MASTER

UCRL 6529

325
12-14-61

UNIVERSITY OF
CALIFORNIA

Ernest O. Lawrence

*Radiation
Laboratory*

SYNTHESIS OF WAVE-FORMING NETWORKS

LIVERMORE SITE

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Livermore, California

Contract No. W-7405-eng-48

SYNTHESIS OF WAVE-FORMING NETWORKS

J. W. LaPatra

August 3, 1961

Printed in USA. Price 50 cents. Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D.C.

SYNTHESIS OF WAVE-FORMING NETWORKS

J. W. LaPatra

Lawrence Radiation Laboratory, University of California

Livermore, California

August 3, 1961

ABSTRACT

The general area of synthesizing wave-forming networks is very broad with many diverse facets. This report is restricted to passive networks whose specified wave forms are given as a function of time. The wave forms may be periodic or a single pulse. Suggestions and techniques are presented which should enable a circuit designer with a basic knowledge of network synthesis procedures to obtain wave-forming networks for a wide variety of cases.

SYNTHESIS OF WAVE-FORMING NETWORKS*

J. W. LaPatra

Lawrence Radiation Laboratory, University of California
Livermore, California

INTRODUCTION

The ability to provide passive circuits to satisfy a variety of specifications is of important concern to network designers. These circuits are composed of linear, bilateral, lumped, and constant elements. The versatility of combinations of these elements is indicated by the volume of network synthesis literature available. The bulk of this literature, however, is concerned with synthesis procedures to meet a particular set of requirements. It is hoped a general theory will become evident when sufficient specialized techniques have been developed.

The synthesis procedures presented in this report attempt to deal with a broad set of specifications, but only at the price of some extended computation.

Briefly, if a periodic wave form or a pulse is specified as a function of time in a circuit containing sources, linear elements, and nonlinear elements, a realizable transfer function may be obtained. The synthesis of this transfer function will provide a one-port passive-wave-forming circuit. The degree of approximation is determined by a decision made during the synthesis procedure.

TIME DOMAIN SYNTHESIS

Particularly in the area of pulse work, circuit specifications are given as a function of time. This is quite contrary to the usual method of specifying frequency response. Time domain or transient synthesis is concerned with determining networks whose inputs and outputs are functions of time.

Using transformation calculus the time function may be converted to a function involving the complex variable s and a synthesis procedure employed.¹

---

* Work performed under the auspices of the U. S. Atomic Energy Commission.

¹ D. K. Cheng, Analysis of Linear Systems, (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts), 1959.

The basic problems are:

1. What approximation to the given wave form is acceptable?
2. How may the wave form be expressed in time so that it transforms into a realizable function of s ?

These two questions are, of course, intimately related. Having chosen the method of expressing the wave form, the approximation technique will follow if a reasonable choice is available.

Expressing the Required Wave Form in Time

There are several approaches to the expression of a wave form with time as the independent variable. These should be considered as alternatives to be used in deciding which transfer function is most suitable for a particular case.

- (a) The wave form may be expressed as a periodic function such as sine or cosine modified by an exponent as a coefficient.
- (b) The pulse or the periodic wave may be represented as combinations of ramps, step functions, and others. They will usually include time translation terms.
- (c) A pulse may be synthesized in time by combining known wave forms, all of which begin at $t = 0$. This insures that the presence of exponentials in the transform is avoided.
- (d) A wave form, pulse or periodic, of any shape may be resolved into a sine or cosine series by means of the proper Fourier expansion.

To help decide which representation would be best for a particular case it is appropriate to consider some of the problems which arise.

In (a) a realizable frequency domain approximate may not exist because the immittance function may contain right-half-plane poles. It is possible these may be discarded without seriously altering the approximation.

For case (b) the time-translation terms cause the transfer function to be transcendental. The exponentials may be handled by finding a rational approximate and again discarding the negative terms.

The most obvious difficulty with (c) is the quality of the approximation of the time function. In addition, this is for all practical purposes an empirical expression which has been developed, and its transform may be very complicated.

In the general time domain synthesis problem, the basic concern is the rationality of the Laplace transforms of the input and output functions. However, the only time functions whose transforms are rational functions are sums of exponentials of the form Ae^{at} . This suggests the exponential Fourier series expansion as indicated in (d). Immediately implied in this method is a graphical expansion involving copious computation; however, a suitable computer program would remove this difficulty.

