A NOTE ON CYLINDRICAL WAVES WHICH PROPAGATE AT THE VELOCITY OF LIGHT

Norman H. Kroll
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305
and
University of California, San Diego†
San Diego, California 92093

INTRODUCTION

The continuous linear acceleration of ultra relativistic particles in free space by an electromagnetic field requires the presence of a cylindrical wave component with phase velocity that differs negligibly from c and with non-vanishing electric field component in the direction of propagation. Lawson and Woodward have pointed out the fact that certain geometries proposed for laser driven acceleration fail to satisfy these requirements. On the other hand, complex wave number plane wave fields which do satisfy these requirements have been constructed by Palmer, who also points out that any cylindrical wave with the required properties can be formed from superposition of plane waves of the form which he has obtained.

The situation is analogous to that which occurs in standard waveguide theory. There it is also true that any waveguide mode can be constructed by superposition of plane waves. Nevertheless, the study of the general properties of cylindrical waves has proved to be a very powerful tool for the analysis of waveguides and similar structures. Because this may also prove to be the case for fields with propagation velocity c we present a brief study of their properties below.

A BRIEF REVIEW OF CYLINDRICAL WAVES

By definition, cylindrical waves with sinusoidal time dependence are solutions of Maxwell's equations in which the fields have the general form

\[ \mathbf{E} = \mathbf{E}(x,y) \exp(i(kz - \omega t)) \]
\[ \mathbf{B} = \mathbf{B}(x,y) \exp(i(kz - \omega t)) \] (1)

* Work supported in part by the Department of Energy, contracts DE-AC03-76SF00515 (SLAC) and DE-AT03-81ER40029 (UCSD).
† Permanent address.

(Presented at the Laser Acceleration of Particles Workshop, Los Alamos, New Mexico, February 18-23, 1982.)
Since each field satisfies the vector wave equation we have
\[ \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \vec{\nabla} \times \left( \frac{\omega^2}{c^2} \vec{E} \right) - \vec{\nabla} \times \left( \frac{\omega^2}{c^2} \vec{B} \right) = 0 \]
and hence the transverse variation of each field component satisfies the two dimensional Helmholtz equation
\[ \nabla^2 f(x,y) + k^2 f(x,y) = 0 \quad (2) \]
where
\[ k^2 = \frac{\omega^2}{c^2} - k^2 \quad (3) \]
writing
\[ \vec{E}(x,y) = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z, \quad \hat{z} \cdot \vec{E} = 0 \quad (4a) \]
\[ \vec{B}(x,y) = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z, \quad \hat{z} \cdot \vec{B} = 0 \quad (4b) \]
and substituting in Maxwell's equations yields (Gaussian units)
\[ \begin{align*}
  ik \hat{z} \times \vec{E}_T - \hat{z} \times \vec{E}_T &= i \frac{\omega}{c} \vec{B}_T \\
  ik \hat{z} \times \vec{B}_T - \hat{z} \times \vec{B}_T &= -i \frac{\omega}{c} \vec{E}_T
\end{align*} \quad (5) \]
\[ \begin{align*}
  \hat{z} \times \vec{E}_T &= i \frac{\omega}{c} B_z \hat{z} \\
  \hat{z} \times \vec{B}_T &= -i \frac{\omega}{c} E_z \hat{z}
\end{align*} \quad (6) \]
From Eq. (3) one obtains
\[ \begin{align*}
  \vec{E}_T &= \frac{4k}{K^2} \vec{E}_z - \frac{4\omega/c}{K^2} \hat{z} \times \vec{B}_z \\
  \vec{B}_T &= \frac{4k}{K^2} \vec{B}_z + \frac{4\omega/c}{K^2} \hat{z} \times \vec{E}_z
\end{align*} \quad (7) \]
The standard TE (TM) modes are obtained by setting \( E_z (B_z) \) equal to zero and choosing a suitable set of solutions of (2) for \( B_z (E_z) \). The transverse fields are then obtained from (7).
THE SPECIAL CASE $k = \omega/c$

For waves which propagate at velocity $c$, the quantity $k^2$, which appears in the denominator of Eq. (7) vanishes, so that the previously described procedure fails. Setting $E_z = B_z = 0$ provides an obvious way out of the difficulty with Eq. (7) and leads to the usually discussed TEM modes. There are additional possibilities, however, which we discuss below.

