution should be real and exactly unity, but it differs from unity by about one part in 10^4 . If the accuracy and stability of the hydrodynamic integration can be improved, though, we may be able to find eigenvalues even closer to unity and reach smaller growth rates. We have already programmed the key parts of the matrix manipulations in double precision to avoid any possible almost exact cancellations.

Our very tentative conclusion is that we see no reason for the large amplitudes that Stellingwerf found in his periodic solutions. Perhaps the deep damping that we have or the time-dependent convection will give us the amplitude limitation that we need.

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TABLE 1
THEORETICAL MASSES, RADII, AND LUMINOSITIES
FOR DOUBLE-MODE δ SCUTI VARIABLES

Variable	Π_0 (d)	Π_1/Π_0	T_e (K)	$M_T/M_\odot$	$R_T/R_\odot$	$L_T/L_\odot$	Q_0 (d)
SX Phe*	0.05496	0.778	7850	1.1 ± 0.1	1.3 ± 0.1	5 ± 1	0.0325
CY Agr**	.06104	.744:	7930	1.4 ± 0.1	1.7 ± 0.1	10 ± 1	.0326
AI Mic	.0654	.763	7500	1.4 ± 0.1	1.8 ± 0.1	10 ± 1	.0327
U Ma	.08602	.773	7500	1.6 ± 0.1	2.2 ± 0.3	14 ± 4	.0328
AI Ari	.09313	.773	7500	1.6 ± 0.1	2.3 ± 0.3	14 ± 4	.0328
AI Peg	.10954	.772	7500	1.8 ± 0.1	3.0 ± 0.3	25 ± 5	.0328
AI Vel	.11157	.773	7620	1.8 ± 0.1	2.9 ± 0.3	25 ± 5	.0328
V703 Sco	.14996	.768	7000	1.9 ± 0.1	3.9 ± 0.5	33 ± 8	.0329
VX Hya	0.22339	0.773	6980	2.2 ± 0.1	4.8 ± 0.4	48 ± 8	0.0330

1st crossing assumed

1st or third crossing

values for SX Phe, CY Agr, AI Vel, and VX Hya from McNamara and Itz (1978).

V703 Sco, T_e is from Jones (1975). For all variables the deep interior composition is $Z = 0.01$ with Y between .2 and .3 except for SX Phe with $Z = 0.001$ with interior Y between .2 and .3.

Table 2
 δ Scuti Variable Star Model Parameters
 $1.8 M_{\odot}$, $23.2 L_{\odot}$, $X=0.70$, $Z=0.02$, 60 mass shells

Model	Deep Radiation	Shallow Radiation	Convection
Effective Temperature (K)	7263	7263	7800
Radius (10^{11} cm)	2.116	2.116	1.835
Outer Shell Mass (g)	6×10^{23}	6×10^{23}	2×10^{23}
Outer Shell Optical Thickness	0.06	0.06	0.02
Central Ball Mass (g)	9.8×10^{32}	2.6×10^{33}	1.18×10^{33}
Central Ball Mass Fraction	0.27	0.73	0.33
Central Ball Radius Fraction	0.08	0.25	0.11
Mixing Length Ratio	—	—	1.768
H Convection Zone Min. T (K)	—	—	7800
H Convection Zone Max. T (K)	—	—	26000
Radiative Luminosity Fract. Min.	—	—	0.022 at 12800 K
He Convection Zone Min. T (K)	—	—	35000
He Convection Zone Max. T (K)	—	—	51000
H Convection Zone Min. q	—	—	5×10^{-11}
H Convection Zone Max. q	—	—	1.5×10^{-9}
H Convection Zone Min. x	—	—	0.9979
H Convection Zone Max. x	—	—	0.9999

Table 3
Radial Mode Periods, Period Ratios, and Growth Rates
Deep Radiative Model

