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SCATTERING OF LONG OCEAN WAVES

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SCATTERING OF LONG OCEAN WAVES

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ABSTRACT

The differential equation describing the scattering of long ocean waves by a localized scatterer such as a seamount is transformed to an integral equation of the form arising in quantum mechanical scattering theory. The application of various approximation techniques developed in quantum mechanics is discussed and, as an example, an analytical expression is obtained for the first Born approximation to the wave scattered by a cylindrically symmetric seamount of arbitrary profile. The range of validity of this approximation is determined.

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INTRODUCTION

A classic and important example of a long ocean wave is the tsunami. Tsunamis, generated by abrupt displacements of the ocean floor associated with seismic or volcanic activity, have a destructive power that is well known. To predict the effects of tsunamis, two separate features of the wave propagation problem are of interest: the time required for the wave to propagate from the source to any given destination, and the wave amplitude at that destination. While the former problem is now adequately treated by ray theory, the amplitude of the incoming wave is not readily obtained. Wave amplitudes at any given point of observation depend upon the distance from the source; the mode of generation at the source; the bottom topography between the source and the observing point; and the individual characteristics of shoreline and shelf at or near the point of observation. As an initial step in understanding one element of this complex process, we consider the deep-water scattering of long ocean waves by localized topographic features on the ocean bottom, such as sea mounts.

THE SCATTERING PROBLEM

The propagation of long ocean waves is described by the equation:⁽¹⁾

$$\frac{\partial^2 \xi(\underline{r}, t)}{\partial t^2} = g \nabla \cdot D(\underline{r}) \nabla \xi(\underline{r}, t) \quad , \quad (1)$$

where $\xi(\underline{r}, t)$ is the displacement of the free surface above mean sea level, $\underline{r}$ is the horizontal position vector, and

$D(\underline{r})$ is the local depth of the water (Figure 1). If the time dependence is assumed to be of the simple $e^{-i\omega t}$ form, and if

$$\xi(\underline{r}, t) = \eta(\underline{r})e^{-i\omega t} \quad , \quad (2)$$

then $\eta(\underline{r})$ must satisfy the equation:

$$\nabla \cdot D(\underline{r}) \nabla \eta(\underline{r}) + \frac{\omega^2}{g} \eta(\underline{r}) = 0 \quad . \quad (3)$$

In practice, Equation (3) has usually been applied to the situation where $D(\underline{r})$ is constant except in a localized region, corresponding to a seamount or island. In this case, the solution has the form of an incoming plane wave, $e^{ik_0 x}$, which impinges on the seamount or island and is scattered (or diffracted) into an outgoing cylindrical wave having the form $f(\theta) \exp(ik_0 r)/\sqrt{r}$ far from the disturbance.

FIGURE 1. *Nomenclature for Scattering Problem*

Analytical solutions have been obtained in highly idealized cases,^(2,3) although Vastano and Reid⁽⁴⁾ have recently presented a numerical technique that should generally apply. However, the numerical approach does not readily lend itself to systematic approximations, and it has a number of other disadvantages (noted below). In fact, we believe that the entire

problem can be approached more fruitfully by exploiting the analogies between the scattering of long ocean waves and quantum-mechanical scattering.

In the quantum-mechanical scattering problem, a beam of particles (plane wave) is incident on a region characterized by a potential, $V(\underline{r})$. Since the wave speed depends upon the potential, the wave is diffracted by $V(\underline{r})$ and the scattered part gives rise to an outgoing spherical wave whose amplitude depends upon the scattering angle. The scattering of long ocean waves by a local variation in the depth is quite similar. Since the wave speed depends upon the depth ($V \sim \sqrt{gD}$), the incident plane wave is diffracted by the local depth variation and gives rise to an outgoing cylindrical wave whose amplitude depends upon the scattering angle.

The analysis of quantum mechanical scattering usually proceeds by formulating the problem as an integral equation rather than a differential equation. This approach offers many advantages, as will be seen below.

