ELECTROMAGNETIC FIELDS IN AN AXIAL SYMMETRIC WAVEGUIDE WITH VARIABLE CROSS SECTION *)

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There are many problems the solution of which can be reduced to a much simpler one if only the eigenfunctions of the corresponding equations were found. The only one known analytical solution of the Maxwell equations for the space inside an axial symmetric waveguide is the solution for the waveguide with a constant circular cross section. The solutions for a stepwise constant cross section waveguide can be obtained by properly matching the solutions for individual steps. This procedure is time and labor consuming. The example of such an approach for a stepwise periodic structure is given in work.¹¹

Exact analytical solution for a waveguide with variable cross section is not known. For the case when the corrugations of the cylindrical wall are smooth and small, the solution can be obtained by means of perturbation theory.^{2,3)} This method was recently applied to problems of finding the wake fields induced by a current moving on axis⁴⁾ and off axis⁴⁾ of a corrugated pipe.

In this paper another method is described.⁽¹⁾ It allows one to find an approximate analytical solution in a waveguide with quite arbitrary wall shape, satisfying certain conditions. The central idea of the method is a conform transformation to coordinates in which the boundary curve appears to be a straight line. In the new orthogonal curvilinear coordinates the variables are separable even in the vector wave equation. Analogously to cylindrical waveguide with constant cross section TM and TE modes are introduced. Next step is then to satisfy the boundary conditions for the electromagnetic field inside the pipe and to find the eigenvalues (the frequencies and the propagation functions).

An example of the application of the derived method as well as the comparison of the calculations with the results of measurements taken from the paper²⁾ are given.

Let us consider the following dependencies of the rectangular coordinates x, y, z on curvilinear coordinates ρ, θ, σ :

$$y = A \left(\rho + \sum_n a_n \sin n\theta \cos n\theta \right) \sin \theta \quad (2)$$

$$z = A \left(0 + \sum_n a_n \operatorname{ch} n\sigma \sin n\tau \right) + z_0 \quad (3)$$

From (1) and (2) we get for the radius in the plane $z = \text{const}$

$$r = A \left(\rho + \sum_n a_n \operatorname{sh} n\rho \cos n\sigma \right) \quad (4)$$

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Symbol $\sum_n$ means either summation or integration over n . Generally speaking, one needs to distinguish between covariant and contravariant components of any vector $\vec{F}$. To avoid this, I will use only projections F_1, F_2, F_3 of any vector $\vec{F}$ on a base system of unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Using Equations (1, 2, 3), it is easy to find the metric tensor g_{ij} .

$$g_{ij} = 0, \quad i \neq j \quad (5)$$

proves the orthogonality of the coordinate system, ρ, θ, z . From (4) follows that $r = 0$ for $\rho = 0$. Constant z_0 in (3) allows the choice of any plane $z = \text{const}$ for placing there the origin $\rho = \theta = 0$. If we put $z_0 = 0$, then the origin will be in the plane where the cross section has the maximum. The constant A is an arbitrary scale factor with the dimension of the length. The ρ, θ and z are dimensionless. In the limit $a_n \rightarrow 0$, ρ, θ and z go into usual cylindrical coordinates r, θ, z . The constants a_n are dimensionless arbitrary quantities limited only by a natural condition $r > 0$ or $\rho > 0$ everywhere. The case $r = 0$ at some value of θ should be considered separately (it represents a closed axial symmetric cavity of an arbitrary shape).