Setting $k^2 = 0$ in (2) we obtain:

$$\nabla^2 f = 0 \quad (8)$$

Setting $k = \omega/c$ in (5) and carrying out a little vector algebra yields

$$\nabla B_z = -\hat{\mathbf{\hat{z}}} \times \hat{E}_z \quad (9)$$

Equation (8) tells us that $B_z$ and $E_z$ satisfy Laplace's equation, and (9) tells us that they are related by the Cauchy-Riemann equations. Thus if we write

$$W(x+iy) = B_z(x,y) + iE_z(x,y) \quad (10)$$

Equations (6) and (9) are satisfied wherever $W$ is analytic. We note that in the general case the modes are neither TE nor TM as Eq. (9) implies that non-vanishing $\nabla E_z$ implies non-vanishing $B_z$. An additional set of useful restrictions on the transverse fields is obtained by taking the divergence of (5), which when combined with (6) yields

$$\nabla \cdot \hat{E}_T = -ikE_z \quad (11a)$$

$$\nabla \cdot \hat{B}_T = -ikB_z \quad (11b)$$

Equations (5), (6), (10), and (11) are a set of necessary relations which the fields must satisfy. The relations are not, however, independent since satisfaction of a suitable subset implies the others. It is sufficient, for example, to satisfy (6a), (11a), (5a) and either (10) or the two dimensional vector Laplace equation for $\hat{E}_T$. The simplest procedure for generating a solution is to begin by specifying any solution of the two dimensional vector Laplace equation for $\hat{E}_T$. If one identifies this solution with $\hat{E}_T$, then (6a), (11a), and (5a), provide explicit formulas for $B_z$, $E_z$, and $\hat{B}_z$, respectively. Alternatively, if one identifies this solution with $\hat{B}_T$, then (6b), (11b), and (5b) provide explicit formulas for $E_z$, $B_z$, and $\hat{E}_T$. Thus each solution of the vector Laplace equation generates two independent solutions.
SOME SIMPLE APPLICATIONS

A solution of the type applied by Palmer to the grating problem is obtained by setting

\[ E_x = E \cos qye^{-qx} \]
\[ E_y = 0 . \]

Then from (6a)

\[ B_z = \frac{i}{k} \frac{3E_x}{k} = -\frac{iq}{k} E \sin qye^{-qx} \]

from (11a)

\[ E_z = \frac{i}{k} \frac{3E_x}{k} = -\frac{iq}{k} E \cos qye^{-qx} \]

and from (5a)

\[ \vec{B}_T = \hat{\phi} \times \left( \vec{E}_T + \frac{1}{k} \hat{\phi} \vec{E}_z \right) \]
\[ = E \left[ \left( 1 - \frac{q^2}{k^2} \right) \hat{\phi} \cos qy + \frac{2}{k^2} \hat{\theta} \sin qy \right] e^{-qy} \]

The cylindrically symmetric solution suitable for acceleration is obtained by setting

\[ E_z = \text{constant} \]
\[ H_z = 0 \]

Then from (6a) and (11a)

\[ \vec{\dot{r}}_T = -\frac{ikr^2}{2} \vec{E}_z \]

and from (5a)

\[ \vec{\dot{r}}_T = -\frac{ik\theta}{2} \vec{E}_z . \]

Finally, we recall that Lawson and Woodward showed that any physical solution independent of y and valid in free space for all \( x > 0 \) must have vanishing \( E_z \). This result is easily obtained in the present context from the fact that for \( y \) independent fields (11a) and (8) imply that \( E_z \) is constant and \( E_x = ikE_x \hat{x} + \text{constant} \).
REFERENCES


3. For a more detailed discussion see, for example, J. D. Jackson, Classical Electrodynamics (Wiley, New York, 2nd Edition 1975), Ch. 8.

4. Related TEM modes are obtained by setting $\mathbf{E}_T = \nabla (\nabla \cdot \mathbf{P})$, $\mathbf{H}_T = \nabla \times \mathbf{E}_T$ (or $\mathbf{H}_T = \nabla (\nabla \cdot \mathbf{P})$, $\mathbf{E}_T = -\nabla \times \mathbf{H}_T$) where $\mathbf{P}$ is the solution to the vector Laplace equation referred to above. Note further that although $\mathbf{E} \times \mathbf{P}$ is also a solution of the vector Laplace equation, the solutions which it generates by the same procedure are linear combinations of those obtained with $\mathbf{P}$.