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HAMILTONIAN GUIDING CENTER DRIFT ORBIT CALCULATION
FOR TOROIDAL PLASMAS OF ARBITRARY CROSS SECTION

By

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ABSTRACT

A Hamiltonian guiding center drift orbit formalism is developed which permits the efficient calculation of particle trajectories in toroidal devices of arbitrary cross section with arbitrary plasma β . The magnetic field is assumed to be a small perturbation from a zero order toroidal "equilibrium" field possessing either axial or helical symmetry. The equilibrium field can be modelled analytically or obtained numerically from equilibrium codes. A numerical code based on the formalism is used to study particle orbits in circular and bean-shaped tokamak configurations.

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I. INTRODUCTION

The magnetic fields in plasma confinement devices approximately describe magnetic surfaces, since this is a condition for good confinement. The exact fields are thus conveniently described in terms of perturbations of an idealized equilibrium field which possesses magnetic surfaces. Present equilibrium and stability analyses use magnetic coordinates based on the existence of these surfaces. In addition, magnetic coordinates are closely related to the canonical Hamiltonian coordinates describing the guiding center particle drift motion.¹ Even in the presence of field perturbations, which to some degree destroy the magnetic surfaces, the coordinates based on these surfaces are the most convenient way to treat the particle orbits.

In this work we use a set of canonical variables for the drift motion which is closely related to the magnetic coordinates used in equilibrium and stability analyses. A Hamiltonian formulation of the guiding center drift motion is developed which can be directly interfaced with equilibrium codes. This is particularly useful because it allows an investigation of particle orbits in equilibria which are highly distorted from circularity either through plasma pressure or special shaping coils. As an example we investigate particle orbits in bean-shaped tokamak configurations.² In Sec. II the choice of magnetic coordinates is discussed and in Sec. III the equilibria are described. In Sec. IV the Hamiltonian guiding center drift equations are given, integral invariants derived, and magnetic islands introduced. The properties of beam particle orbits are given in Sec. V, and in Sec. VI mode induced beam particle loss is examined. Results and implications are summed up in the concluding Sec. VII.

II. MAGNETIC COORDINATES

There are only three situations in which global magnetic surfaces are known to exist, involving translational, axial, and helical symmetry of the magnetic field.³ However, it has been shown by Kolmogorov⁴ that small perturbations of fields which possess global magnetic surfaces produce fields in which magnetic surfaces exist in all but a finite volume proportional to the square root of the perturbation. Thus we begin with a system possessing some symmetry and consider small perturbations of it. Consider for example toroidal symmetry. Introducing the toroidal angle ϕ , a parameter labelling the poloidal flux surfaces ψ_p , and θ , an arbitrary periodic measure of angle across the field lines at fixed ϕ , the field can be represented through the expression

$$\vec{B} = \nabla\phi \times \nabla\psi_p + q(\psi_p)\nabla\phi \quad (2.1)$$

Because of the toroidal symmetry, the magnitude of B is a function of two variables only, $B = B(\psi_p, \theta)$. As is shown explicitly in Appendix A, by the introduction of a new toroidal coordinate ζ with

$$\phi = \zeta + v(\psi_p, \theta) \quad (2.2)$$

with v a periodic function of θ , the field lines can be made straight in the θ, ζ plane of the new coordinate system, i.e.,

$$\frac{\vec{B} \cdot \nabla \zeta}{\vec{B} \cdot \nabla \theta} = q(\psi_p) = \frac{d\psi_p}{d\theta} \quad (2.3)$$

Here $q(\psi_p)$ is the safety factor.

Further, by a particular choice of the coordinate θ , the representation of $\vec{B}$ can be reduced to

$$\vec{B} = g(\phi_p) \nabla \zeta + I(\phi_p) \nabla \theta + \delta(\phi_p, \theta) \nabla \phi,$$

$$\vec{B} = \nabla \psi \times \nabla \theta + \nabla \zeta \times \nabla \phi_p. \quad (2.5)$$

Integrating the flux $\int d\vec{s} \cdot \vec{B}$ over a surface bounded by a curve of constant ψ, ϕ with $d\vec{s} = \nabla \phi d\phi_p d\theta / \nabla \phi_p \times \nabla \theta \cdot \nabla \phi$, we find from Eq. (2.5) that $\int d\vec{s} \cdot \vec{B} = 2\pi\psi$, i.e., $2\pi\psi$ is the toroidal flux. Similarly, choosing a surface bounded by the magnetic axis and a curve with constant θ, ϕ with $d\vec{s} = \nabla \theta d\phi_p d\phi / \nabla \phi_p \times \nabla \theta \cdot \nabla \phi$, we find $\int d\vec{s} \cdot \vec{B} = 2\pi\phi_p$, i.e., $2\pi\phi_p$ is the poloidal flux. The function $g(\phi_p)$ is the toroidal field function, $I(\phi_p)$ is the plasma current contained within the surface ϕ_p , and $\delta(\phi_p, \theta)$ describes plasma pressure and distortion of the magnetic surfaces from circularity. Since I is a surface quantity, the Jacobian for the toroidal coordinate system defined by $J^{-1} = (\nabla \phi_p \times \nabla \theta) \cdot \nabla \phi = \nabla \phi_p \times \nabla \theta \cdot \nabla \zeta$ takes on a particularly simple form in this representation

$$J = \frac{I + g\zeta}{B^2}. \quad (2.6)$$

The simplest example is a zero β large aspect ratio equilibrium,⁵ given by

$$\vec{B} = \frac{r}{q(r)} \vec{\nabla} \phi \times \vec{\nabla} r + R \vec{\nabla} \phi$$

for which $\zeta = \phi$, $g = R$, $\delta = 0$, $I = r^2/Rq$, $B^2 = (R^2/X^2)(1 + r^2/R^2q^2)$, R is the major axis, and X is the distance from the toroidal symmetry axis. The

toroidal flux is $\phi = r^2/2$ and $\phi_p = \int^r dr r/q(r)$. However, as we will see in Sec. III, any consistent modification of the functions q , g , δ , I , B , will also lead to a Hamiltonian system. For example, truncating all quantities to first order in r/R , giving $I \approx 0$, $B/B_0 = 1 - r/R \cos\theta$ in this model leads to very minor modification of the orbits.

