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Stability of High β Large Aspect Ratio Tokamaks

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Abstract

High β ($\beta \gg \epsilon/q^2$) large aspect ratio ($\epsilon \ll 1$) tokamak equilibria are shown to be always stable to ideal M.H.D. modes that are localized about a flux surface. Both the ballooning and interchange modes are shown to be stable. This work uses the analytic high β large aspect ratio tokamak equilibria developed by Cowley et.al. [Phys. Fluids B, **3**, 2069(1991)], which are valid for arbitrary pressure and safety factor profiles. The stability results make no assumption about these profiles or the shape of the boundary.

I. Introduction

It has long been recognized that it may be desirable to operate tokamaks at high β since raising β raises the amount of fusion power that a given magnetic field can produce. Of course, it is not necessarily advantageous. If raising β lowers the confinement time, for instance, then ignition may be harder to achieve at high β than at low β . It is clearly appropriate to study high β tokamaks to ascertain whether high β is desirable or even possible.

In a recent paper,¹ hereafter referred to as CK^3 , we constructed analytically the equilibria of large aspect ratio ($\epsilon \ll 1$, see Fig. 1.) high β ($\beta \gg \epsilon/q^2$) tokamaks. Our solution is valid for arbitrary boundary shape, pressure profile, $p(\psi)$, and safety factor profile, $q(\psi)$. In Sec. II we summarize the equilibrium. The reader is referred to the paper¹ for details and discussion of the equilibrium properties. The obvious next question is, are these equilibria stable? If they are unstable, one wishes to know if the instabilities could terminate the discharge or merely enhance the transport. Ideal Magnetohydrodynamic instabilities are thought to be capable of terminating the discharge. We divide these modes into two types: the ballooning and interchange modes which, are localized about a particular magnetic surface; and the kink modes which are global, i.e., fill a region comparable to the size of the plasma.

In this paper we show that ballooning modes (for $1 \gg \beta \gg \epsilon/q^2$) and interchange modes (for $\beta \gg \epsilon/q^2$) are *stable in the CK^3 equilibria*. These results are independent

of the pressure and safety factor profiles and the boundary shape. The global M.H.D. instabilities, the kink modes, are not considered here.

The stability of localized M.H.D. modes at high β is not entirely unexpected. It has been known for over a decade² that in model equilibria the ballooning mode is stable below some critical β , the so called “first stability region”, and stable above some higher critical β , the “second stability region.” Subsequent analysis of numerically generated equilibria confirmed the existence of both stability regions and demonstrated how shaping (bean shaping) can remove the unstable region altogether.³ This paper demonstrates, for the first time, the general existence of the second stability region of ballooning modes in large aspect ratio tokamaks.

One of the commonly used models for analyzing ballooning mode stability is the $s - \alpha$ model.⁴ This model considers the stability of a circular flux surface on which the poloidal field is assumed to be constant. Although the model is a local solution of the Grad Shafranov equation, it is not necessarily part of a global equilibrium and indeed it is not part of the equilibria considered in CK^3 . The stability boundaries are plotted as functions of the global shear, s ($s = d \ln q / d\psi$), and the normalized pressure gradient, α ($\alpha = -2p' R q^2 / B^2$), of the circular surface. The first and second regions of stability are obtained.⁵ Greene and Chance⁶ generalized this to local (in ψ) variations in shear and pressure gradient in numerically generated low β equilibria. Again, stability boundaries are plotted as functions of s and α . In CK^3 the safety factor, q , and global shear are treated as finite quantities — i.e.,

they scale asymptotically as ϵ^0 as $\epsilon \rightarrow 0$. Thus the CK^3 equilibria have large α and finite s . For large α and finite s the $s - \alpha$ model predicts stability in accordance with our result. Unfortunately we cannot address the accessibility of the second stability region from low β since the stability boundaries are expected to be at $\beta \sim \epsilon/q^2$, outside the region of validity of the equilibrium.

In Sec. II of this paper we summarize the equilibrium results. Interchange stability is demonstrated in Sec. III and Ballooning stability in Sec. IV. The conclusion is Sec. V.