Exponential Fourier Series²

The experience of the author has indicated that the exponential Fourier series offers the greatest possibility for a useful time expression. This will be considered in some detail in anticipation of an example to follow.

The well-known trigonometric Fourier series

$$f(t) = \frac{a_0}{2} + a_1 \cos \omega_1 t + a_2 \cos \omega_2 t \dots \\ + b_1 \sin \omega_1 t + b_2 \sin \omega_2 t \dots$$

may be written in the exponential form

$$f(t) = \dots + A_{-2}e^{-j\omega_2 t} + A_{-1}e^{-j\omega_1 t} + A_0 + A_1e^{j\omega_1 t} + A_2e^{j\omega_2 t} + \dots$$

The exponential form may be obtained by first computing the trigonometric expansion and substituting

$$\cos x = \frac{1}{2} (e^{jx} + e^{-jx}) \quad ,$$

$$\sin x = -\frac{j}{2} (e^{jx} - e^{-jx}) \quad .$$

The coefficients of the exponential series may be obtained directly from

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-jn\omega_1 t} d(\omega_1 t) \quad .$$

To insure minimization of calculation effort, all wave forms could be subjected to a computer programmed to accomplish graphical Fourier expansion in the trigonometric form. In a later part of this report there is a discussion of representing pulses in terms of periodic functions. This is,

---

² H. H. Skilling, Electrical Engineering Circuits, (John Wiley & Sons, Inc., New York, 1957), Chap. 15.

of course, necessary in order to accomplish the Fourier expansion. See Appendix 1 for a discussion and an example of graphical Fourier expansion.

The characteristics of the exponential series may be summarized as follows:

1. For odd functions, all A's are purely imaginary.
2. For even functions, all A's are purely real.
3. If there is half-wave symmetry, $A_n = 0$ for even n.
4. A_{-n} is the conjugate of A_n .

The series may be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_1 t}$$

where n is any integer.

Of primary concern is the form of the Laplace transform of the exponential series.³ Consider a typical case with the constant term present and when positive and negative values of two integers are included.

$$f(t) = A_{-2} e^{-j\omega_2 t} + A_{-1} e^{-j\omega_1 t} + A_0 + A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t},$$

$$\mathcal{L}[f(t)] = \frac{A_{-2}}{s + j\omega_2} + \frac{A_{-1}}{s + j\omega_1} + \frac{A_0}{s} + \frac{A_1}{s - j\omega_1} + \frac{A_2}{s - j\omega_2}.$$

$$\mathcal{L}[f(t)] = \frac{\text{numerator}}{s(s^2 + \omega_1^2)(s^2 + \omega_2^2)}.$$

The numerator has terms of the form

$$A_0 s(s^2 + \omega_1^2)(s^2 + \omega_2^2)$$

and

$$\left\{ 2\operatorname{Re}[A_{\pm n}] s^2 + 2\operatorname{Im}[A_{\pm n}] s \right\} (s^2 + \omega_{n-1}^2).$$

The transfer function in this case will be an immittance function and must be examined in detail for realizability characteristics. If the transfer

³W. R. LePage. Complex Variables and the Laplace Transform for Engineers, (McGraw-Hill Book Company, Inc., New York, 1961).

function is $T(s)$, then

$$T(s) = N/D;$$

N and D, the numerator and denominator of the transfer function, will be polynomials. Their characteristics are fixed and known for physical networks.

Synthesis Considerations

For lossless one-port networks, the immittance functions must have the following characteristics:⁴

1. All transfer functions are quotients of polynomials in s with a constant multiplier.
2. The order of the numerator and denominator polynomials never differs by more than unity.
3. Polynomials will have only even or odd powers of s . Whatever the numerator turns out to be, the denominator will be the converse.
4. In a polynomial with even powers of s , no term in the sequence may be missing. The same is true of odd powers.
5. The zeros and poles of an immittance function are interleaved on an s -plane plot.
6. The function has either a pole or a zero at the origin, and the same is true at infinity.

One-port networks with resistance, inductance, and capacitive elements included must meet the following specifications.⁵

1. $T(s)$ is positive-real
  - (a) $T(s)$ is real for all real values of s
  - (b) $T(s)$ has a positive real part for values of s with a positive real part
2. $T(s)$ is rational
3. Zeros and poles of $T(s)$, if not real, occur in conjugate pairs
4. The coefficients of the various powers of s in $T(s)$ are real numbers
5. No pole or zero of a positive-real $T(s)$ can lie in the right-half plane; they are confined to the left-half plane and the imaginary axis.