Apparently, it has not been realized in the past that the differential equation [Equation (3)] can easily be converted to an integral equation of the same form as that arising in the quantum-mechanical scattering problem. We have succeeded in carrying out this transformation for the case of scattering by a seamount or similar undersea obstacle that does not break the surface. We let $D(\underline{r}) = D_0 - h(\underline{r})$, where D_0 is the (uniform) depth far from the seamount and $h(\underline{r})$ is the local height of the seamount above D_0 . Substituting this expression into Equation (3) yields:

$$(\nabla^2 + k^2) \eta(\underline{r}) = \frac{1}{D_0} \nabla \cdot h(\underline{r}) \nabla \eta(\underline{r}) \quad , \quad (4)$$

where

$$k^2 = \omega^2 / gD_0 \quad . \quad (5)$$

Equation (4) has the same form as the Schroedinger quantum mechanical equation with a "potential" given by:

$$V(\underline{r}) = \frac{1}{D_0} \nabla \cdot h(\underline{r}) \nabla \quad . \quad (6)$$

This corresponds in quantum mechanics to a potential which depends upon both the position and the momentum of the scattered particles. Equation (4) can be converted to an integral equation for $n(\underline{r})$ by introducing the Green's function $G(\underline{r}, k)$ for the operator $(\nabla^2 + k^2)$:

$$(\nabla^2 + k^2) G(\underline{r}, k) = \delta(\underline{r}) \quad , \quad (7)$$

with $G(\underline{r}, k)$ having the form of an outgoing cylindrical wave for large r . [As shown in the Appendix, $G(\underline{r}, k)$ is proportional to the Hankel function $H_0^{(1)}$.] Equation (4) is then equivalent to the integral equation

$$n(\underline{r}) = \exp(i \underline{k}_0 \cdot \underline{r}) + \int G(\underline{r} - \underline{r}', k) V(\underline{r}') n(\underline{r}') d\underline{r}' \quad (8)$$

where the term $\exp(i \underline{k}_0 \cdot \underline{r})$ is a solution of the homogeneous equation

$$(\nabla^2 + k^2) \exp(i \underline{k}_0 \cdot \underline{r}) = 0 \quad , \quad (9)$$

and represents a plane wave incident along the direction $\underline{k}_0$, with $k_0^2 = k^2$. The incoming wave has unit amplitude; thus, $n(\underline{r})$ is measured in units of the incoming wave amplitude.

Equation (8) is completely equivalent to the differential equation, [Equation (3)] with associated boundary conditions. However, we believe that the integral equation approach offers several significant advantages:

- (1) Many systematic approximation techniques have been developed to treat the quantum-mechanical scattering

problem, and many of these should be applicable to the scattering of long ocean waves as well. For example, the approximation of the solution by the first two terms of the Neumann series (the Born approximation) or the replacement of $V(\underline{r})$ by an appropriate generalized function (the pseudopotential approximation) often give good results.

- (2) The method of "partial waves," where the incoming and outgoing waves are represented as a sum of elementary functions, has been used in quantum mechanics to convert the scattering problem to a series of more elementary problems. This method should also be applicable here.
- (3) Even if the problem must be solved numerically, Equation (8) should be a more practical starting point than the differential equation since the exact Green's function $G(\underline{r}, k)$ can be used; hence, the solution $n(\underline{r})$ necessarily has the proper outgoing wave form for large r . When working with the differential equation, it is necessary to calculate the solution well beyond the range where $h(\underline{r})$ vanishes to be assured of the proper form; with the integral equation this is inherent, and the integration extends only over values of $\underline{r}'$ lying in the range where $h(\underline{r}')$ is nonzero. We remark that when a numerical solution is required, a more convenient form of Equation (8) can be obtained by using Green's theorem and the explicit form of $V(\underline{r})$:

$$n(\underline{r}) = \exp(i\mathbf{k}_0 \cdot \underline{r}) + h(\underline{r})n(\underline{r}) + \int d\underline{r}' [\nabla' h(\underline{r}') \cdot \nabla' G(\underline{r} - \underline{r}', k) - k_0^2 G(\underline{r} - \underline{r}', k) h(\underline{r}')] n(\underline{r}')$$

This form of the equation does not require derivatives of the unknown function $n(\underline{r})$ and is, thus, more suitable for a numerical solution.