III. EQUILIBRIA

The numerical equilibria used have been generated by solving the Grad-Shafranov equation in the fixed boundary mode with the shape of the plasma surface prescribed.⁶ A flux conserving sequence of equilibria with increasing pressure is formed, with pressure and safety factor profiles $p(y) = p_0(1 - y^2)^2$ and

$$q(y) = q(0) + y\{q(1) - q(0) + (y-1)[(q'(1) - q'(0) + q(0)](1-y_s)/(y-y_s)\} \quad (3.1)$$

where

$$y_s = [q'(1) - q'(0) + q(0)]/[q'(0) + q'(1) - 2\{q(1) - q(0)\}]$$

and where $y = \phi_p/\Delta\phi_p$, $2\pi\Delta\phi_p$ being the poloidal flux within the plasma. The shape of the plasma surface for circular and bean-shaped equilibria, respectively, are parameterized by $x = R + a \cos t$, $z = a \sin t$, and $x = x_0 + \rho \cos \gamma$, $z = E \rho \sin \gamma$ where $\rho = 1 + b \cos t$, $\gamma = C \sin t$, and $0 < t < 2\pi$. Examples of low and high β circular and bean-shaped equilibria are shown in Fig. 1. The aspect ratio, R/a , the elongation b/a , and the indentation, $d/2a$ of the bean-shaped plasma, are generated by appropriate choices for x_0 , E , B

and C. Two different sequences of equilibria were used, a low shear sequence with $1 < q < 2.3$ and a high shear sequence with $1 < q < 3$. In Fig. 2 is shown, for a low β circular equilibrium, the magnitude of $\mathcal{B}$ as a function of θ for $\psi_p = 1/3 \Delta\psi_p, 2/3 \Delta\psi_p, \Delta\psi_p$. The same plot for a low β bean configuration (Fig. 3) shows that a second magnetic well exists for exterior flux surfaces due to the indentation. Due to the presence of this well there exists a second class of trapped particles, in addition to the outer bananas of circular equilibria. An example of both outer and inner banana orbits is shown in Fig. 4.

Any equilibrium is a solution of the Grad-Shafranov equation

$$\nabla \cdot \frac{1}{x^2} \nabla \psi_p = J \cdot \nabla \phi = -(p' + \frac{gg'}{x^2}) , \quad (3.2)$$

and using this equation a given equilibrium can be scaled to produce a whole family of equilibria. One way of changing the equilibrium without changing this equation is to change $g(\psi_p)$ in such a way that gg' is invariant.⁷ Let the new g be given by $g_N^2 = g^2 + c$. Choosing c such that $g_N(\Delta\psi_p) = sg(\Delta\psi_p)$ we find $g_N^2 = g^2 + (s - 1)^2 g^2(\Delta\psi_p)$. Since the function $f(\psi_p)$ is unchanged (see Appendix A), we have $q_N = (g_N/g)q$. Thus for this family of equilibria the flux surfaces are unchanged, only g and q , and thus the toroidal beta, are modified.

IV. HAMILTONIAN DRIFT EQUATIONS

A. Equations of Motion

In previous work¹ it was demonstrated that the magnetic coordinates of the unperturbed, or equilibrium, field are closely related to the canonical Hamiltonian variables for the guiding center drift motion. The Hamiltonian

for the drift motion is

$$H = \frac{1}{2} \rho_{\parallel}^2 B^2 + \mu B + \Phi \quad (4.1)$$

where $\rho_{\parallel}$ is the normalized parallel gyro radius, $\rho_{\parallel} = v_{\parallel}/B$, μ the normalized magnetic moment, $\mu = (1/2)v_{\perp}^2/B$, and Φ the electric potential. Here the units are given by defining time through the on-axis gyro frequency ω_0 and using the major radius R_0 for the unit of distance. The basic unit of energy is then $m\omega_0^2 R_0^2$. Here R_0 refers to the location of the magnetic axis, not the geometric center (see Fig. 1). For low β circular equilibria $R = R_0$. The dimensionless gyro radius ρ is equal to the particle gyro radius divided by the system size R_0 , $\rho = v/\omega_0 R_0$ and is typically 10^{-2} or smaller. It represents the fundamental small parameter in the description of particle trajectories, since the original Alfvén guiding center drift equations^{8,9} give the cross field drifts only to first order in ρ . The formalism described here is Hamiltonian in character and agrees to first order in ρ with the usual guiding center treatment. Higher order corrections are beyond the scope of this work.

Now introduce magnetic field perturbations of the form

$$\delta \vec{B} = \vec{\nabla} \times \alpha \vec{B} \quad (4.2)$$

with $\alpha = \alpha(\psi_p, \theta, \zeta)$. This form, while not completely general, is sufficient to represent exactly the $\vec{\nabla}\psi$ component of any perturbation. Other components of $\delta \vec{B}$ contribute only nonresonant distortions of the equilibrium and are unimportant.¹⁰ The equations of motion are changed¹ only by a modification of the parallel canonical momentum, i.e., the canonical momentum becomes

$$\rho_C = \rho_H + \alpha \quad (4.3)$$

so that

$$H = \frac{1}{2} (\rho_C - \alpha)^2 B^2 + \mu B + \Phi \quad (4.4)$$

Other variables are left unchanged. Since from Eq. (4.2), α plays the role of a vector potential in $\delta \vec{B}$, this modification is analogous to that of the canonical momentum of a particle in classical electrodynamics.

Canonical variables convenient for dealing with equilibria of arbitrary cross section and plasma pressure are closely related to the magnetic coordinates introduced in Sec. II. As shown in Appendix B, the canonical angular variables are given by ζ and $\theta_C = \theta - \rho_C \delta$, with the associated toroidal canonical momentum

$$P_\zeta = g\rho_C - \psi_P \quad (4.5)$$

and the poloidal canonical momentum

$$P_\theta = I\rho_C + \psi \quad (4.6)$$

Eliminating ρ_C from these equations, we find

$$g\psi = gP_\theta - IP_\zeta - I\psi_P \quad (4.7)$$

and it follows from this equation that $\psi_P = \psi_P(P_\theta, P_\zeta)$ with

$$\frac{\partial \phi_p}{\partial p_\theta} = \frac{g}{D}, \quad (4.8)$$

$$\frac{\partial \phi_p}{\partial p_\zeta} = -\frac{I}{D}, \quad (4.9)$$

where

$$D = \rho_c [gI' - Ig'] + I + gq, \quad (4.10)$$

and primes refer to differentiation with respect to ϕ_p .