---

⁴ M. E. Van Valkenburg, Network Analysis, (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1955), Section on network functions.

⁵ D. F. Tuttle, Network Synthesis, (John Wiley & Sons, Inc., New York, 1959), Vol. 1, Section on lossless networks.

6. Imaginary axis zeros and poles are simple
7. At infinity $T(s)$ must have a simple pole, a simple zero, or a regular point at which it does not vanish
8. The degree of the numerator and denominator of an immittance function can differ at most by unity
9. The minimum number of elements required for a realization of an immittance function is equal to one more than the number of internal critical frequencies
10. The minimum number of elements required for a realization of an immittance function is equal to
  - (a) total number of zeros
  - (b) total number of poles
  - (c) the order of the function.

The function characteristics enumerated for one-port networks are natural results of the positive realness which realizable functions must exhibit. These are described in detail in standard references such as those cited in this report.

After obtaining the $T(s)$, one must examine it with respect to these characteristics, or subject $T(s)$ to one of the standard tests for positive-reality. The third alternative involves a brief inspection and an attempt to realize $T(s)$ by using one of the standard techniques such as Foster or Cauer. The procedure in this report is essentially to put one in a position to realize a physical network, but some knowledge of well-known synthesis procedures is assumed.

Example. The example presented is a classic problem and the answer is apparent by inspection. However, it is ideal for displaying a typical approach and the form of the equations which result.

Presume that a square wave of voltage is applied to a one-port network and it is required that a triangular wave of current will flow. Specify the network.

The trigonometric Fourier expansion for $v(t)$ is:

$$v(t) = \frac{4V}{\pi} \left(\cos \omega_1 t - \frac{1}{3} \cos \omega_3 t \right).$$

Now convert to the exponential form:

$$v(t) = \frac{-2V}{3\pi} e^{-j\omega_3 t} - \frac{2V}{\pi} e^{-j\omega_1 t} + \frac{2V}{\pi} e^{j\omega_1 t} - \frac{2V}{3\pi} e^{j\omega_3 t}.$$

Transform $v(t)$:

$$V(s) = \mathcal{L}[v(t)] = \frac{-2V}{3\pi(s + j\omega_3)} + \frac{2V}{\pi(s + j\omega_1)} + \frac{2V}{\pi(s - j\omega_1)} - \frac{2V}{3\pi(s - j\omega_3)}$$

$$V(s) = \frac{-4Vs(s^2 + \omega_1^2) + 12Vs(s^2 + \omega_3^2)}{3\pi(s^2 + \omega_1^2)(s^2 + \omega_3^2)}.$$

Perform the same operation on $i(t)$:

$$i(t) = \frac{4I}{\pi} \left(\sin \omega_1 t - \frac{1}{9} \sin 3\omega_1 t \right).$$

Note that, for a first approximation of the required network, only two terms of the Fourier expansion are used:

$$i(t) = -\frac{j2I}{9\pi} e^{-j\omega_3 t} + \frac{j2I}{\pi} e^{-j\omega_1 t} - \frac{j2I}{\pi} e^{j\omega_1 t} + \frac{j2I}{9\pi} e^{j\omega_3 t},$$

$$I(s) = \frac{-j2I}{9\pi(s + j\omega_3)} + \frac{j2I}{\pi(s + j\omega_1)} - \frac{j2I}{\pi(s - j\omega_1)} + \frac{j2I}{9\pi(s - j\omega_3)},$$

$$I(s) = \frac{(-12 I \omega_1) \left[(s^2 + \omega_1^2) - 3(s^2 + \omega_3^2) \right]}{9\pi(s^2 + \omega_1^2)(s^2 + \omega_3^2)},$$

$$Z(s) = \frac{V(s)}{I(s)} = \frac{\frac{-4Vs(s^2 + \omega_1^2) + 12Vs(s^2 + \omega_3^2)}{3\pi(s^2 + \omega_1^2)(s^2 + \omega_3^2)}}{\frac{(-12 I \omega_1) \left[(s^2 + \omega_1^2) - 3(s^2 + \omega_3^2) \right]}{9\pi(s^2 + \omega_1^2)(s^2 + \omega_3^2)}}$$

$$Z(s) = \frac{V}{I \omega_1} s.$$

Therefore, the required one-port network is an inductance of magnitude $V/I\omega_1$.