The metric coefficients $h_i = \sqrt{g_{ii}}$ are very complicated and do not give the possibility to solve the corresponding Maxwell equations. They become much simpler in the limit

$$N \rho < 1 \quad (6)$$

where N is the most sufficient harmonic number in the expansion (4). If (6) is fulfilled, then

$$h_1 = h_3 = h(\rho) \quad (7)$$

$$h_2 = \rho h(\rho) \quad (8)$$

where

$$h(\rho) = A \left(1 + \sum_n a_n n \cos n\theta \right) \quad (9)$$

For the coordinate transformation we get

$$r = \rho h(\rho) \quad (10)$$

$$z = \int_0^\rho h(\sigma) d\sigma \quad (11)$$

or excluding σ from (11)

$$r = \rho f(z) \quad (12)$$

If one chooses the function $f(z)$ in such a way as to describe the shape of the wall in the plane $\theta = \text{const}$, then $\rho = \rho_{\text{max}} = \text{const}$ on the boundary. The physical meaning of the condition (6) depends on the actual value of the coefficients a_n (or, in other words, on actual shape of the boundary). But, in the case where all a_n are small, the meaning can be understood easily. In this case, $z \approx A\theta$,

$$\begin{aligned} r_{\text{boundary}} &= \rho_{\text{max}} f(z) \approx \rho_{\text{max}} h(z/A) \\ &= A \rho_{\text{max}} \left(1 + \sum_n a_n n \cos n\theta \right) \end{aligned} \quad (13)$$

and we see that $A = L/2\pi$, where L is the characteristic length of the change of the function $f(z)$ (the period for a periodic function). Now, take the average of equation (13) over z :

$$\langle r_{\text{boundary}} \rangle = b = A \rho_{\text{max}} = L \rho_{\text{max}} / 2\pi \quad (14)$$

or

$$\rho_{\text{max}} = 2\pi b / L \quad (15)$$

where b has the meaning of the average pipe radius. The condition (6) now gives

$$b < L / 2\pi N \quad (16)$$

Since ρ changes in the limits

$$0 \leq \rho \leq \rho_{\text{max}} \quad (17)$$

the condition (6) is fulfilled for all the values of ρ , and the approximation (7)-(9) is valid everywhere inside the pipe.

Since expansion (9) contains only cosines, not all shapes of the boundary can be treated in this way. Namely, we can consider only such boundary curves $r=r(z)$, which allow the representation in the form (10), (11) with $h(\sigma)$ symmetric

$$h(-\sigma) = h(\sigma) \quad (18)$$

For the sake of completeness, it is useful to mention also the restriction of the method by axial symmetric waveguides only.

Let us confine ourselves to the case when the media inside the waveguide is a vacuum. The generalization to a homogeneous dielectric media is straightforward. In the absence of charges and currents, the fields can be defined in terms of two vector functions, $\vec{\Pi}$ and $\vec{\Pi}^*$ (the Hertz vectors); each one of them satisfies homogeneous vector wave equation. Since the coefficients of this equation are constant it is possible to deal with each time Fourier harmonic of $\vec{\Pi}$ and $\vec{\Pi}^*$ separately:

$$\vec{\Pi} = \int \vec{\Pi}_\omega e^{-i\omega t} d\omega \quad (19)$$

$$\vec{\Pi}^* = \int \vec{\Pi}_\omega^* e^{-i\omega t} d\omega \quad (20)$$

Then, $\vec{\Pi}_\omega$ and $\vec{\Pi}_\omega^*$ satisfy the equations:

$$\nabla^2 \vec{\Pi}_\omega + k^2 \vec{\Pi}_\omega = 0, \quad (21)$$

$$\nabla^2 \vec{\Pi}_\omega^* + k^2 \vec{\Pi}_\omega^* = 0, \quad (22)$$

where

$$k = \omega/c.$$

The differential operator ∇^2 here is defined by:

$$\nabla^2 \vec{\Pi}_\omega = \nabla \cdot \nabla \vec{\Pi}_\omega - \nabla \times \nabla \times \vec{\Pi}_\omega \quad (23)$$

In all the further formulae, I omit the subscript ω by Fourier components of the vector functions.

The fields can be found, provided $\vec{\Pi}$ and $\vec{\Pi}^*$ are known:

$$\vec{E}(1) = \nabla \times \nabla \times \vec{\Pi} \quad (24)$$

$$\vec{H}(1) = -ik \nabla \times \vec{\Pi} \quad (25)$$

$$\vec{E}(2) = ik \nabla \times \vec{\Pi}^* \quad (26)$$

$$\vec{H}(2) = \nabla \times \nabla \times \vec{\Pi}^* \quad (27)$$

Vectors $\vec{E}(1,2)$ and $\vec{H}(1,2)$ also satisfy equation analogous to (21).