II. Summary of the Equilibrium

In this section we summarize the relevant features of the equilibrium solution given in CK^3 . We shall not attempt to give the details or the derivation here. The magnetic field is given by the usual form,

$$\mathbf{B} = \frac{1}{R} \nabla \psi \times \mathbf{e}_\phi + \frac{F}{R} \mathbf{e}_\phi. \quad (1)$$

where $\mathbf{e}_\phi$ is a unit vector in the toroidal direction and R is the distance from the symmetry axis, see Fig. 1. The safety factor, $q(\psi)$, is given by

$$q(\psi) = \frac{F(\psi)}{2\pi} \oint \frac{dl_p}{R |\nabla \psi|} \quad (2)$$

where dl_p is a poloidal line element. The integration is performed along poloidal projections of surfaces of constant ψ . The equilibrium is taken to be bounded by a conducting wall of arbitrary shape but large aspect ratio, i.e., $R \gg a$. The geometry is given in Fig.1. Note that noncircular boundaries are considered in Sec. VI of CK^3 as a simple generalization of the circular case. We shall use the same normalizations as CK^3 . These normalizations

are chosen to reflect the ordering $\beta \sim O(1)$ and $q \sim O(1)$, where order refers to order in the inverse aspect ratio expansion. Thus we choose the following notation:

$$\bar{z} = z/a, \quad R = R_0(1 + \epsilon \bar{x}), \quad \epsilon = a/R_0, \quad \bar{\psi} = \psi/\psi_{max}, \quad d\bar{l}_p = dl_p/a.$$

where ψ_{max} is the value of ψ at the magnetic axis and we choose $\psi = 0$ at the boundary.

The scaled flux functions $\bar{p}(\bar{\psi})$, pressure, and $\bar{F}(\bar{\psi})$ are defined by

$$\bar{F} = \frac{a^2}{R_0\psi_{max}} F, \quad \bar{p} = \frac{\mu_0 a^4}{\psi_{max}^2} p. \quad (3)$$

Henceforth all equations in this paper will be expressed in scaled variables except where explicitly stated. We will omit the bars for convenience. In the asymptotic limit $\epsilon \rightarrow 0$, with $p, q \sim O(1)$, the solution of the Grad-Shafranov equation for $\psi(x, z)$ splits into two regions. The *Core Region* where $\psi = \psi(x)$ (i.e., ψ is independent of z), and the *Boundary Layer Region* (with width of order $\epsilon^{1/2}$) adjacent to the conducting wall where $\nabla\psi$ is $O(\epsilon^{-1/2})$ and normal to the wall. Typical flux surfaces (for $dp/d\psi > 0$) are shown in Fig. 2.

When $p' > 0$ the core solution, $\psi(x)$, is given implicitly by¹

$$2\pi \int_0^{\psi_0} \frac{q(\tilde{\psi}_0)}{\sqrt{2(C - p(\tilde{\psi}_0))}} d\tilde{\psi}_0 = \int_{-1}^x \ell(\tilde{x}) d\tilde{x} \quad (4)$$

where $\ell(x)$ is the vertical height of the vacuum vessel at x , see Fig. 1. The constant C is given, again implicitly, by

$$\int_0^1 \frac{q(\tilde{\psi}_0)}{\sqrt{C - p(\tilde{\psi}_0)}} d\tilde{\psi}_0 = A_0, \quad (5)$$

where A_0 is the area of the poloidal cross section of the vacuum vessel. We will assume that β is small enough that a solution to Eq. (5) exists.¹ When a solution to Eq. (5) does *not* exist we have a new kind of solution that we termed “hole” solutions. These solutions are described in CK^3 . The core solution puts the magnetic axis at $x \sim 1 - O(\epsilon^{1/2})$ for $p' > 0$. When $p' < 0$ the Shafranov shift is negative and the magnetic axis is at $x \sim -1 + O(\epsilon^{1/2})$.¹

The boundary layer solution yields¹

$$\frac{1}{2} \left(\frac{\partial \psi_0}{\partial s} \right)^2 = \frac{2}{\epsilon} \int_{\psi_{core}(x)}^{\psi_0} d\psi' (x(\psi') - x) \frac{dp}{d\psi'} \quad (6)$$

where s is the perpendicular distance from the conducting wall. The function $x_0(\psi')$ in Eq. (6) is the x position of the ψ' surface in the core and $\psi_{core}(x)$ is the core solution [i.e., a solution of Eq. (4)]. The boundary layer solution matches the core solution when $s(p'/\epsilon)^{1/2} \gg 1$. The equilibrium solution is valid when $\epsilon \ll 1$ and when the boundary layer is narrow compared to the minor radius; that is when $\beta \gg \epsilon/q^2$. One can integrate Eq. (6) to obtain $s = s(\psi, x)$,¹ however this is not necessary for the following stability analysis.

III. Interchange Stability Criterion

The interchange criterion for stability is commonly known as the Mercier criterion.^{7,8} It is derived by considering stability to perturbations that are confined to an arbitrarily narrow region around a rational surface. These perturbations are naturally interchange in form, i.e., the displacement of the plasma is nearly constant along the magnetic field line.