Note also from Eqs. (4.5) and (4.6) that $\rho_c = \rho_c(p_\zeta, p_\theta)$ with

$$\frac{\partial \rho_c}{\partial p_\zeta} = \frac{q + \rho_c I'}{D} \quad (4.11)$$

and

$$\frac{\partial \rho_c}{\partial p_\theta} = \frac{1 - \rho_c g'}{D}. \quad (4.12)$$

Since the canonical variable $\theta_c = \theta - \rho_c \delta$ is very nearly equal to θ , it is convenient to treat this modification perturbatively. Expanding H in $\rho_c \delta$ we find $H = H_0 + H_\delta$ with $H_\delta = (\rho_\perp^2 B + \mu)(\partial B / \partial \theta) \rho_c \delta$, and canonical variable θ . This term, however, in addition to being third order in ρ , introduces only nonresonant distortions of the equilibrium and has no significant effect on particle trajectories. This has been verified numerically using a large variety of equilibria. Thus we drop H_δ , and the canonical variable becomes θ . We then find for the equations of motion

$$\dot{p}_\zeta = -\frac{\partial H}{\partial \zeta} = \rho_\perp^2 B^2 \frac{\partial \alpha}{\partial \zeta} - \frac{\partial \Phi}{\partial \zeta}, \quad (4.13)$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -(\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \theta} + \rho_\parallel B^2 \frac{\partial \alpha}{\partial \theta} - \frac{\partial \Phi}{\partial \theta}, \quad (4.14)$$

$$\dot{\zeta} = \frac{\partial H}{\partial P_\zeta} = -(\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \zeta} \frac{I}{D} + \rho_\parallel B^2 (q + \rho_c I')/D + \rho_\parallel B^2 \frac{\partial \alpha}{\partial \zeta} \frac{I}{D} - \frac{I}{D} \frac{\partial \Phi}{\partial \zeta}, \quad (4.15)$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = (\mu + \rho_\parallel^2 B) \frac{\partial B}{\partial \zeta} \frac{g}{D} + \rho_\parallel B^2 (1 - \rho_c g')/D - \rho_\parallel B^2 \frac{\partial \alpha}{\partial \zeta} \frac{g}{D} + \frac{g}{D} \frac{\partial \Phi}{\partial \zeta}. \quad (4.16)$$

These four equations suffice to advance all quantities in time. Since functions of position are given in terms of θ, ζ, ϕ_p , the function $\psi_p(P_\theta, P_\zeta)$ must also be evaluated at the new time. Although Eq. (4.7) could be used to solve for ϕ_p , it is more convenient to use Eqs. (4.8) and (4.9) to advance ϕ_p in a differential manner

$$\dot{\phi}_p = \frac{g}{D} \dot{P}_\theta - \frac{I}{D} \dot{P}_\zeta. \quad (4.17)$$

This equation of course describes the particle motion across the magnetic surfaces. Equation (4.5) can now be used to advance ρ_c in time

$$\rho_c = \frac{P_\zeta + \phi_p}{g} \quad (4.18)$$

and also $\rho_\parallel = \rho_c - \alpha(\phi_p, \theta, \zeta)$.

Simple closed expressions for $\dot{\rho}_\parallel$ and $\dot{\phi}_p$ are also useful. Using Eqs. (4.11 - 4.18) we find

$$\dot{\rho}_\parallel = \frac{(\rho_\parallel^2 B + \mu)}{D} \left[(-1 + \rho_c g' + g \frac{\partial \alpha}{\partial \zeta}) \frac{\partial B}{\partial \theta} + (I \frac{\partial \alpha}{\partial \zeta} - g \frac{\partial \alpha}{\partial \theta}) \frac{\partial B}{\partial \phi_p} \right] \quad (4.19)$$

$$\begin{aligned}
 & - \frac{(q + \rho_C I')}{D} \frac{\partial \Phi}{\partial \zeta} - \frac{(1 - \rho_C g')}{D} \frac{\partial \Phi}{\partial \theta} \\
 & + \frac{\alpha'}{D} \left(g \frac{\partial \Phi}{\partial \theta} - I \frac{\partial \Phi}{\partial \phi} \right) + \frac{\Phi'}{D} \left(I \frac{\partial \alpha}{\partial \phi} - g \frac{\partial \alpha}{\partial \theta} \right) - \frac{\partial \alpha}{\partial t} .
 \end{aligned}$$

The last term, which appears when δB is time dependent, represents the forces due to the induced electric field. Also, substituting into Eq. (4.17), we find for the radial motion

$$\begin{aligned}
 \dot{\psi}_p = & - \frac{q}{D} (\rho_{||}^2 B + \mu) \frac{\partial B}{\partial \theta} + \frac{\rho_{||} B^2}{g D} \frac{\partial \alpha}{\partial \theta} - \frac{I \rho_{||} B^2}{D} \frac{\partial \alpha}{\partial \zeta} \\
 & - \frac{g}{D} \frac{\partial \Phi}{\partial \theta} + \frac{I}{D} \frac{\partial \Phi}{\partial \zeta} .
 \end{aligned} \tag{4.20}$$

The functions $I(\psi_p)$, $g(\psi_p)$, $q(\psi_p)$, $B(\psi_p, \theta)$, and $\alpha(\psi_p, \theta, \zeta)$ completely define the particle orbits in the coordinates ψ_p , θ , ζ through Eqs. (4.13) to (4.16). To describe the trajectories in real space, the functions $\phi(\zeta, \theta, \psi_p)$, $X(\psi_p, \theta)$, and $Z(\psi_p, \theta)$ are also necessary, where X is the distance from the vertical axis of the torus and Z the vertical distance from the equatorial plane.

The numerical code ORBIT is described in Appendix C.

B. Integral Invariants

Invariants of the motion are readily found in a Hamiltonian formalism. The magnetic moment μ is of course constant. If the Hamiltonian is time independent, then the energy is conserved. Axisymmetry ($\alpha = 0$) gives $P_\zeta =$ constant. In this case the energy takes the form

$$W = \frac{1}{2g} [P_\zeta + \psi_p(P_\zeta, P_\theta)]^2 B^2 + \mu B \tag{4.21}$$

so that $W = W(P_\theta, \theta)$, and surfaces of constant energy are closed curves in the P_θ, θ plane, independent of ζ . Note that Eqs. (4.6) and (4.4) imply that these surfaces are also closed curves in the ϕ_p, θ plane. In this case any single valued function not explicitly dependent on ζ , integrated in time over a closed orbit in the P_θ, θ plane, is a constant of the motion, since $P_\theta = P_\theta(\theta)$. The first action integral

$$I_1 = \oint P_\theta d\theta, \quad (4.22)$$

which is exactly conserved in the Hamiltonian formalism, is to first order in ρ , equal to the first adiabatic invariant,⁹ J . To see this use $\dot{\phi}_p = g\rho_\parallel$ and the fact that the excursions in ϕ are small to write

$$P_\theta = \phi_0 - gq\rho_0 + D\rho_\parallel \quad (4.23)$$

with ϕ_0, ρ_0 the values of $\phi, \rho_\parallel$ at $t = 0$. Substituting $d\theta = \dot{\theta}dt$ and using $\dot{\theta} = \rho_\parallel B^2/D + O(\rho^2)$, we find

$$I_1 = \oint (\phi_0 - gq\rho_0) d\theta + \oint v_\parallel ds + O(\rho^2) \quad (4.24)$$

The first term is approximately equal to the toroidal flux enclosed inside the orbit for a passing particle, and zero for a trapped particle. The second term is the usual definition of the adiabatic invariant, J .