EXAMPLE

As an illustration of the use of techniques developed in quantum-mechanical scattering theory to solve a long wave scattering problem, we apply the first Born approximation to calculate the scattering due to a cylindrically symmetric seamount located at the origin [so that $h(\underline{r})$ depends only on r]. The conditions under which this approximation accurately represents the scattered wave are also investigated.

By using the asymptotic form of the Hankel function, we find that for large r ,

$$G(\underline{r}-\underline{r}', k) \approx (-i/4)(2/\pi k_o)^{1/2} \exp(-i\pi/4) \exp(-ik_s \cdot \underline{r}') \exp(ik_o r) (r)^{-1/2} \quad (10)$$

where

$$\underline{k}_s = (k_o/r) \underline{r} \quad (11)$$

Thus, for large r , Equation (8) becomes

$$n(\underline{r}) \approx \exp(ik_o \cdot \underline{r}) + f(\theta) \exp(ik_o r) (r)^{-1/2} \quad (12)$$

where

$$f(\theta) = (-i/4)(2/\pi k_o)^{1/2} \exp(-i\pi/4) \int \exp(-ik_s \cdot \underline{r}') V(\underline{r}') n(\underline{r}') d\underline{r}' \quad (13)$$

Here,

$$\theta = \cos^{-1} \left(\frac{\underline{k}_o \cdot \underline{k}_s}{k_o^2} \right) \quad (14)$$

is the angle between the direction of incidence and the vector $\underline{r}$.

Equation (12) gives a representation of the wave far from the scatterer as the sum of an incoming plane wave and a scattered cylindrical wave. The amplitude of the scattered wave, $f(\theta)$, is easily related to the "cross section" for scattering of energy from the incoming wave into a range of angles $d\theta$. To show this, let ϕ_0 denote the energy flux (energy/length·time) of the incoming plane wave and let $\phi(\theta)d\theta$ denote the energy flux (energy/time) scattered into $d\theta$. The scattering cross section* $\frac{d\sigma}{d\theta}$ is defined by

$$\phi_0 \cdot \frac{d\sigma}{d\theta} \cdot d\theta = \phi(\theta)d\theta$$

and since the energy carried by the wave is proportional to the square of the amplitude it is easily shown that

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2$$

The expression for the scattering amplitude $f(\theta)$ given in Equation (13) is quite general. The first Born approximation is obtained by replacing $n(\underline{r}')$ under the integral sign by the unscattered wave, $\exp(i\mathbf{k}_0 \cdot \underline{r}')$. This corresponds to retaining only the zeroth and first terms of the Neumann series solution of Equation (8). For the moment, we assume that this approximation is adequate; the conditions for its validity will be investigated below.

When $h(\underline{r})$ depends only on r , we find that

$$\begin{aligned} V(\underline{r}) &= \frac{1}{D_0} \nabla \cdot h(\underline{r}) \nabla = \frac{1}{D_0} [\nabla h(\underline{r}) \cdot \nabla + h(\underline{r}) \nabla^2] \\ &= \frac{1}{D_0} \left[\frac{1}{r} \frac{dh(r)}{dr} \underline{r} \cdot \nabla + h(r) \nabla^2 \right] \end{aligned} \quad (15)$$

* The term "cross section" is borrowed from three-dimensional scattering problems; in the present context, $\frac{d\sigma}{d\theta}$ has dimensions of length.