The criterion for instability is^{7,8}

$$\begin{aligned}
-\frac{1}{4}F(2\pi\frac{dq}{d\psi})^2(\frac{dp}{d\psi})^{-1} &> \frac{1}{\epsilon^4}F^3\frac{dp}{d\psi}\left[\oint\frac{dl_p R}{|\nabla\psi|^3}\oint\frac{dl_p R}{|\nabla\psi|^3}-\oint\frac{dl_p R^3}{|\nabla\psi|^3}\oint\frac{dl_p}{R|\nabla\psi|^3}\right]+ \\
\frac{1}{\epsilon^2}F\left[F^2\oint\frac{dl_p}{R|\nabla\psi|^3}\frac{d}{d\psi}\left(\oint\frac{dl_p R}{|\nabla\psi|}\right)-F\oint\frac{dl_p R}{|\nabla\psi|^3}\frac{d}{d\psi}\left(F\oint\frac{dl_p}{R|\nabla\psi|}\right)-\frac{dp}{d\psi}\oint\frac{dl_p R^3}{|\nabla\psi|^3}\oint\frac{dl_p}{R|\nabla\psi|}\right] \\
&+ \oint\frac{dl_p}{R|\nabla\psi|}\frac{d}{d\psi}\left(\oint\frac{dl_p R}{|\nabla\psi|}\right). \tag{7}
\end{aligned}$$

where we have used scaled variables (and omitted bars). We can reduce Eq. (7) to a simple expression using the equilibrium properties. First let us write

$$R = \hat{R}(\psi)(1 + \epsilon\delta R), \tag{8}$$

where $\hat{R} = 1 + \epsilon x_0(\psi)$ is the exact position of the ψ surface in the core. Thus δR is zero in the core and finite (and dependant on angle) in the boundary layer where $|\nabla\psi| \sim O(\epsilon^{-1/2})$. It is trivial to show that the $O(\epsilon^{-4})$ term in Eq. (7) reduces to an $O(\epsilon^{-1/2})$ term. The $O(\epsilon^{-2})$ term can be reduced to $O(\epsilon^{-1})$ with the help of the core equilibrium relation $\hat{R}^2(\psi)dp/d\psi + FdF/d\psi \sim O(\epsilon^2)$. Thus, with a little algebra, Eq. (7) reduces to

$$\epsilon > 2\left(\frac{dp}{d\psi}\frac{dx_0}{d\psi}\right)\left(\frac{1}{q}\frac{dq}{d\psi}\right)^{-2}. \tag{9}$$

Note that this is the criterion for instability. The boundary layer does not contribute to this expression. In these high β equilibria $(dp/d\psi)(dx/d\psi)$ is always positive. When $dp/d\psi$ is positive the magnetic axis is at $x = 1$ (thus $dx_0/d\psi > 0$). When $dp/d\psi$ is negative the magnetic axis is at $x = -1$ (a negative Shafranov shift) and thus $dx_0/d\psi < 0$. Consequently

the right hand side of Eq. (9) is always positive and in the limit $\epsilon \rightarrow 0$ larger than the left hand side. We therefore conclude that *these equilibria are always Mercier/interchange stable*. One can trace the physical origin of the stability to two effects. First, the field line spends most of its length in the core which is always the good curvature region (independent of the sign of p'). Second, the shear in the core provides stability. These effects contribute equally.

IV. Ballooning Mode Stability

In this section we examine the ballooning mode stability of large aspect ratio high β equilibria. Here we restrict β so that $\epsilon/q^2 \ll \beta \ll 1$ and $p' > 0$. These modes are high toroidal mode number instabilities. Specifically perturbed quantities have the dependence $e^{in\phi}$ on the toroidal angle ϕ and $n \gg 1$. In this limit the instability analysis reduces to the solution of a one dimensional eigenvalue problem along the field line. The derivation of this equation and the mathematical formalism is well known^{4,9} and we omit it here. We shall, however, describe briefly some aspects which help our intuition.

The perturbed displacement ξ can be constructed from a function $\hat{\xi}$ which is *not* periodic in the poloidal direction. This is achieved by adding $\hat{\xi}$ to itself periodically. Thus,^{4,9}

$$\xi(\theta, \psi, \phi, t) = \sum_{N=-\infty}^{\infty} \hat{\xi}(\theta + 2\pi N, \psi, t) e^{in\phi}. \quad (10)$$