The second invariant⁹

$$I_2 = \oint P_\zeta d\zeta \quad (4.25)$$

is simply equal to $2\pi p_c$, and within a term of order ρ is equal to $-2\pi\phi_p$, the flux contained between the magnetic axis and the drift surface.

C. Toroidal Precession

The average toroidal precession rate of a trapped or passing particle can be expressed in terms of an integral in θ over the flux surface given by the initial conditions.¹¹⁻¹⁴ We are interested in the time average of $\dot{\phi}$, $\langle\dot{\phi}\rangle = \frac{1}{T} \int_0^T \dot{\phi} dt$. Writing $\dot{\phi} = \dot{\zeta} + \dot{\psi}$ and letting the time T correspond to the period for the completion of a closed orbit in the P_θ, θ plane, the $\dot{\psi}$ term vanishes due to the periodicity of ψ . Converting into integration over θ we find, to lowest order¹⁵ in ρ

$$\langle\dot{\phi}\rangle = \frac{q}{T} \oint d\theta - \frac{(q+I/g)}{T} \oint d\theta \frac{\Delta(\theta)}{\rho_\parallel} \quad (4.26)$$

with the period given by $T = \oint d\theta D / \rho_\parallel B^2$ and $\Delta = (2E/B - \mu)(g/B^2) \partial B / \partial \psi_p$. In general, the bounce average of any quantity A is thus seen to be $\bar{A} = (\oint d\theta A D / \rho_\parallel B^2) / (\oint d\theta D / \rho_\parallel B^2)$. The first term in $\langle\dot{\phi}\rangle$ is the toroidal motion produced by following the initial field line, and vanishes for trapped particles. We have neglected corrections proportional to derivatives of g , I and q , and of higher order in ρ in taking these functions outside the integrals. Note $T = O(1/\rho)$, $\Delta = O(\rho^2)$ and thus the second term is of order ρ^2 . Ignoring the toroidal motion due to simply following field lines, Eq. (4.26) gives for the toroidal drift frequency

$$\omega_d = - \frac{(qg+I)}{2T} (2E)^{1/2} \int_{-\theta_b}^{\theta_b} d\theta \frac{[2-3(\theta)/B(\theta_b)]}{[1-\mu B(\theta)/E]^{1/2}} \frac{1}{B^2} \frac{\partial B}{\partial \psi_p} \quad (4.27)$$

with $B(\theta_b) = E/\mu$. Thus the toroidal drift is a negative weighted average of $\partial B / \partial \psi_p$. For low β equilibria $\partial B / \partial \psi_p < 0$ for $|\theta| < \pi/2$. Thus the toroidal

drift is positive except for $\theta_p \approx \pi$, when the major contribution to the integral comes from the region $\theta \approx \pi$. If there exists a magnetic well, $\partial B / \partial \psi_p$ is positive near $\theta = 0$ for ψ greater than the location of the well minimum, and the toroidal drift can be negative. This tendency for the pressure to reverse the sign of $\langle \dot{\phi} \rangle$ was pointed out by Connor et al.¹⁶ In Fig. 5 and Fig. 6 are shown, respectively, the toroidal precession rates, as a function of initial pitch, for 1 keV and for 50 keV deuterium particles in a 30 KG low β circular equilibrium, obtained by numerically advancing the Hamiltonian equations. These cases correspond to $\rho = 1.5 \times 10^{-3}$ and $\rho = 10^{-2}$, respectively. Also shown is the first order expression obtained from Eq. (4.27), and the toroidal precession rate including the leading second order terms, due to finite banana width, calculated in Appendix D. Finite banana width corrections are quite significant. For 1 keV particles (Fig. 5) the second order expression cannot be distinguished from the numerical results, but at 50 keV some error is present. The second order expression is seen to be a significant improvement over the first order Kadomtsev expression except at zero pitch. For the low β circular equilibrium $B = B_0(1 - r \cos\theta)$, Eq. (4.27) reduces to

$$\omega_d = \frac{(2E(k^2) - K'(k^2))E}{K(k^2)r}g \quad (4.28)$$

where $E(k^2)$ and $K(k^2)$ are complete elliptic integrals and $k^2 = (1-r - \mu/E)/2r$. For small bounce angle this reduces to $\omega_d \approx Eq/r$, which in conventional units is $\omega_d = Eq/mrR\omega_0$.

The toroidal drift for a trapped particle can be expressed in terms of the invariant J , as first shown by Kadomtsev.¹¹ To lowest order in ρ , from Eq. (4.23),

$$J = \oint \rho_{\parallel} d\theta . \quad (4.29)$$

Within terms of higher order in ρ we also have, using $\rho_{\parallel}^2 = 2E/B^2 - 2\mu/B$,

$$\left. \frac{\partial \rho_{\parallel}}{\partial E} \right|_{\phi, \mu} = \frac{1}{B^2 \rho_{\parallel}} , \quad (4.30)$$

$$\left. \frac{\partial \rho_{\parallel}}{\partial \phi} \right|_{E, \mu} = - \frac{\Delta}{g \rho_{\parallel}} , \quad (4.31)$$

giving to lowest order in ρ

$$\left. \frac{\partial J}{\partial E} \right|_{\phi, \mu} = \oint \frac{D}{\rho_{\parallel} B} \quad (4.32)$$

and

$$\left. \frac{\partial J}{\partial \phi} \right|_{E, \mu} = - \oint \frac{D \Delta}{g \rho_{\parallel}} d\theta , \quad (4.33)$$

and thus for a trapped particle

$$\langle \dot{\phi} \rangle = \left(\frac{\partial J}{\partial \phi} \right)_{E, \mu} / \left(\frac{\partial J}{\partial E} \right)_{\phi, \mu} . \quad (4.34)$$

Now consider the effect of a perturbation which destroys toroidal symmetry, $\alpha(\phi_p, \zeta)$. For small α the orbit can be treated as a sequence of banana orbits, described by the successive locations of the turning points. From Eqs. (4.15) and (4.20) the changes in ζ and ϕ_p due to α are

$$(\delta\zeta)_\alpha = \int dt \rho_\parallel B^2 \frac{\partial \alpha}{\partial \psi_p} \frac{I}{D}, \quad (4.35)$$

$$(\delta\psi_p)_\alpha = - \int dt \rho_\parallel B^2 \frac{\partial \alpha}{\partial \zeta} \frac{I}{D}. \quad (4.36)$$