In coordinates ρ, θ, z and with the metric coefficients (7), (8) the projections of the equation (21) on axes $\vec{e}_i$ have the following form:

$$\begin{aligned} \frac{1}{h^2} \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \Pi_1 + \frac{1}{\rho^2 h^2} \frac{\partial^2 \Pi_1}{\partial \theta^2} + \frac{1}{h^2} \frac{\partial}{\partial \sigma} \frac{1}{h} \frac{\partial}{\partial \sigma} h \Pi_1 + k^2 \Pi_1 + \frac{1}{h^2} \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial \Pi_1}{\partial \theta} \\ - \frac{1}{\rho^2 h^2} \frac{\partial}{\partial \rho} \rho \frac{\partial \Pi_2}{\partial \theta} + \frac{1}{h^4} \frac{\partial^2}{\partial \sigma \partial \rho} h^2 \Pi_3 - \frac{1}{h^2} \frac{\partial^2 \Pi_3}{\partial \sigma \partial \rho} = 0 \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{1}{h^2} \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \Pi_2 + \frac{1}{\rho^2 h^2} \frac{\partial^2 \Pi_2}{\partial \theta^2} + \frac{1}{h^2} \frac{\partial}{\partial \sigma} \frac{1}{h} \frac{\partial}{\partial \sigma} h \Pi_2 + k^2 \Pi_2 + \frac{1}{h^2} \frac{\partial^2 \rho \Pi_1}{\partial \sigma \partial \rho} \\ - \frac{1}{h^2} \frac{\partial}{\partial \rho} \frac{1}{\rho} \frac{\partial \Pi_1}{\partial \theta} + \frac{1}{\rho h^4} \frac{\partial}{\partial \sigma} h^2 \frac{\partial \Pi_3}{\partial \theta} - \frac{1}{\rho h^2} \frac{\partial^2 \Pi_3}{\partial \sigma \partial \theta} = 0 \end{aligned} \quad (29)$$

$$\frac{1}{\rho h^2} \frac{\partial}{\partial \rho} \rho \frac{\partial \pi_3}{\partial \rho} + \frac{1}{\rho^2 h^2} \frac{\partial^2 \pi_3}{\partial \theta^2} + \frac{1}{h} \frac{\partial}{\partial \theta} \frac{1}{h^2} \frac{\partial}{\partial \theta} h^2 \pi_3 + k^2 \pi_3 + \frac{1}{\rho h} \frac{\partial}{\partial \theta} \frac{1}{h} \frac{\partial}{\partial \rho} \rho \pi_1 - \frac{1}{\rho h^2} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \theta} h \pi_1 + \frac{1}{h^2} \frac{\partial}{\partial \theta} \frac{1}{h} \frac{\partial \pi_2}{\partial \theta} - \frac{1}{\rho h^2} \frac{\partial^2}{\partial \theta^2} h \pi_2 = 0 \quad (30)$$

The same system of equations holds also for the vector $\vec{E}^*$.

To solve equations (28)-(30) we look for the solution in the following form:

$$\pi_1 = J_m(\lambda \rho) F_1(\theta) e^{im\theta} \quad (31)$$

$$\pi_2 = im \frac{J_m(\lambda \rho)}{\rho} F_2(\theta) e^{im\theta} \quad (32)$$

$$\pi_3 = \lambda J_m(\lambda \rho) F_3(\theta) e^{im\theta} \quad (33)$$

A prime from now on means the differentiation with respect to ρ if it is applied to a function of ρ or with respect to θ if it is applied to a function of θ . $J_m(\lambda \rho)$ is the m th order Bessel function, λ is an arbitrary constant, and m is any positive or negative integer including zero. The function F_1 , F_2 , F_3 is unknown up to this time and should be chosen in such a way that the functions (31)-(33) satisfy equations (28)-(30).