Here θ is any periodic poloidal angle variable. The requirement that the stabilizing magnetic field compressional energy be finite (in a $1/n$ expansion) leads to a perpendicular

displacement of the form

$$\hat{\xi}_{\perp}(\theta, \psi, t)e^{in\phi} = \xi_0(\theta, \psi, t)\frac{\mathbf{B} \times \nabla S}{B^2}e^{inS} + \dots + O(1/n), \quad (11)$$

where $|\nabla \xi_0|/n \rightarrow 0$ as $n \rightarrow \infty$. For the purposes of this discussion we use unscaled variables; we return to scaled variables in Eq. (17). The field line bending energy is also finite if we demand

$$\mathbf{B} \cdot \nabla S = 0 \quad (12)$$

and $\mathbf{B} \cdot \nabla \xi_0 \sim O(1)$. Equation (12) and the condition that the ϕ variation is periodic, $R^2 \nabla \phi \cdot \nabla S = 1$, yield

$$\nabla S = \frac{\mathbf{B} \times \nabla \psi}{|\nabla \psi|^2} + I(\psi, \theta) \nabla \psi. \quad (13)$$

$I(\psi, \theta)$ is determined by taking the curl of Eq. (13). Thus,

$$-\mathbf{B} \cdot \nabla I = \frac{\mathbf{B} \times \nabla \psi}{|\nabla \psi|^2} \cdot \nabla \times \left(\frac{\mathbf{B} \times \nabla \psi}{|\nabla \psi|^2} \right). \quad (14)$$

We may integrate Eq. (14) using an arbitrary poloidal angle variable θ to obtain

$$I(\theta, \psi) = \int_{\theta_0(\psi)}^{\theta} \frac{\partial}{\partial \psi} \left(\frac{FJ}{R^2} \right) d\theta + \frac{FJ}{R^2} \frac{\nabla \psi \cdot \nabla \theta}{|\nabla \psi|^2} \Big|_{\theta_0}^{\theta}, \quad (15)$$

where $J = (\nabla \psi \cdot \nabla \theta \times \nabla \phi)^{-1}$ and θ_0 , the constant of integration, is an arbitrary function of ψ . Clearly $I = 0$ when $\theta = \theta_0$. The physical interpretation of these terms is fairly obvious.

Consider a surface of constant S . The surface is, of course, everywhere tangent to $\mathbf{B}$. Let us further consider one particular field line lying on S . If we move along this field line ∇S will rotate relative to the $\nabla \psi$ direction due to the shear. From Eq. (13) we note that the angle, λ , ∇S makes with the $\mathbf{B} \times \nabla \psi$ direction is given by $\tan \lambda = I|\nabla \psi|^3/|\mathbf{B} \times \nabla \psi|$, see Fig. 3.

The righthand side of Eq. (14) is called the local shear,⁶ since it causes ∇S to rotate as one moves along field lines. The global shear, $2\pi q'(\psi)$, is just equal to $I(\theta, \psi) - I(\theta + 2\pi, \psi) -$ i.e., the integral of the local shear over one period. Note that when $\theta = \theta_0$, the displacement is entirely in the $\nabla\psi$ direction.

In Fig. 3 we illustrate the displacement and ∇S in a plane normal to one field line. The twisting mode structure was first discussed by Roberts and Taylor¹⁰ in a slab. They termed it a twisting slice. The radial localization can be found in higher order in the $1/n$ expansion — we shall not be concerned with this here.

Pressure differences across the mode (in the ∇S direction) are balanced by small amounts of magnetic compression. We take an incompressible perturbation, then the perturbed force along the field line is zero. One can show from the energy principle for ideal M.H.D.¹¹ that the stability boundaries for compressible and incompressible perturbations are the same. Curvature and field line bending forces compete in the direction of ξ yielding the stability equation

$$\mathbf{B} \cdot \nabla \left(\frac{|\nabla S|^2}{B^2} \mathbf{B} \cdot \nabla \xi_0 \right) - 2 \frac{dp}{d\psi} \frac{\kappa \cdot \mathbf{B} \times \nabla S}{B^2} \xi_0 = - \frac{\rho \omega^2}{B^2} |\nabla S|^2 \xi_0 \quad (16)$$

where $\kappa = \mathbf{B} \cdot \nabla \mathbf{B} / B^2$ is the curvature and ω is the eigenvalue determining stability. Equation (16) is solved with the boundary conditions that $\xi_0 \rightarrow 0$ as θ (or equivalently the distance along the field line) goes to plus or minus infinity, so that the periodic sum in Eq. (10) converges. Note that θ_0 appears as a parameter in Eq. (16). *For modes for which $n \rightarrow \infty$ a necessary and sufficient condition for stability is that, for all θ_0 , all eigenvalues,*