From Eq. (4.6) the change in the action J due to α is

$$J_\alpha = \int d\theta I_\alpha. \quad (4.37)$$

Combining these equations with Eq. (4.34) for the coroidal drift we find the equations describing the motion of the banana orbit,

$$\dot{\zeta} = \left(\frac{\partial J}{\partial \psi_p} \right)_{E, u, \zeta} \quad (4.38)$$

$$\dot{\psi}_p = - \left(\frac{\partial J}{\partial \zeta} \right)_{E, \mu, \psi_p} \quad (4.39)$$

with the time given in units of the banana period,

$$T = \left(\frac{\partial J}{\partial E} \right)_{\psi_p, \mu}. \quad (4.40)$$

These equations form a Hamiltonian system for the banana-averaged motion, with J serving as Hamiltonian. Thus it is clear that J is conserved even when toroidal symmetry is broken.¹¹

A useful representation of trapped particle motion can be obtained by converting these equations to a discrete time step. The integral defining J is taken over half a period, so that the expressions for $\delta\zeta$, $\delta\psi_p$ refer successively, at each time step, to the upper and lower banana tips. Also, in the discrete step equations, the arguments of the right-hand sides must be chosen appropriately in order that the equations describe an area preserving

map. White et al.¹⁷ found this map in an investigation of ripple induced banana transport without recognizing that J played the role of Hamiltonian.

D. Magnetic Islands

Now examine the effect of a perturbation $\alpha = \alpha_{nm} \sin(n\zeta - m\theta)$ on the flux surfaces. To study magnetic field lines take $\mu = 0$ and the limit $\rho \rightarrow 0$. We then find from Eqs. (4.16), (4.15), and (4.20)

$$d\theta = d\zeta/q, \quad (4.41)$$

$$d\psi = -[mg + nI] \alpha_{nm} \cos(n\zeta - m\theta) d\zeta. \quad (4.42)$$

We wish to find the form of the flux surfaces at fixed ζ . Far from the rational surface $q = m/n$ the ϕ dependence of q can be ignored and we find for $\zeta = q\theta = 2\pi$,

$$\phi - \phi_0 = \frac{[mg+nI]}{(n-m/q)} \alpha_{nm} [\sin(m\theta) + c], \quad (4.43)$$

with c an integration constant. Near the rational surface the ϕ dependence of q cannot be ignored due to the rapid variation of the argument of the cosine. Expand Eq. (4.41) in a Taylor's series about $q = m/n$ and substitute the variable $z = \theta - nq\zeta/m$ to find

$$dz = -\frac{q'}{q} [\phi - \phi_0] d\zeta, \quad (4.44)$$

$$d\psi = -[mg + nI] \alpha_{nm} \cos(mz) dz, \quad (4.45)$$

where $q' = dq/d\psi_p$ and $q(\psi_0) = m/n$. Integrating these equations and evaluating at $\zeta = 0$, we find

$$(\psi - \psi_0)^2 = \frac{2q'}{q'} \left(g + \frac{nI}{m} \right) \alpha_{nm} (\sin m\theta + c). \quad (4.46)$$

The separatrix is described by choosing the integration constant $c = 1$, giving magnetic islands with width

$$\Delta\psi = 4q' \left\{ \frac{q'}{q'} \left(g + \frac{nI}{m} \right) \alpha_{nm} \right\}^{1/2}. \quad (4.47)$$

In Fig. 7 is shown an example of a Poincaré plot showing a $m = 2$, $n = 1$ magnetic island in a bean equilibrium. The presence of such magnetic islands has a particularly large effect on electron transport because the effect of small islands dominates over the diffusion due to the relatively small banana width of electron orbits. The Hamiltonian drift orbit formalism makes possible the study of modifications to particle confinement produced by the presence of various MHD and kinetic plasma modes. A preliminary investigation of this phenomenon has been carried out for circular low β equilibria.¹⁷

V. NEUTRAL BEAM INJECTION

The geometry for neutral beam injection is shown in Fig. 8. Assuming that the particle is ionized in the equatorial plane, $z = 0$, the pitch is initially given by

$$\frac{v_\parallel}{v} = \left(\frac{R}{R} \right) \left(\frac{B}{B} \right) \quad (5.1)$$

In a reactor, near perpendicular injection is necessary for available beam

energies in order to achieve sufficient penetration. The toroidal precession rate of the beam particles is important in the destabilization of some MHD modes.¹⁸ It is modified both by external shaping and by plasma pressure. Shown in Fig. 8 is the toroidal precession rate ω_d of injected beam particles, as a function of major radius X , for different plasma pressures in a circular and a bean configuration. Here the toroidal motion due to simply following field lines is included in ω_d as it is the total toroidal motion which is relevant for beam loss studies. As shown, almost all particles born outside the magnetic axis are trapped. In both configurations negative precession is achieved only near the plasma edge and only for plasma β greater than 3%. The region of negative precession corresponds to values of ϕ_p with $\partial B / \partial \phi_p > 0$, i.e., outside the bottom of the magnetic well.

VI. INDUCED PARTICLE LOSS

The possible stabilizing or destabilizing influence of high energy particles on plasma equilibria is a subject of considerable interest. Of particular importance is the possibility of the ejection of the high energy beam particles and the termination of plasma heating. This can occur if there exists a magnetic perturbation rotating toroidally in synchronization with the toroidal precession of the trapped particles. The internal kink mode, in particular, is destabilized by the trapped particles and its frequency automatically matches the precession rate of the trapped particles,¹⁸ giving rise to the so-called fishbone instability. That this can give rise to secular radial motion, and hence particle loss, can be readily seen from the equations of motion. Consider a single mode,

$$\alpha \sim \alpha_{nm}(\phi_p) \sin(n\zeta - m\theta - n\omega t - \delta) \quad (6.1)$$

The total electric field is given by

$$\vec{E} = - \frac{\partial}{\partial t} \alpha \vec{B} - \nabla \phi \quad (6.2)$$

The large value of the electron parallel conductivity ensures that the electric field is perpendicular to the magnetic field

$$(\vec{B} + \delta \vec{B}) \cdot \vec{E} = 0 \quad (6.3)$$

Writing $\phi = \phi_0(\psi_p) + \phi_1$ and using $\delta \vec{B} = \vec{\nabla} \times \alpha \vec{B}$, Eq. (6.3) can be solved perturbatively for ϕ_1 giving

$$\phi_1 = \frac{[n(gq+I)\omega + \phi'_0(mg+nI)]}{nq - m} \alpha_{nm} \sin(n\zeta - m\theta - n\omega t - \delta) \quad (6.4)$$

The apparent singularity at the rational surface $q = m/n$ is not real. In the ideal limit (zero resistivity) α_{nm} vanishes at this surface, and for finite resistivity this calculation is invalid in the island interior.