so that in the first Born approximation,

$$f(\theta) = (-i/4)(2/\pi k_o)^{1/2} \exp(-i\pi/4) \\ \frac{1}{D_o} \int \left[\frac{i}{r'} h'(r') i \underline{k}_o \cdot \underline{r}' - h(r') k_o^2 \right] \\ \exp(i \underline{k} \cdot \underline{r}') d\underline{r}' \quad , \quad (16)$$

where

$$\underline{k} = \underline{k}_o - \underline{k}_s \quad . \quad (17)$$

The integral in Equation (16) is easily evaluated by taking a polar coordinate system for $\underline{r}'$, with $\theta' = 0$ lying along $\underline{k}$. We suppose that $h(r) = 0$ for $r \geq a$. Then

$$I \equiv \int \left[\frac{1}{r'} h'(r') i \underline{k}_o \cdot \underline{r}' - h(r') k_o^2 \right] \\ \exp(i \underline{k} \cdot \underline{r}') d\underline{r}' \\ = \int_0^a dr' r' \int_0^{2\pi} d\theta' \left[\frac{1}{r'} h'(r') i k_o \cos(\psi - \theta') \right. \\ \left. - k_o^2 h(r') \right] \exp(i k r' \cos \theta') \quad (18)$$

where

$$\psi = \underline{k}_o \cdot \underline{k} / k_o k \quad . \quad (19)$$

By using the addition formula for $\cos(\psi - \theta')$, and the integral representations

$$J_0(\kappa r') = \frac{1}{2\pi} \int_0^{2\pi} \exp(i\kappa r' \cos \theta') d\theta'$$

$$J_1(\kappa r') = \frac{1}{2\pi i} \int_0^{2\pi} \cos \theta' \exp(i\kappa r' \cos \theta') d\theta' \quad , \quad (20)$$

we obtain

$$I = -2\pi \int_0^a \left[k_0 \cos \psi J_1(\kappa r') h'(r') \right. \\ \left. + k_0^2 J_0(\kappa r') h(r') \right] r' dr' \quad . \quad (21)$$

If the first term is integrated by parts, we obtain,

$$I = 2\pi \left(k_0 \kappa \cos \psi - k_0^2 \right) \\ \int_0^a r' J_0(\kappa r') h(r') dr' \quad . \quad (22)$$

Finally, by using the definitions of $\underline{\kappa}$, $\underline{k}_s$ and θ , and substituting the above into Equation (16), we obtain

$$f(\theta) = i(\pi/2k_0)^{1/2} \exp(-i\pi/4) \\ \left[\left(k_0^2/D_0 \right) \int_0^a r' J_0(2k_0 r' \sin(\theta/2)) \right. \\ \left. h(r') dr' \right] \cos \theta \quad , \quad (23)$$

which is the general first Born approximation to $f(\theta)$.

Since the wave lengths of actual tsunamis in deep water are large compared to the radii of seamounts, we will usually have $k_0 a \ll 1$, so that

$$f(\theta) \approx i(\pi/2k_0)^{1/2} \exp(-i\pi/4) \left[\left(k_0^2/D_0 \right) \int_0^a r' h(r') dr' \right] \cos \theta \quad (24)$$

If the first Born approximation is valid, and if $k_0 a \ll 1$, the scattering amplitude of a cylindrically symmetric seamount depends only on the first moment of $h(r)$. Thus, seamounts of differing profile that have the same first moment will scatter long waves identically. It is also interesting to note that the amplitude of the scattered wave is proportional to $\cos \theta$; consequently, the wave scattered at 90° has zero amplitude.

The validity range of the above results can be estimated by calculating the correction resulting from the retention of an additional term in the Neumann series solution of Equation (8). This is the second Born approximation, and the corresponding expression for $f(\theta)$ is obtained by making the replacement,

$$n(\underline{r}') \approx \exp(i\underline{k}_0 \cdot \underline{r}') + \int G(\underline{r}' - \underline{r}'', k) V(\underline{r}'') \exp(i\underline{k}_0 \cdot \underline{r}'') dr'' \quad (25)$$

in Equation (13). Therefore, the correction term to $f(\theta)$ is

$$\Delta f(\theta) = (-i/4)(2/\pi k_0)^{1/2} \exp(-i\pi/4) \int d\underline{r}' \exp(i\underline{k}_s \cdot \underline{r}') V(\underline{r}') \times \int d\underline{r}'' G(\underline{r}' - \underline{r}'', k) V(\underline{r}'') \exp(i\underline{k}_0 \cdot \underline{r}'') \quad (26)$$