A PULSE SYNTHESIS TECHNIQUE⁶

The discussion thus far has dealt with periodic functions. Periodicity is a requirement for Fourier expansion. Since a single pulse and its response may be important, the two requirements must be reconciled. This may be done easily in the following way:

suppose a desired $f(t)$ is as shown,

then $f(t)$ may be represented periodically as

$$f(t) = f_1(t) - f_1\left(t - \frac{T}{2}\right) \text{ or}$$

$$f(t) = f_2(t) + f_2\left(t - \frac{T}{2}\right),$$

where $f_1(t)$ and $f_2(t)$ are defined as even and odd repetitions of $f(t)$.

This representation, however, introduces exponentials in the transformed expression and the difficulty previously discussed must be handled.

⁶ E. A. Guillemin, Synthesis of Passive Networks, (John Wiley & Sons, Inc., New York, 1957), Section on time-domain synthesis.

Using the above pulse representation, it is possible to avoid this by a more complex manipulation.

Let

$$\mathcal{L}[f_1(t)] = h_1(s)$$

$$\mathcal{L}[f_2(t)] = h_2(s).$$

Assume a circuit N_1 to provide $h_1(s)$ and a circuit N_2 for $h_2(s)$. This means then that a unit voltage impulse applied to N_1 would produce a current of the form $f_1(t)$. Likewise, the same voltage applied to N_2 gives a current of the form $f_2(t)$.

It follows then that if a unit impulse of voltage were applied to N_1 and N_2 in series, the current would be of the form $f(t)$ (modified by a constant factor of 2).

Repeating $f(t)$ at half-period intervals forms a periodic function $f_p(t)$. The exponential Fourier series would be of the form

$$f_p(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega t}.$$

Transforming:

$$h_p(s) = \sum_{n=-\infty}^{\infty} \frac{a_n}{s - jn\omega};$$

$h_1(s)$ and $h_2(s)$ may now be formed from $h_p(s)$.

$$h_1(s) = h_p(s) (1 + e^{-sT/2}) = \sum_{n=-\infty}^{\infty} \frac{a_n (1 + e^{-jn\pi})}{s - jn\omega},$$

$$h_2(s) = h_p(s) (1 - e^{-sT/2}) = \sum_{n=-\infty}^{\infty} \frac{a_n (1 - e^{-jn\pi})}{s - jn\omega},$$

$$\begin{aligned} 1 + e^{-jn\pi} &= 2 \text{ for even } n \\ &= 0 \text{ for odd } n \end{aligned}$$

$$\begin{aligned} 1 - e^{-jn\pi} &= 2 \text{ for odd } n \\ &= 0 \text{ for even } n. \end{aligned}$$

Therefore:

$$h_1(s) = 2 \sum_{n=0 \pm (2, 4, \dots)}^{\infty} \frac{a_n}{s - jn\omega} = \frac{p_1(s)}{q_1(s)},$$

$$h_2(s) = 2 \sum_{n=\pm (1, 3, \dots)}^{\infty} \frac{a_n}{s - jn\omega} = \frac{p_2(s)}{q_2(s)}.$$

Since a current $f_1(t)$ applied to N_1 or $f_2(t)$ applied to N_2 produces a unit voltage impulse, the $Z(s)$ for N_1 and N_2 must be $1/h_1(s)$ and $1/h_2(s)$. The admittance of the series combination must be

$$h(s) = \frac{h_1(s) h_2(s)}{h_1(s) + h_2(s)},$$

or

$$h(s) = \frac{p_1(s) p_2(s)}{p_1(s) q_2(s) + p_2(s) q_1(s)}.$$

Example. Assume a single pulse of current, $I(t)$, as shown, is the response of a network when a step function of voltage $v(t)$ is applied. Determine the network