Rigid toroidal plasma rotation can be approximated by the potential ϕ_0 with $\phi'_0 = -\Omega$. Particles moving with $\rho_{\parallel}^2/g = \Omega$ have $\xi \approx \Omega$ and $\delta \approx 0$, i.e., rigid rotation at frequency Ω . Substituting α and ϕ into Eq. (4.20), we find for the radial motion

$$\ddot{\psi}_p = - \frac{g(\rho_{\parallel}^2 + \mu)}{D} \frac{\partial B}{\partial \theta} + A \alpha_{nm} \cos(n\zeta - m\theta - n\omega t - \delta) \quad (6.5)$$

with

$$A = - \frac{(mg+nI)}{D} \left[\rho_{\parallel} B^2 - \frac{(ngq\omega - mgQ + nI\omega - nIQ)}{nq - m} \right] \quad (6.6)$$

Secular radial motion can occur if the mode frequency is in resonance with particle drift motion. From Eq. (4.15) and neglecting I , which is higher order in ϵ , the average precession rate is

$$\langle \dot{\zeta} \rangle \approx \langle \rho_{\parallel} \frac{B^2 q}{D} \rangle \equiv \omega_d + Q \quad (6.7)$$

where ω_d is the toroidal precession rate in the rotating frame, and the angular brackets indicate an average over the orbit. Thus $\zeta = (\omega_d + Q)t + \hat{\zeta}$, with $\hat{\zeta}$ periodic in time. For trapped particles resonance occurs when $\omega = \omega_d + Q$, giving

$$\langle \dot{\phi}_p \rangle \approx A \alpha_{nm} \langle \cos(n\hat{\zeta} - m\theta - \delta) \rangle \quad (6.8)$$

with $A = (mg + nI)m\omega_p / [q(nq - m)]$. Here we have neglected the radial excursion of the orbit in carrying out the bounce average. This expression can thus be regarded as only approximate for high energy particles. For circular flux surfaces the average of the cosine can be calculated and is given by $J_0[(m - q)\theta_p] \cos \delta$. This mechanism is capable of efficient beam expulsion, and associated loss of heating.¹⁹ It is particularly effective for low shear plasmas and near perpendicular injection, where the precession rate ω_d is almost independent of minor radius (see Fig. 8).

In Fig. 9 is shown the beam particle loss, as a function of mode frequency, for low β circular and bean equilibria. For a fixed $\delta B/B$ the losses are smaller in the bean equilibria because of the lower precession frequency. This result is in itself favorable for the bean configuration. In

addition, this configuration is significantly more stable to MHD modes, and thus it is possible that the fishbone mode will be altogether absent.

VII. CONCLUSION

A Hamiltonian guiding center drift formalism has been developed which greatly facilitates the study of particle motion in arbitrary shaped plasmas with arbitrary plasma β . A numerical code has been developed based on this formalism which permits Monte Carlo studies of particle orbits in realistic plasma equilibria, including field perturbations due to field ripple or MHD modes. The formalism allows a simple analysis of toroidal precession and of the response of trapped particles to MHD modes with real frequency. It also permits a study of test particle diffusion in real equilibria including magnetic fluctuations and electric potentials. An example of the use of the code is given for the loss of particles due to a rotating MHD mode, the "fishbone" oscillation.

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APPENDIX A: MAGNETIC COORDINATES

Represent the field as

$$\vec{B} = \nabla\phi \times \nabla\phi_p + g\nabla\phi, \quad (A1)$$

$$\vec{B} = a_1 \nabla\phi_p + b_1 \nabla\theta + c_1 \nabla\phi, \quad (A2)$$

where ϕ is the angular coordinate in the cylindrical system (x, ϕ, z) . The Jacobian for this coordinate system is given by $J^{-1} = \nabla\phi_p \times \nabla\theta$, $\nabla\phi = \vec{B} \cdot \nabla\theta$, and we find

$$a_1 = J \vec{B} \cdot (\nabla\theta \times \nabla\phi) = -J \frac{\nabla\phi_p \cdot \nabla\theta}{x^2}, \quad (A3)$$

$$b_1 = J \vec{B} \cdot (\nabla\phi \times \nabla\phi_p) = J \frac{|\nabla\phi_p|^2}{x^2}, \quad (A4)$$

$$c_1 = J \vec{B} \cdot (\nabla\phi_p \times \nabla\theta) = g, \quad (A5)$$

where $\nabla\phi = \hat{\phi}/x$, i.e., x is the distance from the toroidal axis. Now introduce a new coordinate ζ with $\phi = \zeta + v(\phi_p, \theta)$ with v a periodic function of θ . This gives for B

$$\vec{B} = \left(-\frac{\nabla\phi_p \cdot \nabla\theta}{x^2} J + g v'\right) \nabla\phi_p + \left(\frac{|\nabla\phi_p|^2}{x^2} J + g \frac{\partial v}{\partial \theta}\right) \nabla\theta + g \nabla\zeta, \quad (A6)$$

where primes denote derivatives with respect to ϕ_p . Requiring that the field lines be straight in this coordinate system, i.e.,

$$\frac{B \cdot \nabla \zeta}{B \cdot \nabla \theta} = q(\phi_p) = \frac{gJ}{x^2} - \frac{\partial v}{\partial \theta} \quad (A7)$$

determines the relation between v and J . We thus find

$$\vec{B} = g(\phi_p) \nabla \zeta + I(\phi_p, \theta) \nabla \theta + \delta(\phi_p, \theta) \nabla \phi_p \quad (A8)$$

with

$$\delta = g \frac{\partial v}{\partial \phi_p} - J \frac{\nabla \phi_p \cdot \nabla \theta}{x^2} ,$$

and $I = J B^2 - gq$ since $B^2 = (\nabla \phi_p)^2/x^2 + g^2/x^2$. Substituting $\phi = \zeta + v$ into the first term of Eq. (A1) and $\nabla \phi = J \nabla \phi_p \times \nabla \theta/x^2$ into the second term we find

$$\vec{B} = \nabla(\zeta - q\theta) \times \nabla \phi_p \quad (A9)$$

Solving Eq. (A7) for v we find

$$v = \int_0^\theta \frac{gJ}{x^2} d\theta - q\theta \equiv q(\theta_p - \theta) \quad (A10)$$

giving $\phi = \zeta + q(\theta_p - \theta)$. Thus we find that

$$\delta = - \frac{\nabla \phi_p \cdot \nabla \theta}{x^2} J + g \frac{\partial}{\partial \phi_p} [q(\theta_p - \theta)] \quad (A11)$$

Letting $J = \alpha(\phi_p) h(\phi_p, \theta)$ we have for the element of arc in θ ,

$$d\ell_\theta = \frac{\nabla \phi \times \nabla \phi_p}{\nabla \theta \cdot (\nabla \phi \times \nabla \phi_p)} d\theta = J \frac{|\nabla \phi_p|}{x} d\theta , \quad (A12)$$

and thus integrating $d\theta$ through 2π we find

$$\alpha(\phi_p) = \frac{1}{2\pi} \oint \frac{x d\ell_\theta}{h |\nabla \phi_p|} \quad . \quad (A13)$$