ω , of Eq. (16) are real.^{4,9}

We rewrite Eq. (16) using the scalings Eq. (3) and $\mathbf{B} \cdot \nabla \equiv J^{-1} \partial / \partial \theta$. We shall also drop terms of order β . Thus, to lowest order, in scaled variables,

$$\begin{aligned} \frac{\epsilon}{J} \frac{\partial}{\partial \theta} \left(\left[\frac{1}{|\nabla \psi|^2} + |\nabla \psi|^2 \frac{I^2}{F^2} \right] \frac{1}{J} \frac{\partial \xi_0}{\partial \theta} \right) - 2 \frac{dp}{d\psi} \left[\frac{1}{|\nabla \psi|^2} \frac{\partial \psi}{\partial x} + \frac{I}{F} \hat{\mathbf{e}}_\phi \times \hat{\mathbf{x}} \cdot \nabla \psi \right] \xi_0 + \\ \omega^2 \left[\frac{1}{|\nabla \psi|^2} + |\nabla \psi|^2 \frac{I^2}{F^2} \right] \xi_0 = 0 \end{aligned} \quad (17)$$

where $\hat{\mathbf{x}}$ is the unit vector in the x (∇R) direction,

$$\frac{1}{J} = (\nabla \psi \cdot \nabla \theta \times \hat{\mathbf{e}}_\phi) \quad (18)$$

and Eq. (15)

$$\frac{I}{F} = \int_{\theta_0}^{\theta} \left(\frac{\partial J}{\partial \psi} \right) d\theta + J \left. \frac{\nabla \psi \cdot \nabla \theta}{|\nabla \psi|^2} \right|_{\theta_0}. \quad (19)$$

Again we have dropped the bars on the scaled variables. Only the toroidal field curvature enters Eq. (17). When $\beta \sim O(1)$, the poloidal field curvature in the boundary layer and the variation of F in the shear must be included. The first term in Eq. (17) is the force from bending the field by motions in the normal, $\nabla \psi$, direction. The second term, the one proportional to I^2 , is the force from bending the field line by motions in the surface (the $\mathbf{B} \times \nabla \psi$ direction). The curvature force in the normal direction is the third term and the fourth term is the curvature force in the surface the “geodesic curvature” term. The terms proportional to ω^2 represent inertia. The explicit ϵ in front of the field line bending terms indicates that field line bending is a weak stabilizing effect in the core region. However,

in the boundary layer (where $|\nabla\psi| \sim O(\epsilon^{-1/2})$) field line bending is competitive with curvature.

We shall consider the behavior of $\xi_0(\theta)$ in three regions on a flux surface. These regions are: the core region where $\nabla\psi = d\psi/dx\hat{x} \sim O(1)$; the boundary layer region where $\nabla\psi = d\psi/ds\hat{s} \sim O(\epsilon^{-1/2})$ ($\hat{s}$ is the unit vector normal to the wall); and the transition regions between the boundary layer and core where $\nabla\psi \sim (\partial\psi/\partial s)\hat{s} + (\partial\psi/\partial x)\hat{x} \sim O(1)$ (see Fig. 4). Let us consider these regions separately.

a. The Core Region

In the core region $\hat{y} \cdot \nabla\psi = 0$ and $p'd\psi/dx > 0$ and the curvature term is therefore stabilizing. We expect the amplitude to be small in the core since all terms are stabilizing. This is most easily shown via the quadratic form obtained from Eq. (17):

$$\begin{aligned} \int_{-\infty}^{\infty} d\theta J \left(\frac{\epsilon}{J^2} \left[\frac{1}{|\nabla\psi|^2} + |\nabla\psi|^2 \frac{I^2}{F^2} \right] \left(\frac{\partial\xi_0}{\partial\theta} \right)^2 + 2 \frac{dp}{d\psi} \left[\frac{1}{|\nabla\psi|^2} \frac{\partial\psi}{\partial x} + \frac{I}{F} \hat{e}_\phi \times \hat{x} \cdot \nabla\psi \right] \xi_0^2 \right) \\ = \omega^2 \int_{-\infty}^{\infty} d\theta J \left[\frac{1}{|\nabla\psi|^2} + |\nabla\psi|^2 \frac{I^2}{F^2} \right] \xi_0^2. \quad (20) \end{aligned}$$

Note that the curvature terms are the only possible destabilizing terms. The contribution from the stabilizing curvature term in the core is of order $p'(dx/d\psi)^2 \xi_{core}^2$, the possible destabilizing curvature terms in the boundary layer or transition regions are of order $\epsilon^{1/2} p' \xi_{boundarylayer}^2$ or $\epsilon^{1/2} p' \xi_{transitionregion}^2$ respectively. Thus ξ_0 in the core must be small compared to ξ_0 elsewhere otherwise the stabilizing contribution of the core dominates.