The Hertz vector $\vec{\Pi}$ (31)-(32) allows us to find the electric $\vec{E}^{(1)}$ and magnetic $\vec{H}^{(1)}$ fields from expressions (24) and (25). Using the expressions for differential operators in our coordinates ρ , θ , ϕ , one gets:

$$E_1^{(1)} = J_m' \frac{(hF_1)'}{h^2} e^{im\theta} \quad (34)$$

$$E_2^{(1)} = im \frac{J_m}{\rho} \frac{(hF_1)'}{h^2} e^{im\theta} \quad (35)$$

$$E_3^{(1)} = \lambda^2 J_m \frac{F_3}{h} e^{im\theta} \quad (36)$$

$$H_1^{(1)} = mk \frac{J_m}{\rho} F_2 e^{im\theta} \quad (37)$$

$$H_2^{(1)} = ik \frac{J_m}{\rho} F_2 e^{im\theta} \quad (38)$$

$$H_3^{(1)} = 0 \quad (39)$$

The function $F(\theta)$ is a solution of the following equation:

$$F'' + \frac{h'}{h} F' + F \left(\frac{h''}{h} - \frac{h'^2}{h^2} + k^2 h^2 - \lambda^2 \right) = 0 \quad (40)$$

We see that the third component of the magnetic field, $\vec{H}^{(1)}$ is zero. Analogously to cylindrical coordinates we can call the fields $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ transverse magnetic modes (TM modes). One should remember, of course, that the field $\vec{H}^{(1)}$ is transverse to the direction $\vec{i}_3$, but it is not transverse to the axis of the waveguide $\vec{i}_2$.

The field $\vec{E}^{(1)}$, $\vec{H}^{(1)}$ should satisfy also the boundary conditions. That can be achieved by proper choosing of the value of the constant λ . Namely, it implies:

$$J_m(\lambda \rho_{\max}) = 0 \quad (41)$$

or

$$\lambda_{mk} = v_{mk} / \rho_{\max} \quad (42)$$

where v_{mk} is the k th root of equation $J_m(x) = 0$.

The second independent set of electromagnetic fields inside the waveguide can be obtained from the same vector $\vec{n}$ (31)-(33) by means of equations (26), (27). We can call these modes transverse electric modes (TE modes) since $E_z^{(2)} = 0$. To satisfy the boundary conditions for the TE modes, the value of the constant λ should be chosen in such a way that

$$J_m'(\lambda \rho_{\max}) = 0 \quad (41.a)$$

or

$$\tilde{\lambda}_{mk} = \tilde{\nu}_{mk} / \rho_{\max} \quad (42.a)$$

where $\tilde{\nu}_{mk}$ is the k th root of equation $J_m'(x) = 0$.

Let us look more closely on the σ -dependence of the fields $\vec{E}$ and $\vec{H}$. It is described by the function F which satisfies equation (40). Since z is the function of σ only, we can come back from the σ variable to the z variable.

$$dz = h d\sigma \quad (43)$$

$$h' = h \frac{dh}{d\sigma}$$

It is convenient at the same time to change the function $F(\sigma)$ to a new function u :

$$u(z) = F(\sigma) \cdot h(\sigma) \quad (45)$$

where it is understood that in the right-hand side of expression (45) σ should be substituted by the function $\sigma(z)$ from (11). Then, from the equation (40) we get the following equation for the function u :

$$d^2 u / dz^2 + (h^2 - \nu^2 / r_b^2) u = 0 \quad (46)$$

where $r_b = r_b(z)$ is the boundary curve. For TM modes, we get now:

$$E_1^{(1)} = \nu_m \frac{J_m}{r_b} u' e^{im\theta} \quad (47)$$

$$E_2^{(1)} = i m \frac{J_m}{r_b} u' e^{im\theta} \quad (48)$$

$$E_3^{(1)} = \sqrt{2} \frac{J_m}{r_b} u e^{im\theta} \quad (49)$$

$$H_1^{(1)} = m k \frac{J_m}{r_b} u e^{im\theta} \quad (50)$$

$$H_2^{(1)} = i k \frac{\nu}{r_b} \frac{J_m}{r_b} u e^{im\theta} \quad (51)$$

$$H_3^{(1)} = 0 \quad (52)$$

and analogous expressions for TE modes. In expressions (46)-(52) $J_m = J_m(\nu r / r_b)$, $\nu = \nu_{mk}$ (see (42)), $u = u(z)$ and prime means derivative over whole argument of corresponding function.