Similarly,

$$q(\phi_p) = \frac{1}{2\pi} \int \frac{B \cdot \nabla \phi}{B \cdot \nabla \theta} d\theta = \frac{g}{2\pi} \oint \frac{J d\theta}{x^2} \quad (A14)$$

and thus,

$$q(\phi_p) = \frac{g\alpha}{2\pi} \int \frac{h d\theta}{x^2} = \frac{g}{2\pi} \oint \frac{d\ell_\theta}{x |\nabla \phi_p|} \quad (A15)$$

or finally,

$$q(\phi_p) = \frac{g}{f(\phi_p)} \quad (A16)$$

with

$$f^{-1}(\phi_p) = \frac{1}{2\pi} \oint \frac{d\ell_\theta}{x |\nabla \phi_p|} \quad . \quad (A17)$$

Note that if we choose $h = x^2$ here we have $\alpha = f^{-1}$ giving $v = 0$ or $\theta = \theta_p$, and $\zeta = \phi$. This is the coordinate system used in the PEST code.²⁰ The generalized coordinate system described here is used in the PEST-2 code.²¹ This allows for the possibility of directly interfacing the present ORBIT code to the perturbations obtained from that code. For the ORBIT code we need $h = 1/B^2$ to ensure that $I = I(\phi_p)$. For this choice of coordinate system we find

$$I(\psi_p) = \alpha(\psi_p) - gq \quad (A18)$$

and

$$J = \frac{I + gq}{B^2} \quad (A19)$$

APPENDIX B: CANONICAL COORDINATE TRANSFORMATION

In previous work¹ the canonical Hamiltonian variables for the guiding center drift motion were found for the representation

$$\vec{B} = \nabla\psi \times \nabla\theta_0 = \nabla\chi + \beta^* \nabla\psi, \quad (B1)$$

$$\delta\vec{B} = \vec{\nabla} \times \alpha \vec{B}. \quad (B2)$$

The canonical variables associated with this representation were shown to be $\chi, p_c, \theta_{oc}, \psi$ with $\theta_{oc} = \theta_0 - \beta^* p_c, p_c = p_{||} + \alpha$, and the guiding center drift equations given by

$$\dot{\chi} = \frac{\partial H}{\partial p_c}, \quad \dot{p}_c = -\frac{\partial H}{\partial \chi} \quad (B3)$$

$$\dot{\theta}_{oc} = \frac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial H}{\partial \theta_{oc}},$$

with the Hamiltonian

$$H = \frac{1}{2} p_{\eta}^2 B^2 + \mu B. \quad (B4)$$

Consider the representation of Sec. II,

$$\vec{B} = g\nabla\zeta + I\nabla\theta + \delta\nabla\psi, \quad (B5)$$

$$\vec{B} = \nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\phi_p, \quad (B6)$$

with g, I , and ϕ functions of the poloidal flux ϕ_p only, and $d\phi/d\phi_p = \alpha(\phi_p)$.

By comparing the two representations, we find the coordinate systems to be related through

$$\chi = g\zeta + I\theta , \quad (B7)$$

$$\theta_o = \theta - \zeta/q , \quad (B8)$$

$$\delta = \beta^* + (\zeta g' + \theta I')/q . \quad (B9)$$

Introducing toroidal and poloidal momentum variables and defining $\theta_c = \theta - \rho_c \delta$, we then have the four equations which define the coordinate transformation

$$P_{\zeta} = g\rho_c - \psi_p , \quad (B10)$$

$$P_{\theta} = I\rho_c + \psi , \quad (B11)$$

$$\chi = g\zeta + I(\theta_c + \rho_c \delta) , \quad (B12)$$

$$\theta_{oc} = \theta_c - \zeta/q + \rho_c(\delta - \theta^*) , \quad (B13)$$

and from these relations we find

$$\begin{array}{c|ccc|c}
 d\zeta & 1 & -I & (\beta^* - \delta + I\zeta q'/q^3) & -\theta^* I & d\chi \\
 d\theta_c & 1/q & g & (1/q)(\beta^* - \delta - g\zeta q'/q^2) & g(\beta^* - \delta) - I\delta/q & d\theta_{oc} \\
 dP_\zeta & 0 & 0 & -(g + I/q)/q & g(g + I/q) & d\phi \\
 dP_\theta & 0 & 0 & (g + I/q) & I(g + I/q) & d\rho_c
 \end{array}
 = \frac{1}{g + I/q}$$

(B14)

plus terms of higher order in the gyro radius ρ_H , which are ignorable since all that is desired is a Hamiltonian formalism that reproduces the equations of motion to lowest order in ρ . From these partial derivatives the Poisson brackets for the change of variables can be calculated, giving

$$[\theta_c, P_\theta] = 1$$

$$[\theta_c, P_\zeta] = 0$$

$$[\theta_c, \zeta] = 0$$

(B15)

$$[\zeta, P_\zeta] = 1$$

$$[\zeta, P_\theta] = 0$$

$$[P_\theta, P_\zeta] = 0,$$

and thus the new variables are canonical with

$$\begin{aligned}\dot{\theta}_c &= \frac{\partial H}{\partial P_\theta} & \dot{P}_\theta &= -\frac{\partial H}{\partial \theta_c} \\ \dot{\zeta} &= \frac{\partial H}{\partial P_\zeta} & \dot{P}_\zeta &= -\frac{\partial H}{\partial \zeta}\end{aligned}\quad (B16)$$

This canonical transformation is given by the generating function

$$F(\phi, \rho_\parallel, \theta_c, \zeta) = \rho_\parallel (g\zeta + I\theta_c) + \phi\theta_c - \zeta\phi_p \quad (B17)$$

with

$$\begin{aligned}\chi &= \frac{\partial F}{\partial \rho_\parallel} & \theta_{oc} &= \frac{\partial F}{\partial \phi} \\ P_\zeta &= \frac{\partial F}{\partial \zeta} & P_\theta &= \frac{\partial F}{\partial \theta_c}\end{aligned}\quad (B18)$$

APPENDIX C: THE NUMERICAL CODE ORBIT

The code ORBIT reads equilibrium data in the form of the functions g , I , q , and B , evaluated on the grid used for determining the equilibrium. In addition, the Jacobian is necessary for projecting into real space. The evaluation of these functions at the particle locations is then done using Lagrangian interpolation from the grid points. Alternatively, an analytic equilibrium can be substituted. Since the particles move very slowly across field lines and very rapidly along them, we employ five-point Lagrangian interpolation in θ and three-point interpolation in ϕ_p . A subroutine FIELD evaluates these functions at the particle locations. All operations are vectorized in the particle index to take advantage of the vector handling capability of the CRAY 1 computer.