For $\omega^2 < 0$ Eq. (17) has two solutions in the core region: one that exponentially

grows as θ increases and one that decays as θ increases. The exponentiation distance is $\Delta\theta \sim O(\epsilon^{1/2})$. The solution can be obtained using a W.K.B. ansatz but its form is unimportant. Unstable modes must, therefore, decay into the core from the transition region. Thus we have established boundary conditions for the transition region solutions. Since the nonperiodic displacement $\hat{\xi}$ dies away exponentially (over a short distance) in the core region, the periodic summation in Eq. (10) is dominated by one term (i.e., the overlap of the nonperiodic modes is negligible).

b. The Boundary Layer Solutions

In the boundary layer, $\nabla\psi \sim (d\psi/ds)\hat{s} \sim O(\epsilon^{-1/2})$. We see from Eqs. (11) and (13) that the displacement in the surface is much larger than the displacement normal to the surface (except, of course, near where $I = 0$). Physically this arises because the shear is large in the boundary layer. The mode twists from the upright position (ξ parallel to $\nabla\psi$) at $\theta = \theta_0$ to almost horizontal ($\hat{\xi}$ perpendicular to $\nabla\psi$) at $\theta = \theta_0 \pm O(\epsilon^{1/2})$. Thus Eq. (17) is dominated (except in the small region where $\theta - \theta_0 \sim O(\epsilon^{1/2})$) by field line bending due to motion in the surface and the geodesic curvature term.

The analysis in this region is helped by a convenient cancellation. Let us write

$$\begin{aligned} & \frac{\epsilon}{J} \frac{\partial}{\partial\theta} \left[|\nabla\psi|^2 \frac{I^2}{F^2} \frac{1}{J} \frac{\partial\xi_0}{\partial\theta} \right] = \\ & \epsilon \frac{I}{F} \hat{s} \cdot \nabla\theta \times \hat{e}_\phi \frac{d\psi}{ds} \frac{\partial}{\partial\theta} \left[\left(\frac{d\psi}{ds} \right)^3 \hat{s} \cdot \nabla\theta \times \hat{e}_\phi \frac{\partial}{\partial\theta} \left(\frac{I\xi_0}{F} \right) \right] - \\ & \left(\epsilon \frac{I}{F} \hat{s} \cdot \nabla\theta \times \hat{e}_\phi \frac{d\psi}{ds} \frac{\partial}{\partial\theta} \left[\left(\frac{d\psi}{ds} \right)^3 \hat{s} \cdot \nabla\theta \times \hat{e}_\phi \frac{\partial}{\partial\theta} \left(\frac{I}{F} \right) \right] \right) \xi_0. \end{aligned} \quad (21)$$

We use

$$\frac{\partial}{\partial \theta} \left(\frac{I}{F} \right) = \frac{-1}{\hat{\mathbf{s}} \cdot \nabla \theta \times \hat{\mathbf{e}}_\phi} \left(\frac{d\psi}{ds} \right)^{-3} \frac{d^2\psi}{ds^2} \quad (22)$$

and the equilibrium relation,

$$\frac{d^2\psi}{ds^2} = \frac{2p'}{\epsilon} (x(\psi) - x), \quad (23)$$

to simplify the second term on the righthand side of Eq. (21). Thus

$$\begin{aligned} \epsilon \frac{I}{F} \hat{\mathbf{s}} \cdot \nabla \theta \times \hat{\mathbf{e}}_\phi \frac{d\psi}{ds} \frac{\partial}{\partial \theta} \left[\left(\frac{d\psi}{ds} \right)^3 \hat{\mathbf{s}} \cdot \nabla \theta \times \hat{\mathbf{e}}_\phi \frac{\partial}{\partial \theta} \left(\frac{I}{F} \right) \right] = \\ -2p' \frac{I}{F} \hat{\mathbf{s}} \cdot \hat{\mathbf{x}} \times \hat{\mathbf{e}}_\phi \frac{d\psi}{ds}. \end{aligned} \quad (24)$$

In the boundary layer, substitution of Eqs. (21) and (24) into Eq. (17) yields

$$\begin{aligned} \frac{\epsilon}{J} \frac{I}{F} \frac{\partial}{\partial \theta} \left[\left(\frac{d\psi}{ds} \right)^2 \frac{1}{J} \frac{\partial}{\partial \theta} \left(\frac{I \xi_0}{F} \right) \right] + \\ \frac{\epsilon}{J} \frac{\partial}{\partial \theta} \left[\left(\frac{d\psi}{ds} \right)^{-2} \frac{1}{J} \frac{\partial}{\partial \theta} \xi_0 \right] + \omega^2 \frac{I^2}{F^2} \left(\frac{d\psi}{ds} \right)^2 \xi_0 = 0. \end{aligned} \quad (25)$$

where the geodesic term has cancelled the term given by Eq. (24). In Eq. (25) we have dropped the normal curvature as it is $O(\epsilon^{1/2})$ compared to the first term in Eq. (25). We have, however, retained the small field line bending term from motion in the $\nabla \psi$ direction [the second term in Eq. (25)] since it is important near $\theta = \theta_0$ where $I = 0$.