In the limit of the constant cross section waveguide $r_b = \text{const}$ and the solution of equation (46) is

$$u = \exp(z \sqrt{\lambda^2 - \lambda^2 z}) \quad (53)$$

We can call $u(z)$ a propagation function. When $r_b(z)$ is a periodic function, equation (46) becomes the Hill equation. The solution of it defines positions and widths of stopbands as well as the phase velocity of the wave in passbands. More detailed investigation of the behavior of the propagation function can be done only for a given function $r_b(z)$.

I present here an example of the application of the theory. The propagation function $u(z)$ is calculated for the waveguide with the following boundary:

$$r_b = b(1 + a \cos 2\pi z / L) \quad (54)$$

where b is the average radius of the pipe's cross section, a and L are the amplitude and the period of the boundary variation. The applicability of the theory limits the values

of b and L by the condition $\rho_{\max} = 2\pi b/L < 1$. The general solution of equation (46) in regions of its existence ("passbands") can be expressed in terms of the corresponding ϕ -function of this equation:²⁾

$$u(z) = u_0 \sqrt{B(z)} e^{i\phi(z) - i\omega t}, \quad (55)$$

where

$$\phi(z) = \int_0^{2\pi z/L} d\epsilon/B(\epsilon) \quad (56)$$

and the function $\zeta(z) = \sqrt{B(z)}$ is the solution of the following nonlinear equation:

$$\zeta'' + (k^2 - \sqrt{2}/r_0^2) \zeta = 1/\zeta^3. \quad (57)$$

Outside of the passbands there is no solution of equation (46) ("stopbands"). The electromagnetic wave with corresponding frequencies (or values k) can not propagate in the waveguide under consideration. Inside one of the passbands we can determine the phase velocity v_{ph} of the wave. From (55) one finds

$$v_{ph}/c = (kL/2\pi) \dot{\phi}(z). \quad (58)$$

Fig. 1. The phase velocity of the TM_{01} ($\nu=2.405$) mode vs longitudinal coordinate $y=2\pi z/L$. The waveguide parameters are $\rho_{\max} = 2.87$, $a = 0.111$. K is above the stopband shown on Fig. 2. $K = 1.03$, $\nu_{av} = 3.04$.

The phase velocity is not constant, but it is modulated with the frequency $2\pi/L$. Figure 1 illustrates the dependence of the phase velocity versus the longitudinal coordinate $y = 2\pi z/L$ for different values of the parameters $K = kL/2\pi$, $\rho_{\max} = 2\pi b/L$ and a . For some problems the average (over z) phase velocity is of interest. This quantity is relevant for example to a problem of the coupling between the electromagnetic wave propagating in the waveguide and a particle moving along its axis. The relative average phase velocity v_{ph}/c is also given on this figure. Figure 2 shows dependencies of the average phase velocity on the parameter K as well as position and widths of the stopband.

The case $\rho_{\max} = 2.87$ is far outside the validity of the described method. It was calculated here for the

sake of the comparison with known results. The crosses on Fig. 2 represent the results of calculations according to second order perturbation theory and of the measurements and are taken from the work.²⁾

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Fig. 2. The average relative phase velocity of the TM_{01} mode versus parameter $K=kL/2\pi$. The waveguide parameters are $\rho_{\max} = 2.87$, $a = 0.111$. The crosses represent the results of the calculations and measurements of the work.²⁾ Although this case is outside the applicability region of this work method, the correspondence is not at all bad.