The stepping procedure is carried out using a 5th order Runge-Kutta algorithm, vectorized in the particle index. For many applications the particle parameters are changed after each time step by a collision operator, so information from the previous time step cannot be used in the stepping algorithm. Typically 20 steps per toroidal transit results in particle energy conservation to one part in 10^7 . Computing time is approximately 50μ sec per particle step. It is more accurate to advance $\rho_{||}$ directly, using Eq. (4.19), than to advance it implicitly using Eqs. (4.13) to (4.16). This is because $\dot{\rho}_{||}$ nominally contains terms of first order in ρ , which exactly cancel, leaving only terms of order ρ^2 .

In addition to evaluating the equilibrium and stepping the particles, there are several diagnostic routines for evaluating particle diffusion, induced loss, making Poincare plots of magnetic island structures, etc.

APPENDIX D: HIGHER ORDER TERMS IN THE TOROIDAL PRECESSION

A large second order effect, which can be calculated, is due to the change in $\rho_{\parallel}$ and q caused by orbit excursions in ψ , i.e., due to finite banana width. These two are related, since $\dot{\phi}_p = g \dot{\rho}_{\parallel}$. Use the fact that orbit excursions are small and ignore variations in g to write

$$\Delta\phi_p = g(\rho_{\parallel} - \rho_0) \quad (D1)$$

where ρ_0 is the value of $\rho_{\parallel}$ at $t = 0$. Expanding $\rho_{\parallel}^2 = 2E/B^2 - 2\mu/B$ about ϕ_{p0} we find

$$\rho_{\parallel} = -\Delta \pm (\Delta^2 + 2\rho_0 \Delta + \rho_*^2)^{1/2} \quad (D2)$$

with $\rho_*^2 = 2E/B^2 - 2\mu/B$, and all quantities are evaluated on the initial flux surface ϕ_{p0} . The two signs in Eq. (D2) refer to the two sides of the banana orbit (for a passing particle only the plus sign is relevant). Evaluating the width at the point defined by $t = 0$, where $\rho_*^2 = \rho_0^2$, we find for the banana width

$$\Delta\phi_p = 2g(\rho_0 + \Delta) \quad (D3)$$

For a circular equilibrium with $\psi = r^2/2$ we find

$$\Delta r_b = \frac{2g}{r} (\rho_0 + \Delta) \quad (D4)$$

Keeping second order terms resulting from Eqs. (D2) and (D1), we find for

Eq. (4.27)

$$\omega_d = \frac{q'}{T} \oint \Delta \phi_p d\theta - \frac{[q+(I/q)]}{T} \oint d\theta \frac{\Lambda}{\rho_{\parallel} + \Delta} \quad (D5)$$

with

$$T = \oint d\theta \frac{D}{B^2(\rho_{\parallel} + \Delta)} \quad (D6)$$

and again we have dropped the term representing toroidal motion following the field line, $q/g \oint d\theta$. Here all expressions are evaluated on the initial flux surface. Note that the bounce point, $\theta = 0$, does not correspond to $\rho_{\parallel} = 0$, but rather $\rho_{\parallel} + \Delta = 0$. In Figs. 5 and 6 are shown both the first and second order expressions, Eq. (4.27) and Eq. (D5), respectively, as well as the value obtained numerically by advancing the Hamiltonian equations. The first order expression is accurate only for deeply trapped particles.

It is interesting to note that from Eq. (D2) we have

$$\left. \frac{\partial \rho_{\parallel}}{\partial E} \right|_{\psi, \mu} = \frac{1}{B^2(\rho_{\parallel} + \Delta)} \quad (D7)$$

$$\left. \frac{\partial \rho_{\parallel}}{\partial \psi_p} \right|_{E, \mu} = - \frac{\Lambda}{g(\rho_{\parallel} + \Delta)} \quad (D8)$$

and thus the expression derived by Kadomtsev, Eq. (4.33) is correct also including these second order terms.

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FIGURE CAPTIONS

FIG. 1. Examples of equilibria used. The average β values are zero and 8% for the circular (top) and bean (bottom) equilibria shown.

FIG. 2. Variation of magnetic field strength B with θ and ϕ_p , for a low β circular equilibrium. The curves correspond to $\phi_p = (1/3)\Delta\phi_p$, $(2/3)\Delta\phi_p$ and $\Delta\phi_p$, with $\Delta\phi_p$ the poloidal flux at the plasma edge.

FIG. 3. Variation of magnetic field strength B with θ and ϕ_p for a low β bean equilibrium. For outside surfaces there is a magnetic well near $\theta = \pi$.

FIG. 4. Example of the two classes of banana orbits existing in bean equilibria. Inside bananas exist only on flux surfaces near the plasma edge.

FIG. 5. The average toroidal precession rate ω_d as a function of initial pitch (at $\theta = 0$) in a circular equilibrium, obtained numerically using the Hamiltonian equations (N). Also shown is the first order result (1) from expression Eq. (4.26), and the second order expression (2) derived in Appendix D. This curve is for 1 keV deuterium ions in a 30 kG field. The precession frequency ω_d is normalized to the on-axis gyro frequency $\omega_0 = 4.8 \times 10^7 \text{ sec}^{-1}$. For this case $\rho = 1.5 \times 10^{-3}$.

FIG. 6. The average toroidal precession rate ω_d as a function of initial pitch in a circular equilibrium. 50 keV deuterium ions in a 30 kG field were used. Also shown are first order analytic results (1) from Eq. (4.26) and second order results (2) from Appendix D. For this case $p = 10^{-2}$.

FIG. 7. An example of a magnetic island in a bean configuration.

FIG. 8. Beam injection parameters. Normally the beam is introduced horizontally and in the equatorial plane ($z = 0$).

FIG. 9. Toroidal precession rates versus ionization point X for 50 keV deuterium into bean and circular equilibria for various β . Precession reversal for near perpendicular injection takes place only when there exists a magnetic well, so that $\partial B / \partial \phi_p > 0$. The trapped particle precession rates are, however, significantly lower in bean configurations than in circular ones. Here $R_T = 36$ cm, $R = 143$ cm, $a = 35$ cm, and $B = 30$ kG. ω_d is given in terms of the on-axis gyro frequency. The magnetic axis is located approximately at the transition point from trapped to passing.

FIG. 10. Fraction of beam lost during one period (~ 500 μ sec) for a fixed amplitude fishbone oscillation versus mode frequency. The loss from the bean is significantly less because of the lower average precession rates.

#83T0437

Fig. 1

#83T0439

Fig. 2

#83T0438

Fig. 3

Fig. 4

Fig. 5

#83T0476

Fig. 6

Fig. 7

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Fig. 8

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Fig. 9

#63T0435

Fig. 10

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