Multiplying Eq. (25) by ξ_0 and integrating from θ_1 to θ_2 , where θ_1 and θ_2 are in the

boundary layer, we obtain

$$\begin{aligned} \frac{\epsilon}{2J} \left[\frac{I}{F} \left(\frac{d\psi}{ds} \right)^2 \frac{\partial}{\partial \theta} \left(\frac{I\xi_0}{F} \right)^2 + \left(\frac{d\psi}{ds} \right)^{-2} \frac{\partial}{\partial \theta} \xi_0^2 \right] \Big|_{\theta_1}^{\theta_2} = \\ \epsilon \int_{\theta_1}^{\theta_2} d\theta J \left[\left(\frac{d\psi}{ds} \right)^2 \frac{1}{J^2} \left(\frac{\partial}{\partial \theta} \left(\frac{I\xi_0}{F} \right) \right)^2 + \left(\frac{d\psi}{ds} \right)^{-2} \frac{1}{J^2} \left(\frac{\partial \xi_0}{\partial \theta} \right)^2 \right] - \\ \omega^2 \int_{\theta_1}^{\theta_2} d\theta J \frac{I^2}{F^2} \left(\frac{d\psi}{ds} \right)^2 \xi_0^2. \end{aligned} \quad (26)$$

If the mode were entirely in the boundary layer then it would be possible to find values of θ_1 and θ_2 such that the left hand side of Eq. (26) vanishes. Then it is clear from Eq. (26) that $\omega^2 > 0$, i.e., the mode is stable.

Since we have shown that it is impossible to find an unstable mode that is entirely confined to the boundary layer, we must consider modes which lie partially or entirely in the transition region. Let us, therefore, take θ_1 and θ_2 in the overlap region between the transition regions and the boundary layer region — θ_2 at the top and θ_1 at the bottom, see Fig. 4. Now we examine the solutions in the transition region. From these solutions we may evaluate the left hand side of Eq. (26).

The Transition Region

In the transition region $|\nabla\psi| \sim O(1)$ but, unlike the core, $\hat{\mathbf{y}} \cdot \nabla\psi \neq 0$. The flux surface is turning sharply out of the core into the boundary layer. The radius of curvature of the flux surface is $O(\epsilon^{1/2})$. Despite this small radius of curvature in the poloidal plane, the field line curvature is still dominated by the curvature of the toroidal field (by a factor of order

$O(\epsilon^{-1/2})$). Both normal and “geodesic” curvature terms are important in this region. As we shall see $x - x_0 \sim O(\epsilon^{1/2})$ in this region. In our equilibrium solution¹ the transition region was (correctly) treated as part of the boundary layer solution. The ballooning equation, Eq. (17) does not take the boundary layer form, Eq. (25), in this region. Therefore, we treat the transition region separately.

In this region we may approximate $\nabla\psi$ by

$$\nabla\psi = \hat{s} \sqrt{\frac{2p'}{\epsilon} \frac{d\psi}{dx}} (x - x_0(\psi)) + \hat{x} \frac{d\psi_{core}}{dx} \quad (27)$$

where $\psi_{core}(x)$ is the core solution [i.e., a solution of Eq. (4)] and $x_0(\psi)$ is the inverse function i.e., $x_0[\psi_{core}(x)] = x$. It is convenient to take $\theta = x$ as the angle variable. This choice is not appropriate in the boundary layer because $x = 1$ is a coordinate degeneracy. The coordinate t is defined by

$$t = \frac{1}{\psi'_0} \sqrt{\frac{2p'}{\epsilon} \frac{d\psi}{dx}} [x - x_0(\psi)]. \quad (28)$$

In the transition region $t \sim O(1)$, we pass to the core when $t \ll 1$ and to the boundary layer when $t \gg 1$. The ballooning equation, Eq. (17), after some algebra, reduces to

$$\begin{aligned} t \frac{d}{dt} \left\{ t [(t_0 + w - t)^2 + 1 - w^2] \frac{d\xi_0}{dt} \right\} - t_0(t_0 - t)\xi_0 \\ + w^2 [(t_0 + w - t)^2 + 1 - w^2] \xi_0 = 0 \end{aligned} \quad (29)$$