This expression is most easily evaluated by introducing a Fourier representation of $G(\underline{r}' - \underline{r}'', k)$:

$$G(\underline{r}' - \underline{r}'', k) = - \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int \frac{\exp(i\mathbf{p} \cdot (\underline{r}' - \underline{r}'')) d\mathbf{p}}{p^2 - (k_0^2 + i\epsilon)} \quad (27)$$

The integration over θ' and θ'' in Equation (26) then leads to terms involving the Bessel functions $J_0(r''|\underline{k}_0 - \underline{p}|)$ and $J_0(r'|\underline{p} - \underline{k}_s|)$. These can be expressed in terms of Bessel functions of simpler arguments by using the Gegenbauer addition theorem,

$$J_0(|\underline{u} - \underline{v}|) = J_0(u) J_0(v) + 2 \sum_{n=1}^{\infty} J_n(u) J_n(v) \cos n(\theta_u - \theta_v) \quad (28)$$

With these devices and considerable algebra, $\Delta f(\theta)$ can be evaluated. We find that for $k_0 a \ll 1$,

$$\Delta f(\theta) = -i(\pi/2k_0)^{1/2} \exp(-i\pi/4) (k_0^2/2D_0^2) \int_0^a r' h^2(r') dr' \cos \theta \quad (29)$$

Thus,

$$\left| \frac{\Delta f(\theta)}{f(\theta)} \right| = \frac{\left[\int_0^a r' h^2(r') dr' \right]}{2D_0 \left[\int_0^a r' h(r') dr' \right]} \quad (30)$$

This quantity will generally be small if the seamount is not tall ($h(r)/D_0 \ll 1$); or if it is not broad ($h^2(r)$ will then be

"narrower" than $h(r)$). These conditions are certainly not met for all cases of interest, and if we consider scattering from regions that include both an elevation and a depression of the sea bottom, it is quite possible that the denominator of Equation (30) will be smaller than the numerator. If it is, the first Born approximation will then give very poor results. However, the requirement that the quantity in Equation (30) be small does serve to define a class of physical problems to which the above simplified analysis will apply.

DISCUSSION

The potential advantages of considering long ocean-wave propagation as an analogue of quantum mechanical scattering are illustrated by the above example. Approximate analytical expressions are readily obtained for the scattering amplitude, and the features of the scatterer, which are most important in determining the nature of the scattered wave are identified. It is expected that the application of the Born approximation to more complex scatterers and the application of other techniques from quantum mechanics, such as the pseudopotential approximation and partial wave analysis, will lead to similar results for wide classes of problems of interest.

Future work in this direction will determine the classes of problems to which the various approximate techniques may fruitfully be applied, and should also result in a much better understanding of the features of the scatterer most strongly influencing the scattered wave. Finally, the suitability of the integral equation approach for the numerical solution of complex scattering problems will be investigated.

APPENDIXTHE GREEN'S FUNCTION

It is desired to find the solution of the equation

$$(\nabla^2 + k^2) G(\underline{r}, k) = \delta(\underline{r}) \quad (A-1)$$

for which $G(\underline{r}, k)$ has the form of an outgoing cylindrical wave at large r .

By symmetry, we note that $G(\underline{r}, k)$ can depend only on r and k . Thus, Equation (A-1) reduces to Bessel's equation for $r \neq 0$ and since $G(\underline{r}, k)$ is to represent an outgoing wave, we deduce that

$$G(\underline{r}, k) = C \cdot H_0^{(1)}(kr) \quad , \quad (A-2)$$

where $H_0^{(1)}(kr)$ is the Hankel function of the first kind.⁽⁵⁾

To evaluate the constant C , we integrate Equation (A-1) over a small cylindrical region about $r = 0$ and deduce that

$$\lim_{r \rightarrow 0} 2\pi r \frac{\partial}{\partial r} G(\underline{r}, k) = 1 \quad . \quad (A-3)$$

Using properties of $H_0^{(1)}(kr)$ from the above reference, we find that $C = -i/4$, so that

$$G(\underline{r}, k) = -i/4 H_0^{(1)}(kr) \quad . \quad (A-4)$$

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