Model	Period (day)	Period Ratio	$\Delta KE/KE$ per period
Fundamental	0.130	0.771	0.0000028
First Overtone	0.100	0.816	0.000049
Second Overtone	0.082	0.839	0.00021
Third Overtone	0.069	0.858	0.00037
Fourth Overtone	0.059	0.876	-0.00022
Fifth Overtone	0.052	0.889	-0.0038
Sixth Overtone	0.046	—	-0.0118

Shallow Radiative Model

Model	Period	Period Ratio	$\Delta KE/KE$ per period
Fundamental	0.128	0.762	0.0000035
First Overtone	0.098	0.807	0.000066
Second Overtone	0.070	0.832	0.00028
Third Overtone	0.066	0.854	0.00040
Fourth Overtone	0.056	0.873	-0.00100
Fifth Overtone	0.049	0.886	-0.0070
Sixth Overtone	0.044	—	-0.0179

Convective Model

Model	Period	Period Ratio	$\Delta KE/KE$ per period
Fundamental	0.105	0.769	0.0000070
First Overtone	0.081	0.815	0.000097
Second Overtone	0.066	0.840	0.00053
Third Overtone	0.055	0.857	0.00167
Fourth Overtone	0.047	0.875	0.0040
Fifth Overtone	0.041	0.890	0.0076
Sixth Overtone	0.037	—	0.0096

Captions

Figure 1. Light curves for a typical low amplitude and a typical high amplitude δ Scuti variable star as presented by Breger (1979).

Figure 2. Variables and nonvariables are plotted on the Hertzsprung-Russell diagram from observations by Breger (1979). The limits of the variables in temperature and luminosity define the δ Scuti variable instability strip.

Figure 3. The abundance of helium in the pulsation driving regions of the stellar envelope gives pulsation, no pulsation and metallicity.

Figure 4a). The growth rate for radial and nonradial pulsation modes for δ Scuti star models is plotted by Lee (1985) for the low order p-modes. b). The kinetic energy growth rate is plotted for the $l=0$ and 1 modes for similar models constructed by Guzik (private communication).

Figure 5. The logarithmic derivative of the opacity with respect to temperature is plotted versus zone for the 60 mass shell deep radiative model. Three "bumps" are shown that correspond to pulsation driving regions.

Figure 6. The work for each mass shell each pulsation cycle is plotted versus the zone number. This nonradial p_1 mode is for $l=1$. The driving is determined by the opacity behavior as well as the mass in each shell and the timescale of the radiation flow compared to the pulsation period.

Figure 7. The convection luminosity is plotted versus time for the 7800 K model. Note that for a brief time each cycle, the convection carries about 90 % of the total luminosity.

Figure 8. Several cycles of the deep radiation model are shown with the trajectory for a zone in the P-V diagram. As the periodic solution is reached the cycle repeats. This hydrogen driving zone has the looping in the clockwise direction.

Light Curves of two Delta Scuti stars: BN Cnc ($A_V = 0^m.014$) and AD CMi ($A_V = 0^m.294$). Most Delta Scuti stars resemble BN Cnc. However, a continuous range of amplitudes exist from $0^m.01$ to $0^m.8$.

FIG. 6.—The surface helium abundance Y_0 gives pulsation, nonpulsation, and metallicity for T_e values across the pulsation instability strip. The mass range is $1.5-2.5 M_{\odot}$ at 0.11 . At the cooler temperatures in the right third of the strip, pulsation and metallicity can occur simultaneously, but for hotter temperatures one excludes the other. Nonpulsating stars need not show metallicity in the hotter two thirds of the strip, but they should show metallicity for the cool third with no He convection zone if there is no other mixing process. The luminosity of the models is given on the right-hand side scale.

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1.8 Solar Mass Delta Scuti Star Unstable Modes

Fig. 2. Growth rates of unstable radial and low- l nonradial modes. The growth rates of nonradial modes are connected by solid lines.

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delta set 1.8 solar mass

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