where $w = \hat{s} \cdot \hat{x}$ evaluated at $x = x_0$ (clearly $-1 < w < 1$) and when $t = t_0$, $\theta = \theta_0$ and $I = 0$. When θ_0 is in the transition region t_0 is finite. When θ_0 is in the boundary layer, $t_0 \rightarrow \infty$. As we have discussed (in Sec.IVa), Eq. (29) must be solved with the boundary

condition $\xi_0 \rightarrow 0$ as $t \rightarrow 0$, i.e., in the core. Let us suppose, in order to get a contradiction, that we have an unstable mode with $\omega^2 = -\gamma^2$. When $|t - t_0| \gg 1$ and $t \rightarrow \infty$, Eq. (29) has the solution

$$\xi_0 \sim \xi_1(\gamma) \frac{t^{\alpha_+}}{t - t_0} + \xi_2(\gamma) \frac{t^{\alpha_-}}{t - t_0} \quad (30)$$

where $\alpha_{\pm} = \pm\sqrt{1 + \gamma^2}$. The ratio of the constants $\xi_1(\gamma)$ and $\xi_2(\gamma)$ is determined by the integration through the region where $t \sim O(1)$. Note that the ξ_1 term dominates for large t . There are, of course, two transition regions: one at the top and one at the bottom with, in general, different values of w and ξ_1 . We use the solutions in Eq. (30) to evaluate the left hand side of Eq. (26). One may simply verify by substitution that unless $\xi_1(\gamma) = 0$, for both transition regions, the left hand side of Eq. (26) is stabilizing. Thus our assumption that there is an unstable mode is contradicted unless $\xi_1(\gamma) = 0$. The condition $\xi_1(\gamma) = 0$ implies that $\xi_0 \rightarrow 0$ as $t \rightarrow \infty$ in the transition region.

The preceding, rather involved, reasoning leads us to the following conclusion. Equation (26) with the boundary conditions $\xi_0 \rightarrow 0$ when $t \rightarrow 0$ and $t \rightarrow \infty$ defines an eigenvalue problem (ω is the eigenvalue) for each pair of parameters t_0 and w . Ballooning modes are stable if and only if all ω are real for *all* t_0 and w . Note that w essentially labels the surface.

We have tested Eq. (29) for stability numerically. Rather than solve the eigenvalue problem we use the standard¹² Sturmian oscillation theorem as follows. We solve Eq. (29) with $\omega = 0$ and the boundary condition $\xi_0 \rightarrow 0$ at $t = 0$. If this $\xi_0(t)$ crosses zero for any $t > 0$ then an unstable mode exists. One may show analytically that if either $t_0 < 0$ or

$w < 0$ the mode is stable.

For all values of t_0 and w we found no instability. We tested values of t_0 between zero and ten in steps of 0.1 and w between zero and one in steps of 0.1. Thus we have demonstrated stability in the transition region and completed the proof of stability on the whole surface, for all surfaces. We therefore conclude that *Ballooning modes are stable* in large aspect ratio equilibria when $\epsilon/q^2 \ll \beta \ll 1$. Our conclusion does not depend on the shape of the boundary or the q or p profiles.

Conclusion

The sole conclusion of this paper is that high β ($\beta \gg \epsilon/q^2$) large aspect ratio ($\epsilon \ll 1$) tokamaks are stable to localized M.H.D. modes — i.e., interchange and ballooning modes. This conclusion gives one hope that high β tokamaks will be stable to the most destructive instabilities. The stability to kink modes is, as yet, undetermined. Low β tokamaks can be stable to ideal M.H.D. However, they always seem to be unstable to drift type instabilities (trapped particle modes, η_i modes etc.). The effects that stabilize the ballooning mode — strong local shear and short connection length in the bad curvature region — also help stabilize the drift type modes. There are some indications of increased stability of these modes in the second stability region.¹³ It is, however, too early to draw any conclusions about improved confinement properties at high β . Resistive modes could also be problematic, although once again there are qualitative reasons to be optimistic, specifically the average good curvature that produces interchange stability is also strongly stabilizing for

resistive modes.¹⁴

Future work will address the global M.H.D. modes, kink modes, and aspects of the drift wave physics in the CK^3 equilibria.

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Figures

FIG. 1. Geometry of the equilibrium for an unusual boundary shape.

FIG. 2. Equilibrium flux surface, illustrating the core region and the boundary layer region.

FIG. 3. Motion in a plane perpendicular to a field line for the ballooning mode.

FIG. 4. The three regions of the ballooning mode calculation. The core region where $s \gg \epsilon^{1/2}$, $x_0(\psi) = x$, the boundary layer region where $s \ll 1$ and $|\nabla\psi| \sim O(\epsilon^{-1/2})$ and the transition regions where $x - x_0 \sim O(\epsilon^{1/2})$.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

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