Represent $I(t)$ as $I_p(t)$, a wave which repeats $I(t)$ at half-period intervals. The coefficients of the exponential Fourier expansion of $I_p(t)$ are determined by

$$A_n = \frac{1}{2\pi} \int_0^{\pi} \left[-\frac{2I}{\pi} \omega_1 t + I \right] e^{-jn\omega_1 t} d\omega_1(t),$$

$$A_n = \frac{2I}{n\pi} \quad \text{for } n = \pm(1, 3, \dots),$$

$$A_n = \frac{-jI}{n\pi} \quad \text{for } n = 0 \pm(2, 4, \dots).$$

The general form of $I_p(t)$ is

$$I_p(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_1 t}.$$

$$h_p(s) = \sum_{n=-\infty}^{\infty} \frac{A_n}{s - jn\omega_1},$$

$$h_1(s) = 2 \sum_{n=0 \pm(2, 4, \dots)}^{\infty} \frac{A_n}{s - jn\omega_1} = \sum_{n=0, 2, 4, \dots}^{\infty} \frac{2n\omega_1 |A_n|}{(s^2 + n^2\omega_1^2)},$$

$$h_2(s) = 2 \sum_{n=\pm(1, 3, \dots)}^{\infty} \frac{A_n}{s - jn\omega_1} = \sum_{n=1, 3, \dots}^{\infty} \frac{2n\omega_1 |A_n|}{(s^2 + n^2\omega_1^2)}.$$

The transform of the voltage pulse may be given in the form

$$b(s) \frac{V}{s}.$$

The admittances of the networks N_1 and N_2 are now determined by modifying $h_1(s)$ and $h_2(s)$ by the excitation function and forming a general admittance term

$$Y(s) = \frac{\frac{2n\omega_1}{V} |A_n| s}{s^2 + n^2\omega_1^2} \quad n = 0, 1, 2, 3, \dots$$

By inspection it is seen that this equation describes networks of the following forms:

CONCLUSIONS

In this report general methods have been outlined whereby a circuit designer with minimum network synthesis experience may obtain useful circuits for particular specifications. The emphasis has been on providing sufficient information so that, to some extent, an informed "cut and try" technique may be utilized.

It is hoped that with the alternatives presented and the exercise of some ingenuity a wide variety of problems of this nature may be handled.

APPENDIX 1

Graphical Fourier Analysis

Sometimes it is necessary to obtain a Fourier expansion for a wave form for which no analytic function is known. A graphical or numerical technique may be used to obtain the coefficients of the Fourier series.⁷

Assume the wave to be analyzed is the following periodic form:

The formal method of evaluating b_1 , the coefficient of the fundamental frequency sine component, is to evaluate

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin x \, dx.$$

With half-wave symmetry the equation becomes

$$b_1 = \frac{2}{\pi} \int_0^{\pi} y \sin x \, dx.$$

In the above figure the curve $y \sin x$ is shown, and the desired integral is the area under this curve. Divide the half-period into k parts forming rectangles whose area is $(y \sin x) \pi/k$. It follows then that

⁷ R. M. Kerchner and G. F. Corcoran, Alternating-Current Circuits, (John Wiley & Sons, Inc., New York, 1951), Section on non-sinusoidal waves.

$$b_1 \approx \frac{2}{\pi} \sum (y \sin x) \frac{\pi}{k}$$

or
$$b_1 \approx \frac{2}{k} \sum y \sin x.$$

Therefore, if the numerical values of $y \sin x$ are known at the k specified points along the base line of the wave, b_1 may be found by summing the values and multiplying by $2/k$. This method continues until all coefficients have been evaluated. The technique is straightforward, however, the bookkeeping problem may be tedious unless a carefully designed table is used.

The following example shows the first few entries for the expansion of the wave form of the exciting current of a transformer taken from an oscillogram.

Fundamental

$\sin x$	Products $y \sin x$ (+) (-)	Ordinate Number	Angle x to Ordinate	Measure Ordinate y	$\cos x$	Products $y \cos x$ (+) (-)
0.0872	0.5	1	5°	5.9	0.9962	5.9
0.1736	1.7	2	10°	10.0	0.9848	9.8
0.2588	3.5	3	15°	13.4	0.9659	13.0
0.3420	5.6	4	20°	16.4	0.9397	15.4
0.4226	8.2	5	25°	19.4	0.9063	17.6

$$a_1 = 2 \frac{\text{Sum of products } (y \sin x)}{\text{Number of ordinates}}$$

$$b_1 = 2 \frac{\text{Sum of products } (y \cos x)}{\text{Number of ordinates}}$$

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.